PHYSICAL REVIEW D **73,** 103520 (2006)

Cosmologies with energy exchange

John D. Barrow* and T. Clifton†

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, United Kingdom
(Received 17 April 2006; published 30 May 2006)

We provide a simple mathematical description of the exchange of energy between two fluids in an expanding Friedmann universe with zero spatial curvature. The evolution can be reduced to a single nonlinear differential equation which we solve in physically relevant cases and provide an analysis of all the possible evolutions. Particular power-law solutions exist for the expansion scale factor and are attractors at late times under particular conditions. We show how a number of problems studied in the literature, such as cosmological vacuum-energy decay, particle annihilation, and the evolution of a population of evaporating black holes, correspond to simple particular cases of our model. In all cases we can determine the effects of the energy transfer on the expansion scale factor. We also consider the situation in the presence of ''antidecaying'' fluids and so-called ''phantom'' fluids which violate the dominant energy conditions.

DOI: 10.1103/PhysRevD.73.103520 PACS numbers: 98.80.−k

I. INTRODUCTION

There are many cosmological situations where the transfer of energy between two fluids is important. The interaction between matter and radiation [1], the decay of massive particles into radiation [2], matter creation [3], the formation and evaporation of primordial black holes [4], the annihilations of particle-antiparticle pairs [5], particle or string production [6,7], inflaton decay [8] and the decay of some scalar field [9] or vacuum energy [10] are all particular examples which have been studied in general-relativistic cosmology. The situation in Brans-Dicke cosmology has also been investigated [11], as have the cases of two arbitrary interacting fluids [12] and more than two interacting fluids [13]. In some cases, as in the example of accreting and evaporating black holes, there will be a two-way transfer of energy occurring as, say, a spectrum of radiation inhomogeneities collapse under their self-gravity in the early universe to form a population of primordial black holes but the products of the Hawking evaporation of the black holes add to the cosmological population of interacting relativistic particles [4]. The different studies of these particular situations have often identified the existence of special power-law scaling solutions. In this paper we consider a general problem of this sort, describe its general behavior, relate it to the existence of special power-law solutions, and describe its general solution succinctly in terms of the parameters defining the energy exchanges. The examples in the literature can then be shown to be particular examples of these solutions and the conditions for their stability are made clear.

We will consider the mutual exchange of energy between two fluids at rates that are proportional to a linear combination of their individual densities and the expansion rate of the universe. In the absence of any interaction the fluids reduce to two separate perfect fluids.

*Email address: J.D.Barrow@damtp.cam.ac.uk
†Email address: T.Clifton@damtp.cam.ac.uk

II. DECAYING FLUIDS

Consider a flat Friedmann Robertson Walker (FRW) universe with expansion scale factor $a(t)$ containing two fluids with equations of state

$$p = (\gamma - 1)\rho, \qquad p_1 = (\Gamma - 1)\rho_1,$$

where the γ and Γ are constants, and the evolution of the Hubble parameter $H = \dot{a}/a$ is governed by the Friedmann equation

$$3H^2 = \rho + \rho_1, \tag{1}$$

where $8\pi G \equiv 1$. Assume that the two fluids exchange energy but the total energy is conserved so that

$$\dot{\rho}_1 + 3H\Gamma\rho_1 = -\beta H\rho_1 + \alpha\rho H, \tag{2}$$

$$\dot{\rho} + 3H\gamma\rho = \beta H\rho_1 - \alpha\rho H, \tag{3}$$

where α and β are constants parametrizing the energy exchanges between the two fluids. Generalizations of this simple cosmology to spatially curved or anisotropic universes can be made in an obvious way if required [7,14]. In an expanding universe ($H > 0$) this scenario corresponds to ρ and ρ_1 ''decaying'' into each other in proportion to their energy densities if α and β are positive. The degenerate case $\gamma = \Gamma$ can be seen to be trivially equivalent to the standard scenario without energy exchange, by considering the fluid $\rho_2 = \rho + \rho_1$.

Using the last three equations we can eliminate the densities to obtain a single master equation for the Hubble expansion, $H(t)$:

$$\ddot{H} + H\dot{H}(\alpha + \beta + 3\gamma + 3\Gamma) + \tfrac{3}{2}H^3(\alpha\Gamma + \beta\gamma + 3\Gamma\gamma) = 0. \tag{4}$$

Let us rewrite Eq. (4) as

$$\ddot{H} + AH\dot{H} + BH^3 = 0 \tag{5}$$

with

$$A \equiv \alpha + \beta + 3\gamma + 3\Gamma, \qquad B \equiv \tfrac{3}{2}(\alpha\Gamma + \beta\gamma + 3\Gamma\gamma).$$

This equation is a special case of the more general differential equation[1] considered by Chimento [15]. In [15] Chimento investigates the mathematical structure of this equation by showing that it has a form invariance, which is subsequently used to find solutions.

Simple self-similar solutions to Eq. (5) exist with

$$H = \frac{h}{t} \tag{6}$$

where, for $h \neq 0$,

$$2 - Ah + Bh^2 = 0,$$

and there are two nontrivial solutions, $H_+(t)$ and $H_-(t)$, with h values

$$h_\pm = \frac{A \pm \sqrt{A^2 - 8B}}{2B}. \tag{7}$$

These real power-law solutions for $H(t)$ exist iff $A^2 \geq 8B$. For α, β, γ, $\Gamma \geq 0$ and $\gamma \neq \Gamma$, this inequality is always satisfied. We can see this by defining

$$\delta \equiv \frac{B}{A^2} = \frac{3(\alpha\Gamma + \beta\gamma + 3\Gamma\gamma)}{2(\alpha + \beta + 3\gamma + 3\Gamma)^2}, \tag{8}$$

so that $A^2 \geq 8B$ iff $\delta \leq 1/8$. We see that the denominator in Eq. (8) is always positive, so δ is always nonsingular and positive for finite and semidefinite positive values of α, β, γ and Γ. It can also be seen that $\delta \to 0$ as either α or $\beta \to \infty$. The maximum value of δ must therefore occur at finite values of α and β. If this maximum exists when α and β are both nonzero then there must exist a point at which

$$\frac{\partial\delta}{\partial\alpha} = \frac{\partial\delta}{\partial\beta} = 0.$$

Using (8) we can see that this condition is never met, so the maximum value of δ must exist when $\alpha = 0$, when $\beta = 0$, or when $\alpha = \beta = 0$. For $\alpha = 0$ we will have a maximum at nonzero β when

$$\left(\frac{\partial\delta}{\partial\beta}\right)_{\alpha=0} = 0 \quad \text{and} \quad \left(\frac{\partial^2\delta}{\partial\beta^2}\right)_{\alpha=0} \neq 0,$$

which occurs iff $\beta = 3(\gamma - \Gamma)$, for α, β, γ, $\Gamma \geq 0$ and $\gamma \neq \Gamma$. Similarly, the maximum can occur at nonzero α when $\beta = 0$ iff $\alpha = 3(\Gamma - \gamma)$. We can choose, without loss of generality, $\Gamma > \gamma$ so that the maximum value of δ occurs when $\alpha = 3(\Gamma - \gamma)$ and $\beta = 0$, and we have the conclusion that

$$\delta \leq \delta_{\max} = \tfrac{1}{8}$$

for all γ, $\Gamma \geq 0$ and $\gamma \neq \Gamma$.

[1] $\ddot{y} + \alpha f \dot{y} + \beta \int f dy + \gamma f = 0$, where $y = y(x)$, $f = f(y)$ and overdots denote differentiation with respect to x. α, β and γ are constants.

Having established the existence of the power-law solutions (7), we can now show that they behave as attractors of the general solution by solving (4). For $A^2 > 8B$ we find the solution

$$H^2 = a^{-A/2}(c_1 a^{\sqrt{A^2-8B}/2} + c_2 a^{-\sqrt{A^2-8B}/2}), \tag{9}$$

where c_1 and c_2 are constants. This solution to (5) was previously found by Chimento in [15]. As $a \to \infty$, we then have

$$H^2 \to a^{-(A-\sqrt{A^2-8B})/2}$$

and, as $a \to 0$,

$$H^2 \to a^{-(A+\sqrt{A^2-8B})/2}.$$

These two equations can be integrated to obtain

$$a_\pm \propto t^{(A\pm\sqrt{A^2-8B})/2B}, \tag{10}$$

which are the power-law solutions (7), found earlier in Eq. (6). By integrating (9) we can show explicitly the existence of the above power-law attractors, and the smooth evolution of a between them. It is possible to integrate (9) to get a solution in terms of t and the hypergeometric function $2F_1(\tilde{a}, \tilde{b}; \tilde{c}; x)$. An expression in terms of more transparent functions can be found by defining a new time coordinate $d\tau \equiv a^{-(A+\sqrt{A^2-8B})/4}dt$ and integrating in terms of τ. This gives the solution

$$a \propto e^{\sqrt{c_2}(\tau-\tau_0)}(1 - e^{\sqrt{c_2}\sqrt{(A^2-8B)}(\tau-\tau_0)})^{-2/\sqrt{(A^2-8B)}}. \tag{11}$$

This is the same form for the evolution of a that was found by Chimento and Lazkoz in their investigation of phantom fluids in k-essence [16]. It can be seen from this expression that $a \to 0$ as $a \sim e^{\sqrt{c_2}\sqrt{(A^2-8B)}(\tau-\tau_0)}$ when $\tau \to -\infty$. In terms of the coordinate t, this corresponds to the solution a_- above. As $a \to \infty$, the solution smoothly approaches $a \sim (\tau - \tau_0)^{-2/\sqrt{A^2-8B}}$ as $\tau \to \tau_0$, which corresponds to the solution a_+.

We have now shown that the two power-law solutions (10) exist and for all α, β, γ, $\Gamma \geq 0$ and are the attractors of the smoothly evolving general solution at late and early times when $A^2 > 8B$.

It remains to investigate the limiting case $A^2 = 8B$. The exact solution to Eq. (4) when $A^2 = 8B$ is [15]

$$H^2 = a^{-A/2}(c_3 + c_4 \ln a), \tag{12}$$

where c_3 and c_4 are constants. For $c_4 = 0$ this solution corresponds to power-law expansion described by the degenerate case where $a_+ \equiv a_-$. For $c_4 \neq 0$ this solution is more complicated and is bounded by $a = e^{-c_3/c_4}$ while approaching $H^2 \sim a^{-A/2} \ln a$ as $a \to 0$ or ∞, which does not describe a power-law behavior.

An illustrative special exact solution to Eq. (4) exists when $B = A^2/9$, as was shown by Chimento [15]. In this case (4) can be linearized to $\dddot{\psi} = 0$ by the substitution $H = (3/A)\dot{\psi}/\psi$. Hence, for this special value of B

$$H = \frac{3(c_5 + 2c_6 t)}{A(1 + c_5 t + c_6 t^2)}, \quad (13)$$

where c_5, c_6 are constants of integration. This expression can be integrated to

$$\left(\frac{a}{a_0}\right)^{A/3} = 1 + c_5 t + c_6 t^2, \quad (14)$$

where a_0 is a constant and the early- and late-time behavior is clear and has the same form as the power-law solutions (7) when $B = A^2/9$.

III. EVOLUTION OF THE ENERGY DENSITIES

The conservation equations (2) and (3) can be used to construct the second-order differential equation

$$\frac{\rho''}{\rho} + A\frac{\rho'}{\rho} + 2B = 0 \quad (15)$$

where A and B are defined as before and primes denote differentiation with respect to the variable $\eta = \ln a$. This equation can be solved for ρ and the corresponding solution for ρ_1 can then be found from (3). Substituting these solutions into the Friedmann equation (1) gives, for $A^2 > B$, the solution (9) that was previously found by solving the master equation (5).

The advantage of considering the evolution of ρ directly is that a particularly interesting behavior can be observed in the evolution of the ratio ρ/ρ_1 for the self-similar solutions (6). To find this behavior we first note that a solution to Eq. (15) is given by

$$\rho = \rho_0 a^N$$

where ρ_0 is a constant and $2N = -A \pm \sqrt{A^2 - 8B}$. Substituting this into Eq. (3) gives the corresponding solution for ρ_1,

$$\rho_1 = \rho_{10} a^N$$

where $\rho_{10} = (N + 3\gamma + \alpha)\rho_0/\beta$ is constant. These solutions for ρ and ρ_1, when substituted into the Friedmann equation (1), correspond to the self-similar solutions for H given by (6). It is immediately apparent that ρ and ρ_1 evolve at the same rate and so the ratio ρ/ρ_1 is a constant quantity

$$\frac{\rho}{\rho_1} = \frac{\beta}{(N + 3\gamma + \alpha)}$$

during a period described by the power-law evolution (10). It is this constant ratio in the energy density of two fluids with different barotropic indices γ and Γ that has been used by a number of authors in an attempt to alleviate the coincidence problem concerning the present-day values of the vacuum and matter energy densities [9,10].

IV. THREE EXAMPLES

The exact solutions found in the last sections provide us with extensions of the analysis of several cosmological problems that have been studied in the past, which can be defined by particular choices of the two parameters A and B. As we have seen, the overall dynamical behavior is determined by the behavior of the combination $\delta \equiv B/A^2$.

A. Particle-antiparticle annihilation

Consider the problem of the long-term evolution of a universe containing equal numbers of electron-positron pairs [5]. If these particles are assumed to be the lightest massive charged leptons then they cannot decay, and can only disappear by means of the mutual annihilations $e^- e^+ \to 2\gamma$. Page and McKee set up a model for the $e^- e^+$ annihilation into radiation that corresponds to taking the special case $\alpha = 0$, $\beta > 0$, $\Gamma = 1$, $\gamma = 4/3$ in Eqs. (2) and (3) and the definition of $\beta_{\rm PM}$ by Page and McKee is given in terms of our β by $\beta \equiv 3\beta_{\rm PM}/(2 - \beta_{\rm PM})$. They find the power-law solution with $h = h_- = 2/(\beta + 3) = (2 - \beta_{\rm PM})/3$ which reduces to the usual dust FRW model when $\beta = \beta_{\rm PM} = 0$ and there is no annihilation into radiation. The effect of the annihilations is to push the expansion away from the dust-dominated form with $a = t^{2/3}$ towards the radiation-dominated evolution with $a = t^{1/2}$. The other power-law solution corresponds to the pure radiation case with $h = h_+ = 1/2$. We can verify that this power-law solution is an attractor by evaluating δ, since for the $e^- e^+ \to 2\gamma$ annihilation $\beta_{\rm PM} = (13 - \sqrt{105})/8 = 0.3441$ so $\delta \equiv B/A^2 = 0.1247 < 1/8$.

B. Primordial black-hole evolution

A more complicated energy exchange problem was formulated by Barrow, Copeland, and Liddle [4] who consider the problem of a power-law mass spectrum of primordial black holes forming in the early universe and then evolving under the effects of Hawking evaporation of the part of the mass spectrum with Hawking lifetimes less than the expansion age. This has two effects. The radiation background is supplemented by input from the black-hole evaporation products and the fall in the total black-hole density goes faster than the adiabatic $\rho_{\rm bh} \propto a^{-3}$ that occurs in the absence of decays because the black-hole population is a pressureless gas to a very good approximation, since $p/\rho \sim v^2 \sim T/M_{\rm bh} \sim (m_{\rm pl}/M_{\rm bh})(t_{\rm pl}/t)^{1/2} \approx 0$ for masses less than the Planck mass $m_{\rm pl}$ at times greater than the Planck time $t_{\rm pl}$. Accretion of background radiation in the radiation era of the universe by the black holes could be included, but is negligible. This corresponds to our model in the special case $\Gamma = 1$, $\gamma = 4/3$, $\alpha = 0$ and

$$\beta = \frac{3(n-2)}{8-n},$$

where the initial number density spectrum of black holes with masses between m and $m + \delta m$ at time t is given by $N(m, t) \propto m^{-n}$ and $n > 2$.

A power-law solution was found in Ref. [5] with $h = (8 - n)/9$, so long as $2 < n < 7/2$, and the black-hole evaporations have a significant effect on the expansion rate of the universe during the radiation era. We have

$$A = \frac{2(25 - 2n)}{8 - n}, \qquad B = \frac{36}{8 - n},$$

and so

$$\delta \equiv \frac{B}{A^2} = \frac{9(8 - n)}{(25 - 2n)^2}.$$

We see that the allowed range of $n \in (2, 7/2)$ corresponds to $\delta \in (6/49, 1/8)$ and the expansion scale factor evolves as $a \propto t^{(8-n)/9}$. The $n = 2$ limit corresponds to a pure dust-dominated expansion, with $a \propto t^{2/3}$, while the $n = 7/2$ limit corresponds to a pure radiation-dominated evolution, with $a \propto t^{1/2}$. Again we see that the power-law solution is an attractor for the general solution with n in this range. When the expansion of the universe becomes dominated by cold dark matter, with $\rho_{\rm cdm} \propto a^{-3}$, a power-law scaling solution no longer exists because the radiation products from the black-hole evaporations now make a negligible contribution to the total density of the universe, which is dominated by $\rho_{\rm cdm} > \rho_{\rm bh} \gg \rho_\gamma$, and $a \propto t^{2/3}$ becomes the attractor for the evolution of the expansion scale factor.

C. Vacuum decay

The cosmological evolution created by the decay of a vacuum stress ($\rho_1 = \rho_v$) into equilibrium radiation was considered by Freese *et al.* and many other authors [10]. It is described by a special case of our equations (2) and (3) with $\Gamma = 1$, $\gamma = 4/3$, $\alpha = 0$ and $\beta > 0$. It represents the decay of a scalar field stress with $p \approx -\rho$ into radiation. In this case we have

$$A = \beta + 4, \qquad B = 2\beta, \qquad \delta \equiv \frac{B}{A^2} = \frac{2\beta}{(\beta + 4)^2}, \tag{16}$$

with $h_+ = 1/2$ and $h_- = 2/\beta$. We see that the first of these corresponds to the degenerate situation with pure radiation. The second solution has $a \propto t^{2/\beta}$ and requires $\beta > 3$ if the evolution of the universe is to have a matter-dominated era following a radiation era. As the value β increases, the dominance of the vacuum contribution slows the expansion whereas in the limit $\beta \to 0$ the expansion rate increases without bound and the dynamics approaches the usual vacuum-energy dominated de Sitter expansion with $a \propto \exp(t\sqrt{\rho_v}/3)$. Again we see that this simple solution can be generalized by using the full analysis provided above. We see from (16) that we always have $\delta \leq 1/8$ with the maximum of δ achieved when $\beta = 4$. The solution (13) and (14) arises for $\delta = 1/9$ which occurs when $\beta = 2$ or $\beta = 8$.

V. ANTIDECAYING FLUIDS

It was shown in Sec. II that for $\alpha, \beta, \gamma, \Gamma \geq 0$ and $\gamma \neq \Gamma$ the maximum value that δ can take is $1/8$. If we relax these assumptions, then δ can take values greater than $1/8$ and the qualitative character of the solutions to (4) is significantly altered. We will begin by investigating the conditions required for $\delta > 1/8$.

In the previous section it was shown that for $\alpha, \beta, \gamma, \Gamma \geq 0$ and $\gamma \neq \Gamma$ the only point at which $\delta = 1/8$ is at $\alpha = 3(\Gamma - \gamma)$ and $\beta = 0$, and at all other points in this parameter range we have $\delta < 1/8$. It can be seen from (8) that $\delta = 1/8$ when

$$\alpha = (\sqrt{3(\Gamma - \gamma)} \pm \sqrt{-\beta})^2 \geq 0$$

and that the first derivatives of δ are nonzero at any point where this condition is satisfied. These values of α therefore separate regions where $\delta < 1/8$ from those where $\delta > 1/8$. It can also be seen that $\delta > 1/8$ only if $\alpha > 0$ and $\beta < 0$. These conditions correspond to the fluid ρ decaying and the fluid ρ_1 antidecaying. (By "antidecaying" we mean gaining energy in proportion to its energy density, instead of losing it.) An example of an antidecaying fluid is a ghost field which radiates away energy; here the energy density of the ghost is negative, so a negative value of β is required for the radiation to carry away energy.

For $\delta > 1/8$ the exact solution to Eq. (9) is

$$H^2 = a^{-A/2} \cos\left(\frac{1}{2}\sqrt{8B - A^2} \ln a\right), \tag{17}$$

where integration constants have been rescaled into a and H. Again, this equation is difficult to solve in terms of the coordinate t. By introducing the new coordinate $d\tau \equiv a^{A/4} dt$, we get

$$\frac{d \ln a}{d\tau} = \cos\left(\frac{1}{2}\sqrt{8B - A^2} \ln a\right),$$

which can be integrated to obtain a closed form for the expansion scale factor:

$$a(\tau) = \exp\left\{\frac{4}{\sqrt{8B - A^2}} \mathrm{am}\left(\frac{\sqrt{8B - A^2}}{4}(\tau - \tau_0) \,\Big|\, 2\right)\right\}, \tag{18}$$

where τ_0 is constant and $\mathrm{am}(\hat{a}|\hat{b})$ is the Jacobi amplitude, shown in Fig. 1. The form of $a(\tau)$ in (18) is an always-positive oscillatory function of the time τ, with constant amplitude. The corresponding solution in terms of the coordinate t will therefore also be oscillatory with constant amplitude.

While in the previous section we found that, for $\delta < 1/8$, the scale factor evolves as a smooth function with early- and late-time power-law behavior, we have found for

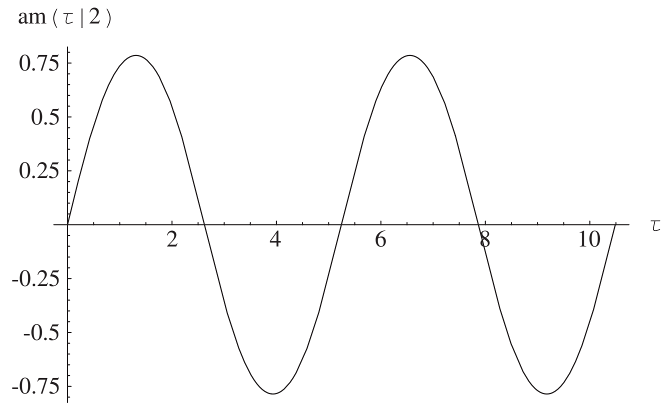

FIG. 1. The Jacobi amplitude am$(\tau|2)$.

$\delta < 1/8$ a substantially different behavior. The scale factor now oscillates in time and does not display the simple power-law behavior found in the $\delta \leq 1/8$ situations.

VI. PHANTOM FLUIDS

We have so far only discussed the cases where $\gamma, \Gamma \geq 0$ and $\gamma \neq \Gamma$. This assumption is useful as it means that δ is nonsingular in the parameter range $\alpha, \beta \geq 0$, for which it was shown in Sec. II that the maximum value of δ is $1/8$. This result was subsequently used in Sec. V to show that there exists a parameter range with $\alpha > 0$ and $\beta < 0$ for which $\delta > 1/8$. In this section we will relax the positive semidefinite assumption on the parameters γ and Γ, extending the analysis we have so far performed to the case of so-called ''phantom'' fluids. We begin by showing that in the parameter range $\alpha, \beta > 0$ there exist no points at which $\delta \rightarrow +\infty$. For this to occur we would require the simultaneous satisfaction of the conditions

$$\alpha\Gamma + \beta\gamma + 3\Gamma\gamma > 0 \quad \text{and} \quad \alpha + \beta + 3\gamma + 3\Gamma = 0.$$

Using the second of these conditions, we can eliminate α in the first to find

$$\beta(\gamma - \Gamma) - 3\Gamma^2 > 0 \quad \text{or} \quad \beta < -\frac{3\Gamma^2}{(\Gamma - \gamma)} \leq 0,$$

where we still assume $\Gamma > \gamma$, without loss of generality. Similarly, we can obtain for α the expression

$$\alpha > \frac{3\gamma^2}{(\Gamma - \gamma)} \geq 0.$$

These two inequalities show that $\delta \rightarrow +\infty$ can only occur in the parameter space $\alpha > 0$ and $\beta < 0$. Therefore, in the range $\alpha, \beta > 0$ the only singularities in δ that can occur are those in which $\delta \rightarrow -\infty$. In this case we can again show, using the arguments in Sec. II, that the maximum value of δ when $\alpha, \beta \geq 0$ is $1/8$. The argument showing the existence of a region where $\delta > 1/8$ in the range $\alpha > 0$ and $\beta < 0$ now follows in exactly the same way as for the $\gamma, \Gamma \geq 0$ case, given in Sec. V.

The form of the solutions in the regions where $\delta < 1/8$ and $\delta > 1/8$ are the same as in the nonphantom case, and are given by Eqs. (9) and (17).

VII. DISCUSSION

We have determined the general solution of a simple model with the exchange of energy between two fluids in an expanding Friedmann universe of zero spatial curvature. The total energy of the exchange is conserved and the model allows energy inputs and outflows proportional to the densities of the two fluids. A number of simple examples of this sort already exist, such as particle decays or particle-antiparticle annihilations into radiation, particle production, the evaporation of a population of primordial black holes, the decay of a cosmological vacuum or cosmological ''constant,'' and energy exchanges between quintessence and ordinary matter or radiation. However, these examples are restricted to one-way energy exchange and do not prove that the scaling solutions that they employ are attractors for the general solution. We have established the existence and form of simple power-law solutions for the expansion scale factor in the case of two-way energy exchange between fluids and determined that they are attractors for the late-time evolution in situations that are usually regarded as generic. If we allow one fluid to be antidecaying then we can move into a domain where these power-law solutions are no longer attractors. Again, we find the general behavior for these cosmologies. These solutions provide a simple model for the study of a wide range of energy exchange problems in cosmology and also reveal the conditions under which power-law solutions previously used to solve some of these problems are stable attractors. They provide a simple model for many future studies of a variety of interacting fluid cosmologies.

ACKNOWLEDGMENTS

We would like to thank Luis Chimento for helpful references. T. Clifton acknowledges support from PPARC.

[1] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon Press, London, 1934), section 165; W. Davidson, Mon. Not. R. Astron. Soc. **124**, 79 (1962); M. Szydlowski, Phys. Lett. B **632**, 1 (2006); J.-P. Uzan, Classical Quantum Gravity **15**, 1063 (1998).
[2] J. A. S. Lima, Phys. Rev. D **54**, 2571 (1996); J. A. S. Lima

and M. Trodden, Phys. Rev. D **53**, 4280 (1996).
[3] J. A. S. Lima, A. S. M. Germano, and L. R. W. Abramo, Phys. Rev. D **53**, 4287 (1996).
[4] J. D. Barrow, E. J. Copeland, E. J. Kolb, and A. R. Liddle, Phys. Rev. D **43**, 984 (1991); J. D. Barrow, E. J. Copeland, and A. R. Liddle, Mon. Not. R. Astron. Soc. **253**, 675 (1991).
[5] J. D. Barrow and F. J. Tipler, Nature (London) **276**, 453 (1978); *The Anthropic Cosmological Principle* (Oxford University Press, Oxford, 1986); D. N. Page and M. R. McKee, Nature (London) **291**, 44 (1981); Phys. Rev. D **24**, 1458 (1981).
[6] N. Turok, Phys. Rev. Lett. **60**, 549 (1988).
[7] J. D. Barrow, Phys. Lett. B **180**, 335 (1986); Nucl. Phys. **B310**, 743 (1988).
[8] A. N. Taylor and A. Berera, Phys. Rev. D **62**, 083517 (2000); J. P. Mimoso, A. Nunes, and D. Pavón, Phys. Rev. D **73**, 023502 (2006).
[9] L. P. Chimento, A. S. Jakubi, and D. Pavón, Int. J. Mod. Phys. D **9**, 43 (2000); J. M. F. Maia and J. A. S. Lima, Phys. Rev. D **65**, 083513 (2002); W. Zimdahl, J. Triginer, and D. Pavón, Phys. Rev. D **54**, 6101 (1996); A. Nunes and J. P. Mimoso, Phys. Lett. B **488**, 423 (2000); L. P. Chimento, A. S. Jakubi, D. Pavón, and W. Zimdahl, Phys. Rev. D **67**, 083513 (2003); W. Zimdahl, D. Pavón, and L. Chimento, Phys. Lett. B **521**, 133 (2001); L. Amendola, Phys. Rev. D **62**, 043511 (2000); L. Amendola and D. Tocchini-Valentini, Phys. Rev. D **64**, 043509 (2001); D. J. Holden and D. Wands, Phys. Rev. D **61**, 043506 (2000); A. P. Billyard and A. A. Coley, Phys. Rev. D **61**, 083503 (2000); A. de la Macorra, Phys. Lett. B **585**, 17 (2004); D. Wands, E. S. Copeland, and A. Liddle, Ann. N.Y. Acad. Sci. **688**, 647 (1993); W. Zimdahl, Int. J. Mod. Phys. D **14**, 2319 (2005); M. Gasperini, F. Piazza, and G. Veneziano, Phys. Rev. D **65**, 023508 (2002); G. Mangano, G. Miele, and V. Pettorino, Mod. Phys. Lett. A **18**, 831 (2003); R. R. Khuri, Phys. Lett. B **568**, 8 (2003); W. Zimdahl and D. Pavón, Gen. Relativ. Gravit. **35**, 413 (2003); X. Zhang, Phys. Lett. B **611**, 1 (2005); W. Zimdahl and D. Pavón, Gen. Relativ. Gravit. **36**, 1483 (2004); R.-G. Cai and A. Wang, J. Cosmol. Astropart. Phys. 03 (2005) 002; M. Szydlowski, Phys. Lett. B **632**, 1 (2006); M. Szydlowski, T. Stachowiak, and R. Wojtak, Phys. Rev. D **73**, 063516 (2006).
[10] K. Freese, F. C. Adams, J. A. Friedman, and E. Mottola, Nucl. Phys. **B287**, 797 (1987); M. Ozer, Phys. Lett. B **404**, 20 (1997); J. S. Alcaniz and J. A. S. Lima, Phys. Rev. D **72**, 063516 (2005); W. Chen and Y-S. Wu, Phys. Rev. D **41**, 695 (1990); M. S. Berman, Phys. Rev. D **43**, 1075 (1991); D. Pavón, Phys. Rev. D **43**, 375 (1991); J. C. Carvalho, J. A. S. Lima, and I. Waga, Phys. Rev. D **46**, 2404 (1992); J. A. S. Lima and J. M. F. Maia, Phys. Rev. D **49**, 5597 (1994); J. A. S. Lima and M. Trodden, Phys. Rev. D **53**, 4280 (1996); A. I. Arbab and A. M. M. Abdel-Rahman, Phys. Rev. D **50**, 7725 (1994); J. M. Overduin and F. I. Cooperstock, Phys. Rev. D **58**, 043506 (1998); J. M. Overduin, Astrophys. J. **517**, L1 (1999); M. V. John and K. B. Joseph, Phys. Rev. D **61**, 087304 (2000); O. Bertolami and P. J. Martins, Phys. Rev. D **61**, 064007 (2000); R. G. Vishwakarma, Gen. Relativ. Gravit. **33**, 1973 (2001); A. S. Al-Rawaf, Mod. Phys. Lett. A **16**, 633 (2001); M. K. Mak, J. A. Belinchon, and T. Harko, Int. J. Mod. Phys. D **11**, 1265 (2002); M. R. Mbonye, Int. J. Mod. Phys. A **18**, 811 (2003); J. V. Cunha and R. C. Santos, Int. J. Mod. Phys. D **13**, 1321 (2004); S. Carneiro and J. A. S. Lima, Int. J. Mod. Phys. A **20**, 2465 (2005); I. L. Shapiro, J. Sola, and H. Stefancic, J. Cosmol. Astropart. Phys. 01 (2005) 012; E. Elizalde, S. Nojiri, S. D. Odintsov, and P. Wang, Phys. Rev. D **71**, 103504 (2005); J. Sola and H. Stefancic, Mod. Phys. Lett. A **21**, 479 (2006); P. Wang and X. Meng, Classical Quantum Gravity **22**, 283 (2005); C. Espana-Bonet, P. Ruiz-Lapuente, I. L. Shapiro, and J. Sola, J. Cosmol. Astropart. Phys. 02 (2004) 006; I. L. Shapiro, J. Sola, C. Espana-Bonet, P. Ruiz-Lapuente, Phys. Lett. B **574**, 149 (2003); M. C. Bento, O. Bertolami, and A. A. Sen, Phys. Rev. D **70**, 083519 (2004); O. Bertolami, Nuovo Cimento B **93**, 36 (1986).
[11] T. Clifton and J. D. Barrow, Phys. Rev. D **73**, 104022 (2006).
[12] A. Gromov, Y. Baryshev, and P. Teerlkorpl, Astron. Astrophys. **415**, 813 (2004); L. P. Chimento and D. Pavón, Phys. Rev. D **73**, 063511 (2006).
[13] A. Nunes, J. P. Mimoso, and T. C. Charters, Phys. Rev. D **63**, 083506 (2001).
[14] J. D. Barrow, in *The Formation and Evolution of Cosmic Strings* (Cambridge University Press, Cambridge, 1990), pp. 449–462.
[15] L. P. Chimento, J. Math. Phys. (N.Y.) **38**, 2565 (1997).
[16] L. P. Chimento and R. Lazkoz, astro-ph/0604090.