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Tunnelling, temperature, and Taub-NUT black holes

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We investigate quantum tunnelling methods for calculating black hole temperature, specifically the null-geodesic method of Parikh and Wilczek and the Hamilton-Jacobi Ansatz method of Angheben *et al.* We consider application of these methods to a broad class of spacetimes with event horizons, inlcuding Rindler and nonstatic spacetimes such as Kerr-Newman and Taub-NUT. We obtain a general form for the temperature of Taub-NUT-AdS black holes that is commensurate with other methods. We examine the limitations of these methods for extremal black holes, taking the extremal Reissner-Nordstrom spacetime as a case in point.

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I. INTRODUCTION

There are several methods for deriving Hawking radiation [1–17] and for calculating its temperature. The original method considered the creation of a black hole in the context of a collapse geometry, calculating the Bogoliubov transformations between the initial and final states of incoming and outgoing radiation. The more popular method of analytic continuation to a Euclidean section (the Wick Rotation method) emerged soon after. Relying on the methods of finite-temperature quantum field theory, an analytic continuation $t \to i\tau$ of the black hole metric is performed and the periodicity of τ (denoted by β) is chosen in order to remove a conical singularity that would otherwise be present at fixed points of the U(1) isometry generated by $\partial/\partial\tau$ (the event horizon in the original Lorentzian section). The black hole is then considered to be in equilibrium with a scalar field that has inverse temperature β at infinity.

Recently a semiclassical method of modeling Hawking radiation as a tunneling effect was proposed [5–17]. This method involves calculating the imaginary part of the action for the (classically forbidden) process of s-wave emission across the horizon (first considered by Kraus and Wilczek [5–7]), which in turn is related to the Boltzmann factor for emission at the Hawking temperature. Using the WKB approximation the tunneling probability for the classically forbidden trajectory of the s-wave coming from inside to outside the horizon is given by:

$$\Gamma \propto \exp(-2\,\mathrm{Im}I) \tag{1}$$

where I is the classical action of the trajectory to leading order in $\hbar$ (here set equal to unity) [6]. Expanding the action in terms of the particle energy, the Hawking temperature is recovered at linear order. In other words for $2I = \beta E + O(E^2)$ this gives

$$\Gamma \sim \exp(-2I) \simeq \exp(-\beta E) \tag{2}$$

which is the regular Boltzmann factor for a particle of energy E where β is the inverse temperature of the horizon. The higher order terms are a self-interaction effect resulting from energy conservation [6,9]; however, for calculating the temperature, expansion to linear order is all that is required. Two different methods have been employed to calculate the imaginary part of the action—one used by Parikh and Wilczek [9] and the other by Angheben, Nadalini, Vanzo, and Zerbini [16] (which is an extension from the method used by Srinivasan and Padmanabhan [15]).

The former method considers a null s-wave emitted from the black hole. Based on previous work analyzing the full action in detail [5–8], the only part of the action that contributes an imaginary term is $\int_{r_{\rm in}}^{r_{\rm out}} p_r dr$, where p_r is the momentum of the emitted null s-wave. Then by using Hamilton's equation and knowledge of the null geodesics it is possible to calculate the imaginary part of the action. We will refer to this approach as the null-geodesic method.

The latter method involves consideration of an emitted scalar particle, ignoring its self-gravitation, and assumes that its action satisfies the relativistic Hamilton-Jacobi equation. From the symmetries of the metric one picks an appropriate ansatz for the form of the action. We will refer to this method as the Hamilton-Jacobi ansatz.

In this paper we examine these two methods in the context of a broader class of spacetimes than has previously been studied. One of our prime motivations is to understand the applicability of the method to stationary black hole spacetimes such as the Kerr-Newman and Taub-NUT spacetimes. The Taub-NUT metric is a generalization of the Schwarzschild metric and has played an important role in the conceptual development of general relativity and in the construction of brane solutions in string theory and M-theory. [18] The NUT charge plays the role of a magnetic mass, inducing a topology in the Euclidean section at infinity that is a Hopf fibration of a circle over a 2-sphere. "A counter example to almost anything" [19], Taub-NUT spaces have been of particular interest in recent years because of the role they play in furthering our understanding of the AdS-CFT correspondence [20–22]. Along

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these lines, the thermodynamics of various Taub-NUT solutions has been a subject of intense study in recent years. Their entropy is not proportional to the area of the event horizon and their free energy can sometimes be negative [20,22–24]. Solutions of Einstein equations with a negative cosmological constant Λ and a nonvanishing NUT charge have a boundary metric that has closed timelike curves. The behavior of quantum field theory is significantly different in such spaces, and it is of interest to understand how AdS-CFT correspondence works in these sorts of cases [25].

All such thermodynamic calculations have thus far been carried out in the Euclidean section, using Wick rotation methods. For most Taub-NUT spaces the Lorentzian section has closed timelike curves. As a consequence, determination of the temperature via the original method of Hawking—while mathematically clear—is somewhat problematic in terms of its physical interpretation. It is straightforward enough to analytically continue the time coordinate and various metric parameters to render the metric Euclidean. Regularity arguments then yield a periodicity for the time coordinate that can then be interpreted as a temperature. However the Lorentzian analogue of this procedure is less than clear, though it has been established that a relationship between distinct analytic continuation methods exists [26]. An independent method of computing the temperature associated with event horizons in NUT-charged spacetimes is certainly desirable.

Our goal in this paper is to address this question, and to more generally investigate the tunnelling approach outside of the spherically symmetric ansatz. To this end, we compare the null-geodesic method and the Hamilton-Jacobi ansatz for obtaining the imaginary part of the action. We then apply these methods to a variety of spacetimes, and derive a general formula for the temperature from this method. We then consider specific cases of interest, beginning with Rindler space and moving on to charged and rotating black hole spacetimes. Turning to Taub-NUT spaces, we obtain a general expression for the temperature for a subclass of Taub-NUT spacetimes without closed timelike curves (CTCs) that we can compare to those obtained via Wick rotation methods. We find agreement in all relevant cases.

Our paper is structured as follows. The next section will outline the two methods, starting with a discussion and generalization of the null-geodesic method and followed by a discussion of the Hamilton-Jacobi ansatz. We demonstrate that knowledge of the total mass or energy is not essential by showing the direct application of these methods to Rindler spacetime. We then apply these methods to stationary spacetimes, considering in turn the Kerr-Newman class of metrics and then Taub-NUT spacetimes. In each case we obtain results commensurate with other methods, concentrating in the latter case on the subclass of Taub-NUT-AdS spacetimes that do not have closed timelike curves [25]. We finish with a preliminary discussion of issues that occur when applying the method to extremal black holes, concentrating on the specific case of the extremal Reissner-Nordstrom spacetime.

II. CALCULATING THE IMAGINARY PART OF THE ACTION FOR AN OUTGOING S-WAVE

A. Null-Geodesic method

We begin by reviewing the null-geodesic method [9]. The general static spherically metric can be written in the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2 \tag{3}$$

which covers a broad range of black hole metrics[1]. We want to write it in Painlevé form [27] so that there is no singularity at the horizon. This is easily accomplished via the transformation

$$t \to t - \int \sqrt{\frac{1 - g(r)}{f(r)g(r)}} dr \tag{4}$$

yielding

$$ds^2 = -f(r)dt^2 + 2\sqrt{f(r)}\sqrt{\frac{1}{g(r)} - 1} dr dt + dr^2 + r^2 d\Omega^2 \tag{5}$$

This coordinate system has a number of interesting features. At any fixed time the spatial geometry is flat. At any fixed radius the boundary geometry is the same as that of the metric (3).

The radial null geodesics for this metric correspond to

$$\dot{r} = \sqrt{\frac{f(r)}{g(r)}}(\pm 1 - \sqrt{1 - g(r)}) \tag{6}$$

where the plus/minus signs correspond to outgoing/ingoing null geodesics.

The basic idea behind this approach is to regard Hawking radiation as a quantum tunnelling process. However unlike other tunnelling processes in which two separated classical turning points are joined by a trajectory in imaginary time, the tunnelling barrier is created by the outgoing particle itself, whose trajectory is from the inside of the black hole to the outside, a classically forbidden process. The probability of tunnelling is proportional to the exponential of minus twice the imaginary part of the action for this process in the WKB limit. Because of energy conservation, the radius of the black hole shrinks as a function of the energy of the outgoing particle; in this sense the particle creates its own tunnelling barrier.

[1]For previous work generalizing the null-geodesic method to these black hole metrics see Ref. [12].

In the spherically symmetric case the emitted particle is taken to be in an outgoing s-wave mode, and so we use the plus sign in (6). At the horizon (where $g(r) = f(r) = 0$) then $\dot{r} = 0$ provided $\frac{f(r)}{g(r)}$ is well defined there. The imaginary part of the action for an outgoing s-wave from $r_{\rm in}$ to $r_{\rm out}$ is expressed as

$$\mathrm{Im}\, I = \mathrm{Im} \int_{r_{\rm in}}^{r_{\rm out}} p_r dr = \mathrm{Im} \int_{r_{\rm in}}^{r_{\rm out}} \int_0^{p_r} dp'_r dr \tag{7}$$

where $r_{\rm in}$ and $r_{\rm out}$ are the respective initial and final radii of the black hole. The trajectory between these two radii is the barrier the particle must tunnel through.

We assume that the emitted s-wave has energy $\omega' \ll M$ and that the total energy of the spacetime was originally M. Invoking conservation of energy, to this approximation the s-wave moves in a background spacetime of energy $M \to M - \omega'$. In order to evaluate the integral, we employ Hamilton's equation $\dot{r} = \frac{dH}{dp_r}|_r$ to switch the integration variable from momentum to energy ($dp_r = \frac{dH}{\dot{r}}$), giving

$$I = \int_{r_{\rm in}}^{r_{\rm out}} \int_M^{M-\omega} \frac{dr}{\dot{r}} dH = \int_0^{\omega} \int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}}(-d\omega') \tag{8}$$

where $dH = -d\omega'$ because total energy $H = M - \omega'$ with M constant. Note that $\dot{r}$ is implicitly a function of $M - \omega'$. For the special cases where this function is known (eg. Schwarzschild) the integral in Eq. (8) can be solved exactly in terms of ω [9]. In another generalization of the null-geodesic method [12] spacetimes with a well defined ADM mass are considered (since dependence of $M - \omega'$ is explicitly known) in order to obtain self-gravitation effects; for our considerations self gravitation will be ignored.[2]

In general we can always perform a series expansion in ω in order to find the temperature. To first order this gives

$$\begin{aligned} I &= \int_0^{\omega} \int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}(r, M - \omega')}(-d\omega') \\ &= -\omega \int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}(r, M)} + O(\omega^2) \simeq \omega \int_{r_{\rm out}}^{r_{\rm in}} \frac{dr}{\dot{r}(r, M)} \end{aligned} \tag{9}$$

To proceed further we will need to estimate the last integral. First we note that $r_{\rm in} > r_{\rm out}$ because the black hole decreases in mass as the s-wave is emitted; consequently the radius of the event horizon decreases. We therefore write $r_{\rm in} = r_0(M) - \epsilon$ and $r_{\rm out} = r_0(M - \omega) + \epsilon$ where $r_0(M)$ denotes the location of the event horizon of the original background spacetime before the emission of particles. Henceforth the notation r_0 will be used to denote $r_0(M)$. Note that with this generalization no explicit knowledge of the total energy or mass is required since r_0 is simply the radius of the event horizon before any particles are emitted.

There is a pole at the horizon where $\dot{r} = 0$. For a nonextremal black hole $f'(r_0)$ and $g'(r_0)$ are both finite and nonzero at the horizon, so for these cases $\frac{1}{\dot{r}}$ only has a simple pole at the horizon with a residue of $\frac{2}{\sqrt{f'(r_0)g'(r_0)}}$. Hence the imaginary part of the action will be

$$\mathrm{Im}\, I = \frac{2\pi\omega}{\sqrt{f'(r_0)g'(r_0)}} + O(\omega^2) \tag{10}$$

Therefore the tunnelling probability is

$$\Gamma = \exp(-2\, \mathrm{Im} I) = \exp(-\beta\omega) \tag{11}$$

and so the Hawking temperature $T_H = \beta^{-1}$ is

$$T_H = \frac{\sqrt{f'(r_0)g'(r_0)}}{4\pi} \tag{12}$$

It is easy to confirm that for a Schwarzschild black hole the correct result of $T_H = \frac{1}{8\pi M}$ follows. Situations in which the horizon does not have a simple pole correspond to extremal black holes, and need to be handled separately. One conceptual issue that arises when applying either the Hamilton-Jacobi or null-geodesic methods to the extremal case is due to the fact that the model is dynamic, so emission of a neutral particle from the black hole implies a naked singularity, in violation of cosmic censorship. We will discuss the extremal case in Sec. IV.

B. Hamilton-Jacobi ansatz

We next consider an alternate method for calculating the imaginary part of the action making use of the Hamilton-Jacobi equation [16]. We assume that the action of the outgoing particle is given by the classical action I that satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + m^2 = 0 \tag{13}$$

To leading order in the energy we can neglect the effects of the self-gravitation of the particle.

For a metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + C(r)h_{ij}dx^i dx^j \tag{14}$$

the Hamilton-Jacobi Eq. (13) is

$$-\frac{(\partial_t I)^2}{f(r)} + g(r)(\partial_r I)^2 + \frac{h^{ij}}{C(r)}\partial_i I \partial_j I + m^2 = 0 \tag{15}$$

There exists a solution of the form

$$I = -Et + W(r) + J(x^i) \tag{16}$$

where

$$\partial_t I = -E, \qquad \partial_r I = W'(r), \qquad \partial_i I = J_i$$

and that the J_i's are constant. Solving for $W(r)$ yields

[2]See Ref. [17] for a discussion of self-gravitation effects in the context of the information-loss problem.

$$W(r) = \int \frac{dr}{\sqrt{f(r)g(r)}} \sqrt{E^2 - f(r)\left(m^2 + \frac{h^{ij}J_iJ_j}{C(r)}\right)} \quad (17)$$

(for an outgoing particle) and the imaginary part of the action can only come from the pole at the horizon. It is important to parameterize in terms of the proper spatial distance in order to get the correct result [16].

For the first method the Painlevé coordinate r was the proper spatial distance. In this case the proper spatial distance between any two points at some fixed t is given by

$$d\sigma^2 = \frac{dr^2}{g(r)} + C(r)h_{ij}dx^i dx^j \quad (18)$$

As with the null-geodesic method we are only concerned with radial rays, and so the only proper spatial distance we are concerned with is radial

$$d\sigma^2 = \frac{dr^2}{g(r)}$$

Employing the near horizon approximation

$$\begin{aligned} f(r) &= f'(r_0)(r - r_0) + \ldots \\ g(r) &= g'(r_0)(r - r_0) + \ldots \end{aligned} \quad (19)$$

we find that

$$\sigma = \int \frac{dr}{\sqrt{g(r)}} \simeq 2\frac{\sqrt{r - r_0}}{\sqrt{g'(r_0)}} \quad (20)$$

is the proper radial distance. So for particles emitted radially

$$\begin{aligned} W(\sigma) &= \frac{2}{\sqrt{g'(r_0)f'(r_0)}} \int \frac{d\sigma}{\sigma} \\ &\quad \times \sqrt{E^2 - \frac{\sigma^2}{4} g'(r_0)f'(r_0)\left(m^2 + \frac{h^{ij}J_iJ_j}{C(r_0)}\right)} \\ &= \frac{2\pi i E}{\sqrt{g'(r_0)f'(r_0)}} \end{aligned} \quad (21)$$

and from this point the computation is the same as for the previous method, yielding

$$T_H = \frac{\sqrt{f'(r_0)g'(r_0)}}{4\pi} \quad (22)$$

for the temperature.

III. APPLICATIONS

A. Rindler space

We first illustrate how these methods apply for the horizon of an accelerated observer. We shall employ different coordinate systems for 2D Rindler space to show that the same temperature results from applying the two tunneling methods directly.

The forms of the Rindler metric being used are:

$$ds^2 = -(a^2x^2 - 1)dt^2 + \frac{a^2x^2}{a^2x^2 - 1}dx^2 \quad (23)$$

$$ds^2 = -a^2x^2dt^2 + dx^2 \quad (24)$$

where a is the proper acceleration of the hyperbolic observer. Here there is no well defined total mass or energy as with Schwarzschild, but there are well defined horizons. The metric (23) locates the horizon at $x = \frac{1}{a}$, whereas for the metric (24) it is at $x = 0$.

We consider a null particle to be emitted from the Rindler horizon, and it is reasonable to assume the emitted particle will have a Hamiltonian associated with it. However providing an explicit definition for the total energy of the spacetime is less than clear, though it has been claimed recently [28] that one can associate a surface energy density $\sigma = \frac{a}{4\pi}$ with a Rindler horizon and a total energy $E = \frac{1}{4a}$ with the spacetime. In the context of the null-geodesic method we expect that the Hamiltonian of the spacetime will correspond to the total energy E (perhaps with respect to some reference energy via a limiting procedure) so as long as the emitted particles have $\omega \ll \frac{1}{4a}$, in which case the method is applicable. We shall proceed under the assumption that we can use Hamilton's equation and follow through the derivation for the null-geodesic method as before. We shall find that these assumptions are justified a-posteriori.

The null geodesics for (24) in the x-direction are given by

$$\dot{x} = \pm ax$$

and so

$$\mathrm{Im}\, I = \omega \int_{x_{\rm in}}^{x_{\rm out}} \frac{dx}{ax} = \frac{\pi\omega}{a}$$

yielding a temperature of

$$T_H = \frac{a}{2\pi}$$

which is the expected result for the temperature [29].

We now employ the Hamilton-Jacobi ansatz[3] for the Rindler metric (23).

Here $f = a^2x^2 - 1$, $g = \frac{a^2x^2 - 1}{a^2x^2}$ and at the horizon $f'(1) = g'(1) = 2a$ so using (21)

$$W = \frac{E\pi i}{a}$$

again giving a temperature of $T_H = \frac{a}{2\pi}$.

We see that we can recover the expected value for the temperature of Rindler space given our assumptions. This

[3]For earlier work in the Rindler context along these lines see Ref. [15].

could perhaps be regarded further evidence that a total energy $E = \frac{1}{4a}$ can be associated with Rindler space.

B. Charged-Kerr black hole

We consider next the Kerr-Newman solution. The Kerr-Newman metric and vector potential are given by

$$ds^2 = -f(r,\theta)dt^2 + \frac{dr^2}{g(r,\theta)} - 2H(r,\theta)dtd\phi + K(r,\theta)d\phi^2 + \Sigma(r,\theta)d\theta^2 \tag{25}$$

$$A_a = -\frac{er}{\Sigma(r)}[(dt)_a - a^2\sin^2\theta(d\phi)_a]$$

$$f(r,\theta) = \frac{\Delta(r) - a^2\sin^2\theta}{\Sigma(r,\theta)},$$

$$g(r,\theta) = \frac{\Delta(r)}{\Sigma(r,\theta)},$$

$$H(r,\theta) = \frac{a\sin^2\theta(r^2 + a^2 - \Delta(r))}{\Sigma(r,\theta)}$$

$$K(r,\theta) = \frac{(r^2 + a^2)^2 - \Delta(r)a^2\sin^2\theta}{\Sigma(r,\theta)}\sin^2(\theta)$$

$$\Sigma(r,\theta) = r^2 + a^2\cos^2\theta$$

$$\Delta(r) = r^2 + a^2 + e^2 - 2Mr$$

We assume a nonextremal black hole so that $M^2 > a^2 + e^2$ so that there are two horizons at $r_\pm = M \pm \sqrt{M^2 - a^2 - e^2}$.

There is a technical issue in applying these methods because the metric functions depend on the angle θ. In order to account for this we can no longer just look a generic spherical wave; instead we will examine rings of emitted photons for arbitrary fixed $\theta = \theta_0$. In the end we will discover our temperature is independent of θ_0 (as it should be).

A naive first attempt utilizing the null-geodesic method would be to consider the transformation

$$dt = dT - \sqrt{\frac{1 - g(r,\theta_0)}{g(r,\theta_0)f(r,\theta_0)}}dr$$

This gives the equation

$$ds^2 = -f(r,\theta_0)dT^2 + 2\sqrt{f(r,\theta_0)}\sqrt{\frac{1}{g(r,\theta_0)} - 1}drdT + dr^2 - 2Hd\phi\left(dT - \frac{\sqrt{\frac{1}{g(r,\theta_0)} - 1}}{\sqrt{f(r,\theta_0)}}dr\right) + Kd\phi^2 \tag{26}$$

whose radial null geodesics correspond to

$$\dot{r} = \sqrt{\frac{f(r,\theta_0)}{g(r,\theta_0)}}(\pm 1 - \sqrt{1 - g(r,\theta_0)}) \tag{27}$$

There remain divergences in the $dtdr$ and $drd\phi$ terms at the horizon, and $\frac{f(r,\theta_0)}{g(r,\theta_0)}$ is not well behaved there. Only for $\sin\theta_0 = 0$ are these eliminated. Restricting further the calculation to $\theta_0 = 0$ or π (in which case $\frac{f}{g} = 1$), the outgoing radial null geodesics along the z axis are

$$\dot{r} = 1 - \sqrt{1 - g(r,\theta_0)|_{\sin\theta_0 = 0}} \tag{28}$$

which yields

$$I = \omega\int_{r_{\rm out}}^{r_{\rm in}}\frac{dr}{\dot{r}} = \frac{2\pi\omega}{g'(r_+,\theta_0)|_{\sin\theta=0}} = 2\pi\omega\frac{r_+^2 + a^2}{2(r_+ - M)}$$

for the imaginary part of the action. This in turn results in the temperature

$$T_H = \frac{1}{2\pi}\frac{r_+ - M}{r_+^2 + a^2} = \frac{1}{2\pi}\frac{(M^2 - a^2 - e^2)^{1/2}}{2M(M + (M^2 - a^2 - e^2)^{1/2}) - e^2} \tag{29}$$

which is the same as the found for the Kerr-Newman black hole by other means.

The restriction to two specific values of θ_0 is because of the presence of the ergosphere. The calculation breaks down because $f(r,\theta)$ is actually negative elsewhere at the horizon (i.e. inside the ergosphere) and ∂_T is not properly timelike there. The two values $\theta_0 = 0$ or π correspond to where the event horizon and ergosphere coincide.

To address this issue, we note that the original charged-Kerr metric can be rewritten as

$$ds^2 = -F(r,\theta)dt^2 + \frac{dr^2}{g(r,\theta)} + K(r,\theta)\left(d\phi - \frac{H(r,\theta)}{K(r,\theta)}dt\right)^2 + \Sigma(r)d\theta^2$$

$$F(r,\theta) = f(r,\theta) + \frac{H^2(r,\theta)}{K(r,\theta)} = \frac{\Delta(r)\Sigma(r,\theta)}{(r^2 + a^2)^2 - \Delta(r)a^2\sin^2\theta} \tag{30}$$

where at the horizon

$$\frac{H(r_+,\theta)}{K(r_+,\theta)} = \frac{a}{r_+^2 + a^2} = \Omega_H$$

So the metric near the horizon for fixed $\theta = \theta_0$ is

$$ds^2 = -F_r(r_+,\theta_0)(r - r_+)dt^2 + \frac{dr^2}{g_r(r_+,\theta_0)(r - r_+)} + K(r_+,\theta_0)\left(d\phi - \frac{H(r_+,\theta_0)}{K(r_+,\theta_0)}dt\right)^2 \tag{31}$$

and defining $d\chi = d\phi - \frac{H(r_+,\theta_0)}{I(r_+,\theta_0)}dt$.

$$ds^2 = -F_r(r_+,\theta_0)(r-r_+)dt^2 + \frac{dr^2}{g_r(r_+,\theta_0)(r-r_+)} + K(r_+,\theta_0)(d\chi)^2 \tag{32}$$

The metric (32) is well behaved for all θ_0 and is of the same form as (3) with $f(r) = F_r(r_+,\theta_0)(r-r_+)$ and $g(r) = g_r(r_+,\theta_0)(r-r_+)$. Hence we easily obtain the final result (12)

$$T_H = \frac{\sqrt{F_r(r_+,\theta_0)g_r(r_+,\theta_0)}}{4\pi}$$

Explicit calculation of $F_r(r_+,\theta_0)$ and $g_r(r_+,\theta_0)$ yields

$$g_r(r_+,\theta_0) = \frac{\Delta_r(r_+)}{\Sigma(r_+,\theta_0)} = \frac{2r_+ - 2M}{r_+^2 + a^2\cos^2(\theta_0)}$$

$$F_r(r_+,\theta_0) = \frac{\Delta_r(r_+)\Sigma(r_+,\theta_0)}{(r_+^2+a^2)^2} = \frac{(2r_+-2M)(r_+^2+a^2\cos^2(\theta_0))}{(r_+^2+a^2)^2}$$

Although $F_r(r_+,\theta_0)$ and $g_r(r_+,\theta_0)$ each depend on θ_0, their product

$$F_r(r_+,\theta_0)g_r(r_+,\theta_0) = \frac{(2r_+-2M)^2}{(r_+^2+a^2)^2}$$

is independent of this quantity. Hence the temperature is

$$T_H = \frac{1}{2\pi}\frac{r_+-M}{r_+^2+a^2} = \frac{1}{2\pi}\frac{(M^2-a^2-e^2)^{1/2}}{2M(M+(M^2-a^2-e^2)^{1/2})-e^2}$$

for any angle.

We turn next to the Hamilton-Jacobi method to find the temperature. The action is assumed to be of the form

$$I = -Et + J\phi + W(r,\theta_0)$$

and rewriting this in terms of $\chi(r_+) = \phi - \Omega_H t$ we find

$$I = -(E - \Omega_H J)t + J\chi + W(r,\theta_0)$$

where it is assumed that $E - \Omega_H J > 0$. This demonstrates a nuance overlooked in the null-geodesic method; the transformation to χ implies that E should be replaced by $E - \Omega_H J$ for the emitted particle. The reason for this is the presence of the ergosphere. The Killing field that is timelike everywhere is $\chi = \partial_t + \Omega_H\partial_\phi$. A particle can escape to infinity only if $p_a\chi^a < 0$, and so $-E + \Omega_H J < 0$ where E and J are the energy and angular momentum of the particle.

Employing the metric in the near horizon form (31), the final result for $W(r,\theta_0)$ is the same as (21) with E replaced by $E - \Omega_H J$:

$$W(r,\theta_0) = \frac{2\pi i(E-\Omega_H J)}{\sqrt{F_r(r_+,\theta_0)g_r(r_+,\theta_0)}} = (E-\Omega_H J)\frac{\pi i(r_+^2+a^2)}{(r_+-M)} \tag{33}$$

again yielding the temperature over the full surface of the Black Hole

$$T_H = \frac{1}{2\pi}\frac{r_+-M}{r_+^2+a^2} = \frac{1}{2\pi}\frac{(M^2-a^2-e^2)^{1/2}}{2M(M+(M^2-a^2-e^2)^{1/2})-e^2}$$

in full agreement with the previous method and with Euclidean space techniques.

C. Taub-NUT-AdS solutions

The general Taub-NUT-AdS solutions with cosmological constant $\Lambda = -3/\ell^2$ are given by [25]

$$ds^2 = -F(r)\left(dt + 4n^2 f_k^2\left(\frac{\theta}{2}\right)d\varphi\right)^2 + \frac{dr^2}{F(r)} + (r^2+n^2)(d\theta^2 + f_k^2(\theta)d\varphi^2) \tag{34}$$

where

$$F(r) = k\frac{r^2-n^2}{r^2+n^2} + \frac{-2Mr + \frac{1}{\ell^2}(r^4 + 6n^2r^2 - 3n^4)}{r^2+n^2} \tag{35}$$

and k is a discrete parameter that takes the values $1, 0, -1$ and defines the form of the function $f_k(\theta)$

$$f_k(\theta) = \begin{cases} \sin\theta & \text{for } k=1 \\ \theta & \text{for } k=0 \\ \sinh\theta & \text{for } k=-1 \end{cases} \tag{36}$$

One of the interesting properties of Taub-NUT spaces is the existence of closed timelike curves (CTCs) [19]. For these cases it is not clear how to apply the null-geodesic method, since the emission of an s-wave particle would have to recur in a manner consistent with the presence of CTCs.

However there exists a special subclass of Hyperbolic Taub-NUT solutions (for $4n^2/\ell^2 \leq 1$) that do not contain CTCs. A discussion of Taub-NUT space and the special cases without CTCs appears in the Appendix. We shall consider these cases in what follows.

The temperature can be successfully calculated using the metric in the following form:

$$ds^2 = -Hdt^2 + \frac{dr^2}{F} + G\left(d\varphi - \frac{F4nf_k^2(\frac{\theta}{2})}{G}dt\right)^2 + (r^2+n^2)d\theta^2 \tag{37}$$

where:

$$H(r,\theta) = \left(F + F^2 \frac{16n^2 f_k^4(\frac{\theta}{2})}{G}\right) \quad (38)$$

$$G(r,\theta) = 4f_k^2\left(\frac{\theta}{2}\right)\left(r^2 + n^2 - f_k^2\left(\frac{\theta}{2}\right)(4n^2F + k(r^2+n^2))\right) \quad (39)$$

As before, we will consider rings at constant θ_0 and use the near horizon approximation.

At the horizon

$$\frac{G(r_+,\theta_0)}{f_k^2(\frac{\theta_0}{2})} = \begin{cases} 4\left((r_+^2+n^2)\cosh^2\left(\frac{\theta_0}{2}\right)\right), & k=-1 \\ 4(r_+^2+n^2), & k=0 \\ 4\left((r_+^2+n^2)\cos^2\left(\frac{\theta_0}{2}\right)\right), & k=1 \end{cases}$$

Only when $k=1$ (for which CTCs are present) and $\theta_0 = \pi$ (i.e. when $\cos(\frac{\theta_0}{2}) = 0$) are there any potential divergences at the horizon. Since

$$H_r(r_+,\theta_0) = F_r(r_+)$$

the metric near the horizon for fixed $\theta = \theta_0$ is

$$ds^2 = -F_r(r_+)(r-r_+)dt^2 + \frac{dr^2}{F_r(r_+)(r-r_+)} + G(r_+,\theta_0)\left(d\varphi - \frac{F_r(r_+)4nf_k^2(\frac{\theta}{2})}{G(r_+,\theta_0)}(r-r_+)dt\right)^2 \quad (40)$$

$$= -F_r(r_+)(r-r_+)dt^2 + \frac{dr^2}{F_r(r_+)(r-r_+)} + G(r_+,\theta)d\varphi^2 \quad (41)$$

Notice that defining $\chi = \varphi - \Omega_H t$ as with the charged-Kerr case is pointless since $\Omega_H = 0$. From this point the steps are the same as for the general procedures outlined for either the null-geodesic method or the Hamilton-Jacobi ansatz. Inserting this into the final result for temperature (either (12) or (22)) yields

$$T_H = \frac{F_r(r_+)}{4\pi} \quad (42)$$

which is the same form found using the Wick rotation method [25,26].

To demonstrate this is straightforward. Consider the hyperbolic case ($k=-1$). The mass parameter can be written in terms of the other metric parameters upon recognition that $F(r_+) = 0$ yielding

$$M = \frac{r_+^4 + (6n^2 - \ell^2)r_+^2 - n^2(3n^2 - \ell^2)}{2\ell^2 r_+}$$

Using this mass in (43) yields an expression for the hyperbolic Taub-NUT temperature of

$$T_H = \frac{4\pi\ell^2 r_+}{3(r_+^2 + n^2) - \ell^2} \quad (43)$$

Comparing this to the result [25] for the hyperbolic Taub-NUT temperature obtained from Wick rotation methods

$$T_H = \frac{4\pi\ell^2 r_+}{3(r_+^2 - N^2) - \ell^2} = \frac{F_r(r_+)}{4\pi} \quad (44)$$

(where N is the Wick rotated NUT charge) we obtain agreement upon recognizing that $n^2 = -N^2$ due to analytic continuation. Note however that there is an implicit analytic continuation in the definition of r_+, since $F(r_+, n) \to F(r_+, iN)$ [26].

We close by commenting that although we considered only the $k=-1$ case to avoid problems with CTCs, both the $k = 0$, 1 cases can be formally carried through, yielding the result (42). In the context of the null-geodesic method this situation could perhaps be interpreted by noting that Hawking radiation yields a thermal bath of particles, whose existence can statistically be reconciled with the presence of CTCs. In the context of the Hamilton-Jacobi ansatz the physical interpretation is less problematic provided the classical action for the particle can be considered to obey the Hamilton-Jacobi equation in the presence of CTCs. Our results suggest a-posteriori the answer is yes, but the matter merits further study. In this context we note recent work [30] demonstrating that there are no SU(2)-invariant (time-dependent) tensorial perturbations of asymptotically flat Lorentzian Taub-NUT space, calling into question the possibility that a physically sensible thermodynamics can be associated to Lorentzian Taub-NUT spaces without cosmological constant. Whether or not such results extend to Taub-NUT spaces without CTCs is an interesting question.

IV. EXTREMAL BLACK HOLES

Extremal black holes need to be treated separately from the other generalizations, since the integrand no longer has a single pole. The general results derived above are no longer valid and even the self-gravitating terms may play a very important role. One of the properties that occurs in extremal case is the presence of a divergent real component in the action. Although such a term does not contribute to the imaginary part of the action, this may be an indication that the tunnelling approach is breaking down and the calculation is becoming pathological. Unlike the Wick-rotation method, which involves finding an equilibrium temperature, the tunnelling approach describes a dynamical system. In this latter context when a black hole is extremal the possibility exists that an emitted neutral particle may cause the creation of a naked singularity, in violation of cosmic censorship.

Such a pathological situation would be prevented if the tunnelling barrier had infinite height. However we do not find this to be the case, and an evaluation of the imaginary part of the action yields a finite temperature. This is consistent with the proposal that extremal black holes can be in thermal equilibrium at any temperature [31].

For concreteness, we shall consider the particular case of the Reissner-Nordstrom metric, though we note that a diverging real component has also been seen to occur with the extremal GHS solution [16].

Extremal Reissner-Nordstrom black hole

The Reissner-Nordstrom spacetime is described by the metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} + \frac{Q^2}{r^2})} + r^2 d\Omega^2 \quad (45)$$

The black hole is nonextremal when $M^2 > Q^2$ and extremal when $Q = M$. For the nonextremal case when the tunnelling approach yields a temperature of

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{(M^2 - Q^2)})^2} \quad (46)$$

using either of (12) or (22). Note that the limit $Q \to M$ gives a temperature of zero.

For the Reissner-Nordstrom case self-gravitating effects have been calculated exactly [9] and the full emission rate is

$$\Gamma \sim e^{-2I}$$
$$= e^{-2\pi(2\omega(M-(\omega/2))-(M-\omega)\sqrt{(M-\omega)^2-Q^2}+M\sqrt{M^2-Q^2})} \quad (47)$$

Expanding this emission rate in powers of ω yields the temperature (46) to leading order. Note that setting $Q = M$ yields a contradictory result, since the second term in the exponent becomes imaginary. This unphysical situation corresponds to an extremal black hole emitting a particle, a situation in violation of cosmic censorship.

Consider a nearly extremal black hole that emits a particle so that the resulting black hole is extremal. This corresponds to substitution of $Q = (M - \omega)$ where the black hole emits a null particle of energy ω. Insertion of this value of Q into (46) yields

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - (M - \omega)^2}}{(M + \sqrt{(M^2 - (M - \omega)^2)})^2}$$
$$= \frac{1}{2\pi} \frac{\sqrt{2M\omega}}{M^2} + O(\omega) \quad (48)$$

Comparing this to the temperature obtained from the emission rate using (47) gives

$$\Gamma = e^{-2\pi(2\omega(M-(\omega/2))+M\sqrt{M^2-(M-\omega)^2})}$$
$$= e^{-2\pi(M\sqrt{2M\omega}+2\omega M+O(\omega^{3/2}))}$$

From the definition (11) we find that the temperature that is $O(\sqrt{\omega})$ and again approaches zero the closer the original black hole is to extremality. Explicitly

$$T = \frac{1}{2\pi} \frac{\omega}{M\sqrt{2M\omega}} = \frac{1}{4\pi} \frac{\sqrt{2M\omega}}{M^2} \quad (49)$$

which differs from the value given in (48) by a factor of 1/2. This discrepancy arises due to an inappropriate expansion implicitly used in obtaining (48), which assumes that $\omega \ll \frac{M^2-Q^2}{2M}$, an invalid assumption for $Q = (M - \omega)$. In this context we note earlier work demonstrating that the transition probability of emitting such a particle, that will make the black hole extremal, is zero [7].

We obtain a temperature that depends on the energy of the emitted particle. We pursue the extremal case further by considering a direct attempt to find the temperature from the metric in its extremal form

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \frac{dr^2}{(1 - \frac{M}{r})^2} + r^2 d\Omega^2 \quad (50)$$

Using the Hamilton-Jacobi Ansatz as a first attempt at the calculation yields only a diverging real component. i.e.

$$f(r) = g(r) = \frac{1}{M^2}(r - M)^2 + O((r - M)^3)$$

so that

$$\sigma = \int \frac{dr}{\sqrt{g(r)}} = M \int \frac{dr}{(r - M)} \simeq M \ln(r - M) \quad r - M = e^{\sigma/M}$$

where $M < r < \infty$ implies that bounds on σ are now $-\infty < \sigma < \infty$. Rather than considering an observer at infinity, we will consider an observer outside the horizon at some r_1 corresponding to $\sigma(r_1)$. From (17)

$$W(r) = \int \frac{dr}{\frac{1}{M^2}(r - M)^2} \times \sqrt{E^2 - \frac{1}{M^2}(r - M)^2\left(m^2 + \frac{h^{ij}J_iJ_j}{C(r)}\right)}$$
$$= M \int_{-\infty}^{\sigma(r_1)} \frac{d\sigma}{e^{\sigma/M}} \sqrt{E^2 - \frac{1}{M^2} e^{2(\sigma/M)}\left(m^2 + \frac{h^{ij}J_iJ_j}{C(r_0)}\right)}$$
$$= M \int_{-\infty}^{\sigma(r_1)} d\sigma \sqrt{E^2 e^{-2\sigma/M} - \frac{1}{M^2}\left(m^2 + \frac{h^{ij}J_iJ_j}{C(r_0)}\right)} \quad (51)$$

For convenience we choose $\sigma(r_1)$ so that the term under the root is never negative. This integral is diverging and real, suggesting that no particles are emitted [16]. However this result is suspect in that it may be contingent on employing the near horizon approximation in the early stages of this method.

We turn next to the null-geodesic method. The outward radial null geodesic is given by

$$\dot{r} = 1 - \sqrt{1 - \left(1 - \frac{M}{r}\right)^2} \tag{52}$$

$$= \frac{1}{2M^2}(r-M)^2 - \frac{1}{M^3}(r-M)^3 + O((r-M)^4) \tag{53}$$

Insertion of this into (9) yields

$$\begin{aligned}
\mathrm{Im} I &\simeq \mathrm{Im}\left[-\omega \int_0^{\pi} \frac{\epsilon i e^{i\theta} d\theta}{\frac{1}{2M^2}\epsilon^2 e^{2i\theta}(1 + \frac{2}{M}\epsilon e^{i\theta})}\right] \\
&= -2\omega M^2 \,\mathrm{Im}\left[\int_0^{\pi}\left(\frac{i}{\epsilon e^{i\theta}} + \frac{2i}{M(1+\frac{2}{M}\epsilon e^{i\theta})}\right)d\theta\right] \\
&= \mathrm{Im}\left[O\left(\frac{1}{\epsilon}\right) + 4M\omega\left[\ln\left(e^{-i\theta} + \frac{2\epsilon}{M}\right)\right]\Big|_0^{\pi}\right] \\
&= 4M\omega\,\mathrm{Im}\left[\ln\left(\frac{-1+\frac{2\epsilon}{M}}{1-\frac{2\epsilon}{M}}\right)\right] \\
&= (2n+1)4\pi M\omega
\end{aligned} \tag{54}$$

where we have written $(r-M) = -\epsilon e^{i\theta}$ and n is an integer. The first part of the integral is a real contribution of $O(\frac{1}{\epsilon})$ that diverges as $\epsilon \rightarrow 0$. It does not contribute to the imaginary part of the action. The imaginary part of the action leads to a nonzero finite temperature

$$T_H = \frac{1}{8\pi M(2n+1)} \tag{55}$$

for any integer n. The extremal temperature is quantized in units of the temperature of a Schwarzschild black hole!

Note that this result depends crucially on the inclusion of the third order term, whose evaluation depends upon assumptions of the choice of Riemannian sheet. Had we expanded the integral for small ϵ, we would have obtained a value for the temperature given by $n = -1$ in Eq. (55), ie a negative temperature for the extremal black hole.

Obtaining many (finite-valued) results for the temperature is reminiscent of the proposal that an extremal black hole can be in thermal equilibrium at any finite temperature [31]. However we can see that these strange results arise due to an inappropriate use of the WKB approximation in the null-geodesic method. Although writing $(r-M) = -\epsilon e^{i\theta}$ is consistent with the the assumptions $r_{\mathrm{in}} = r_0(M) - \epsilon$ and $r_{\mathrm{out}} = r_0(M-\omega) + \epsilon$ (where $r_0(M)$ denotes the location of the event horizon of the original background spacetime) for a nonextremal black hole, in fact the quantity r_{out} does not exist, since the extremal black hole cannot retain an event horizon upon emitting any neutral quantum of energy—its only option for future evolution would appear to be that of evolving into a naked singularity, which cosmic censorship forbids.

These results seem to imply that for black holes near extremality one must consider the full self-gravitating results, where the emitted particle drives the hole toward extremality. For an already extremal spacetime both methods yield a diverging real component in the action. This could be taken to imply that no particle can be emitted (since the alternative is creation of a naked singularity).

Based on the results of this calculation it would be interesting to consider the emission of a specific charged particle that would cause the black hole to go from one extremal black hole to another extremal black hole. In that case there would be well defined horizons before and after emission.

V. CONCLUSIONS

We have examined and compared the two different approaches to the tunnelling method for finding the black hole temperatures. Our results indicate that the method is particularly robust for nonextremal black holes, yielding results commensurate with other methods for Rindler space, rotating black holes, and Taub-Nut black holes. In this latter instance we have provided independent verification of the temperatures obtained for Taub-NUT spaces without CTCs via analytic continuation methods. Indeed it is not too difficult to show that the temperatures even match when CTCs are present, though in this case an a-priori justification for the method is unclear.

We also investigated extremal black holes, for which the tunnelling method is somewhat more problematic due to its dynamic nature. We found that the temperature is proportional to the energy of the emitted particles for black holes close to extremality. We also found that both methods yield a divergent real part to the action for extremal black holes, which is suggestive of a full suppression of particle emission. However the null-geodesic method has a nonzero finite imaginary part, whose value yields a countably infinite number of possible finite temperatures for an extremal Reissner-Nordstrom black hole. This rather strange result arises because of a breakdown of the WKB method in the null-geodesic approximation. This suggests limitations on the method, whose study would make an interesting subject for future work. An interesting test case would be that of emission of charged particles from an extremal black hole.

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APPENDIX

CTC's and Taub-NUT space

The presence of closed timelike curves in Taub-NUT space can be seen by considering the curve generated by the Killing vector ∂_φ and by examining $g_{\varphi\varphi}$

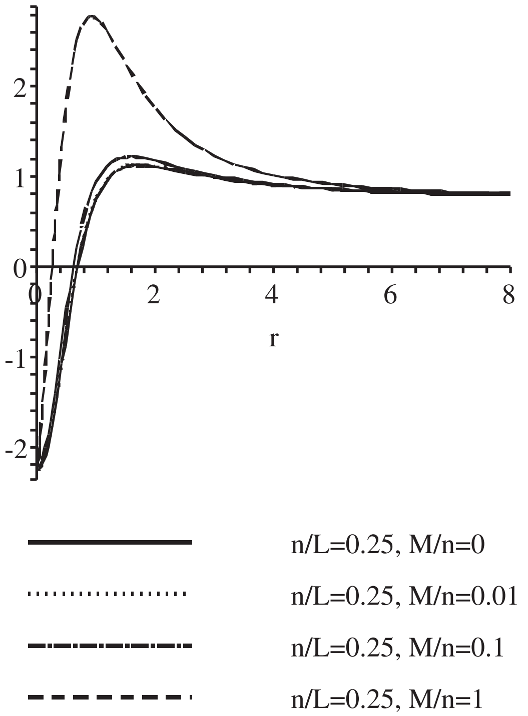

FIG. 1. Plots when $4n^2/L^2 < 1$ for a range of masses.

$$g_{\varphi\varphi} = 4f_k^2\left(\frac{\theta}{2}\right)\left(r^2 + n^2 - f_k^2\left(\frac{\theta}{2}\right)(4n^2F + k(r^2 + n^2))\right)$$

So for $k = 1$, 0, and $k = -1$ with $4n^2/\ell^2 > 1$ the quantity $g_{\varphi\varphi} < 0$, yielding a timelike ∂_φ; the curve $r = r_0$, $t = t_0$, and $\theta = \theta_0$ becomes a CTC.

However there is a range of hyperbolic Taub-NUT solutions that occur when $4n^2/\ell^2 \le 1$ that do not contain CTC's. Now it is possible for $g_{\varphi\varphi}$ to be negative when $4n^2/\ell^2 < 1$ but this occurs for small values of r_0 and happens inside the horizon. Explicitly when $k = -1$ then $g_{\varphi\varphi}$ is given by

$$g_{\varphi\varphi} = 4\sinh^2\left(\frac{\theta}{2}\right)(r^2 + n^2)\left(\cosh^2\left(\frac{\theta}{2}\right) - \frac{4n^2F}{r^2 + n^2}\sinh^2\left(\frac{\theta}{2}\right)\right)$$

So $g_{\varphi\varphi} \ge 0$ will always be true as long as $\frac{4n^2F}{r^2+n^2} \le 1$. Figs. 1–3 are plots of $1 - \frac{4n^2F}{r^2+n^2}$ for a range of mass and NUT-charge. On the plots the x-axis is r/n. The $k = -1$ case corresponds to hyperbolic solutions whose event horizon has radius $r_b > n$. Since $g_{\varphi\varphi}$ only becomes negative when $r < n$ (within $4n^2/\ell^2 \le 1$) any CTCs are contained within the horizon (provided the mass is positive). So no CTC's are present outside of the horizon for the hyperbolic case when $4n^2/\ell^2 \le 1$.

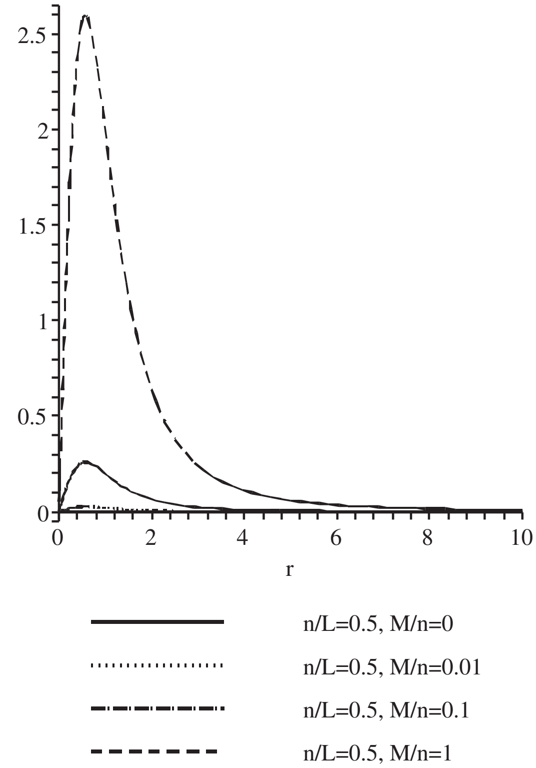

FIG. 2. Plots for when $4n^2/L^2 = 1$ for range of masses.

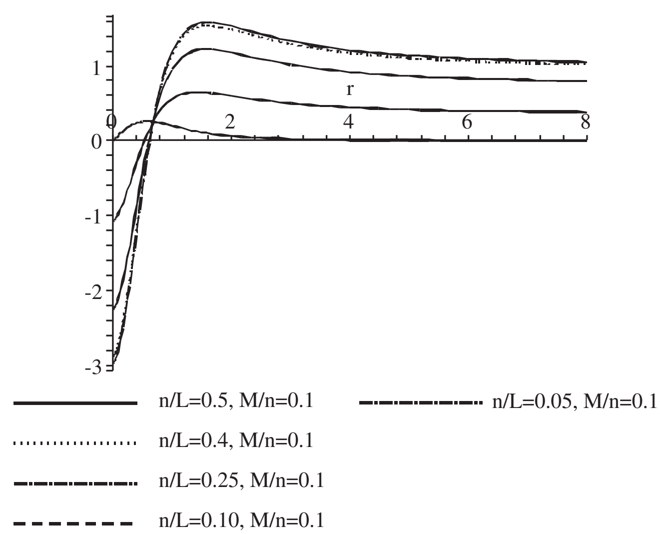

FIG. 3. Plots for fixed mass and a range of n^2/L^2.

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