

Inhomogeneous magnetic seed fields and gravitational waves within the magnetohydrodynamic limit

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In this paper we apply second-order gauge-invariant perturbation theory to investigate the possibility that the coupling between gravitational waves (GWs) and a large-scale *inhomogeneous* magnetic field acts as an amplification mechanism in an “almost” Friedmann-Lemaître-Robertson-Walker Universe. The spatial inhomogeneities in the magnetic field are consistently implemented using the *magnetohydrodynamic* (MHD) approximation, which yields an additional source term due to the interaction of the magnetic field with velocity perturbations in the plasma. Comparing the solutions with the corresponding results in our previous work indicates that, on superhorizon scales, the interaction with the spatially inhomogeneous field in the dust regime induces the same boost as the case of a homogeneous field, at least in the ideal MHD approximation. This is attributed to the observation that the MHD induced part of the generated field effectively only contributes on scales where the coherence length of the initial field is less than the Hubble scale. At subhorizon scales, the GW induced magnetic field is completely negligible in relation to the MHD induced field. Moreover, there is no amplification found in the long-wavelength limit.

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I. INTRODUCTION

Astrophysical observations indicate that almost all environments in the Universe are magnetized, and the more we search for extragalactic fields, the more pervading they are revealed to be. Such cosmological magnetic fields are found in galaxy clusters, disk and spiral galaxies as well as in high-redshift condensations (see [1] for reviews). A fascinating and as yet unsolved question is how did these fields originate? The current properties of magnetic fields should, in principle, reflect their past and give clues to their origins, so we rely on observations to contribute to finding an answer to this important question.

Faraday rotation and Zeeman splitting measurements indicate that galactic magnetic fields at high redshifts exist with roughly the same strength, 10^{-7} to 10^{-5} G, as those found in the Milky Way [2,3]. The common properties of large-scale fields in different galaxies indicates that their origins may be intrinsically connected to the cosmological repercussions of the interplay between gravitational and gauge interactions [4]. This suggests that their origin may be primordial, in which case their presence could be related to the physics of the very early Universe. One example is big bang nucleosynthesis, where the interaction of a magnetic field with the magnetic moment of a neutrino may have given rise to a spin-flip and change of its handedness, introducing an additional neutrino degree of freedom [5].

Depending on their spectrum, magnetic fields existing in proto-galactic clouds with strengths of 10^{-12} to 10^{-9} G may have played a significant role in structure formation [6]. The Lorentz force that acts on charges in an inhomogeneous (i.e. $\text{curl}B \neq 0$) magnetic field has been shown to induce peculiar velocities [7] which seed density perturbations, and in so doing alter the gravitational instability picture. Moreover, hyper-magnetic fields, hypothesized to emerge during the electroweak phase, have also been identified as a possible source of the observed baryon asymmetry of the Universe [8]. For these reasons, the determination of the origin and properties of cosmic magnetic fields is of extreme importance in cosmology. This makes magnetogenesis, the determination of a self-consistent theory for the generation of cosmological magnetic fields with the strengths and on the scales measured today, one of the “hot” topics in modern cosmology. The most popular theories include the amplification of a small field by the galactic dynamo and the adiabatic proto-galactic collapse at the start of structure formation. Although these mechanisms are shown to yield substantial enhancement, they are not self-sufficient as they presuppose the existence of seed fields. In addition, these seed fields must satisfy very stringent strength and size criteria in order for the generated fields to agree with the magnitudes observed today. It follows that the problem we face is to provide a mechanism that induces a large enough amplification of a weak *preexisting* seed field, so that the aforementioned mechanisms are physically viable.

In this paper, we extend the work of Betschart *et al.* [9], which investigated the coupling between a large-scale

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homogeneous magnetic field and the gravitational wave spectrum which accompanies most inflationary scenarios. This work built on earlier work by Tsagas *et al.* [10] in which the same interaction was studied within the weak field approximation [11,12]. The analysis in [9] demonstrated that this coupling can lead to an amplification, provided the dimensionless shear anisotropy σ/H at the end of inflation is larger than 10^{-40} .

In this investigation, we consider the general case where the original magnetic field is *inhomogeneous* over a typically observed coherence scale. By comparing this analysis with our treatment of homogeneous fields in [9], we aim to determine the implications of placing restrictions (such as homogeneity) on the properties of primordial magnetic seed fields.

The highly nonlinear nature of the Einstein's field equations makes finding exact solutions as well as applying numerical techniques extremely difficult. In order to solve them analytically, severe symmetry assumptions are often required to simplify the physical models, which then restricts their applicability. For cosmological applications, using a perturbative approach, which entails decomposing the real physical Universe into a family of spacetimes, yields surprisingly good results. This allows us to encode the inhomogeneities we see today as perturbations expanded around a fictitious idealized background model, most commonly taken to be the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes. Before we can implement perturbation theory in a self-consistent way, we must give due attention to the issue of gauge-invariance which has historically plagued such studies [13].

If we consider a weak large-scale magnetic field residing in a background FLRW model as done by Tsagas *et al.* [10], the interaction with linearised gravitational waves manifests itself as a first-order perturbation of this background. The problem with this approach is that it is not gauge-invariant in a strict mathematical sense, due to the fact that the magnetic field introduces a preferred direction and therefore breaks the isotropy of the background FLRW model. This problem is partially overcome by assuming that the magnetic field is weak and that its contribution to the energy-momentum tensor is such that it does not disturb the isotropy of the FLRW background [12].

A completely self-consistent solution to this problem is obtained by treating any seed magnetic fields as a first-order perturbation and including the interaction with gravitational perturbations by going to second order in perturbation theory [9]. Such a seed field has been shown to arise during the inflationary era via various mechanisms, one being the gauge-invariant coupling of the Maxwell field to the accompanying scalar field [14].

The introduction of magnetic spatial gradients at linear order requires a more subtle treatment of the associated spatial currents, requiring the use of the magnetohydrodynamic approximation (MHD) to provide a framework in

order to obtain a tractable solution. This single-component fluid model provides an accurate description of a two-species plasma when effects occurring over much larger time and length scales than those characteristic of plasma effects are studied. This allows one to handle low-frequency phenomena in a magnetized plasma using the standard machinery of fluid dynamics. This reduced description is achieved by defining appropriate one-fluid variables representing the bulk quantities, while Ohm's law provides a consistent treatment of the associated electric field. It is common practice to take advantage of the high conductivity, σ , of the young cosmic plasma and employ the ideal MHD limit ($\sigma \rightarrow \infty$). In this limit, the flux lines are effectively glued to the plasma fluid elements. In the nonideal MHD limit [15], where σ is assumed to be large but not infinite, the electric field enters at linear order in the case of a first-order magnetic field. Here we focus on the *ideal* MHD case which is shown to be equivalent to assuming that the observed electric field vanishes in the rest frame of the fluid.

The mathematical framework we use is the 1 + 3 covariant approach [16–18] to perturbation theory which allows Maxwell's and Einstein's equations to be written in an intuitive and simple fashion [15]. The covariant definition of the variables ensures that their connection to physically and geometrically significant quantities is immediately transparent and their exact presentation gives them meaning in any spacetime. Most importantly, the identification of gauge-invariant (GI) perturbation variables at a given order is relatively straightforward.

It is found that the generated magnetic field consists of contributions from two sources. The first stems from the interaction between GWs and the “background” magnetic field, the second comes from the rotation of the induced electric field, which is caused in MHD by the background magnetic field interacting with the velocity perturbations in the plasma. Comparing the solutions with the corresponding results in [9] reveals that, at superhorizon scales, the interaction of GWs with a spatially inhomogeneous magnetic field in the dust regime yields an amplification of the same order of magnitude as found in the case of a homogeneous field, at least in the ideal MHD approximation. The MHD induced part of the magnetic field becomes important only at subhorizon scales, where the GW induced contribution is negligible. In other words, the contribution stemming from the GWs dominates over plasma effects in the ideal MHD limit at superhorizon scales, whereas the roles are interchanged at subhorizon scales.

The units employed in this paper are $c = h = 1$ and $\kappa = 8\pi G = 1$.

II. PERTURBATION SCHEME

We employ the same perturbative scheme as in Betschart *et al.* [9]. Since the presence of a magnetic vector field in the FLRW background does not yield a strictly gauge-

invariant system, we introduce such fields as a perturbation of the FLRW background. Similarly, the gravitational and velocity perturbations are required to vanish in this background for them to be gauge-invariant. Since we are interested in the interaction of magnetic fields with linear gravitational wave distortions, we need to treat this problem at second-order in perturbation theory. Using the 1 + 3 covariant approach [16–18], we expand the physically relevant variables in terms of two smallness parameters to distinguish between the magnitudes of the inhomogeneous magnetic field ($\sim \epsilon_B$) and the amplitude of the GWs ($\sim \epsilon_g$). It follows that the magnitude of the interaction of interest is of order $\mathcal{O}(\epsilon_B \epsilon_g)$. Since we are only interested in the cross interaction of the magnetic and gravitationally sourced perturbations, we only retain mixed terms of order $\mathcal{O}(\epsilon_B \epsilon_g)$ and neglect those of order $\mathcal{O}(\epsilon_g^2)$ and $\mathcal{O}(\epsilon_B^2)$. In fact, such terms always appear in the calculations concerning the induced magnetic field, multiplied by terms of order $\mathcal{O}(\epsilon_B)$ and therefore do not lead to inconsistencies in the perturbation scheme. [19]

The perturbation spacetimes may be split as follows [9]:

- (i) \mathcal{B} = Exact FLRW as the background spacetime, $\mathcal{O}(\epsilon^0)$;
- (ii) \mathcal{F}_1 = Exact FLRW perturbed by an inhomogeneous magnetic field whose energy density and anisotropic stress are neglected, $\mathcal{O}(\epsilon_B)$;
- (iii) \mathcal{F}_2 = Exact FLRW with gravitational and velocity perturbations [20] $\mathcal{O}(\epsilon_g)$;
- (iv) $\mathcal{S} = \mathcal{F}_1 + \mathcal{F}_2$ allows for inclusion of interactions terms of order $\mathcal{O}(\epsilon_B \epsilon_g)$.

We will generally refer to terms of order $\mathcal{O}(\epsilon_B)$ and $\mathcal{O}(\epsilon_g)$ appearing in \mathcal{F} as ‘‘first-order’’ and to the cross terms $\mathcal{O}(\epsilon_B \epsilon_g)$ appearing in \mathcal{S} as ‘second-order’.

As we will discuss in more detail in the next section, the presented hierarchy of spacetimes given above is only justified if the electric field is at least of second order. However, we will assume the ideal MHD case, which indeed implies that the electric field must be second order.

Before we turn to the MHD approximation, we first describe the basic equations describing the background and first-order spacetimes keeping the equations as general as possible.

A. FLRW background

In order to perform a 1 + 3 decomposition of any spacetime, we need to introduce a universal reference 4-velocity field u^a relative to which all motion is defined and quantified. In accordance with the observed average recession of the galaxies, we assume that the matter in the Universe has a locally well-defined preferred motion that can be represented by a unique 4-velocity vector field u^a , satisfying $u_a u^a = -1$. Based on the Copernican principle, we can assume that this holds at each point in the Universe. We then introduce a family of observers, called the *fundamental observers*, traveling such that this field represents the

congruence of their world lines. In so doing, any observations made are those relative to this preferred frame. In choosing this field to coincide with the average velocity of the matter in the Universe, it acquires an ‘‘invariant significance’’ such that the covariant quantities at every point, which are defined with respect to u^a , can be decomposed uniquely [18].

FLRW models are spacetimes that are spatially isotropic and homogeneous about every point. Relative to the congruence of fundamental observers with 4-velocity u^a , the kinematics are assumed to be locally isotropic. This implies that all tensorial quantities such as the acceleration vector $\dot{u}_a \equiv u^b \nabla_b u_a$, the shear $\sigma_{ab} \equiv D_{(a} u_{b)}$ and the vorticity $\omega_{ab} \equiv D_{[a} u_{b]}$ must vanish to eliminate preferred direction in the spatial sections:

$$0 = \dot{u}_a = \sigma_{ab} = \omega_{ab}, \quad (1)$$

The constraints

$$\pi_{ab} = q_a = 0, \quad (2)$$

ensure that the energy-momentum tensor and thus the Ricci tensor are isotropic and indicate that a perfect fluid is a necessary requirement of this model. These restrictions mean that the electric and magnetic components of the Weyl tensor and hence the Weyl tensor itself, vanish identically

$$E_{ab} = H_{ab} = 0 \Rightarrow C_{abcd} = 0. \quad (3)$$

We can then infer that these models are conformally flat. Furthermore, the spatial uniformity forces the spatial gradients of the energy density μ , the pressure p and the expansion Θ to vanish

$$0 = D_a \mu = D_a \Theta = D_a p. \quad (4)$$

As usual, the spatial derivative $D_a \equiv h_a{}^b \nabla_b$ is obtained by projection of the spacetime covariant derivative ∇_a onto the 3-space (with metric $h_{ab} \equiv g_{ab} + u_a u_b$) orthogonal to the observer’s world line. As a consequence, the key background equations are the energy conservation equation

$$\dot{\mu} + \Theta(\mu + p) = 0, \quad (5)$$

the Raychaudhuri equation

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p) + \Lambda, \quad (6)$$

and the Friedmann equation

$$\mu + \Lambda = \frac{1}{3}\Theta^2 + \frac{3K}{a^2}, \quad (7)$$

describing the intrinsic curvature of the homogeneous and isotropic 3-spaces. The curvature constant K indicates the geometry of the Universe and can be normalized to $K = -1, +1, 0$ for spatially open, closed and flat Universes.

B. First-order perturbations

1. The inhomogeneous magnetic field \tilde{B}_a

We assume that the seed magnetic field \tilde{B}_a residing in the \mathcal{F}_1 spacetime is inhomogeneous over its typical coherence scale. The spatial gradients $D_b \tilde{B}_a$ are thus of order $\mathcal{O}(\epsilon_B)$. Given that the magnetic field is a first-order perturbation on the background, the magnetic anisotropy $\pi_{ab} = -\tilde{B}_{\langle a} \tilde{B}_{b \rangle} \sim \mathcal{O}(\epsilon_B^2)$ can be neglected [21]. This means that under this approximation, the additional sourcing of gravitational waves by a magnetic field induced stress-tensor within the plasma can be neglected. Since the associated electric field is perturbatively smaller than the magnetic field and enters only in \mathcal{S} as will be argued in the next section, the magnetic induction equation has the form

$$\dot{\tilde{B}}_{\langle a} + \frac{2}{3}\Theta \tilde{B}_a = 0. \quad (8)$$

Thus the magnetic field decays as

$$\tilde{B}_a = \tilde{B}_a^0 \left(\frac{a_0}{a}\right)^2, \quad (9)$$

where a denotes the scale factor, i.e., $\Theta = 3\dot{a}/a = 3H$ and H denotes the inverse Hubble length. The adiabatic decay evident in Eq. (9) arises from the expansion of the Universe which conformally dilutes the field lines due to flux conservation [22]. By taking the spatial gradient of Eq. (8), it is easy to show that the gradient of the magnetic field evolves as $D_b \tilde{B}_a \sim a^{-3}$. Note also that the induction equation does not discriminate between homogeneous and inhomogeneous magnetic fields.

2. Gravitational waves

In the covariant approach to cosmology, linearized gravitational waves are purely tensorial and are monitored via the electric (E_{ab}) and magnetic (H_{ab}) Weyl constituents, which are not sourced by rotational (vector) and density (scalar) perturbations [23,24]. The transverse (divergence-free) nature of these projected, symmetric, trace-free (PSTF) tensors means that we only need to eliminate their vector parts in order for them to characterize frame-invariant GWs. We isolate the linear tensorial modes by imposing the constraints

$$0 = D_a \mu = D_a p = D_a \Theta = \omega_a = \dot{u}_a. \quad (10)$$

These restrictions ensure that the sources of vector modes (spatial gradients and vector perturbations themselves) vanish and lead to the constraints [25]

$$0 = D^a \sigma_{ab} = D^a E_{ab} = D^a H_{ab} = H_{ab} - \text{curl} \sigma_{ab}. \quad (11)$$

Since the shear tensor is coupled to H_{ab} and E_{ab} , it can also be used as a measure of gravitational waves. The tensorial gravitational waves are governed completely by a closed wave equation for the shear. At linear order, this wave equation is given by [9]

$$\ddot{\sigma}_{\langle ab \rangle} - D^2 \sigma_{ab} + \frac{5}{3}\Theta \dot{\sigma}_{\langle ab \rangle} + \left(\frac{1}{9}\Theta^2 + \frac{1}{6}\mu - \frac{3}{2}p + \frac{5}{3}\Lambda\right) \sigma_{ab} = 0. \quad (12)$$

Observe that in general the right-hand side of Eq. (12) is nonzero (see [26] for the case of irrotational dust spacetimes).

III. BASICS OF IDEAL MHD

Since the mean free path between the electron-ion collisions in a typical plasma can be short compared to the characteristic MHD length scale, it is not always obvious that a fluid description is indeed valid. On small scales these interactions are frequent and cause the two species to move relative to each other, generating charge separation effects referred to as plasma oscillations. If we consider much larger scales on which the individual collisions are not explicitly seen (i.e. low-frequency phenomena), the different species are observed to move together with a common average velocity, allowing an effective single fluid description of this two-component system. If, in addition, the characteristic MHD length scale is much greater than the plasma Debye length and the gyro radius, then MHD gives an accurate description of low-frequency phenomena in a magnetized plasma (see, for example, Refs. [15,27]).

We now pay attention to how the currents, which are established due to the net motion of the fluid by induction, modify the field and in so doing couple the hydrodynamical equations to Maxwell's equations via Ohm's law.

We perform the calculations in an irrotational Universe with vanishing cosmological constant Λ , and assume $p = 0$ from now on since it has been argued that the reduced MHD description of a plasma is only valid in the cold limit [28] (i.e. $p = 0$ and nonrelativistic motion of the plasma particles). These assumptions simplify the analysis tremendously while still allowing us to make a meaningful comparison with the results obtained in the homogeneous case for dust [9].

As a further simplification, we adopt the *geodesic* frame, in which the acceleration of the fluid frame \dot{u}^a vanishes to all orders.

Covariant theory

We now turn to the evolution equations of magneto-hydrodynamic variables. We assume charge quasineutrality of the plasma (i.e. the number densities of the electrons, n_e , and ions, n_i , are roughly equal such that the total charge ρ_c effectively vanishes, $\rho_c = -e(n_e - n_i) \approx 0$). In the spirit of the MHD approximation, we choose to formulate the equations using the magnetic field as our primary variable and use Maxwell's equations to express the electric field and currents in terms of B^a . In order to identify the perturbative order of the terms which we can eliminate without excluding physically important effects, the exact

equations that are necessary to fully describe this system are stated.

1. Maxwell's equations

Under the assumption of charge neutrality and vanishing vorticity, the behavior of electromagnetic fields and currents in curved spacetimes is governed by Maxwell's equations in the following form (see, e.g., [15,17,29]):

$$\dot{B}_{\langle a} + \frac{2}{3}\Theta B_a = \sigma_{ab}B^b - \text{curl}E_a, \quad (13)$$

$$\dot{E}_{\langle a} + \frac{2}{3}\Theta E_a = \text{curl}B_a - j_{\langle a} + \sigma_{ab}E^b, \quad (14)$$

$$D^a B_a = 0, \quad (15)$$

$$D^a E_a = 0, \quad (16)$$

where the last two constraint equations are supposed to hold at all orders.

2. Equations of motion

We assume that the interactions of the ions and electrons collectively isotropize their motions, such that in a chosen frame the properties of the fluid on macroscopic scales can be described in terms of an average velocity v^a . We take this mean motion to be the center of mass velocity of the electron-ion system, defined by

$$v^a = \frac{\mu_e v_e^a + \mu_i v_i^a}{\mu_e + \mu_i}, \quad (17)$$

where v^a coincides with the velocity of the fundamental observer at zeroth order, that is, $v^a = 0$ relative to the fundamental observer at lowest order. At higher orders, the behavior of the electromagnetic fields also depends on this bulk velocity.

Ohm's law is generally formulated in the local rest frame of the conducting fluid and is assumed to hold at all orders:

$$j_{\langle a} = \sigma(E_a + \epsilon_{abc}v^b B^c), \quad (18)$$

where the conductivity σ is taken to be a constant for simplicity. The 3-vector E^a represents the field as observed from the rest-space of the fluid and the second term in (18) is the apparent electric field associated with the bulk velocity v^a . We make the standard assumption that the cosmic medium is infinitely conducting, and consequently apply the ideal MHD limit. This is valid considering the early epoch in which the interaction takes place and is also consistent with the treatment of a homogeneous magnetic field in a *spatially flat* Universe in Betschart *et al.* [9]; the gravito-magnetic interaction was shown in section (IV.B) of [9] to generate the same magnetic field irrespective of conductivity. By letting $\sigma \rightarrow \infty$ in Ohm's law (18), we find that $E_a + \epsilon_{abc}v^b B^c$ must tend towards zero in order for the spatial current $j^{(a)} = \rho_e v_e^a + \rho_i v_i^a$ to remain finite. The electric field is now determined jointly by the fluid velocity

and the magnetic field:

$$E_a = -\epsilon_{abc}v^b B^c. \quad (19)$$

It follows that the electric field is at least of second order and vanishes at lower orders. At first order, the magnetic field \tilde{B}^a determines the current via $\text{curl}\tilde{B}^a = j^{(a)}$, while at second order the current follows from Eqs. (14) and (19). Using charge neutrality, the evolution of the total energy density $\mathcal{E} \equiv \mu_e + \mu_i$ and the center of mass velocity follow from the corresponding total energy and total momentum conservation equations [15,29]:

$$\dot{\mathcal{E}} + \Theta\mathcal{E} = -D_a(\mathcal{E}v^a), \quad (20)$$

$$\mathcal{E}(\dot{v}_{\langle a} + \frac{1}{3}\Theta v_a) = -\mathcal{E}(v^b D_b v_a + \sigma_{ab}v^b) - \epsilon_{abc}j^b B^c, \quad (21)$$

which hold in the cold plasma limit in the geodesic frame.

IV. SECOND-ORDER PERTURBATIONS: THE INTERACTION

We look to Maxwell's equation to determine the nature of the interaction between GWs and the first-order magnetic field \tilde{B}^a . If the backreaction of the induced field with the shear and the center of mass velocity is ignored, the induction equation takes the form

$$\dot{B}_{\langle a} + \frac{2}{3}\Theta B_a = \sigma_{ab}\tilde{B}^b + 2D^b(v_{[a}\tilde{B}_{b]}), \quad (22)$$

where the second term on the right-hand side— $\text{curl}E_a$ describes the dragging of the field lines by the fluid. We are faced with the problem of removing the primary magnetic field component of B^a from the left-hand side of (22) to ensure that it is truly second order. Given that the magnetic spatial gradients are now retained in \mathcal{F}_1 , the commutation relation used as an example in the homogeneous case in Betschart *et al.* [9], is now consistently satisfied when the power series expansion of B^a ,

$$B^a = \epsilon_B B_1^a + \epsilon_g \epsilon_B B_2^a + \mathcal{O}(\epsilon_g^2, \epsilon_B^2),$$

is applied. The reason why an inconsistency arose previously seems to stem from the requirement that the first-order magnetic field is homogeneous, which messed up the commutation relations. A similar inconsistency was encountered when studying the gravito-magnetic interaction in the vicinity of a Schwarzschild black hole, where staticity was imposed upon the first-order magnetic field (see [30] for details). It is interesting to remark, however, that constraining the magnetic field to be solenoidal does not pose any problems whatsoever.

Although the above expansion does not immediately appear to be invalid, in the event that an inconsistency might exist, we choose to represent the magnetic field using the second-order gauge invariant (SOGI) variable

$$\beta_a \equiv \dot{B}_{\langle a} + \frac{2}{3}\Theta B_a, \quad (23)$$

first identified in Betschart *et al.* [9]. We select the interaction variable $I_a = \sigma_{ab}\tilde{B}^b$ and define the variable $F_a \equiv -\text{curl}E_a = 2D^b(v_{[a}\tilde{B}_{b]})$. We note that the electric field is of the order $\mathcal{O}(\epsilon_B\epsilon_g)$ and thus enters the \mathcal{S} spacetime. We can now restate Maxwell's equations in \mathcal{S} as a system of differential equations in terms of these SOGI variables

$$\beta_a = I_a + F_a, \quad (24)$$

$$D^a E_a = 0, \quad (25)$$

$$D^a B_a = 0. \quad (26)$$

To close the system, we use the velocity propagation Eq. (21). Given that both v^a and \tilde{B}^a are individually regarded as first order, only the linear part of this equation is needed:

$$\dot{v}_{\langle a} + \frac{1}{3}\Theta v_a = 0. \quad (27)$$

From Eq. (24), we see that the generated magnetic field B^a can be found directly by integrating a linear combination of the I^a and F^a solutions once they are found.

It is convenient to rescale the magnetic field variable by defining $\mathcal{B}_a = B_a(a/a_0)^2$. The time dependence of \mathcal{B}^a found in the final solutions then describes the evolution of the *generated* field relative to the background field. The main variables become

$$\beta_a = \left(\frac{a_0}{a}\right)^2 \dot{\mathcal{B}}_{\langle a}, \quad I_a = \sigma_{ab}\tilde{\mathcal{B}}^b, \quad (28)$$

$$\mathcal{F}_a = 2D^b(v_{[a}\tilde{\mathcal{B}}_{b]}).$$

Using $H = \Theta/3 = \dot{a}/a$ we can restate the important Eq. (24) in terms of these variables as

$$\dot{\mathcal{B}}_{\langle a} = I_a + \mathcal{F}_a. \quad (29)$$

Integrating this equation with respect to proper time yields \mathcal{B}^a ; the constant of integration is determined by the physical condition that at the time t_0 , when the interaction begins, there is no generated magnetic field, so we have $\mathcal{B}_0^a = B_0^a = 0$ initially.

V. EVOLUTION EQUATIONS FOR THE MAIN VARIABLES

It is evident from Eq. (29) that the SOGI magnetic field \mathcal{B}^a can be extracted directly by integrating a linear combination of the solutions for the interaction I^a and the electric field rotation \mathcal{F}^a . We turn to find the evolution equations for these variables, maintaining generality.

A. Harmonics

We employ the standard harmonic decomposition [13,31] to deal with the Laplacian operator present in the wave equation of the shear. It is standard procedure to assume that the time and spatial dependence of each vari-

able is separable, so that the variable can be expressed as the product of the time and spatial parts. This operation effectively decomposes the differential equation for the time evolution of a perturbation variable into separate equations describing the time evolution of each harmonic component which characterized by a comoving wave number k . Since any perturbation of a quantity can be expressed as the superposition of normal modes, we can decompose the spatial part into the summation over a series of harmonics $Q^{(k)}$ which are covariantly constant $\dot{Q}^{(k)} = 0$ and are chosen to be the eigenfunctions of the Laplace-Beltrami operator

$$D^2 Q^{(k)} = -\frac{k^2}{a^2} Q^{(k)}. \quad (30)$$

We can then define a comoving scale $\lambda = 2\pi a/k$ for each perturbation associated with a harmonic function $Q^{(k)}$. In our application, the harmonic decomposition is particularly useful as it allows us to distinguish the specific situation where the wavelengths of the perturbations are much larger than the Hubble scale ($2\pi a/k \gg H^{-1}$), in which case the Laplacian operator in Eq. (30) that is proportional to k^2 , can be eliminated, yielding differential equations which are easier to solve. Although the use of a plane wave description is mathematically incorrect in curved space, it turns out that the only difference that arises is the allowable values of the wave numbers. For a flat geometry ($K = 0$) the eigenvalues form a continuous spectrum where $k^2 \geq 0$. The spectrum for an open model ($K = +1$) is discrete with $k^2 = \alpha(\alpha + 2)$ where $\alpha = 1, 2, 3, \dots$. A spacetime with negative curvature ($K = -1$) can accommodate the eigenvalues $k^2 = 1 + \alpha^2$ where $\alpha^2 \geq 0$ [31].

B. Governing equation for the interaction variable I^a

In Sec. (B.1) of Betschart *et al.* [9], the solution for I^a obtained by solving its wave equation was found to agree with the result calculated from the multiplication of the time dependencies of the shear and background magnetic field, determined individually. Solving the wave equation for the interaction variable, however, requires a harmonic decomposition. Since the interaction variable is not necessarily divergence-free, $D^a I_a = \sigma_{ab}D^a \tilde{B}^b \neq 0$ in general, it has a nonzero scalar contribution. This contribution is however exactly equal to $D_a B^a$ and therefore drops out of Maxwell's equation (13) rendering it purely solenoidal. Although decomposing the interaction term I^a as a pure vector (as in the homogeneous case) is therefore incorrect, it is still possible to find the solution for I^a : either by solving for σ_{ab} and the first-order part of \tilde{B}^b separately and then multiplying the solutions obtained, or by proceeding in a similar manner as in [9].

GWs are purely tensorial and so we expand the representative shear variable with the help of tensor harmonics $Q_{ab}^{(k)}$,

$$\sigma_{ab} = \sum_k \sigma^{(k)} Q_{ab}^{(k)}, \quad (31)$$

where as usual $\dot{Q}_{(a)}^{(k)} = 0$ and $D^2 Q_{ab}^{(k)} = -(k^2/a^2) Q_{ab}^{(k)}$ hold. Each gravitational wave mode is associated with the physical wavelength

$$\lambda_{\text{GW}} = 2\pi a/k. \quad (32)$$

The expansion of the magnetic field in pure vector (solenoidal) harmonics is

$$\tilde{B}_a = \sum_n \tilde{B}^{(n)} Q_a^{(n)}, \quad (33)$$

and these vector harmonics obey the relations $\dot{Q}_{(a)}^{(n)} = 0$ and $D^2 Q_a^{(n)} = -(n^2/a^2) Q_a^{(n)}$. Similarly as above, we associate with a given wave number n characterizing a magnetic perturbation a characteristic length scale,

$$\lambda_{\tilde{B}} = 2\pi a/n, \quad (34)$$

which we relate to the size of the magnetized region.

To simplify the treatment of the interaction between GWs and the magnetic field, we proceed as follows. First, we assume the magnetized plasma region has a finite size $\lambda_{\tilde{B}} = 2\pi a/m$ corresponding to some wave number m , which encodes the magnetic inhomogeneity over this region. We therefore write $\tilde{B}_a = \sum_{n \geq m > 0} \tilde{B}^{(n)} Q_a^{(n)} \approx \tilde{B}^{(m)} Q_a^{(m)}$. Since the interaction is most effective if the gravitational wavelength λ_{GW} matches the size of the magnetic field region $\lambda_{\tilde{B}}$ [9], we may restrict ourselves to the resonant case where the gravitational and magnetic wave numbers agree. This means that the comoving scale of the magnetic field perturbation is the same as that of the GWs, i.e. $k = m$. Consequently, the m -mode of the shear is the main contribution to the interaction, which now reduces to a single mode-mode term: $I_a \approx \sigma^{(m)} \tilde{B}^{(m)} Q_{ab}^{(m)} Q_a^{(m)}$. Making use of these considerations, it is now a straightforward task to obtain a closed equation for the interaction variable I_a by combining, as in [9], the standard evolution equations for the shear and the electric Weyl tensor, and using $\dot{\tilde{B}}_{(a)} = 0$. In this way we readily arrive at

$$\dot{I}_{(a)} + \frac{m^2}{a^2} I_a + \frac{5}{3} \Theta \dot{I}_{(a)} + \left(\frac{1}{9} \Theta^2 + \frac{1}{6} \mu - \frac{3}{2} p + \frac{5}{3} \Lambda \right) I_a = 0. \quad (35)$$

Here, the second term on the left-hand side stems from the expression $(-D^2 \sigma_{ab}) \tilde{B}^b$ which emerges during the calculation. As expected, Eq. (35) is equivalent to multiplying the m -mode shear Eq. (12) with the magnetic field \tilde{B}^a . We remind the reader that the derivation of Eq. (35) does not rely on the MHD approximation and is indeed valid for all values of the curvature index K and is also independent of the equation of state.

C. Governing equation for \mathcal{F}^a

In the ideal MHD limit, the electric field is expressed as the cross-product of the primary magnetic field and the velocity, $E_a = -\epsilon_{abc} v^b \tilde{B}^c$. Using the linear velocity propagation Eq. (27) together with the linear evolution Eq. (8) for the magnetic field, one easily finds $\dot{E}_{(a)} + \Theta E_a = 0$, and therefore

$$(\text{curl} E_a)_\perp = \text{curl} \dot{E}_{(a)} - \frac{1}{3} \Theta \text{curl} E_a = -\frac{4}{3} \Theta \text{curl} E_a. \quad (36)$$

This further implies that the evolution equation of the MHD contribution encoded in \mathcal{F}^a is simply given by

$$\dot{\mathcal{F}}_{(a)} + \frac{2}{3} \Theta \mathcal{F}_a = 0. \quad (37)$$

It follows that the term \mathcal{F}_a evolves like the electric field E_a , and decays as $\sim a^{-2}$.

VI. SOLUTIONS FOR SPATIALLY FLAT UNIVERSES

Because the derived MHD equations above are only valid in the cold plasma limit, we only investigate the dust solutions for spatially flat models with zero cosmological constant. It is convenient to use the dimensionless time variable $\tau \equiv \frac{3}{2} H_0 (t - t_0) + 1$ introduced in [9]. In terms of this variable, the Hubble parameter evolves simply as H_0/τ and the scale factor obeys $a = a_0 \tau^{2/3}$, where the zero index indicates evaluation at some arbitrary initial time t_0 .

Employing the time variable τ , the equation for the interaction variable (35) may be written as

$$\frac{9}{4} I''_{(a)} + \frac{15}{2\tau} I'_{(a)} + \left[\frac{3}{2\tau^2} + \left(\frac{m}{a_0 H_0} \right)^2 \tau^{-(4/3)} \right] I_{(a)} = 0, \quad (38)$$

while the Eq. (37) for the MHD term transforms into

$$\mathcal{F}'_{(a)} + \frac{4}{3\tau} \mathcal{F}_a = 0, \quad (39)$$

whose solution is simply

$$\mathcal{F}_a = \mathcal{F}_a^0 \tau^{-(4/3)} = \mathcal{F}_a^0 (a/a_0)^{-2}. \quad (40)$$

The generated magnetic field will typically depend upon $x \equiv m/(a_0 H_0) = 2\pi(\lambda_H/\lambda_{\tilde{B}})_0 = 2\pi(\lambda_H/\lambda_{\text{GW}})_0$, which is the ratio between the size of the magnetized field region and the horizon size when the interaction begins.

A. Limiting case where $x \rightarrow 0$

If the value of x is so small that we can drop the term x^2 in Eq. (38), the general solution for the interaction variable is

$$I_a(\tau) = C_1 \tau^{-(1/3)} + C_2 \tau^{-2}, \quad (41)$$

where C_1 and C_2 are integration constants. Since for the

generation of magnetic fields the dominant mode is more important, we set $C_2 = 0$ and obtain the solution

$$I_a(\tau) = I_a^0 \tau^{-(1/3)} = \sigma_{ab}^0 \tilde{B}_0^b \tau^{-(1/3)}. \quad (42)$$

It is now a very simple exercise to integrate the induction Eq. (29) to find

$$\frac{3}{2} H_0 \mathcal{B}_a = \frac{3}{2} \mathcal{I}_a^0 (\tau^{2/3} - 1) - \mathcal{F}_a^0 (\tau^{-(1/3)} - 1), \quad (43)$$

where the integration constant was determined by requiring the generated magnetic field to vanish initially. It follows that the total magnetic field measured by the fundamental observer due to the zero-zero mode interaction becomes

$$B^{(0-0)}(a) = \tilde{B}_0^{(0)} \left(\frac{a_0}{a} \right)^2 \left[1 + \frac{\sigma_0^{(0)}}{H_0} \left\{ \frac{a}{a_0} - 1 \right\} - \frac{2}{3} \frac{\mathcal{F}_0^{(0)}}{H_0 \tilde{B}_0^{(0)}} \left\{ \left(\frac{a}{a_0} \right)^{-(1/2)} - 1 \right\} \right], \quad (44)$$

where the second term in the square bracket originates from the interaction of the magnetic field \tilde{B}_a with the shear (already obtained in [9]), while the third term represents the MHD contribution, which is the interaction of the magnetic field \tilde{B}_a with the plasma velocity perturbation v_a . Note that the MHD contribution slowly decays away as the scale factor increases, in contrast to the gravito-magnetic part, which linearly grows with the scale factor. However, in the long-wavelength limit the contribution due to the GWs is negligible since one typically finds the shear anisotropy to be very small, $(\sigma/H)_0 \ll 1$. Moreover, since we may approximate $\mathcal{F}_0 = F_0 \approx (v\tilde{B}/\lambda_{\tilde{B}})_0$ due to $F_a = -\text{curl}E_a$ and the MHD relation (19), we see that the factor $\mathcal{F}_0/(H_0\tilde{B}_0)$ in (44) is proportional to $x \ll 1$, and the MHD contribution turns also out to be negligible. Consequently, in the long-wavelength limit, there is no amplification of the initial magnetic field by GWs or velocity perturbations and one needs to consider the general case in order to look for an amplification.

B. General case with $x \neq 0$

When x is not negligible (the magnetized region is strictly finite), the general solution to the Eq. (38) is found to be

$$I_a(\tau) = \tau^{-(7/6)} [D_1 J_1(\frac{5}{2} 2x\tau^{1/3}) + D_2 J_2(\frac{5}{2} 2x, \tau^{1/3})], \quad (45)$$

where D_1, D_2 are integration constants and J_1, J_2 denote Bessel functions of the first and second kind, respectively. Since we are only interested in the dominant contribution, we set $D_2 = 0$ as before, noting that the Bessel function of the second kind is decaying on superhorizon scales $x \ll 1$. The remaining integration constant takes then the value

$$D_1 = \frac{\sqrt{\pi} x^{5/2}}{4x^2 \sin(2x) - 3 \sin(2x) + 6x \cos(2x)} \sigma_0 \tilde{B}_0. \quad (46)$$

Assuming that the induced magnetic field is zero initially when the interaction begins ($\tau = 1$), the solution for the rescaled magnetic field then becomes

$$\mathcal{B}(\tau) = \frac{\sigma_0 \tilde{B}_0}{H_0 y} \left[\frac{1}{2} \sin(2x) - x \cos(2x) + x \cos(2x\tau^{1/3}) \tau^{-(2/3)} - \frac{1}{2} \sin(2x\tau^{1/3}) \tau^{-1} \right] - \frac{2}{3} \frac{F_0}{H_0} (\tau^{-(1/3)} - 1), \quad (47)$$

where we defined $y \equiv 4x^2 \sin(2x) - 3 \sin(2x) + 6x \cos(2x)$ and made use of (39). Notice that in the limit $x \rightarrow 0$ one recovers the result (43). Had we instead focused on the other branch of the solution (45), we would find (47) again but with the sin and cos functions as well as some signs interchanged. Hence, the $m-m$ mode interaction generated magnetic field as seen by the fundamental observer moving with 4-velocity u^a is given by the expression

$$B^{(m-m)}(a) = \tilde{B}_0^{(m)} \left(\frac{a_0}{a} \right)^2 \times \left[\frac{\sigma_0^{(m)}}{H_0}, \frac{\frac{1}{2} \sin(2x) - x \cos(2x)}{4x^2 \sin(2x) - 3 \sin(2x) + 6x \cos(2x)} + \frac{2}{3} \frac{F_0^{(m)}}{H_0 \tilde{B}_0^{(m)}} + \mathcal{O}(a^{-(1/2)}) \right]; \quad (48)$$

here, the nondisplayed terms are decaying with time and therefore irrelevant for the amplification process but can be inferred easily from (47) if required.

What happens if we take the full solution (45) of the interaction variable into account instead of just looking at one branch? Since both the Bessel functions of the first as well as the second kind are merely sin and cos functions modified by the *same* damping envelopes, one should in principle consider both branches even though their asymptotic behavior is different. In this case, the integration constants D_1 and D_2 are determined by requiring that initially one has $I_0 = I(\tau = 1) = \sigma_0^{(m)} \tilde{B}_0^{(m)}$ together with $I_0' = \sigma_0'^{(m)} \tilde{B}_0^{(m)}$. The exact solution for the rescaled magnetic field is then found in analogy with the above example [cf. (47)], but it is too large to display it in full. However, the dominant contribution to the generated magnetic field can be written down in simple terms as

$$B^{(m-m)}(a) = \tilde{B}_0^{(m)} \left(\frac{a_0}{a} \right)^2 \left[\frac{3}{2x^2} \frac{2\sigma_0^{(m)} + \sigma_0'^{(m)}}{H_0} + \frac{2}{3} \frac{F_0^{(m)}}{H_0 \tilde{B}_0^{(m)}} + \mathcal{O}(a^{-(1/2)}) \right], \quad (49)$$

where it was again assumed that there is no generated magnetic field initially. If we employ the natural length scale $\lambda_{\tilde{B}}$, inherent to the problem under considera-

tion, we may estimate $\dot{\sigma}_0 \approx (\sigma/\lambda_{\tilde{B}})_0$, implying $\sigma_0' \approx 2/3(\sigma/H_0\lambda_{\tilde{B}})_0$, and also $F_0 \approx (v\tilde{B}/\lambda_{\tilde{B}})_0$ due to $F_a = -\text{curl}E_a$ and the MHD relation (19). Remembering further the definition of $x = 2\pi(\lambda_H/\lambda_{\tilde{B}})_0$, we can finally write down the expression for the *total* magnetic field in a more convenient form:

$$B^{(m-m)}(a) \simeq \tilde{B}_0^{(m)} \left(\frac{a_0}{a}\right)^2 \left[1 + \frac{3}{4\pi^2} \left(\frac{\lambda_{\tilde{B}}}{\lambda_H}\right)_0^2 \frac{\sigma_0^{(m)}}{H_0} \left\{ 1 + \frac{1}{3} \left(\frac{\lambda_H}{\lambda_{\tilde{B}}}\right)_0 \right\} + \frac{2}{3} v_0 \left(\frac{\lambda_H}{\lambda_{\tilde{B}}}\right)_0 + \mathcal{O}(a^{-(1/2)}) \right]. \quad (50)$$

This is our main result—it shows in detail how the magnetic field, resulting from the interaction of the background magnetic field \tilde{B}_0 with GWs and velocity perturbations v_0 in the plasma, depends on the initial conditions.

Observe that at super-Hubble scales the MHD contribution becomes completely negligible mirroring the observation that plasma effects are typically more important on small scales. Our main result directly generalizes our previous result (49) in [9] derived for the case of a homogeneous magnetic field to the inhomogeneous case. It should be stressed that the use of ideal MHD in the cold plasma limit allowed for a self-consistent treatment of the electric fields and plasma currents.

VII. DISCUSSION

If we look only at *superhorizon* scales and divide the result (50) through the energy density of the background radiation, μ_γ , (which decays in the same manner as the original magnetic field), the dominant contribution is then given by

$$\frac{B}{\mu_\gamma^{1/2}} \simeq \left[1 + \frac{1}{10} \left(\frac{\lambda_{\tilde{B}}}{\lambda_H}\right)_0^2 \left(\frac{\sigma}{H}\right)_0 \right] \left(\frac{\tilde{B}}{\mu_\gamma^{1/2}}\right)_0, \quad (51)$$

where the wave number indices have been suppressed and the zero suffix indicates the time when the interaction begins. This result (51) was already found in Betschart *et al.* [9] and previously reported by Tsagas *et al.* [10], a paper which employed the weak field approximation. The result (51) can be applied to the reheating phase of the Universe at the end of inflation, for which the effective equation of state was that of dust (cf. [9,10] for an application).

On the other hand, at subhorizon scales the main part of the magnetic field is given by

$$\frac{B}{\mu_\gamma^{1/2}} \simeq \left[1 + \frac{2}{3} v_0 \left(\frac{\lambda_H}{\lambda_{\tilde{B}}}\right)_0 \right] \left(\frac{\tilde{B}}{\mu_\gamma^{1/2}}\right)_0, \quad (52)$$

which could be applied to the matter-dominated phase of the Universe. In order to obtain an order-of-magnitude estimate we assume that the velocity perturbations (resulting from Thomson scattering) in the primordial plasma in effect start to interact with the preexisting magnetic field \tilde{B}

somewhat after matter-radiation equality with a redshift of roughly ($z_{\text{eq}} \approx 10^4$). The horizon at matter-radiation decoupling was $\lambda_H^{\text{eq}} = H_0^{-1}(1+z_{\text{eq}})^{-3/2} \approx 10^{-2}$ Mpc, where H_0 denotes today's Hubble constant. A typical size of a seed field required for the dynamo mechanism is $\lambda_{\tilde{B}} \approx 10$ kpc on a comoving scale today, hence $\lambda_{\tilde{B}}^{\text{eq}} = (1+z_{\text{eq}})^{-1}\lambda_{\tilde{B}} \approx 1$ pc. From cosmic microwave background (CMB) measurements we know that at decoupling ($z_{\text{dec}} \approx 10^3$) the velocity perturbations satisfy $v_{\text{dec}} \leq 10^{-5}$, while we read off from Eq. (27) that they decay like a^{-1} . It follows that $v_{\text{dec}} = v_{\text{eq}}(a_{\text{eq}}/a) = v_{\text{eq}}\tau^{-2/3} = (1+z_{\text{dec}}) \times (1+z_{\text{eq}})^{-1}v_{\text{eq}} \approx 10^{-1}v_{\text{eq}}$, and whence $v_{\text{eq}} \approx 10^{-4}$. Taking everything together, one obtains the boost factor in (52) to be of order unity, that is, $(v\lambda_H/\lambda_{\tilde{B}})_{\text{eq}} \approx 1$. Given that we can rely on the MHD approximation at the early stages of the matter-dominated scenario, the velocity perturbations in the plasma will lead at best to a doubling of the initial magnetic field strength.

Comparing the result (51) with the final solution presented in Betschart *et al.* indicates that the magnitude of the amplification due to the interaction between GWs and magnetic fields is proportional to the square of the ratio of the coherence length $\lambda_{\tilde{B}}$ of the initial magnetic field and the initial size of the horizon λ_H in both the homogeneous and inhomogeneous magnetic field cases. The additional MHD part of the field in (50) above arises from the forcing term $F^a \equiv -\text{curl}E_a = 2D^b(v_{[a}B_{b]})$ whose time behavior is obtained from a first-order propagation equation in which a Laplacian does not appear, its general solution therefore being independent of wave number and scale (of course, the initial conditions still depend on the size of the interaction region). It was found that the seed field's interaction with GWs is only important at superhorizon scales, while the interaction with plasma velocity perturbations dominates at subhorizon scales. It is worth pointing out that there is no amplification at all in the long-wavelength limit.

In contrast with the results in Betschart *et al.* [9], the generated magnetic field modes now have wave numbers that are constructed from those of the interacting field and GWs and thus differ from those of the GWs alone. If the GW has wave vector k_a and the “background” magnetic field a wave vector m_a (where $\lambda_{\tilde{B}} = 2\pi a/m$ corresponds to the size of the magnetic inhomogeneity), the wave number ℓ of the induced field satisfies $\ell^2 = (k_a + m_a) \times (k^a + m^a) = k^2 + m^2 + 2m_a k^a$. If for simplicity the first-order wave vectors are assumed to be orthogonal, $k^a m_a = 0$, then the $m-m$ interaction yields a field with wave number $\ell^2 = 2m^2$ ($4m^2$ in the parallel case). It follows that the characteristic wavelength $\lambda = 2\pi a/\ell$ of the induced field is somewhat shorter than the original B -field by the superposition of the corresponding magnetic and GW wave numbers.

It is worth noting that the presence of a spatially homogeneous field is only consistent in the flat Universe [9], whereas no restrictions on the spatial geometry arise at any

stage in the derivation of the evolution equations with $(D^b B^a) \neq 0$ in this paper. This makes sense when we consider the spectrum of allowed wave numbers for different geometries. For open models ($K = +1$), the lowest wave number is $n = 3$. For closed models ($K = -1$), we find that $n \geq 1$. However, the spectrum of wave numbers in a flat spacetime ($K = 0$) is continuous, with $n \geq 0$. Given that the spatial homogeneity of the background field in [9] restricts its associated wave number to $n = 0$, we see that this eigenvalue can only be accommodated in a flat Universe.

More interesting is the relationship between curvature and magnetic fields. Einstein's theory is geometrical which implies that vectors are directly coupled to the spacetime curvature via the Ricci identity [32]. In [33], Tsagas and Maartens find that the evolution equation of the spatial gradient of the magnetic field contains a term $\epsilon_{acd} B^c H_b^d \sim \mathcal{O}(\epsilon_B \epsilon_g)$, which alludes to a nonlocal coupling to curvature. This result is confirmed in [11] with the appearance of a term containing the spatially projected Riemann curvature tensor in the propagation equation of $(D_a B_b)$ and indicates that the curvature sources magnetic inhomogeneities. In the case of a homogeneous magnetic field in Betschart *et al.*, the spatial gradients of the magnetic field are at least second order. In order to preserve the spatial uniformity of the first-order field through time, the spatially projected Riemann tensor may have to vanish to prevent it from sourcing $(D_a B_b)$ so that the spatial gradients remain small and continue to contribute at higher-order only. In the analysis of the inhomogeneous magnetic field presented here, magnetic spatial eddies exist at first order and for this reason, the boost from the coupling between the field and the curvature need not necessarily be eliminated. In fact, such couplings are explicitly retained in our approach via the standard commutation relations (cf. the appendix).

VIII. CONCLUSION

Although the focus of magnetogenesis in recent years has been the generation of *large-scale* magnetic fields, a self-sufficient mechanism still evades us. The galactic dynamo is indeed physically feasible and has been shown to generate fields with strengths matching current observations, but requires a reasonably strong seed field to work. To make this theory more robust, we need to find a way of producing seed fields that are suitable for subsequent amplification by the dynamo. Cosmological perturbations have been identified as a possible source of primordial magnetic field amplification which is present in the pre-recombination era. Considering second-order couplings between electrons, photons and protons, the electric current induced by the plasma vorticity (a known source of magnetic fields) and the additional contribution of the photon anisotropic stress are found to yield a magnetic field that is a sufficient seed for the dynamo to work [34].

The coupling of density and velocity perturbations that are naturally occurring in the early Universe are shown to lead to similar resultant fields in the context of a relativistic charged multifluid [29].

In this paper we built on previous work in Betschart *et al.*, in which the full set of equations determining the evolution of the gravitational waves and the generated electromagnetic fields was presented, initially for the case of a homogeneous magnetic field, and generalized the analysis to the case of a spatially inhomogeneous magnetic field using the magnetohydrodynamic approximation (restricting ourselves to the dust case). Analyzing the equations for a spatially flat dust FLRW Universe, we were able to confirm our previous results and, in particular, we do not find any amplification in the case of the long-wavelength limit. Our previous work indicated the amplification from this interaction was negligible. It is interesting to consider work in [35] which challenges the assumption that magnetic fields residing within clusters necessarily exist with a coherence lengths of the clusters. Since numerical simulations indicate only a dependence of the final field in a collapsing magnetized cluster on the magnetic energy density only, and not the initial scale, it has been argued that the primordial fields on scales of kpc can exist with strengths as much as 10^{-12} G. Subsequent amplification by adiabatic collapse would generate fields matching current observations, without needing to invoke the galactic dynamo. A physical mechanism to support this argument has however, yet to be proposed. Our final result is also proportional to the square of the field's initial scale length, which means that assuming a smaller magnetic coherence scale prior to collapse leads to an even smaller amplification from the field's coupling to the gravitational waves.

Over and above the presentation of a physically viable mechanism for primordial magnetogenesis, this paper also establishes a formalism which guides the choice of proper second-order gauge-invariant variables. Using this methodology, one is able to obtain results in terms of clearly defined quantities, with no ambiguity concerning the physical validity of the variables.

The possibility of obtaining seed magnetic fields of sufficient magnitude via a combination of inflationary physics and standard MHD theory is an exciting prospect. However, many of the parameters involved in estimating the size of the effect are not well known. For instance the spectrum of gravitational waves predicted by inflationary theory still has to be verified, as well as the large-scale structure of cosmological magnetic fields. For this purpose, more precise and complete measurements (such as the Planck mission) of the, e.g., polarization of the CMB would give much needed information [36]. Moreover, studies of interactions between incoherent gravitational wave distributions and turbulent magnetic fields could also be done in order to obtain a more detailed picture. This is left for future research.

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APPENDIX: COMMUTATION RELATIONS

Here we present various commutator relations which have been used in the text. The relations are given up to second order in our perturbation scheme. The vanishing of vorticity, $\omega_{ab} = 0$, is assumed throughout in conjunction with the constraints $D_a \mu = D_a p = 0$ which isolate the pure tensor modes. All appearing tensors are PSTF, $S_{ab} = S_{(ab)}$, and all vectors V_a, W_a are purely spatial.

Commutators for first-order vectors V_a :

$$(D_a V_b)_{\perp} = D_a \dot{V}_b - \frac{1}{3} \Theta D_a V_b - \sigma_a^c D_c V_b + H_a^d \epsilon_{dbc} V^c \quad (\text{A1})$$

$$(\text{curl} V_a)_{\perp} = \text{curl} \dot{V}_a - \frac{1}{3} \Theta \text{curl} V_a - \sigma_a^b \text{curl} V_b - H_{ab} V^b \quad (\text{A2})$$

$$\begin{aligned} D_{[a} D_{b]} V_c &= [\frac{1}{9} \Theta^2 - \frac{1}{3} (\mu + \Lambda)] V_{[a} h_{b]c} \\ &+ (\frac{1}{3} \Theta \sigma_{c[a} - E_{c[a} V_{b]}) \\ &+ h_{c[a} (E_{b]d} - \frac{1}{3} \Theta \sigma_{b]d}) V^d \end{aligned} \quad (\text{A3})$$

Commutators for first-order tensors S_{ab} :

$$\begin{aligned} (D_a S_{bc})_{\perp} &= D_a \dot{S}_{bc} - \frac{1}{3} \Theta D_a S_{bc} - \sigma_a^d D_d S_{bc} \\ &+ 2H_a^d \epsilon_{de(b} S_{c)}^e \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} (D^b S_{ab})_{\perp} &= D^b \dot{S}_{ab} - \frac{1}{3} \Theta D^b S_{ab} - \sigma^{bc} D_c S_{ab} \\ &+ \epsilon_{abc} H^b_d S^{cd} \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} (\text{curl} S_{ab})_{\perp} &= \text{curl} \dot{S}_{ab} - \frac{1}{3} \Theta \text{curl} S_{ab} - \sigma_e^c \epsilon_{cd(a} D^e S_{b)}^d \\ &+ 3H_{c(a} S_{b)}^c \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} \text{curl curl} S_{ab} &= -D^2 S_{ab} + (\mu + \Lambda - \frac{1}{3} \Theta^2) S_{ab} \\ &+ \frac{3}{2} D_{(a} D^c S_{b)c} + 3S_{c(a} (E_{b)}^c - \frac{1}{3} \Theta \sigma_{b)}^c) \end{aligned} \quad (\text{A7})$$

Commutators for second-order vectors W_a :

$$(D_a W_b)_{\perp} = D_a \dot{W}_b - \frac{1}{3} \Theta D_a W_b \quad (\text{A8})$$

$$D_{[a} D_{b]} W_c = [\frac{1}{9} \Theta^2 - \frac{1}{3} (\mu + \Lambda)] W_{[a} h_{b]c} \quad (\text{A9})$$

$$\begin{aligned} \text{curl curl} W_a &= -D^2 W_a + D_a (\text{div} W) \\ &+ \frac{2}{3} (\mu + \Lambda - \frac{1}{3} \Theta^2) W_a \end{aligned} \quad (\text{A10})$$

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- [21] Here the angle bracket represents the projected symmetric trace-free (PSTF) part of any tensor: $A_{(ab)} \equiv h^c_{(a} h^d_{b)} A_{cd} - \frac{1}{3} h_{ab} A^c_c$.
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