

***S*p-brane accelerating cosmologies**

Viktor Baukh* and Alexander Zhuk†

Department of Theoretical Physics, Odessa National University, 2 Dvoryanskaya Street, Odessa 65026, Ukraine

(Received 7 February 2006; published 12 May 2006)

We investigate time dependent solutions (S-brane solutions) for product manifolds consisting of factor spaces where only one of them is non-Ricci-flat. Our model contains a minimally coupled free scalar field as a matter source. We discuss a possibility of generating late-time acceleration of the Universe. The analysis is performed in conformally related Brans-Dicke and Einstein frames. Dynamical behavior of our Universe is described by its scale factor. Since the scale factors of our Universe are described by different variables in both frames, they can have different dynamics. Indeed, we show that with our S-brane ansatz in the Brans-Dicke frame the stages of accelerating expansion exist for all types of the external space (flat, spherical, and hyperbolic). However, applying the same ansatz for the metric in the Einstein frame, we find that a model with flat external space and hyperbolic compactification of the internal space is the only one with the stage of the accelerating expansion. A scalar field can prevent this acceleration. It is shown that the case of hyperbolic external space in the Brans-Dicke frame is the only model which can satisfy experimental bounds for the fine-structure constant variations. We obtain a class of models where a pair of dynamical internal spaces have fixed total volume. This results in a fixed fine-structure constant. However, these models are unstable and external space is nonaccelerating.

DOI: [10.1103/PhysRevD.73.104016](https://doi.org/10.1103/PhysRevD.73.104016)

PACS numbers: 04.50.+h

I. INTRODUCTION

Recent astronomical observations abundantly evidence that our Universe underwent stages of accelerating expansion during its evolution. There are at least two such stages: early inflation and late-time acceleration. The latter began approximately at the redshift $z \sim 1$ and continues until now. Thus, the construction and investigation of models with stages of acceleration is one of the main challenges of the modern cosmology.

Among such models, the models originating from fundamental theories (e.g. string/M theory) are of the most of interest. For example, it was shown that some of the space-like brane (S-brane) solutions have a stage of the accelerating expansion. We remind the reader that in the D -dimensional manifold S_p -branes are time dependent solutions with $(p + 1)$ -dimensional Euclidean world volume and, apart from time, they have $(D - p - 2)$ -dimensional hyperbolic, flat, or spherical spaces as transverse/additional dimensions [1]:

$$ds_D^2 = -e^{2\gamma(\tau)}d\tau^2 + a_0^2(\tau)(dx_1^2 + \dots + dx_{p+1}^2) + a_1^2(\tau)d\Sigma_{(D-p-2),\sigma}^2 \quad (1.1)$$

where $\gamma(\tau)$ fixes the gauge of time, $a_0(\tau)$ and $a_1(\tau)$ are time dependent scale factors, and $\sigma = -1, 0, +1$ for hyperbolic, flat, or spherical spaces, respectively.¹

*Electronic address: bauch_vGR@ukr.net†Electronic address: zhuk@paco.net

Also at Department of Advanced Mathematics, Odessa National Academy of Telecommunication, 1 Kuznechnaya Street, Odessa 65069, Ukraine.

¹A slightly generalized ansatz where the $(D - p - 2)$ -dimensional transverse space consists of the k -dimensional hyperspace $\Sigma_{k,\sigma}$ and $(q - k)$ -dimensional Euclidean space was considered in [2]. Here, $D - p - 2 = k + q$.

Obviously, $p = 2$ if the branes describe our 3-dimensional space. These branes are known as SM2-branes if original theory is 11-dimensional M theory and SD2-branes in the case of 10-dimensional Dirichlet strings. For this choice of p , the evolution of our Universe is described by the scale factor a_0 . In general, the scale factor a_1 can also determine the behavior of our 3-dimensional Universe. Hence, $D - p - 2 = 3$ and we arrive at the SM6-brane in the case of M theory and the SD5-brane for the Dirichlet string. Usually, S_p -brane models include form fields (fluxes) and massless scalar fields (dilaton) as matter sources. If SD_p -branes are obtained by dimensional reduction of 11-dimensional M theory, then the dilaton is associated with the scale factor of a compactified 11th dimension.

Starting from [1], the S-brane solutions were also found, e.g., in Refs. [2–5]. It was quite natural to test these models for the accelerating expansion of our Universe. Really, it was shown in [6] that the SM2-brane and the SD2-brane have stages of the accelerating behavior. This result generalizes conclusions of [7] for models with hyperbolic compact internal spaces. Here, the cosmic acceleration (in the Einstein frame) is possible due to a negative curvature of the internal space that gives a positive contribution to an effective potential. This acceleration is not eternal but has a short period and the mechanism of such short acceleration was explained in [8]. It was indicated in [6] that the solution of [7] is the vacuum case (the zero flux limit) of the S-branes. It was natural to suppose that, if the acceleration takes place in the vacuum case, it may also happen in the presence of fluxes. Indeed, it was confirmed for the case of the compact hyperbolic internal space. Even more, it was found that periods of acceleration occur in the cases of flat and spherical internal spaces due to the positive contributions of fluxes to the effective potential.

It is worth noting that accelerating multidimensional cosmological models have been widely investigated for the last few years for different types of models. In general, such models can be divided into two main classes.² The first class consists of models where the internal spaces are stabilized and the acceleration is achieved due to a positive minimum of an effective potential which plays the role of a positive cosmological constant. The general method for stabilization of the internal spaces was proposed in [10] and numerous references can be found, e.g., in Refs. [11–13]. Models where both external (our) space and internal spaces undergo dynamical behavior constitute the second class of models. These models were considered, e.g., in [14–16] where a perfect fluid plays the role of a matter source and the cosmic acceleration happens in the Brans-Dicke frame. Obviously, the S-brane accelerating solutions belong to the second class of models. Along with Ref. [6] mentioned above, the accelerating S-brane cosmologies (in the Einstein frame) were obtained and investigated, e.g., in Refs. [17–21]. Accelerating solutions closely related to them were also found in Refs. [22,23] (see also the general discussion on inflationary cosmologies with the sum of exponential effective potentials in [24]; the complete classification of solutions for such models according to their late-time behavior is given in [25]). It should be noted that some of these solutions are not new ones but either rediscovered or written in different parametrizations (see corresponding comments in Refs. [3,23]). For example, the first vacuum solution for a product manifold [consisting of $(n - 1)$ Ricci-flat spaces and one Einstein space with non-zero constant curvature] was found in [26].³ This solution was generalized to the case of a massless scalar field in Refs. [28,29]. Obviously, solutions in Refs. [26,28,29] are the zero flux limit of the Sp -branes and the result of [7] is the special case of [29].⁴ Some of the solutions in [20,22] coincide with corresponding solutions in Refs. [26,28–30]. An elegant minisuperspace approach for the investigation of the product space manifolds consisting of Einstein spaces was proposed in [31]. Here, it was shown that the equations of motion have the most simple form in a harmonic time gauge⁵ because the minisuperspace metric is flat in this gauge. Even if the authors of the above-mentioned papers were not aware of it, they intuitively used this gauge to get exact solutions. New solutions also can be generated (from the known solutions) with the help of a topological splitting when the Einstein space with

nonzero curvature is split into a number of Einstein spaces of the same sign of the curvature (see Refs. [32,33]). This kind of solution was found, e.g., in Refs. [20,22].

Our paper is devoted to a model with the product of n Einstein spaces where all of them are Ricci-flat except for one with positive and negative curvature. We include a massless scalar field as a matter source. As we mentioned above, the general solutions for this model were found in our papers [28,29]. Here, all factor spaces are time dependent. Obviously, these solutions are the zero flux limit of the Sp -branes. The aim of the present investigations is twofold.

First, we give the detailed analysis for the accelerating behavior of the external (our) space. At this stage, both the Ricci-flat space and nonzero curvature space may play the role of our Universe. The investigation is conducted in the Einstein frame as well as the Brans-Dicke frame. The transition between these two frames is performed with the help of the conformal transformation of the metric of the external space-time. Such transformation does not destroy either the factorizable structure of the D -dimensional metric ansatz or the topology of factor spaces. However, scale factors of our Universe are described by different variables in the Brans-Dicke and Einstein frames. These variables are connected with each other via conformal transformation (see the Appendix). Moreover, synchronous times are also different in both frames. Obviously, these different scale factors may behave differently with corresponding synchronous times. It is precisely this interpretation we bear in mind when we write about the different behavior of our Universe in different frames. For example, we show that in the Brans-Dicke frame, stages of the accelerating expansion exist for all types of the external space (flat, spherical, and hyperbolic). However, in the Einstein frame, the model with flat external space and hyperbolic compactification of the internal space is the only one with the stage of the accelerating expansion, in agreement with the results of Refs. [7,20]. A new result here is that the scalar field can prevent the acceleration in the Einstein frame.

Second, we investigate the variation of the fine-structure constant in our model. It is well known that dynamical internal spaces result in the variations of the fundamental constants (see, e.g., Refs. [11,15] and references therein). For example, the fine-structure constant is inversely proportional to the volume of the internal space. However, there are strong experimental restrictions for the variations of the fundamental constants (see, e.g., [34]). Thus, any multidimensional cosmological models with time dependent internal spaces should be tested from this point of view. In our paper, we show that the examined models run into significant problems related to the too large variations of the fine-structure constant. The case of the hyperbolic external space in the Brans-Dicke frame is the only possibility to avoid this problem, if there is no other way to

²Apart from these models, interesting accelerating cosmologies following from nonlinear models were proposed in [9].

³The first quantum solutions as well as the Euclidian classical solution for this model in the presence of a massless minimally coupled scalar field were obtained in [27].

⁴However, the period of accelerating expansion was not singled out in [29].

⁵For Eq. (1.1), it reads $\gamma = (p + 1) \ln a_0 + (D - p - 2) \ln a_1$. In the harmonic time gauge, time satisfies the equation $\Delta[g]\tau = 0$ [31].

explain the constancy of the effective four-dimensional fundamental constants in multidimensional models. For example, we propose models with the hyperbolic or spherical external space and two Ricci-flat internal spaces where the total volume of the internal spaces is the constant. Here, the dynamical factors of the internal spaces mutually cancel each other in the total volume element. Thus, the effective fundamental constants remain indeed constant in spite of the dynamical behavior of the internal spaces. However, these models are unstable and the external space is nonaccelerating. Anyway, such models are of special interest because they indicate a possible way to avoid the fundamental constant variations in higher-dimensional theories.

The paper is structured as follows. In Sec. II, we explain the general setup of our model and present the exact solutions for a product manifold consisting of two factor spaces where only one of them is non-Ricci-flat. These solutions are carefully investigated in Sec. III (spherical factor space) and Sec. IV (hyperbolic factor space) for the purpose of the accelerating behavior of the external space. In Sec. V, we compare the rate of variations of the fine-structure constant in our accelerating models with the experimental bounds. In Sec. VI, we obtain and discuss a solution with three factor spaces where two dynamical internal spaces have the fixed total volume. The main results are summarized in the conclusion in Sec. VII.

II. THE MODEL AND SOLUTIONS

In this section we present our model and give a sketchy outline of the derivation of exact solutions. A more detailed description can be found in our papers [27–29].

We consider a cosmological model with a slightly generalized metric (1.1) in the form

$$g = -e^{2\gamma(\tau)} d\tau \otimes d\tau + \sum_{i=0}^{n-1} e^{2\beta^i(\tau)} g^{(i)}, \quad (2.1)$$

which is defined on a multidimensional manifold M with product topology

$$M = \mathbb{R} \times M_0 \times \dots \times M_{n-1}. \quad (2.2)$$

Let manifolds M_i be d_i -dimensional Einstein spaces with metric $g^{(i)}$, i.e.

$$R_{m_i n_i} [g^{(i)}] = \lambda^i g_{m_i n_i}^{(i)}, \quad m_i, n_i = 1, \dots, d_i \quad (2.3)$$

and

$$R[g^{(i)}] = \lambda^i d_i \equiv R_i. \quad (2.4)$$

In the case of constant curvature spaces, parameters λ^i are normalized as $\lambda^i = k_i(d_i - 1)$ with $k_i = \pm 1, 0$.

With total dimension $D = 1 + \sum_{i=0}^{n-1} d_i$, κ_D^2 a D -dimensional gravitational constant, φ a massless minimally coupled scalar field, and S_{YGH} the standard York-Gibbons-Hawking boundary term, we consider an action of

the form

$$S = \frac{1}{2\kappa_D^2} \int_M d^D x \sqrt{|g|} (R[g] - g^{MN} \partial_M \varphi \partial_N \varphi) + S_{\text{YGH}}. \quad (2.5)$$

This action encompasses the truncated bosonic sectors of various supergravity theories. For example, for $D = 11$ and in the absence of a scalar field, it represents the low energy limit of the M theory, and for $D = 10$, it relates to the 10-dimensional supergravity. However, for generality, we perform the analysis with arbitrary D in the presence of a scalar field, specifying the value of D only for illustration of particular examples. For our cosmological model, the scalar field is homogeneous and depends only on time.

We restrict our consideration to the case when only one of the spaces M_i is not Ricci-flat: $R_0 \neq 0$, $R_i = 0$, $i = 1, \dots, n - 1$. Taking into account the homogeneity of our model, the action S is reduced to the form

$$S = \mu \int L d\tau = \mu \int d\tau \left\{ \frac{1}{2} e^{-\gamma + \gamma_0} [G_{ij} \dot{\beta}^i \dot{\beta}^j + \dot{\varphi}^2] - e^{\gamma - \gamma_0} U \right\}, \quad (2.6)$$

where

$$U = -\frac{1}{2} e^{2\gamma_0} R_0 e^{-2\beta^0} \quad (2.7)$$

is the potential, $\gamma_0 = \sum_{i=0}^{n-1} d_i \beta^i$, $G_{ij} = d_i \delta_{ij} - d_i d_j$ ($i, j = 0, \dots, n - 1$) is the minisuperspace metric, $\mu = \prod_{i=0}^{n-1} V_i / \kappa^2$, and $V_i = \int_{M_i} d^{d_i} y (\det(g_{m_i n_i}^{(i)}))^{1/2}$ is the volume of M_i (modulo the scale factor of the internal space).

It can be easily seen that the Euler-Lagrange equations for the Lagrangian (2.6) as well as the constraint equation $\partial L / \partial \gamma = 0$ have the most simple form in the harmonic time gauge $\gamma = \gamma_0 = \sum_{i=0}^{n-1} d_i \beta^i$ [31]. The corresponding solutions can be found in [27–29]. For simplicity we consider a model with two factor spaces ($n = 2$). All our conclusions can be easily generalized to a model with $n > 2$ factor spaces. For the two-component cosmological model, the explicit expressions for the scale factors and scalar field as functions of harmonic time read

$$\begin{aligned} a_0(\tau) &= \exp(\beta^0(\tau)) = a_{(c)0} \exp\left(-\frac{\xi_1}{d_0 - 1} \tau\right) \times \frac{1}{g_{\pm}(\tau)}, \\ a_1(\tau) &= \exp(\beta^1(\tau)) = A_1 \exp\left(\frac{\xi_1}{d_1} \tau\right), \\ \varphi(\tau) &= p^2 \tau + q, \end{aligned} \quad (2.8)$$

where

$$g_{+} = \cosh^{1/(d_0-1)}(\xi_2 \tau), \quad (-\infty < \tau < +\infty), \quad (2.9)$$

for $R_0 > 0$ and

$$g_{-} = \sinh^{1/(d_0-1)}(\xi_2 |\tau|), \quad (|\tau| > 0), \quad (2.10)$$

for $R_0 < 0$. Here, $a_{(c)0} = A_0(2\varepsilon/|R_0|)^{1/2(d_0-1)}$, $\xi_1 = [d_1(d_0 - 1)/(D - 2)]^{1/2} p^1$, $\xi_2 = [(d_0 - 1)/d_0]^{1/2}(2\varepsilon)^{1/2}$, and $2\varepsilon = (p^1)^2 + (p^2)^2$. Parameters A_0 , A_1 , p^1 , p^2 , and q are the constants of integration and A_0 , A_1 satisfy the following constraint: $A_0^{d_0} A_1^{d_1} = A_0$. It was shown in [27] that p^1 and p^2 are the momenta in the minisuperspace (p^1 is related to the momenta of the scale factors and p^2 is responsible for the momentum of the scalar field) and ε plays the role of energy.

In what follows, we consider the case of positive ε and, without loss of generality, we choose $2\varepsilon = 1 \Rightarrow (p^1)^2 + (p^2)^2 = 1$. We also put $q = 0$. It is also convenient to consider the dimensionless analogs of the scale factors: $a_0(\tau) \rightarrow a_0(\tau)/a_{(c)0}$ and $a_1(\tau) \rightarrow a_1(\tau)/A_1$. This choice does not affect the results but simplifies the analysis. So, below we investigate these dimensionless scale factors denoting them by the same letters as the dimensional scale factors.

The solution (2.8) is written in the harmonic time gauge. The synchronous time gauge (in other words, the proper time gauge) corresponds to $\gamma = 0$. This choice takes place in the Brans-Dicke frame. In the Einstein frame the synchronous gauge is different. The relation between these gauges in different frames is presented in the Appendix and it depends on the choice of the external and internal spaces. In our analysis both M_0 and M_1 can play the role of the external and internal spaces.

The dynamical behavior of the factor spaces is characterized by the Hubble parameter

$$H_i(t) = \frac{\dot{a}_i(t)}{a_i(t)}, \quad i = 0, 1 \quad (2.11)$$

and the deceleration parameter

$$q_i(t) = -\frac{\ddot{a}_i(t)}{a_i(t)}, \quad i = 0, 1, \quad (2.12)$$

where the overdots denote the differentiation with respect to the synchronous time t which is connected with the harmonic time τ as follows:

$$dt = f(\tau)d\tau \Rightarrow t(\tau) = \int_{-\infty}^{\tau} f(\tau)d\tau, \quad (2.13)$$

where the function $f(\tau)$ is defined in accordance with Eqs. (A6) and (A7) and we fix the constant of integration in such a way that $t \rightarrow 0$ for $\tau \rightarrow -\infty$. In Eqs. (2.11), (2.12), and (2.13), the quantities a_i and t are related to both the Brans-Dicke and the Einstein frames and the exact form of $f(\tau)$ depends on the choice of the frame (in the Einstein frame it depends also on the choice of the external space). Since in our model $f(\tau) > 0$, the synchronous time $t(\tau)$ is a monotone increasing function of the harmonic time. The expressions for the parameters H_i and q_i can be rewritten with respect to the harmonic time:

$$H_i(t(\tau)) = \frac{1}{a_i} \frac{d}{dt} a_i = \frac{1}{f(\tau)a_i(\tau)} \frac{da_i(\tau)}{d\tau} \quad (2.14)$$

and

$$-q_i(t(\tau)) = \frac{1}{f^2(\tau)a_i(\tau)} \left(\frac{d^2 a_i(\tau)}{d\tau^2} - \frac{1}{f(\tau)} \frac{df(\tau)}{d\tau} \frac{da_i(\tau)}{d\tau} \right). \quad (2.15)$$

With the help of these equations we can get a qualitative picture of the dynamical behavior of the factor spaces in synchronous time via the solutions (2.8) in the harmonic time gauge. More detailed information can be found from the exact expressions for $a_i(t)$. To get this, we should calculate the integral (2.13) which provides the connection between harmonic and synchronous times. However, the function $f(\tau)$ is a transcendental function and the integral (2.13) is not expressed in elementary functions. Hence, we shall analyze Eqs. (2.14) and (2.15) and asymptotic expressions for $a_i(t)$ to get information about the dynamics of the factor spaces in synchronous time. To confirm our conclusions graphically, we shall use MATHEMATICA 5.0 to draw the dynamical behavior of $a_i(t)$ for a full range of time t (for a particular choice of parameters of the model).

III. SPHERICAL FACTOR SPACE

In this section we investigate models where the factor space M_0 has the positive curvature $R_0 > 0$. We split our consideration into two separate subsections where calculation will be done in the Brans-Dicke and Einstein frames, respectively.

A. Brans-Dicke frame

In the case of spherical space M_0 the scale factors (2.8) have the following asymptotic forms:

$$a_0(\tau)|_{\tau \rightarrow \pm\infty} \simeq 2^{1/(d_0-1)} \exp\left(-\frac{\xi_1 \pm \xi_2}{d_0 - 1} \tau\right), \quad (3.1)$$

$$a_1(\tau)|_{\tau \rightarrow \pm\infty} = \exp\left(\frac{\xi_1}{d_1} \tau\right), \quad (3.2)$$

where we use the condition $\xi_2 > 0$. It can be easily seen that the asymptotic behavior depends on signs of $\xi_1 \pm \xi_2$ and ξ_1 .

The comparison of Eqs. (2.13) and (A6) gives the expression for the function $f(\tau)$:

$$f(\tau) = f_{+BD}(\tau) = e^{\gamma_0} = a_0^{d_0} a_1^{d_1} = \frac{\exp(-\frac{\xi_1}{d_0-1}\tau)}{g_+^{d_0}(\tau)}, \quad (3.3)$$

$$\tau \in (-\infty, +\infty)$$

with the asymptotes

$$f_{+BD}(\tau)|_{\tau \rightarrow \pm\infty} \simeq 2^{d_0/d_0-1} \exp\left(-\frac{\xi_1 \pm d_0 \xi_2}{d_0 - 1} \tau\right). \quad (3.4)$$

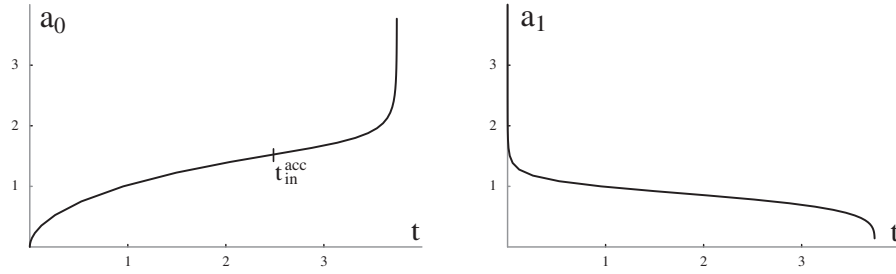


FIG. 1. Typical form of the external (left panel) and internal (right panel) scale factors in the Brans-Dicke frame in the case of the spherical external space M_0 . Specifically, it represents the zero flux limit of the SD5-brane with $d_0 = 3$, $d_1 = 6$, and $p^1 = -0.8$.

Thus, from Eq. (2.13) we obtain the asymptotic expression for the synchronous time

$$t - t_0|_{\tau \rightarrow +\infty} \simeq -2^{d_0/(d_0-1)} \frac{d_0 - 1}{\xi_1 + d_0 \xi_2} e^{-[(\xi_1 + d_0 \xi_2)/(d_0-1)]\tau}, \quad (3.5)$$

which enables us to rewrite the asymptotes (3.1) and (3.2) in the synchronous time gauge:

$$a_0(t)|_{t \rightarrow t_0} \simeq 2^{1/(d_0-1)} \left[\frac{(d_0 \xi_2 + \xi_1)(t_0 - t)}{2^{d_0/(d_0-1)}(d_0 - 1)} \right]^{(\xi_1 + \xi_2)/(\xi_1 + d_0 \xi_2)}, \quad (3.6)$$

$$a_1(t)|_{t \rightarrow t_0} \simeq \left[\frac{(d_0 \xi_2 + \xi_1)(t_0 - t)}{2^{d_0/(d_0-1)}(d_0 - 1)} \right]^{-[\xi_1(d_0-1)/d_1(\xi_1 + d_0 \xi_2)]}. \quad (3.7)$$

Additionally, it can be easily seen that conditions

$$-d_0 \xi_2 < \xi_1 < d_0 \xi_2 \quad (3.8)$$

provide the convergence of the integral (2.13) for any value of τ from the range $(-\infty, +\infty)$. Thus, the infinite range of τ is mapped onto the finite range of t . We remind the reader also that the synchronous time $t(\tau)$ is a monotone increasing function of the harmonic time.

Now, with the help of the expression (3.3) for $f_{+BD}(\tau)$, the Hubble and the deceleration parameters are easily obtained from Eqs. (2.14) and (2.15):

$$H_0 = -\frac{1}{f_{+BD}(\tau)} \frac{\xi_1 + \xi_2 \tanh(\xi_2 \tau)}{d_0 - 1}, \quad (3.9)$$

$$q_0 = \frac{\xi_2}{f_{+BD}^2(\tau)} \frac{\xi_2 + \xi_1 \tanh(\xi_2 \tau)}{d_0 - 1} \quad (3.10)$$

for the factor space M_0 and

$$H_1 = \frac{1}{f_{+BD}(\tau)} \frac{\xi_1}{d_1}, \quad (3.11)$$

$$-q_1 = \frac{\xi_1}{f_{+BD}^2(\tau)} \frac{(D-2)\xi_1 + d_0 d_1 \xi_2 \tanh(\xi_2 \tau)}{d_1^2(d_0 - 1)} \quad (3.12)$$

for the Ricci-flat factor space M_1 .

The following analysis depends on the choice of the external space. Therefore, we consider two separate cases.

1. Spherical external space (SM6- and SD5-branes)

As it was marked in the Introduction, solutions in this case describe the vacuum SM6-brane if $D = 11$, $d_0 = 3$ and scalar field is absent ($p^2 = 0 \rightarrow |p^1| = 1$) and the zero flux limit of the SD5-brane in the presence of scalar/dilaton field if $D = 10$, $d_0 = 3$ and $|p^1| \leq 1$.

Since we are looking for a solution with the dynamical compactification of the internal space M_1 , the parameter ξ_1 should be negative: $\xi_1 < 0 \rightarrow p^1 < 0$ [see Eq. (3.11)]. Then, Eqs. (3.9) and (3.10) show that the accelerating expansion of the external space M_0 takes place for harmonic times

$$\frac{\xi_2}{|\xi_1|} < \tanh(\xi_2 \tau) < \frac{|\xi_1|}{\xi_2} \quad (3.13)$$

which leads to the inequality

$$|\xi_1| > \xi_2 \Rightarrow \xi_1 < -\xi_2. \quad (3.14)$$

Additionally, it can be easily proven that the inequalities (3.8) are also valid for this case (the right inequality is obvious for negative ξ_1 and the left inequality follows from the condition $|p^1| \leq 1$). Therefore, the range of the synchronous time t is finite. Thus, with the help of the inequalities (3.8) and (3.14) we arrive at the following conclusions. First, from the asymptote (3.1) it follows that the factor space M_0 expands from zero ($\tau \rightarrow -\infty$) to infinity ($\tau \rightarrow +\infty$) and it occurs for the finite range of the synchronous time. This is the typical big rip scenario. At the same time, the internal space M_1 contracts from infinity to zero [see (3.2)]. Second, starting from the time $\tanh(\xi_2 \tau) = \xi_2/|\xi_1|$, the acceleration never stops, lasting until the big rip⁶ [because $|\tanh(\xi_2 \tau)| \leq 1 \forall \tau \in (-\infty, +\infty)$]. For example, the accelerating expansion of the M_0 at late synchronous times can be directly observed from (3.6) because $(\xi_1 +$

⁶In the case of a pure imaginary scalar field the parameters ξ_1 and ξ_2 can satisfy the inequality $\xi_1 + d_0 \xi_2 < 0$ because $|p^1| > 1$. Then, starting from the time $\tanh(\xi_2 \tau) = \xi_2/|\xi_1|$ the external space undergoes the eternal accelerating expansion. Here, the synchronous time t runs to $+\infty$.

$\xi_2) < 0$. The typical behavior of the scale factors of the external $[a_0(t)]$ and internal $[a_1(t)]$ factor spaces in the synchronous time gauge is illustrated in Fig. 1. $t_{\text{in}}^{\text{acc}}$ denotes the time of the beginning of the external space acceleration.

2. Ricci-flat external space (SM2- and SD2-branes)

Let us consider now the factor space M_1 as the external one. Solutions in this case describe the vacuum SM2-brane if $D = 11$, $d_1 = 3$, and $p^2 = 0$ ($|p^1| = 1$), and the zero flux limit of the SD2-brane if $D = 10$, $d_1 = 3$, and $|p^1| \leq 1$.

The demand of the external space M_1 expansion results in the positivity of the parameter ξ_1 [see Eq. (3.11)]: $\xi_1 > 0 \rightarrow 0 < p^1 \leq 1$. We remind the reader that the parameter ξ_2 is also positive. It is not difficult to verify that the inequalities (3.8) are also valid for the considered case. Thus, the infinite range $(-\infty, +\infty)$ of the harmonic time τ is mapped onto the finite range of the synchronous time t . According to Eq. (3.2), for this finite synchronous time, the external space M_1 expands from zero value to infinity. So, we have again the big rip scenario. The acceleration of the external space begins at the time

$$\tanh(\xi_2 \tau_a) = -\frac{D-2}{d_0 d_1} \frac{\xi_1}{\xi_2} = -\sqrt{\frac{D-2}{d_0 d_1}} p^1. \quad (3.15)$$

Starting from this time, the acceleration of M_1 never stops, lasting until the big rip. For example, the accelerating expansion of the M_1 at late synchronous times can be directly seen from (3.7) because of the negative sign of the exponent.

As it follows from the asymptote (3.1), concerning the internal factor space M_0 we have two different scenarios depending on the relation between ξ_1 and ξ_2 :

- (1) $\xi_1 > \xi_2 \Rightarrow [\frac{D-2}{d_0 d_1}]^{1/2} < p^1 \leq 1$.

Here, the internal space contracts from plus infinity to zero for a finite synchronous time. This scenario is realized e.g. for the case of the absence of a scalar field: $p^1 = 1$ (see Fig. 2, solid lines).

- (2) $0 < \xi_1 \leq \xi_2 \Rightarrow 0 < p^1 \leq [\frac{D-2}{d_0 d_1}]^{1/2}$.

In this case, the internal scale factor a_0 begins to

expand either from zero value for $\xi_1 < \xi_2$ or from the finite value $2^{1/(d_0-1)}$ for $\xi_1 = \xi_2$ until its turning point at a maximum [at the time $\tanh(\xi_2 \tau) = -\xi_1/\xi_2$; see Eq. (3.9)] and then contracts to zero value (see Fig. 2, dashed lines). Obviously, this scenario takes place in the presence of a scalar field because $p^1 < 1$.

B. Einstein frame

Now, we investigate the dynamical behavior of the corresponding S_p -branes in the Einstein frame. Similarly to the Brans-Dicke frame case, we perform our consideration for two separate cases depending on the choice of the external factor space.

1. Spherical external space (SM6- and SD5-branes)

In this case the conformal factor reads [see Eq. (A2)]

$$\Omega = a_1^{-d_1/(d_0-1)} = \exp\left(-\frac{\xi_1}{d_0-1}\tau\right). \quad (3.16)$$

Making use of Eqs. (A5) and (A7) we obtain the function $f(\tau)$,

$$f(\tau) = f_{+E(0)}(\tau) = \Omega^{-1} e^{\gamma_0} = [\cosh(\xi_2 \tau)]^{-d_0/(d_0-1)} \quad (3.17)$$

and the scale factor of the external space,

$$\tilde{a}_0(\tau) = \Omega^{-1} a_0 = g_+^{-1} = [\cosh(\xi_2 \tau)]^{-1/(d_0-1)}. \quad (3.18)$$

Substituting these expressions in Eqs. (2.14) and (2.15), we obtain the Hubble and the deceleration parameters

$$\tilde{H}_0(\tau) = -\frac{\xi_2}{(d_0-1)f_{+E(0)}(\tau)} \tanh(\xi_2 \tau), \quad (3.19)$$

$$\tilde{q}_0(\tau) = \frac{\xi_2^2}{(d_0-1)f_{+E(0)}^2(\tau)}. \quad (3.20)$$

Equations (3.19) and (3.20) clearly show that $\tilde{H}_0(\tau) < 0$ for positive τ and $\tilde{q}_0(\tau) > 0 \forall \tau \in (-\infty, +\infty)$. Therefore, the external factor space M_0 contracts at late times and never has the stage of the acceleration. Obviously, this

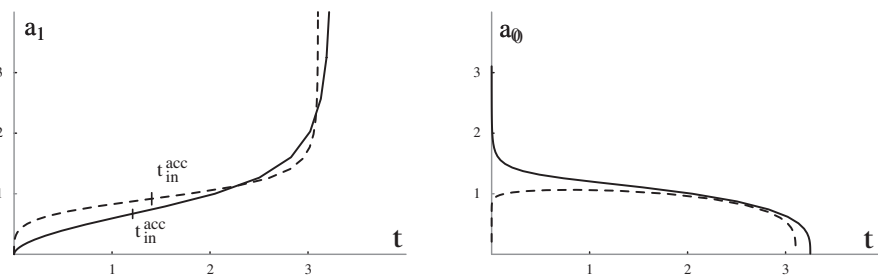


FIG. 2. Typical form of the external (left panel) and internal (right panel) scale factors in the Brans-Dicke frame for the Ricci-flat external space in the cases $\xi_1 > \xi_2$ (solid lines) and $\xi_1 \leq \xi_2$ (dashed lines). Specifically, it represents the vacuum limit of the SM2-brane with $d_0 = 7$, $d_1 = 3$, and $p^1 = 1$ (solid lines) and the zero flux limit of the SD2-brane with $d_0 = 6$, $d_1 = 3$, and $p^1 = 0.5$ (dashed lines).

model contradicts the observations. Here, the SM6-brane corresponds to the choice of $d_1 = 7$, and for the SD5-brane we should take $d_1 = 6$.

2. Ricci-flat external space (SM2- and SD2-branes)

Let the factor space M_1 be the external space. In this case the conformal factor is

$$f(\tau) = f_{+E(1)}(\tau) = \Omega^{-1} e^{\gamma_0} = e^{[(1-d_0-d_1)/(d_0-1)(d_1-1)]\xi_1\tau} [\cosh(\xi_2\tau)]^{-d_0 d_1 / (d_0-1)(d_1-1)}, \quad (3.22)$$

$$\tilde{a}_1(\tau) = \Omega^{-1} a_1 = e^{[(1-d_0-d_1)/d_1(d_0-1)(d_1-1)]\xi_1\tau} [\cosh(\xi_2\tau)]^{-d_0 / (d_0-1)(d_1-1)}. \quad (3.23)$$

Thus, the Hubble and the deceleration parameters of the external factor space M_1 read

$$\tilde{H}_1(\tau) = -\frac{1}{d_1(d_1-1)(d_0-1)f_{+E(1)}(\tau)} ((D-2)\xi_1 + d_1 d_0 \xi_2 \tanh(\xi_2\tau)), \quad (3.24)$$

$$\tilde{q}_1(\tau) = \frac{[(D-2)\xi_1 + d_0 d_1 \xi_2 \tanh(\xi_2\tau)]^2 + d_0(d_0-1)d_1^2 \xi_2^2 \cosh^{-2}(\xi_2\tau)}{d_1^2(d_1-1)(d_0-1)^2 f_{+E(1)}^2(\tau)}. \quad (3.25)$$

Therefore, the deceleration parameter $\tilde{q}_1(\tau) > 0 \forall \tau \in (-\infty, +\infty)$ and the external space M_1 does not undergo the acceleration. Similarly to the previous case, the external space M_1 contracts at late times [it follows from Eq. (3.24) and the condition $|p^1| \leq 1$]. Hence, this model is not of interest to us either. For this case, the SM2-brane corresponds to the choice of $d_0 = 7$ and for the SD2-brane we should take $d_0 = 6$.

IV. HYPERBOLIC FACTOR SPACE

In this section we investigate models where the factor space M_0 has the negative curvature $R_0 < 0$. If this factor space is treated as the internal one we suppose that M_0 is compact (see e.g. [35]). Similarly to the previous section, we split our consideration into two separate subsections where calculation will be done in the Brans-Dicke and Einstein frames, respectively.

A. Brans-Dicke frame

As apparent from Eqs. (2.8) and (2.10), the function $a_0(\tau)$ is divergent at $\tau = 0$. This point divides the range of τ into two separate parts: $(-\infty, 0]$ and $[0, +\infty)$. We choose the interval $(-\infty, 0]$ because the dynamical picture in both of these intervals is equivalent up to the replacement $p^1 \rightarrow -p^1$ [29].

To begin with, let us first define the function $f(\tau)$,

$$f(\tau) = f_{-BD}(\tau) = e^{\gamma_0} = a_0^{d_0} a_1^{d_1} = \frac{\exp(-\frac{\xi_1}{d_0-1}\tau)}{g^{d_0}(|\tau|)}, \quad (4.1)$$

$$\tau \in (-\infty, 0],$$

and its asymptotes,

$$f_{-BD}(\tau) \simeq \begin{cases} 2^{d_0/(d_0-1)} e^{[1/(d_0-1)](\xi_1-d_0\xi_2)|\tau|}, & \tau \rightarrow -\infty, \\ (\xi_2|\tau|)^{-d_0/(d_0-1)}, & \tau \rightarrow -0. \end{cases} \quad (4.2)$$

$$\Omega(\tau) = a_0^{-d_0/(d_1-1)} = e^{[d_0/(d_0-1)(d_1-1)]\xi_1\tau} [\cosh(\xi_2\tau)]^{d_0/(d_0-1)(d_1-1)} \quad (3.21)$$

and for the function $f(\tau)$ and the external scale factor we obtain the following expressions:

The first asymptote $f_{-BD}(\tau) \rightarrow 0$ in the limit $\tau \rightarrow -\infty$ because $(\xi_1 - d_0\xi_2) < 0^7$ and the second asymptote $f_{-BD}(\tau) \rightarrow +\infty$ in the limit $\tau \rightarrow -0$. Thus, it can be easily seen that the harmonic time interval $\tau \in (-\infty, 0]$ is mapped onto the synchronous time interval $t \in [0, +\infty)$ correspondingly. These asymptotes make it possible to connect the synchronous and harmonic times in the corresponding limits. For example, at late times we get the following relation:

$$\xi_2 t \simeq (d_0 - 1)(\xi_2|\tau|)^{-1/(d_0-1)}, \quad \tau \rightarrow -0 \Rightarrow t \rightarrow +\infty. \quad (4.3)$$

It is also useful to present the asymptotes for the scale factors. For the factor space M_0 we get

$$a_0(\tau)|_{\tau \rightarrow -\infty} \simeq 2^{1/(d_0-1)} \exp\left(\frac{\xi_1 - \xi_2}{d_0 - 1}|\tau|\right), \quad (4.4)$$

$$a_0(\tau)|_{\tau \rightarrow -0} \simeq (\xi_2|\tau|)^{-1/(d_0-1)} \rightarrow +\infty. \quad (4.5)$$

The first asymptote demonstrates that there are two different scenarios depending on the sign of the difference $\xi_1 - \xi_2$. If $\xi_1 > \xi_2$, the factor space M_0 begins to contract from plus infinity to a finite value and then to expand again to plus infinity [see (4.5)]. If $\xi_1 < \xi_2$, the factor space M_0 expands for all time, going from zero to infinity.⁸ The substitution of (4.3) into (4.5) shows that the Milne-type

⁷It is obvious for negative ξ_1 and also true for positive ξ_1 because of $|p^1| \leq 1$.

⁸In the exceptional case $\xi_1 = \xi_2 := \xi$, the scale factor a_0 reads $a_0(\tau) = [(1 - e^{-2\xi|\tau|})/2]^{-1/(d_0-1)}$. This formula shows that the scale factor starts from the finite value $(1/2)^{-1/(d_0-1)}$ and expands to infinity.

behavior of M_0 at late times is the attractor solution⁹ (see e.g. [29]):

$$a_0(t)|_{t \rightarrow +\infty} \simeq \frac{1}{d_0 - 1} \xi_2 t. \quad (4.6)$$

Concerning the factor space M_1 we have the following asymptotes:

$$a_1(\tau)|_{\tau \rightarrow -\infty} = \exp\left(\frac{\xi_1}{d_1} \tau\right), \quad (4.7)$$

$$a_1(\tau)|_{\tau \rightarrow 0} \rightarrow 1. \quad (4.8)$$

Here, we also have two scenarios depending on the sign of ξ_1 . If $\xi_1 > 0$, the factor space M_1 contracts from infinity with the subsequent freezing at late times. If $\xi_1 < 0$, the factor space M_1 expands from zero, freezing again at late times. Thus, the freezing of the factor space M_1 is the attractor behavior at late times (see [29]).

Let us define now the Hubble and the deceleration parameters. For the factor spaces M_0 and M_1 we obtain, respectively,

$$H_0 = -\frac{1}{f_{-BD}(\tau)} \frac{\xi_1 + \xi_2 \coth(\xi_2 \tau)}{d_0 - 1}, \quad (4.9)$$

$$q_0 = \frac{\xi_2}{f_{-BD}^2(\tau)} \frac{\xi_2 + \xi_1 \coth(\xi_2 \tau)}{d_0 - 1} \quad (4.10)$$

and

$$H_1 = \frac{1}{f_{-BD}(\tau)} \frac{\xi_1}{d_1}, \quad (4.11)$$

$$-q_1 = \frac{\xi_1}{f_{-BD}^2(\tau)} \frac{(D-2)\xi_1 + d_0 d_1 \xi_2 \coth(\xi_2 \tau)}{d_1^2 (d_0 - 1)}. \quad (4.12)$$

With the help of these expressions we can analyze the factor spaces from the point of their acceleration. Again, the analysis depends on the choice of the external space.

1. Hyperbolic external space (SM6- and SD5-branes)

Usually, we are looking for a model with expanding external space and a contracting (or static) internal one. As it follows from Eqs. (4.9) and (4.11), the choice $\xi_1 \leq 0$ guarantees these conditions. However, the external factor space is a decelerating one at all times because $q_0 > 0 \forall \tau \in (-\infty, 0]$ [see Eq. (4.10)]. Therefore, in the rest of this subsection we investigate the case of positive $\xi_1 > 0 \rightarrow p^1 > 0$ with expanding internal space. In spite of the expanding character of the internal space, Eq. (4.8) shows that this space goes asymptotically to a constant value

⁹It can be easily verified that the dimensional scale factor a_0 has the exact Milne asymptote, $a_0(t)|_{t \rightarrow +\infty} \simeq t$, and for dimensional a_1 we obtain $a_1(\tau)|_{\tau \rightarrow 0} \rightarrow A_1$.

(“frozen out”) at late times. We suppose that this value is less than the Fermi length $L_F \sim 10^{-17}$ cm. It makes the internal space unobservable at late times.

Obviously, for positive ξ_1 we have two scenarios:

- (1) $\xi_1 > \xi_2$.

Here, the external space M_0 after the contraction from infinity to a finite value starts to expand at the time

$$\coth(\xi_2 \tau_e) = -\frac{\xi_1}{\xi_2} = -\sqrt{\frac{d_0 d_1}{D-2}} p^1 \quad (4.13)$$

asymptotically approaching the attractor $a_0 \sim t(t \rightarrow +\infty)$. At all stages of its evolution the factor space M_0 has the accelerating behavior $q_0 < 0 \forall \tau \in (-\infty, 0]$. This scenario is realized e.g. for the case of the absence of a scalar field: $p^1 = 1$ (see Fig. 3, solid lines, where the convex curve a_0 has positive second derivative/acceleration for all $t \in [0, +\infty)$).

- (2) $0 < \xi_1 \leq \xi_2$.

Here, the external space M_0 expands for all time $\tau \in (-\infty, 0]$ starting from zero (for $\xi_1 < \xi_2$) or from a finite value (for $\xi_1 = \xi_2$), asymptotically approaching the attractor $a_0 \sim t(t \rightarrow +\infty)$. The acceleration begins at the time

$$\coth(\xi_2 \tau_a) = -\frac{\xi_2}{\xi_1} = -\sqrt{\frac{D-2}{d_0 d_1}} \frac{1}{p^1}. \quad (4.14)$$

This equation is satisfied for $p^1 < \sqrt{(D-2)/d_0 d_1} < 1$, i.e. in the presence of a sufficiently dynamical scalar field. The typical behavior of the scale factors in the synchronous time gauge for this type of scenario is illustrated in Fig. 3 (dashed lines).

It is worth noting that, to draw the graphics in synchronous time, we use in the integral (2.13) the exact expressions for the function $f(\tau)$ rather than its asymptotes. This can result in a proper shift between an analytic estimate (for the late times) and a graphical plotting. For example, the corresponding shift for the linear asymptote (4.6) has the form of $a_0(t)|_{t \rightarrow +\infty} \simeq \frac{1}{d_0 - 1} \xi_2 (t + t_0)$ where $t_0 = \lim_{\tau \rightarrow 0} \int_{-\infty}^{\tau} [f(\eta) - (\xi_2 |\eta|)^{-(d_0/d_0 - 1)}] d\eta$. Because the function $f(\tau)$ depends on the parameter ξ_1 , the solid and dashed lines in the left panel of Fig. 3 acquire the late-time relative shift with respect to each other.

2. Ricci-flat external space (SM2- and SD2-branes)

It can be easily seen from Eq. (4.11) that the external space M_1 expands only in the case $\xi_1 > 0 \rightarrow p^1 > 0$. Because $p^1 \leq 1 \Rightarrow [(D-2)/d_0 d_1](\xi_1/\xi_2) = [(D-2)/d_0 d_1]^{1/2} p^1 < 1$, the deceleration parameter of the external space $q_1 > 0$ for all times $\tau \in (-\infty, 0]$ [see Eq. (4.12)] and the acceleration is absent. Additionally, the internal space M_0 expands to infinity at late times

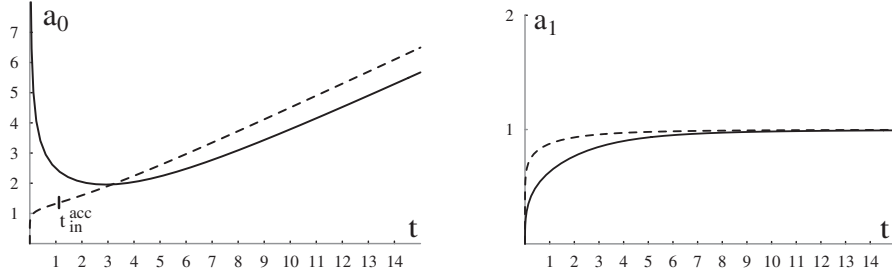


FIG. 3. Typical form of the external (left panel) and internal (right panel) scale factors in the Brans-Dicke frame for the hyperbolic external space M_0 in the case $\xi_1 > \xi_2$ (solid lines) and $\xi_1 < \xi_2$ (dashed lines). Specifically, it represents the vacuum limit of the SM6-brane with $d_0 = 3$, $d_1 = 7$, and $p^1 = 1$ (solid lines) and the zero flux limit of the SD5-brane with $d_0 = 3$, $d_1 = 6$, and $p^1 = 0.5$ (dashed lines).

which obviously contradicts the observations. Thus, this case is not of interest to us.

B. Einstein frame

Now, we investigate the dynamical behavior of the corresponding S_p -branes in the Einstein frame, splitting our consideration into two separate cases depending on the choice of the external factor space.

1. Hyperbolic external space (SM6- and SD5-branes)

In this case we obtain the following expressions:

$$\Omega = a_1^{-[d_1/(d_0-1)]} = \exp\left(-\frac{\xi_1}{d_0-1}\tau\right) \quad (4.15)$$

for the conformal factor,

$$f(\tau) = f_{-E(0)}(\tau) = \Omega^{-1}e^{\gamma_0} = [\sinh(\xi_2|\tau|)]^{-[d_0/(d_0-1)]} \quad (4.16)$$

for the function $f(\tau)$, and

$$\tilde{a}_0(\tau) = \Omega^{-1}a_0 = g^{-1} = [\sinh(\xi_2|\tau|)]^{-[1/(d_0-1)]} \quad (4.17)$$

for the scale factor of the external space. Here, we consider the interval $(-\infty, 0]$ of the harmonic time τ which is mapped onto the interval $[0, +\infty)$ of the synchronous time \tilde{t} . Thus, the Hubble and the deceleration parameters of the external factor space M_0 read

$$\tilde{H}_0 = \frac{\xi_2}{(d_0-1)f_{-E(0)}} \coth(\xi_2|\tau|), \quad (4.18)$$

$$\tilde{q}_0 = \frac{\xi_2^2}{(d_0-1)f_{-E(0)}^2}. \quad (4.19)$$

These equations clearly show that the expanding external space is a decelerating one because $\tilde{H}_0 > 0$, $\tilde{q}_0 > 0 \forall \tau \in (-\infty, 0]$.

2. Ricci-flat external space (SM2- and SD2-branes)

Now let the factor space M_1 be the external space. For this choice of the external space the conformal factor reads

$$\begin{aligned} \Omega(\tau) &= a_0^{-[d_0/(d_1-1)]} \\ &= \exp\left(\frac{d_0}{(d_0-1)(d_1-1)}\xi_1\tau\right) \\ &\quad \times [\sinh(\xi_2|\tau|)]^{d_0/(d_0-1)(d_1-1)}. \end{aligned} \quad (4.20)$$

With the help of this expression we can define the function $f(\tau)$,

$$\begin{aligned} f(\tau) &= f_{-E(1)}(\tau) = \Omega_1^{-1}e^{\gamma_0} \\ &= \exp\left(\frac{1-d_0-d_1}{(d_0-1)(d_1-1)}\xi_1\tau\right) \\ &\quad \times [\sinh(\xi_2|\tau|)]^{-[d_0d_1/(d_0-1)(d_1-1)]}, \end{aligned} \quad (4.21)$$

and the scale factor $\tilde{a}_1(\tau)$,

$$\begin{aligned} \tilde{a}_1(\tau) &= \Omega^{-1}a_1 \\ &= \exp\left(\frac{1-d_0-d_1}{d_1(d_0-1)(d_1-1)}\xi_1\tau\right) \\ &\quad \times [\sinh(\xi_2|\tau|)]^{-[d_0/(d_0-1)(d_1-1)]}. \end{aligned} \quad (4.22)$$

As for the internal space scale factor $a_0(\tau)$, it has the form (2.8) with the asymptotes (4.4) and (4.5).

Similarly to the previous case, we choose the interval $(-\infty, 0]$ of the harmonic time τ . It can be easily verified that this interval is mapped onto the interval $[0, +\infty)$ of the synchronous time \tilde{t} . It is of interest to get the late-time asymptotes for the scale factors. To get them, we first obtain the relation between the synchronous and harmonic times at late stages:

$$\begin{aligned} \xi_2\tilde{t} &= \frac{(d_0-1)(d_1-1)}{d_0+d_1-1}(\xi_2|\tau|)^{-[(d_0+d_1-1)/(d_0-1)(d_1-1)]}, \\ \tau \rightarrow -0 &\Rightarrow t \rightarrow +\infty, \end{aligned} \quad (4.23)$$

which enable us to write the late-time asymptotes in both gauges:

$$\begin{aligned} \tilde{a}_1 &\simeq (\xi_2|\tau|)^{-d_0/(d_0-1)(d_1-1)} \\ &\simeq \left[\frac{d_0+d_1-1}{(d_0-1)(d_1-1)}\xi_2\tilde{t}\right]^{d_0/(d_0+d_1-1)}, \end{aligned} \quad (4.24)$$

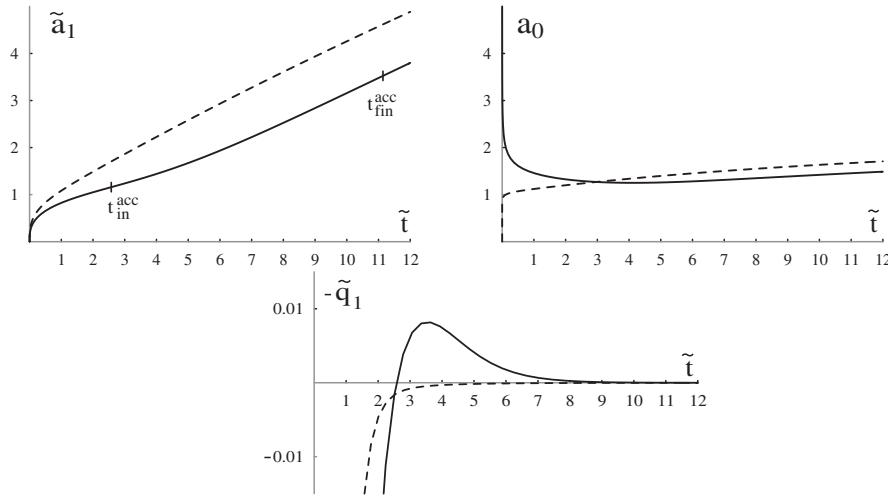


FIG. 4. Typical form of the external \tilde{a}_1 and internal a_0 scale factors as well as the deceleration parameter $-\tilde{q}_1$ of the external Ricci-flat factor space M_1 in the Einstein frame (synchronous time gauge). Specifically, the solid lines represent the vacuum limit of the accelerating SM2-brane with $d_0 = 7$, $d_1 = 3$, and $p^1 = 1$, and the dashed lines correspond to the zero flux limit of the decelerating SD2-brane with $d_0 = 6$, $d_1 = 3$, and $p^1 = 0.5$.

$$a_0 \simeq (\xi_2 |\tau|)^{-1/(d_0-1)} \simeq \left[\frac{d_0 + d_1 - 1}{(d_0 - 1)(d_1 - 1)} \xi_2 \tilde{t} \right]^{(d_1-1)/(d_0+d_1-1)}. \quad (4.25)$$

Thus, both the external and the internal scale factors expand at late times. However, the rate of the expansion of the internal space M_0 is less than for the external space M_1 . For example, in the case $d_1 = 3$, $d_0 = 6$ we get $\tilde{a}_1 \sim \tilde{t}^{3/4}$ and $a_0 \sim \tilde{t}^{1/4}$. So, in spite of this expansion, we suppose that the internal scale factor is still less than the Fermi length which makes it unobservable at present time.

To investigate the accelerating behavior of the external space M_1 , let us define its Hubble and deceleration parameters:

$$\begin{aligned} \tilde{H}_1(\tau) &= -\frac{((D-2)\xi_1 + d_0 d_1 \xi_2 \coth(\xi_2 \tau))}{d_1(d_1-1)(d_0-1)f_{-E(1)}(\tau)} \\ &= -\frac{1}{(d_1-1)f_{-E(1)}(\tau)} \sqrt{\frac{D-2}{d_1(d_0-1)}} m(\tau) \end{aligned} \quad (4.26)$$

and

$$\tilde{q}_1(\tau) = \frac{[(D-2)\xi_1 + d_0 d_1 \xi_2 \coth(\xi_2 \tau)]^2 - d_0(d_0-1)d_1^2 \xi_2^2 \sinh^{-2}(\xi_2 \tau)}{d_1^2(d_1-1)(d_0-1)^2 f_{-E(1)}^2(\tau)} = \frac{(D-2)m^2(\tau) - (d_0-1)d_1 \sinh^{-2}(\xi_2 \tau)}{d_1(d_1-1)(d_0-1)f_{-E(1)}^2(\tau)}, \quad (4.27)$$

where (see also Refs. [7,17])

$$m(\tau) := p^1 + \sqrt{\frac{d_0 d_1}{D-2}} \coth(\xi_2 \tau). \quad (4.28)$$

It can be easily seen that this function is negative: $m(\tau) < 0 \forall \tau \in (-\infty, 0]$ because $|p^1| \leq 1$. Thus, starting from zero value¹⁰ the external space M_1 expands for all times [see Eq. (4.26)]. From the other side, the condition of its acceleration reads

¹⁰For $\tau \rightarrow -\infty$ irrespective of the sign of ξ_1 Eq. (4.22) has the asymptote $\tilde{a}_1 \sim \exp\{[d_0/(d_0-1)(d_1-1)]\xi_2|\tau|[(D-2)/d_0 d_1]\xi_1/\xi_2 - 1\} \rightarrow 0$.

$$\begin{aligned} &\frac{d_1}{D-2} \coth^2(\xi_2 \tau) + 2\sqrt{\frac{d_0 d_1}{D-2}} p^1 \coth(\xi_2 \tau) \\ &+ (p^1)^2 + \frac{(d_0-1)d_1}{D-2} < 0. \end{aligned} \quad (4.29)$$

Because $\coth(\xi_2 \tau) < 0$ for $\tau \in (-\infty, 0]$, this inequality is possible only for positive values of the parameter p^1 : $p^1 > 0$. Moreover, the corresponding quadratic equation should have two roots defining the harmonic time of the beginning ($\tau_{(a)\text{start}}$) and ending ($\tau_{(a)\text{fin}}$) of the acceleration. For these roots we obtain the following relation:

$$\begin{aligned} &\coth(\xi_2 \tau_{(a)\text{start}}) - \coth(\xi_2 \tau_{(a)\text{fin}}) \\ &= 2\sqrt{\frac{(d_0-1)(D-2)}{d_1}} \sqrt{(p^1)^2 - \frac{d_1}{D-2}}. \end{aligned} \quad (4.30)$$

This difference is positive because \coth is a monotone decreasing function. So the stage of the acceleration takes place only if the parameter p^1 satisfies the inequality

$$(p^1)^2 > \frac{d_1}{D-2}. \quad (4.31)$$

For $p^1 = 1$ we restore the results of the paper [7]. However, a new result is that the scalar field with $(d_0 - 1)/(D - 2) \leq (p^2)^2 \leq 1$ prevents the acceleration. In Fig. 4, we present different behavior of the external \tilde{a}_1 and internal a_0 scale factors as well as the deceleration parameter $-\tilde{q}_1$ of the external space M_1 depending on the choice of the parameter p^1 . The solid lines correspond to the values of p^1 satisfying the condition of the acceleration (4.31). $t_{\text{in}}^{\text{acc}}$ and $t_{\text{fin}}^{\text{acc}}$ denote, respectively, the times of the beginning and ending of the external space acceleration. The dashed lines correspond to the case when the parameter p^1 does not satisfy the condition (4.31) and the stage of the acceleration is absent.

V. VARIATION OF THE FINE-STRUCTURE CONSTANT

Above, we considered the model with the dynamical internal spaces. It is well known that the internal space dynamics results in the variation of the fundamental constants such as the gravitational constant and the fine-structure constant (see, e.g., Refs. [34,36]). For example, the effective four-dimensional fine-structure constant is inversely proportional to the volume of the internal space (see, e.g., Refs. [11,15,36]): $\alpha \sim V_{(I)}^{-1} \sim a_{(I)}^{-d_{(I)}}$. Here, the indices “*I*” and “*E*” denote the internal and external spaces, respectively. The origin of such dependence can be easily seen if we add a higher-dimensional electromagnetic action (which should not affect the above-investigated dynamics of the model) and perform the dimensional reduction to an effective four-dimensional theory. It results in the term of the form $\sqrt{g^{(E)}}(V_{(I)}/e^2)F^2$ [11,36] which leads to the above-indicated dependence for the effective fine-structure constant. Thus, if $V_{(I)}$ is a dynamical function which varies with time, then the effective four-dimensional constants will vary as well. For the fine-structure constant, such variations take place in both frames because the quantity $\sqrt{g^{(E)}}(V_{(I)}/e^2)F^2$ is invariant in four-dimensional space-time with respect to the conformal transformation of the metric $g^{(E)}$. Therefore, in both frames we arrive at the following expression for the variation of α :

$$\left| \frac{\dot{\alpha}}{\alpha} \right| = \left| \frac{\dot{V}_{(I)}}{V_{(I)}} \right| = |d_{(I)}H_{(I)}|, \quad (5.1)$$

where the dot denotes the synchronous time derivatives and $H_{(I)} = \dot{a}_{(I)}/a_{(I)}$.

There are strong constraints on $\dot{\alpha}/\alpha$ from a number of experimental and observational considerations [34]. For

our calculations, we use the estimate $|\dot{\alpha}/\alpha| \lesssim 10^{-15} \text{ yr}^{-1}$ [37] which follows from observations of the spectra of quasars. Combining this with the accepted value for the current Hubble rate $H_{(E)} = \dot{a}_{(E)}/a_{(E)} \sim 10^{-10} \text{ yr}^{-1}$ leads to

$$\left| \frac{H_{(I)}}{H_{(E)}} \right| \lesssim 10^{-5}. \quad (5.2)$$

Let us now test the models from Secs. III and IV for the purpose of their satisfaction of the condition (5.2). We perform this investigation only for the cases with the acceleration of the external space.

A. Brans-Dicke frame

1. Spherical space

In this case, the Hubble parameters for the factor spaces are given by Eqs. (3.9) and (3.11). Therefore, depending on the choice of the external space, we obtain the following results:

- (1) Spherical external space (Sec. III A 1):

$$\left| \frac{H_{(I)}}{H_{(E)}} \right| = \left| \frac{H_1}{H_0} \right| = \frac{d_0 - 1}{d_1 |1 + \frac{\xi_2}{\xi_1} \tanh(\xi_2 \tau)|} \sim \mathcal{O}(1). \quad (5.3)$$

This estimate arises from the condition (3.14). Therefore, in this case, we arrive at the obvious contradiction with the experimental bounds.

- (2) Ricci-flat external space (Sec. III A 2):

$$\left| \frac{H_{(I)}}{H_{(E)}} \right| = \left| \frac{H_0}{H_1} \right| = \frac{d_1 \left| 1 + \frac{\xi_2}{\xi_1} \tanh(\xi_2 \tau) \right|}{d_0 - 1} \sim \mathcal{O}(1). \quad (5.4)$$

This estimate is valid for all times $\tau \in (-\infty, +\infty)$. For $\xi_1/\xi_2 < 1$, the only exclusion is a very short period of time in the vicinity of the turning point $\tanh(\xi_2 \tau) = -\xi_1/\xi_2$ of the internal space M_0 , i.e. for the times $\tanh(\xi_2 \tau) \in [-\xi_1/\xi_2 - \delta, -\xi_1/\xi_2 + \delta]$ with $\delta \sim (\xi_1/\xi_2)10^{-5}$. In general, this model conflicts with the observations.

2. Hyperbolic space

In this case the Hubble parameters for the factor spaces are given by Eqs. (4.9) and (4.11). Here, the acceleration takes place only in the case of the hyperbolic external space.

Hyperbolic external space (Sec. IVA 1)—With the help of Eqs. (4.9) and (4.11), the ratio between the Hubble parameters is given by

$$\left| \frac{H_{(I)}}{H_{(E)}} \right| = \left| \frac{H_1}{H_0} \right| = \frac{d_0 - 1}{d_1 |1 + \frac{\xi_2}{\xi_1} \coth(\xi_2 \tau)|}. \quad (5.5)$$

As we have seen in Sec. IVA 1, there are two distinguishing scenarios in this case. The first scenario corresponds to $\xi_1 > \xi_2$ (it happens, e.g., in the case of the absence of a scalar field: $p^1 = 1$). As for this particular case $\xi_2/\xi_1 \sim \mathcal{O}(1)$, we can achieve the necessary smallness of the ratio (5.2) for late times:

$$\left| \frac{H_1}{H_0} \right| < 10^{-5} \quad \text{for } |\xi_2 \tau| < 10^{-5} \quad (\xi_1 > \xi_2). \quad (5.6)$$

The second scenario takes place if $0 < \xi_1 \leq \xi_2$. It can be easily seen that the condition (5.2) is satisfied for the small parameter ξ_1 :

$$\left| \frac{H_1}{H_0} \right| < 10^{-5} \quad \forall \tau \in (-\infty, 0] \quad (\xi_1/\xi_2 \lesssim 10^{-5}). \quad (5.7)$$

We can weaken the condition $\xi_1/\xi_2 \lesssim 10^{-5}$ if we demand the execution of the condition (5.2) from the time τ_a of the beginning of the acceleration [see Eq. (4.14)]:

$$\left| \frac{H_1}{H_0} \right| < 10^{-5} \quad \text{for } \tau_a \leq \tau \leq 0 \quad (\xi_1/\xi_2 \lesssim 10^{-5/2}). \quad (5.8)$$

Therefore, in the hyperbolic external case we can satisfy the condition (5.2) either for sufficiently late times $|\xi_2 \tau| < 10^{-5}$ or for a very dynamical scalar field which results in the smallness of the parameter p^1 : $\sqrt{d_0 d_1 / (D-2)} p^1 \lesssim 10^{-5}$ for (5.7) or $\sqrt{d_0 d_1 / (D-2)} p^1 \lesssim 10^{-5/2}$ for (5.8).

B. Einstein frame

In the Einstein frame, there is only one case with the accelerating stage for the external space. It describes the model with Ricci-flat external and hyperbolic internal spaces.

1. Hyperbolic internal space (Sec. IV B 2)

In this case, the Hubble parameter of the external space M_1 is defined by Eq. (4.26). Concerning the Hubble parameter of the internal factor space M_0 , it is necessary to perform the evident substitution $f_{-BD}(\tau) \rightarrow f_{-E(1)}(\tau)$ in formula (4.9) because in the Einstein frame the function $f(\tau)$ in Eq. (2.14) is defined by $f_{-E(1)}(\tau)$. Thus, the ratio of the Hubble parameters reads

$$\begin{aligned} \left| \frac{H_{(I)}}{H_{(E)}} \right| &= \left| \frac{H_0}{\tilde{H}_1} \right| \\ &= \frac{d_1(d_1 - 1)}{D - 2} \left| \frac{\frac{\xi_2}{\xi_1} + \tanh(\xi_2 \tau)}{\frac{d_0 d_1}{D-2} \frac{\xi_2}{\xi_1} + \tanh(\xi_2 \tau)} \right| \sim \mathcal{O}(1). \end{aligned} \quad (5.9)$$

This estimate is valid for all times $\tau \in (-\infty, 0]$. For $\xi_2/\xi_1 < 1$, the only exclusion is a very short period of time in the vicinity of the turning point $\tanh(\xi_2 \tau) =$

$-\xi_2/\xi_1$ of the internal space M_0 , i.e. for the times $\tanh(\xi_2 \tau) \in [-\xi_2/\xi_1 - \delta, -\xi_2/\xi_1 + \delta]$ with $\delta \sim 10^{-5}$. Therefore, in general, this model conflicts with the experimental bounds.

C. Static internal space

It is clear that the effective fundamental constants do not vary if the internal space is static (“frozen”). Additionally, this results in the equivalence between the Brans-Dicke and Einstein frames. In our model this takes place only if the parameter $\xi_1 = 0 \Rightarrow p^1 = 0$ [see, e.g., Eqs. (2.8), (2.9), and (2.10)], i.e. when the factor space M_1 plays the role of the internal space. Let us investigate this possibility in more detail.

First, we consider the *spherical* external space. It follows from Eq. (3.10) that the external space M_0 is decelerating because $q_0 > 0$ for $\xi_1 = 0$. Moreover, the static solution is unstable. To see this, let us suppose that the internal space scale factor a_1 is frozen up to an arbitrary time τ_0 . Then, small fluctuations $\delta \xi_1 = [d_1(d_0 - 1)/(D - 2)]^{1/2} \delta p^1$ result in the following dynamics:

$$a_1(\tau)|_{\tau \geq \tau_0} = \exp[(\delta \xi_1/d_1)(\tau - \tau_0)], \quad \tau \in [\tau_0, +\infty) \quad (5.10)$$

[see Eq. (2.8)]. Thus, the scale factor a_1 goes from the constant value to either $+\infty$ (for positive δp^1) or zero (for negative δp^1). At the same time, the external scale factor a_0 remains decelerating because the small fluctuation $\delta \xi_1$ does not satisfy the acceleration condition (3.14). Therefore, this case is not of interest to us.

Second, we turn to the *hyperbolic* external space. Here, the external space M_0 is again decelerating [see Eq. (4.10) for $\xi_1 = 0$]. Because of small fluctuations $\delta \xi_1$ at an arbitrary moment $\tau_0 < 0$, the scale factor a_1 acquires the dynamics:

$$a_1(\tau)|_{\tau \geq \tau_0} = \exp[(\delta \xi_1/d_1)(\tau - \tau_0)], \quad \tau \in [\tau_0, 0]. \quad (5.11)$$

Thus, for negative $\delta \xi_1$ the internal scale factor asymptotically approaches the value $\exp[-(|\delta \xi_1|/d_1)|\tau_0|]$ and the external space remains decelerating. In this case the internal space varies in finite limits of the order of $\mathcal{O}(1)$ (from this point we can call this case “quasi stable”). For positive $\delta \xi_1$, the internal scale factor asymptotically approaches the value $\exp[(\delta \xi_1/d_1)|\tau_0|]$ and the external space starts to accelerate at the time $\coth(\xi_2 \tau_a) = -\xi_2/\delta \xi_1$ [see Eq. (4.14)]. The case of the positive $\delta \xi_1$ is of interest because, first, the external space begins to accelerate and, second, the variations of the fundamental constant do not contradict the observations if the ratio $\delta \xi_1/\xi_2$ satisfies the conditions similar to those for the ratio ξ_1/ξ_2 in the expressions (5.7) and (5.8). However, the scale factor a_1 can considerably increase if $(\delta \xi_1/d_1)|\tau_0| > 1$. In this case the solution is unstable.

VI. FIXATION OF THE FINE-STRUCTURE CONSTANT

Let us consider now the case of three factor spaces with the topology of the manifold of the form $M = \mathbb{R} \times M_0 \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ where M_0 is d_0 -dimensional spherical (S^{d_0}) or hyperbolic (H^{d_0}) space.

Here, the solution (in the Brans-Dicke frame) is (see Refs. [28,29])

$$a_0(\tau) = A_0 \exp\left(-\frac{\xi_1}{d_0 - 1} \tau\right) \times \frac{1}{\tilde{g}_\pm(\tau)}, \quad (6.1)$$

$$a_1(\tau) = A_1 \exp\left(\left(\frac{\xi_1}{d_1 + d_2} - \frac{\xi_3}{d_1}\right) \tau\right), \quad (6.2)$$

$$a_2(\tau) = A_2 \exp\left(\left(\frac{\xi_1}{d_1 + d_2} + \frac{\xi_3}{d_2}\right) \tau\right), \quad (6.3)$$

$$\varphi(\tau) = p^3 \tau + q, \quad (6.4)$$

where

$$\tilde{g}_+(\tau) = \left(\frac{2\varepsilon}{R_0}\right)^{-1/2(d_0-1)} \cosh^{1/(d_0-1)}(\xi_2 \tau), \quad (6.5)$$

$$-\infty < \tau < +\infty, \quad \varepsilon > 0$$

and

$$\tilde{g}_-(\tau) = \left(\frac{2\varepsilon}{|R_0|}\right)^{-1/2(d_0-1)} \sinh^{1/(d_0-1)}(\xi_2 |\tau|), \quad (6.6)$$

$$|\tau| > 0, \quad \varepsilon \geq 0.$$

As usual in this paper, the index + (−) indicates that the considered formula is related to the spherical (hyperbolic) factor space M_0 . In the case $\varepsilon = 0$, Eq. (6.6) is reduced to the form

$$\tilde{g}_-(\tau) = [(d_0 - 1)|\tau|]^{1/(d_0-1)}, \quad (6.7)$$

where we used the formula $|R_0| = d_0(d_0 - 1)$.

In Eqs. (6.1), (6.2), (6.3), (6.4), (6.5), and (6.6),

$$\xi_1 = \sqrt{\frac{(d_1 + d_2)(d_0 - 1)}{(D - 2)}} p^1, \quad \xi_2 = \sqrt{\frac{d_0 - 1}{d_0}} 2\varepsilon, \quad (6.8)$$

$$\xi_3 = \sqrt{\frac{d_1 d_2}{d_1 + d_2}} p^2$$

and

$$2\varepsilon = (p^1)^2 + (p^2)^2 + (p^3)^2. \quad (6.9)$$

Parameters $A_0, A_1, A_2, p^1, p^2, p^3$, and q are the constants of integration with the following constraint: $A_0^{d_0} A_1^{d_1} A_2^{d_2} = A_0$.

For this solution, the Hubble and deceleration parameters read

$$H_{\pm 0}(\tau) = -\frac{1}{f_\pm(\tau)} \frac{\xi_1 + \xi_2 h_\pm(\tau)}{d_0 - 1}, \quad (6.10)$$

$$q_{\pm 0}(\tau) = \frac{\xi_2}{f_\pm^2(\tau)} \frac{\xi_2 + \xi_1 h_\pm(\tau)}{d_0 - 1}, \quad (6.11)$$

$$H_{\pm 1}(\tau) = \frac{1}{f_\pm(\tau)} \left(\frac{\xi_1}{d_1 + d_2} - \frac{\xi_3}{d_1} \right), \quad (6.12)$$

$$H_{\pm 2}(\tau) = \frac{1}{f_\pm(\tau)} \left(\frac{\xi_1}{d_1 + d_2} + \frac{\xi_3}{d_2} \right), \quad (6.13)$$

where the transition function $f(\tau)$ [see Eq. (2.13)] is

$$f_\pm(\tau) = e^{\gamma_0} = a_0^{d_0} a_1^{d_1} a_2^{d_2}$$

$$= A_0 \exp\left(-\frac{\xi_1}{d_0 - 1} \tau\right) \times \frac{1}{\tilde{g}_\pm(\tau)^{d_0}} \quad (6.14)$$

and

$$h_\pm(\tau) = \begin{cases} \tanh(\xi_2 \tau), & \tau \in (-\infty, +\infty), \quad R_0 > 0, \\ \coth(\xi_2 \tau), & \tau \in (-\infty, 0], \quad R_0 < 0. \end{cases} \quad (6.15)$$

In this section, the factor space M_0 is treated as the external one. This choice is justified below. As it follows from Eqs. (6.10), (6.11), (6.12), (6.13), (6.14), and (6.15), the dynamics of the model is similar to that described in Secs. III A 1 and IV A 1. For example, the *spherical* external space M_0 undergoes the accelerating expansion [during the period (3.13)] and both internal spaces M_1 and M_2 contract if $\xi_1 < 0$ and $\xi_3 < (d_2/(d_1 + d_2))|\xi_1|$ for positive $\xi_3 > 0$ or $|\xi_3| < (d_1/(d_1 + d_2))|\xi_1|$ for negative $\xi_3 < 0$. In the case of the *hyperbolic* external space, the accelerating expansion of M_0 is possible only if ξ_1 is positive: $\xi_1 > 0$. Here, the acceleration of M_0 is either eternal (if $\xi_1 > \xi_2$) or starts at the time (4.14) (if $0 < \xi_1 \leq \xi_2$). Concerning the internal spaces M_1 and M_2 we can say that at least one of them expands approaching the finite value A_1 or A_2 .

As to the variations of the effective fine-structure constant, we obtain

$$\left| \frac{\dot{\alpha}}{\alpha} \right| = \left| \frac{\dot{V}_{(l)}}{V_{(l)}} \right| = |d_1 H_1 + d_2 H_2|, \quad (6.16)$$

where $V_{(l)} \sim a_1^{d_1} a_2^{d_2}$. Since the combination $d_1 H_1$ in the case of one internal space gives exactly the same expression as the combination $d_1 H_1 + d_2 H_2$ in the case of two internal spaces [see Eqs. (3.11), (4.11), (6.12), and (6.13)] we arrive at the conclusions with respect to the variations of α similar to those obtained in Secs. VA 1 and VA 2: the spherical model is in conflict with the observations [see Eq. (5.3)] and the hyperbolic model can be in agreement with the experimental bounds either at very late times [see Eq. (5.6)] or for very small ξ_1 [see Eqs. (5.7) and (5.8)].

Obviously, the effective four-dimensional fundamental constants are fixed if the total volume of the internal spaces is constant. Now, we try to answer the following question. Is it possible to construct the model with dynamical scale factors but fixed total volume of the internal spaces? The simple analysis of Eqs. (6.1), (6.2), and (6.3) shows that such possibility exists only if we choose the Ricci-flat factor spaces \mathbb{R}^{d_1} and \mathbb{R}^{d_2} as the internal ones and put $p^1 = 0$. In this case

$$V_{(I)} \sim a_1^{d_1} a_2^{d_2} |_{p^1=0} = A_1^{d_1} A_2^{d_2} = \text{const.} \quad (6.17)$$

Hence, in spite of the dynamical behavior of the internal scale factors, first, the Brans-Dicke and Einstein frames are equivalent to each other and, second, the fundamental constants are fixed. This was the main reason to choose the factor space M_0 as the external one. At first sight, this model looks very promising. However, it has a number of drawbacks. First, the external space M_0 is the decelerating one: $q_{\pm 0}(\tau) > 0$ [see Eq. (6.11)].

Additionally, it is necessary to investigate this model for the purpose of its stability with respect to the fluctuations of the parameter p^1 . It can be easily seen that due to small fluctuations $\delta\xi_1 = [(d_1 + d_2)(d_0 - 1)/(D - 2)]^{1/2} \delta p^1$ at an arbitrary moment τ_0 the internal volume acquires the following dynamics:

$$V_{(I)} = A_1^{d_1} A_2^{d_2} \exp(\delta\xi_1(\tau - \tau_0)), \quad (6.18)$$

where $\tau \in [\tau_0, +\infty)$ for the spherical M_0 and $\tau \in [\tau_0, 0]$ for the hyperbolic M_0 . Thus, the stability analysis can be performed in full analogy with Sec. VC of the static internal space. We obtain that the case of the spherical external space is unstable with the decelerating behavior and the case of the hyperbolic external space is “quasi stable” for $\delta\xi_1 < 0$ and unstable for $\delta\xi_1 > 0$. In the latter case the factor space M_0 can acquire the stage of the acceleration without too much variation of α .

To conclude this section, we consider a particular model with fixed internal volume (6.17) and an additional condition $\varepsilon = 0$. It takes place if the scalar field is imaginary, i.e. φ is a phantom field (see, e.g., [38–40] and numerous references therein). For the hyperbolic¹¹ external space M_0 the solution (in the harmonic time gauge) is given by Eqs. (6.1), (6.2), (6.3), and (6.4) with the following substitution: $\xi_1 = 0$, $p^3 = ip^2$ and \tilde{g}_- from Eq. (6.7). This particular model is of interest because of its integrability in the synchronous time gauge where the solution reads

$$a_0(t) = t, \quad (6.19)$$

$$a_1(t) = A_1 \exp\left(\frac{\xi_3}{d_1(d_0 - 1)} \left(\frac{A_0}{t}\right)^{d_0-1}\right), \quad (6.20)$$

¹¹Classical Lorentzian solutions with $\varepsilon = 0$ exist only for the hyperbolic M_0 .

$$a_2(t) = A_2 \exp\left(-\frac{\xi_3}{d_2(d_0 - 1)} \left(\frac{A_0}{t}\right)^{d_0-1}\right), \quad (6.21)$$

$$\varphi(\tau) = i \frac{p^2}{d_0 - 1} \left(\frac{A_0}{t}\right)^{d_0-1} + q, \quad (6.22)$$

and $t \in [0, +\infty)$. Hence, the scale factor of the external space behaves as in the case of the Milne solution with zero acceleration. This is a transitional case between the accelerating and decelerating behavior. Any perturbations δp^1 result in nonzero $2\varepsilon = (\delta p^1)^2 > 0$. The behavior of such perturbed models is described by Eqs. (6.1), (6.2), and (6.3) with $\xi_1 \rightarrow \delta\xi_1 = [(d_1 + d_2)(d_0 - 1)/(D - 2)]^{1/2} \delta p^1$ and $2\varepsilon = (\delta p^1)^2$. In this case, $|\delta\xi_1/\xi_2| = \sqrt{d_0(d_1 + d_2)/(D - 2)} > 1$. Thus, for positive fluctuations $\delta\xi_1$ the external space M_0 undergoes the eternal acceleration in accordance with the results of Sec. IVA 1. However, the variations of α are in agreement with the experimental data only at very late times, as we have seen in Sec. VA 2. Additionally, the internal space volume $V_{(I)}$ can considerably increase if $\delta\xi_1|\tau_0| \gg 1$ [see Eq. (6.18)]. Therefore, this solution is unstable.

VII. CONCLUSION

In the present paper we investigated the possibility of generating the late-time acceleration of the Universe from gravity on product spaces with only one non-Ricci-flat factor space. The model contains a minimally coupled free scalar field as a matter source. Dynamical solutions for this model are called S-brane (spacelike brane) solutions. The analysis was performed in the Brans-Dicke and Einstein frames. We found that, in the context of the considered models, non-Einsteinian gravity provides more possibilities for accelerating cosmologies than the Einsteinian one. As we already mentioned in the Introduction, such different behavior of the external space scale factors in both of these frames is not surprising because these scale factors are described by different variables connected with each other via the conformal transformation [see, e.g., Eq. (A5) in the Appendix]. Moreover, the synchronous times in both of these frames are also different. As a consequence of these discrepancies, the scale factors of the external space in both frames behave differently. In the Brans-Dicke frame, stages of the accelerating expansion exist for all types of the external space (flat, spherical, and hyperbolic). However, in the Einstein frame, the model with flat external space and hyperbolic compactification of the internal space is the only one with the stage of the accelerating expansion. The reason for this acceleration is rather clear. After dimensional reduction of the considered models and conformal transformation to the Einstein frame, we obtain an effective potential of the form $U = -(1/2)e^{2\gamma_0} R_0 e^{-2\beta^0}$ [see Eq. (2.7)], which plays the role of an effective cosmological “constant.” Thus, the acceleration is possible only if the internal space curvature

$R_0 < 0$. The presence of a minimally coupled free scalar field does not help the acceleration because this field does not contribute to the potential. Nevertheless, it makes sense to include such a field in the model because it results in more reach and interesting dynamical behavior.¹² Moreover, we have seen in Sec. IV B 2 that a scalar field can prevent the acceleration in the Einstein frame. This is a new result in comparison with Refs. [7,20].

It is well known that the dynamical behavior of the internal spaces results in the variation of the effective four-dimensional fundamental constants. Therefore, we investigated the rate of variation of the fine-structure constant for the cases of the accelerating external spaces. It was shown that the case of the hyperbolic external space in the Brans-Dicke frame is the only model which can satisfy the experimental bounds for the fine-structure constant variations.

It is clear that the fundamental constant variations are absent if the total volume of the internal spaces is constant. Obviously, there is no difference between the Brans-Dicke and Einstein frames in this case. Such particular solutions exist in the cases of one or two internal Ricci-flat spaces. The latter model is of special interest because the internal spaces undergo the dynamical evolution and, at the same time, the internal space total volume is fixed. However, these models have a number of drawbacks. First, the external space is nonaccelerating and, second, these models are unstable.

Thus, in many cases, the considered S-brane solutions admit stages of the accelerating expansion of the external space. However, they have a significant problem with the experimental bounds for the variations of the fine-structure constant.

APPENDIX: BRANS-DICKE AND EINSTEIN FRAMES

In this appendix we derive the connection between different quantities in the Einstein and Brans-Dicke frames. Since the result depends on the choice of the external space and both M_0 and M_1 can be the external one, we redefine by letter “*E*” the external space and letter “*I*” the internal one, dropping the indices 0 and 1. Further, we can perform the dimensional reduction of action (2.5) integrating over the coordinates of the internal space [41]:

$$S = \frac{V_{0(I)}}{2\kappa_D^2} \int_{\tilde{M}_{(E)}} d^{D_{(E)}x} \sqrt{|\tilde{g}^{(E)}|} e^{d_{(I)}\beta^{(I)}} \{R[\tilde{g}^{(E)}] - G_{II} \tilde{g}^{(E)\mu\nu} \partial \beta_\mu^{(I)} \partial \beta_\nu^{(I)} + R[g^{(I)}] e^{-2\beta^{(I)}}\}, \quad (A1)$$

¹²We have seen that the dynamical picture of the model considerably depends on the relation between the parameters ξ_1 and ξ_2 introduced in Eqs. (2.8), (2.9), and (2.10). If a scalar field is absent, $|\xi_1|/\xi_2 = [d_1 d_0 / (D - 2)]^{1/2} > 1$. However, in the presence of a scalar field this ratio is not fixed but varies in the limits $0 \leq |\xi_1|/\xi_2 \leq [d_1 d_0 / (D - 2)]^{1/2}$.

where $V_{0(I)}$ is the constant volume of the internal space (modulo the scale factor), $\tilde{g}^{(E)}$ is the external space-time metric on the manifold $\tilde{M}_{(E)} = \mathbb{R} \times M_{(E)}$ of the dimension $D_{(E)} = 1 + d_{(E)}$, $G_{II} = d_{(I)}(1 - d_{(I)})$, and we allow the internal space scale factor to depend on all external coordinates $x \in \tilde{M}_{(E)}$. We also dropped the scalar field because it does not affect our results. This reduced action is written in the Brans-Dicke frame. In the next step, we remove the explicit coupling term in (A1) by conformal transformation

$$\tilde{g}_{\mu\nu}^{(E)} = \Omega^2 \tilde{g}_{\mu\nu}^{(E)} := \exp\left(-\frac{2d_{(I)}\beta^{(I)}}{D_{(E)} - 2}\right) \tilde{g}_{\mu\nu}^{(E)} \quad (A2)$$

and obtain the reduced action in the Einstein frame:

$$S = \frac{V_{0(I)}}{2\kappa_D^2} \int_{\tilde{M}_{(E)}} d^{D_{(E)}x} \sqrt{|\tilde{g}^{(E)}|} \{R[\tilde{g}^{(E)}] - \tilde{g}^{(E)\mu\nu} \partial \psi_\mu \partial \psi_\nu + R[g^{(I)}] e^{2A\psi}\}, \quad (A3)$$

where $\psi = -Ad_{(I)}\beta^{(I)}$, and $A = \pm[(D - 2)/(d_{(I)}(D_{(E)} - 2))]^{1/2}$.

Thus, the metric (2.1) in different gauges reads

$$\begin{aligned} g &= -e^{2\gamma_0} d\tau \otimes d\tau + a_{(E)}^2 g^{(E)} + a_{(I)}^2 g^{(I)} \\ &= -dt \otimes dt + a_{(E)}^2 g^{(E)} + a_{(I)}^2 g^{(I)} \\ &= \Omega^2 (-d\tilde{t} \otimes d\tilde{t} + \tilde{a}_{(E)}^2 g^{(E)} + a_{(I)}^2 g^{(I)}), \end{aligned} \quad (A4)$$

where the first line is the metric in the harmonic time gauge in the Brans-Dicke frame, the second line is the metric in the synchronous time gauge in the Brans-Dicke frame, and the third line is the metric in the synchronous time gauge in the Einstein frame. Equations (A4) show that the external scale factors in the Einstein and Brans-Dicke frames are related as follows:

$$\tilde{a}_{(E)} = \Omega^{-1} a_{(E)} \quad (A5)$$

and there exists the following correspondence between different times¹³:

$$dt = e^{\gamma_0(\tau)} d\tau \Rightarrow t = \int e^{\gamma_0(\tau)} d\tau + \text{const}, \quad (A6)$$

$$d\tilde{t} = \Omega^{-1} e^{\gamma_0(\tau)} d\tau \Rightarrow \tilde{t} = \int \Omega^{-1} e^{\gamma_0(\tau)} d\tau + \text{const}. \quad (A7)$$

Additionally, it is worth noting that Eq. (A3) explicitly indicates the possibility of the external space acceleration (in the Einstein frame) in the case of the hyperbolic compactification. The fact is that an effective potential $U_{\text{eff}} := -(1/2)R[g^{(I)}] \exp(2A\psi)$ is positive for $R[g^{(I)}] < 0$. Similarly to the positive cosmological constant, such positive effective potentials can result in the accelerating stages of the Universe.

¹³To have the same directions of the arrows of time, we choose the plus sign for the square root.

- [1] M. Gutperle and A. Strominger, *J. High Energy Phys.* **04** (2002) 018.
- [2] C.-M. Chen, D. V. Gal'tsov, and M. Gutperle, *Phys. Rev. D* **66**, 024043 (2002).
- [3] V.D. Ivashchuk, *Classical Quantum Gravity* **20**, 261 (2003).
- [4] N. Ohta, *Phys. Lett. B* **558**, 213 (2003).
- [5] C.P. Burgess, F. Quevedo, S.-J. Rey, G. Tasinato, and I. Zavala, *J. High Energy Phys.* **10** (2002) 028; C.P. Burgess, C. Nunez, F. Quevedo, G. Tasinato, and I. Zavala, *J. High Energy Phys.* **08** (2003) 056; D. Astefanesei and G.C. Jones, *J. High Energy Phys.* **06** (2005) 037.
- [6] N. Ohta, *Phys. Rev. Lett.* **91**, 061303 (2003); *Prog. Theor. Phys.* **110**, 269 (2003).
- [7] P.K. Townsend and M.N.R. Wohlfarth, *Phys. Rev. Lett.* **91**, 061302 (2003).
- [8] R. Emparan and J. Garriga, *J. High Energy Phys.* **05** (2003) 028.
- [9] S. Nojiri and S.D. Odintsov, *Phys. Rev. D* **68**, 123512 (2003); *Gen. Relativ. Gravit.* **36**, 1765 (2004).
- [10] U. Günther and A. Zhuk, *Phys. Rev. D* **56**, 6391 (1997).
- [11] U. Günther, A. Starobinsky, and A. Zhuk, *Phys. Rev. D* **69**, 044003 (2004).
- [12] U. Günther, P. Moniz, and A. Zhuk, *Phys. Rev. D* **68**, 044010 (2003).
- [13] U. Günther and A. Zhuk, "Can gravexcitons solve the problem of dark matter and dark energy" (unpublished).
- [14] J.-A. Gu and W.-Y.P. Hwang, *Phys. Rev. D* **66**, 024003 (2002).
- [15] J.M. Cline and J. Vinet, *Phys. Rev. D* **68**, 025015 (2003).
- [16] N. Mohammadi, *Phys. Rev. D* **65**, 104018 (2002).
- [17] S. Roy, *Phys. Lett. B* **567**, 322 (2003).
- [18] M.N.R. Wohlfarth, *Phys. Lett. B* **563**, 1 (2003).
- [19] M. Gutperle, R. Kallosh, and A. Linde, *J. Cosmol. Astropart. Phys.* **07** (2003) 001.
- [20] C.-M. Chen, P.-M. Ho, I.P. Neupane, N. Ohta, and J.E. Wang, *J. High Energy Phys.* **10** (2003) 058.
- [21] I.P. Neupane, *Classical Quantum Gravity* **21**, 4383 (2004).
- [22] C.-M. Chen, P.-M. Ho, I.P. Neupane, and J.E. Wang, *J. High Energy Phys.* **07** (2003) 017.
- [23] V.D. Ivashchuk, V.N. Melnikov, and A.B. Selivanov, *J. High Energy Phys.* **09** (2003) 059.
- [24] M.N.R. Wohlfarth, *Phys. Rev. D* **69**, 066002 (2004).
- [25] L. Jarv, T. Mohaupt, and F. Saueressig, *J. Cosmol. Astropart. Phys.* **08** (2004) 016.
- [26] V.D. Ivashchuk, *Phys. Lett. A* **170**, 16 (1992).
- [27] A.I. Zhuk, *Phys. Rev. D* **45**, 1192 (1992); *Classical Quantum Gravity* **9**, 2029 (1992).
- [28] U. Bleyer and A. Zhuk, *Gravitation Cosmol.* **1**, 37 (1995).
- [29] U. Bleyer and A. Zhuk, *Gravitation Cosmol.* **1**, 106 (1995).
- [30] U. Bleyer and A. Zhuk, *Astron. Nachr.* **317**, 161 (1996).
- [31] V.D. Ivashchuk, V.N. Melnikov, and A.I. Zhuk, *Nuovo Cimento Soc. Ital. Fis. B* **104**, 575 (1989).
- [32] V.R. Gavrillov, V.D. Ivashchuk, and V.N. Melnikov, *Classical Quantum Gravity* **13**, 3039 (1996).
- [33] A. Zhuk, *Gravitation Cosmol.* **3**, 24 (1997).
- [34] J.-P. Uzan, *Rev. Mod. Phys.* **75**, 403 (2003).
- [35] N. Kaloper, J. March-Russell, G.D. Starkman, and M. Trodden, *Phys. Rev. Lett.* **85**, 928 (2000).
- [36] P. Loren-Aguilar, E. Garcia-Berro, J. Isern, and Yu. A. Kubyshev, *Classical Quantum Gravity* **20**, 3885 (2003).
- [37] J.K. Webb *et al.*, *Phys. Rev. Lett.* **87**, 091301 (2001).
- [38] G.W. Gibbons, hep-th/0302199.
- [39] P. Singh, M. Sami, and N. Dadhich, *Phys. Rev. D* **68**, 023522 (2003).
- [40] M.P. Dabrowski, T. Stachowiak, and M. Szydlowski, *Phys. Rev. D* **68**, 103519 (2003).
- [41] M. Rainer and A. Zhuk, *Phys. Rev. D* **54**, 6186 (1996).