Characterizing the Larkin-Ovchinnikov-Fulde-Ferrel phase induced by the chromomagnetic instability

Kenji Fukushima

RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA

(Received 4 April 2006; published 19 May 2006)

We discuss possible destinations from the chromomagnetic instability in color superconductors with Fermi surface mismatch $\delta\mu$. In the two-flavor superconducting (2SC) phase we calculate the effective potential for color vector potentials $\boldsymbol{A}_\alpha$ which are interpreted as the net momenta $\boldsymbol{q}$ of pairing in the Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) phase. When $1/\sqrt{2} < \delta\mu/\Delta < 1$ where Δ is the gap energy, the effective potential suggests that the instability leads to a LOFF-like state which is characterized by color-rotated phase oscillations with small $\boldsymbol{q}$. In the vicinity of $\delta\mu/\Delta = 1/\sqrt{2}$ the magnitude of $\boldsymbol{q}$ continuously increases from zero as the effective potential has negative larger curvature at vanishing $\boldsymbol{A}_\alpha$ that is the Meissner mass squared. In the gapless 2SC (g2SC) phase, in contrast, the effective potential has a minimum at $g\boldsymbol{A}_\alpha \sim \delta\mu \sim \Delta$ even when the negative Meissner mass squared is infinitesimally small. Our results imply that the chromomagnetic instability found in the gapless phase drives the system toward the LOFF state with $\boldsymbol{q} \sim \delta\mu$.

DOI: 10.1103/PhysRevD.73.094016 PACS numbers: 12.38.−t, 12.38.Aw

Quark matter has a rich phase structure in the high baryon or quark density region. In a decade we have witnessed tremendous developments in theory, particularly in superconductivity of quark matter [1]. Color superconductivity is inevitable from the Cooper instability in cold and dense quark matter. In the asymptotic density where the perturbative technique is applicable, the color-flavor locked (CFL) phase [2] where all quarks are gapped is concluded from the first-principle calculations of Quantum Chromodynamics (QCD).

The lower density region we explore, the more complicated phase possibilities we have to encounter. The main reason why the situation is perplexing at intermediate density is that a "stress" between quarks which would form a Cooper pair is substantial when the quark chemical potential, μ_q, is comparable to the strange quark mass, M_s. Such an energy cost by the stress, or the Fermi energy mismatch $\delta\mu$, is necessary to bind two quarks into a pair with zero net momentum, $\boldsymbol{q} = 0$. The stress can be reduced by making a pair between quarks sitting on different Fermi surfaces, which results in $\boldsymbol{q} \neq 0$. If the energy gain by easing the stress is greater than the kinetic energy loss coming from nonzero net momentum, the color superconducting phase with $\boldsymbol{q} \neq 0$ would be realized. Since such a state breaks rotational symmetry, this *crystalline color superconducting phase* [3], that is, a QCD analogue of the Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) phase [4], takes a crystal structure [5].

There is another different possibility to consider while keeping $\boldsymbol{q} = 0$; once $\delta\mu$ exceeds the gap energy, Δ, the Cooper pair tends to decay into two quarks. In other words the corresponding quarks have the energy dispersion relation which is gapless. Such a phase is called the *gapless superconducting phase* [6–8], that is, a QCD analogue of the Sarma phase [9]. It would need a careful comparison of energies to see which is favored in reality [10,11]. Interestingly enough, recently, these two different candidates, the crystalline and gapless superconducting phases, have turned out to be closely related through instability.

The gapless superconducting phase is known to be unstable in fact and has to give way to some other stable states. In QCD the negative (color) Meissner mass squared exhibits what is commonly referred to as the chromomagnetic instability [12–15]. This is the central issue we address. The chromomagnetic instability is to be interpreted as instability toward the single plane-wave LOFF state [16], as we will closely discuss later. Also it has been revealed that the two-flavor and three-flavor LOFF phases are chromomagnetic stable [17,18] (see also Ref. [19]). However, it does not necessarily mean that the instability problem has already been resolved. The question we raise is as follows; can we simply identify the instability-induced state with the stable LOFF phase? It is certain that the instability tends to favor a LOFF-like state, but such a LOFF-like state might exist separated from the LOFF phase (if it exists) as sketched in Fig. 1(a), and then the proposed stable LOFF phase is not the *destination* from the instability but one *alternative* free from the instability like a mixed phase. Of course, it might be possible that the instability is directly connected to the LOFF phase as sketched in Fig. 1(b).

To address this question we have to define the qualitative difference between the LOFF and LOFF-like phases. We shall distinguish them by their characteristic wave numbers. That is, if the net momentum $\boldsymbol{q}$ is given as of order $\delta\mu$, then we regard the system as going to the conventional LOFF state that has $|\boldsymbol{q}| \simeq 1.2\delta\mu$. If $\boldsymbol{q}$ is small enough to be well separated from $\boldsymbol{q} \sim \delta\mu$ inherent to the LOFF phase, we consider that the system is then in the LOFF-like state. For the purpose of clarifying which situation of Figs. 1(a) and 1(b) is more relevant, we will calculate the free energy as a function of $\boldsymbol{q}$, or the color vector potential $\boldsymbol{A}_\alpha$ (α being

 © 2006 The American Physical Society

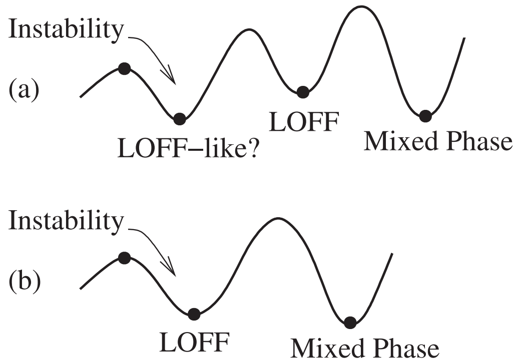

FIG. 1. Schematic energy landscape: (a) The state falls from the unstable gapless phase down toward a LOFF-like phase which is separated from the LOFF phase (if it exists). There are alternatives such as the LOFF phase, mixed phase, and so on, which should be energetically compared to the LOFF-like phase. (b) The instability directly leads to the LOFF phase.

the adjoint color index) which is related to $\boldsymbol{q}$ through the covariant derivative.

As a preparation for our discussions we shall briefly look over the color superconducting phases of our interest, the Meissner mass in respective phases, and associated chromomagnetic instability. The predominant pairing is

$$\Delta_\eta \sim \epsilon_{\eta ab}\epsilon_{\eta ij}\langle \bar{\psi}_{ai}\gamma_5\psi^C_{bj}\rangle \tag{1}$$

with $\psi^C = C\bar{\psi}^T$ and a, b and i, j being the color and flavor indices, respectively. Under this color-flavor locked ansatz, $\Delta_1 \neq 0$, $\Delta_2 \neq 0$, and $\Delta_3 \neq 0$ defines the CFL phase, while the two-flavor superconducting (2SC) phase has $\Delta_3 \neq 0$ and $\Delta_1 = \Delta_2 = 0$, that means only *ru-gd* and *rd-gu* quarks make a pair. The gapless 2SC and CFL phases (abbreviated as the g2SC and gCFL phases) occur when $\delta\mu \approx \mu_e/2 > \delta\mu_c^{\mathrm{g}} = \Delta_3$ and $\delta\mu \approx M_s^2/2\mu_q > \delta\mu_c^{\mathrm{g}} = \Delta_1$, respectively, where μ_e is the electron chemical potential. In the single plane-wave LOFF ansatz the gap parameters are augmented as

$$\Delta_\eta \xrightarrow{\mathrm{LOFF}} \exp[-2i\boldsymbol{q}_\eta\cdot\boldsymbol{x}]\Delta_\eta. \tag{2}$$

The Meissner mass is the screening mass for transverse gauge fields. The individual mass is a quantity dependent on the gauge choice; we can arbitrarily shuffle eight gluon fields $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_8$ by a gauge rotation. It should be noted, however, that the choice of the diquark condensate (1) specifies a gauge direction and then the Meissner mass is uniquely determined. The finite Meissner mass arises associated with spontaneous symmetry breaking and the Higgs-Anderson mechanism in superconductors [20].

In the 2SC phase $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{A}_3$ remain massless and the rest $\boldsymbol{A}_4, \ldots, \boldsymbol{A}_8$ earn a finite Meissner mass. The electromagnetic field $\boldsymbol{A}_\gamma$ has mixing with $\boldsymbol{A}_8$ which ends up with two eigen-fields $\boldsymbol{A}_{\tilde{8}}$ and $\boldsymbol{A}_{\tilde{\gamma}}$. The system has symmetry among $\boldsymbol{A}_4, \ldots, \boldsymbol{A}_7$ and thus the resulting Meissner mass is common for them, while $\boldsymbol{A}_{\tilde{8}}$ has a different Meissner mass. Because the modified (i.e. color-mixed) electromagnetic U(1) symmetry is unbroken, $\boldsymbol{A}_{\tilde{\gamma}}$ stays massless. It is known in the CFL phase at $M_s = 0$, on the other hand, that all eight gluons $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_8$ have a common and nonvanishing Meissner mass. In the presence of mixing with $\boldsymbol{A}_\gamma$ the massive eigen-field $\boldsymbol{A}_{\tilde{8}}$ pulls away from the others and $\boldsymbol{A}_{\tilde{\gamma}}$ is massless.

In the two-flavor case at finite $\delta\mu$ away from the ideal 2SC phase, there are still only two independent Meissner masses [12]; one is for $\boldsymbol{A}_4, \ldots, \boldsymbol{A}_7$ and the other is for $\boldsymbol{A}_{\tilde{8}}$. The Meissner mass squared for $\boldsymbol{A}_4, \ldots, \boldsymbol{A}_7$ becomes negative (i.e. the Meissner mass is imaginary) for $\delta\mu > \delta\mu_c^{4-7} = \delta\mu_c^{\mathrm{g}}/\sqrt{2}$, that means an instability occurs not only in the g2SC phase ($\delta\mu > \delta\mu_c^{\mathrm{g}}$) but in the 2SC phase ($\delta\mu_c^{4-7} < \delta\mu < \delta\mu_c^{\mathrm{g}}$) also. The Meissner mass squared for $\boldsymbol{A}_{\tilde{8}}$ is negatively divergent at the gapless onset, $\delta\mu = \delta\mu_c^{\mathrm{g}}$, and remains on negative in the entire g2SC side.

The three-flavor case with finite M_s has a more complicated pattern and there are five independent Meissner masses [15]. With nonzero M_s one should take account of mixing among $\boldsymbol{A}_3$, $\boldsymbol{A}_8$, and $\boldsymbol{A}_\gamma$ properly, from which two massive eigen-fields, $\boldsymbol{A}_{\tilde{3}}$ and $\boldsymbol{A}_{\tilde{8}}$, and one massless $\boldsymbol{A}_{\tilde{\gamma}}$ result. The Meissner mass is degenerated for $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ due to symmetry, and so is for $\boldsymbol{A}_4$ and $\boldsymbol{A}_5$, and for $\boldsymbol{A}_6$ and $\boldsymbol{A}_7$. No instability takes place until the system reaches the gCFL phase. At $\delta\mu = \delta\mu_c^{\mathrm{g}}$ negatively divergent Meissner masses squared appear for $\boldsymbol{A}_1$-$\boldsymbol{A}_2$ and for $\boldsymbol{A}_{\tilde{8}}$. As for $\boldsymbol{A}_4$-$\boldsymbol{A}_5$ and $\boldsymbol{A}_6$-$\boldsymbol{A}_7$, when $\delta\mu$ gets larger than critical values $\delta\mu_c^{4-5}$ and $\delta\mu_c^{6-7}$ respectively, they eventually have negative Meissner masses squared, which is presumably related to the instability in the two-flavor calculation. We shall summarize the instability patterns in Table I.

Now let us consider what the negative Meissner mass squared signifies. It is a textbook knowledge that in the ϕ^4-theory, for the simplest example, a nonzero expectation value $\langle\phi\rangle \neq 0$ grows when the screening mass squared for ϕ is negative. In the language of the effective potential the negative mass squared means that a state lies in a maximum of the potential and a true ground state should exist somewhere down away from $\langle\phi\rangle = 0$. Therefore it is quite

TABLE I. Chromomagnetic instability for each gluon in the two-flavor and three-flavor cases at zero temperature. Here $\delta\mu_c^{4-7} = \delta\mu_c^{\mathrm{g}}/\sqrt{2}$ and $\delta\mu_c^{4-5} \simeq \delta\mu_c^{6-7}$ has been numerically estimated as $\sim 2.3\delta\mu_c^{\mathrm{g}}$ in Ref. [15].

	two-flavor	three-flavor
$\boldsymbol{A}_1$, $\boldsymbol{A}_2$	massless	unstable $\delta\mu > \delta\mu_c^{\mathrm{g}}$
$\boldsymbol{A}_{\tilde{3}}$	massless	stable
$\boldsymbol{A}_4$, $\boldsymbol{A}_5$	unstable $\delta\mu > \delta\mu_c^{4-7}$	unstable $\delta\mu > \delta\mu_c^{4-5}$
$\boldsymbol{A}_6$, $\boldsymbol{A}_7$	unstable $\delta\mu > \delta\mu_c^{4-7}$	unstable $\delta\mu > \delta\mu_c^{6-7}$
$\boldsymbol{A}_{\tilde{8}}$	unstable $\delta\mu > \delta\mu_c^{\mathrm{g}}$	unstable $\delta\mu > \delta\mu_c^{\mathrm{g}}$
$\boldsymbol{A}_{\tilde{\gamma}}$	massless	massless

natural to anticipate that the chromomagnetic instability is cured by nonzero color vector potentials $\langle \boldsymbol{A}_\alpha \rangle$. Actually the Meissner mass squared is the coefficient of the quadratic terms in the kinetic energy expanded in $\boldsymbol{A}_\alpha$,

$$m^2_{M\alpha\beta} = \frac{1}{3} \frac{\partial^2 \Omega_{\rm kin}}{\partial A^i_\alpha \partial A^i_\beta} \bigg|_{A=0}, \tag{3}$$

where the kinetic energy term takes a form of

$$\begin{aligned}\Omega_{\rm kin}[\Delta, A] = \kappa^{\eta\sigma}_{\eta'\sigma'} &[(\partial^i \delta_{\eta\eta'} + igA^{*i}_{\eta\eta'})\Delta^*_{\eta'}] \\ &\times [(\partial^i \delta_{\sigma\sigma'} - igA^i_{\sigma\sigma'})\Delta_{\sigma'}] \\ &+ \text{higher-order terms in } A \end{aligned} \tag{4}$$

due to symmetry. In the following discussions we shall ignore mixing with $\boldsymbol{A}_\gamma$ for simplicity. Here we should remark that the stiffness parameter κ depends on the "flavor" indices η' and σ' as well as the "color" indices η and σ. This assignment is understood from that η of Δ_η contains the information on both color and flavor as fixed in (1). The covariant derivative acting on a color-triplet rotates η' into η, while the flavor is intact as η'.

In the two-flavor case only $\kappa^{\eta\sigma}_{33}$ is relevant and we can forget about flavor, as parametrized in Refs. [21,22]. Then the chromomagnetic instability with two independent Meissner masses in this case can be expressed by a combination of two parameters $\kappa^{(1)}$ and $\kappa^{(2)}$ where $\kappa^{\eta\sigma}_{33} = \kappa^{(1)}\delta_{\eta\sigma} + \kappa^{(2)}\Delta_\eta\Delta^*_\sigma$ which makes $\Omega_{\rm kin}$ a color-singlet. The single plane-wave LOFF state characterized by (2) with only Δ_3 nonvanishing is sensitive to $\kappa^{33}_{33} = \kappa^{(1)} + \kappa^{(2)}|\Delta_3|^2$ and thus such a gap parameter ansatz cannot separate two distinct instabilities for $\boldsymbol{A}_8$ with $m^2_{M88} \propto \kappa^{33}_{33}$ and for $\boldsymbol{A}_4, \ldots, \boldsymbol{A}_7$ with $m^2_{M4-7} \propto \kappa^{(1)}$ if they coexist.

It is intriguing to look into the three-flavor case next. The stiffness parameter can be decomposed as

$$\kappa^{\eta\sigma}_{\eta'\sigma'} = \kappa^\lambda_{\rm off}|\epsilon^{\lambda\eta\eta'}|\delta_{\eta\sigma}\delta_{\eta'\sigma'} + \kappa^{\eta\sigma}_{\rm diag}\delta_{\eta\eta'}\delta_{\sigma\sigma'}. \tag{5}$$

This decomposition is justified by the color and flavor structure in the quark one-loop calculations [15]. Then the first term involving $\kappa^\lambda_{\rm off}$ is relevant to the Meissner mass squared for the color off-diagonal gluons;

$$\begin{aligned} m^2_{M1-2} &\propto \kappa^3_{\rm off}(|\Delta_1|^2 + |\Delta_2|^2), \\ m^2_{M4-5} &\propto \kappa^2_{\rm off}(|\Delta_3|^2 + |\Delta_1|^2), \\ m^2_{M6-7} &\propto \kappa^1_{\rm off}(|\Delta_2|^2 + |\Delta_3|^2). \end{aligned} \tag{6}$$

The Meissner mass squared for the diagonal gluons, on the other hand, comes from the second term involving $\kappa^{\eta\sigma}_{\rm diag}$, that is, m^2_{M33}, m^2_{M38}, and m^2_{M88} are written as a linear combination of six components of the symmetric 3×3 matrix $\kappa^{\eta\sigma}_{\rm diag}$. We note that, when the ansatz (2) is substituted into (4), the instability toward finite $\boldsymbol{q}_\eta$ does not reflect the information of $\kappa^\lambda_{\rm off}$, that is, $\partial^2\Omega_{\rm kin}/\partial q_\eta \partial q_\sigma \propto \kappa^{\eta\sigma}_{\rm diag}$.

From (4) we can immediately understand how the chromomagnetic instability is generally transformed to the instability toward the single plane-wave LOFF phase, which has been shown for $\boldsymbol{A}_8$ in explicit calculations in two-flavor quark matter in Ref. [16]. The point is that the gauge fields in the quark sector appear only in the covariant derivative, so that they can be absorbed as a phase factor of the gap parameters.

Now we assume that we have an instability only for $\boldsymbol{A}_8$. The rotational symmetry is broken and we choose the n-direction in three-dimensional spatial space in which $\boldsymbol{A}_8$ acquires an expectation value. Then the covariant derivative is equivalently rewritten as

$$\begin{aligned} &[\partial^i \delta_{\eta\eta'} - \delta^{in} igA^n_8(t^8)_{\eta\eta'}]\Delta_{\eta'} \\ &\quad = \exp[igt^8 \boldsymbol{A}_8 \cdot \boldsymbol{x}]_{\eta\eta'}\partial^i\{\exp[-igt^8 \boldsymbol{A}_8 \cdot \boldsymbol{x}]_{\eta'\eta''}\Delta_{\eta''}\}, \end{aligned} \tag{7}$$

where t_α's are the color group generators in the fundamental representation. The color rotation results in

$$\exp[-igt^8 \boldsymbol{A}_8 \cdot \boldsymbol{x}] \cdot \Delta = \begin{pmatrix} \exp[-\frac{ig}{2\sqrt{3}} \boldsymbol{A}_8 \cdot \boldsymbol{x}]\Delta_1 \\ \exp[-\frac{ig}{2\sqrt{3}} \boldsymbol{A}_8 \cdot \boldsymbol{x}]\Delta_2 \\ \exp[+\frac{ig}{\sqrt{3}} \boldsymbol{A}_8 \cdot \boldsymbol{x}]\Delta_3 \end{pmatrix}, \tag{8}$$

which is nothing but the diquark condensate peculiar to the three-flavor single plane-wave LOFF state. [We assumed that $\boldsymbol{A}_\alpha$ is a constant, but the generalization to inhomogeneous $\boldsymbol{A}_\alpha(\boldsymbol{x})$ [23] is easy; the exponential part is then the Wilson line.] From the above rewriting, it is apparent that the non-LOFF (ordinary) superconducting phase with a color vector potential $\boldsymbol{A}_8$ is equivalent to the LOFF phase whose spatial oscillation is characterized by $\boldsymbol{A}_8$ with no vector potential. Of course, this general argument works in the two-flavor case as well; $\Delta_1 = \Delta_2 = 0$ and a phase factor emerges for Δ_3 alone, so one could interpret such an overall phase as associated with the baryon number [24], though such an interpretation has only a limited meaning.

One has to be careful when this argument is applied for the off-diagonal gluons, $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_4$, $\boldsymbol{A}_5$, $\boldsymbol{A}_6$, and $\boldsymbol{A}_7$. For instance, if the instability occurs in $\boldsymbol{A}_4$, then the phase factor is no longer in the form of the single plane-wave. In the same way as in the previous case we have

$$\begin{aligned} &\exp[-igt^4 \boldsymbol{A}_4 \cdot \boldsymbol{x}] \cdot \Delta \\ &\quad = \begin{pmatrix} -i\sin[\frac{g}{2}\boldsymbol{A}_4 \cdot \boldsymbol{x}]\Delta_3 + \cos[\frac{g}{2}\boldsymbol{A}_4 \cdot \boldsymbol{x}]\Delta_1 \\ \Delta_2 \\ \cos[\frac{g}{2}\boldsymbol{A}_4 \cdot \boldsymbol{x}]\Delta_3 - i\sin[\frac{g}{2}\boldsymbol{A}_4 \cdot \boldsymbol{x}]\Delta_1 \end{pmatrix} \end{aligned} \tag{9}$$

This represents not a single but rather multiple plane-wave LOFF state, or *color-rotated* single plane-wave LOFF. In the two-flavor case we keep Δ_3 alone and then, interestingly, (9) indicates that we definitely need to have not only the third component $\cos[\frac{g}{2}\boldsymbol{A}_4 \cdot \boldsymbol{x}]\Delta_3$ but also the first component $-i\sin[\frac{g}{2}\boldsymbol{A}_4 \cdot \boldsymbol{x}]\Delta_3$ which is not considered at all in the conventional two-flavor treatment. Our analysis agrees with the conclusion of Ref. [19] that the single plane-wave LOFF state would still have instability for the off-diagonal gluons.

From the discussions so far we can establish the *qualitative* (apart from a color rotation) correspondence between the vector potentials and the net momenta of the single plane-wave ansatz as

$$\frac{g\boldsymbol{A}_8}{2\sqrt{3}} \longleftrightarrow 2\boldsymbol{q}, \qquad \frac{g\boldsymbol{A}_4}{2} \longleftrightarrow 2\boldsymbol{q}, \tag{10}$$

Now we shall estimate the magnitude of characteristic $\boldsymbol{q}$ as a result of the instability using the above relations.

For that purpose we need to know the higher-order terms in $\boldsymbol{A}_\alpha$ in the expansion (4). As we will explain shortly, however, such an expansion is no longer valid in the gapless phase. Thus we must evaluate the $\boldsymbol{A}_\alpha$-dependent part of the free energy without expansion. For simplicity we will limit our discussions only to the two-flavor calculations from now on.

We write down the 48×48 (two-flavors, three-colors, two-spins, particle-antiparticle, and two-Nambu-Gorkov-doublers) quasiquark propagator with either $\boldsymbol{A}_4$ (that we arbitrarily chose among $\boldsymbol{A}_4, \ldots \boldsymbol{A}_7$) or $\boldsymbol{A}_8$ and calculate the quasiquark energy $\epsilon(\boldsymbol{p})$ which depends on the momentum angle to the vector potential, i.e., $\boldsymbol{p} \cdot \boldsymbol{A}_\alpha$. The free energy is available as integration of the sum over all 48 $|\epsilon(\boldsymbol{p})|$'s with respect to $\boldsymbol{p}$. We regulate the momentum integration by the ultraviolet cutoff $\Lambda = 1$ GeV and subtract the free energy at $\Delta = \delta\mu = 0$ to get rid of the cutoff artifact. The gap parameter is fixed at $\Delta = 100$ MeV. It should be noted that the analytical formulae utilized in the two-flavor LOFF calculations [17,19] do not work for $\boldsymbol{A}_4$ which is color off-diagonal.

We present the numerical results in Figs. 2 and 3. The potential curvature at $\boldsymbol{A}_\alpha = 0$ corresponds to the Meissner screening mass squared. In Fig. 2 the Meissner mass is real finite for $\delta\mu < \delta\mu_c^{4-7}$, while the origin $\boldsymbol{A}_4 = 0$ becomes unstable when $\delta\mu > \delta\mu_c^{4-7}$, as is manifest from the results at $\delta\mu = 0.9\delta\mu_c^{4-7}$ (dotted curve) and $\delta\mu = 1.1\delta\mu_c^{4-7}$ (solid curve). This instability occurs continuously and we can see from the $\delta\mu = 1.2\delta\mu_c^{4-7}$ results (dashed curve) that the expected $\boldsymbol{A}_4$ grows as $\delta\mu$ approaches $\delta\mu_c^{\mathrm{g}}$. It is known [12] that the negative Meissner mass for $\boldsymbol{A}_4$ becomes small again when $\delta\mu$ is larger than $\delta\mu_c^{\mathrm{g}}$. Certainly our calculations for $\delta\mu = 2.0\delta\mu_c^{4-7} = 1.4\delta\mu_c^{\mathrm{g}}$ (dot-dashed curve) result in smaller potential curvature and thus smaller Meissner mass than those for $\delta\mu = 1.2\delta\mu_c^{4-7}$. Nevertheless, the expected $\boldsymbol{A}_4$ is *larger* and we find a potential minimum at $g|\boldsymbol{A}_4|/4 \simeq 1.39\Delta = 0.98\delta\mu$. In this way the results for $\delta\mu > \delta\mu_c^{\mathrm{g}}$ make a sharp contrast to the nature of the instability for $\delta\mu \sim \delta\mu_c^{4-7} < \delta\mu_c^{\mathrm{g}}$. In the gapless region where $\delta\mu > \delta\mu_c^{\mathrm{g}}$, the expected $g|\boldsymbol{A}_4|/4$ (and thus $\boldsymbol{q}$) is of order $\delta\mu$ *however small the potential curvature (Meissner mass squared) is.*

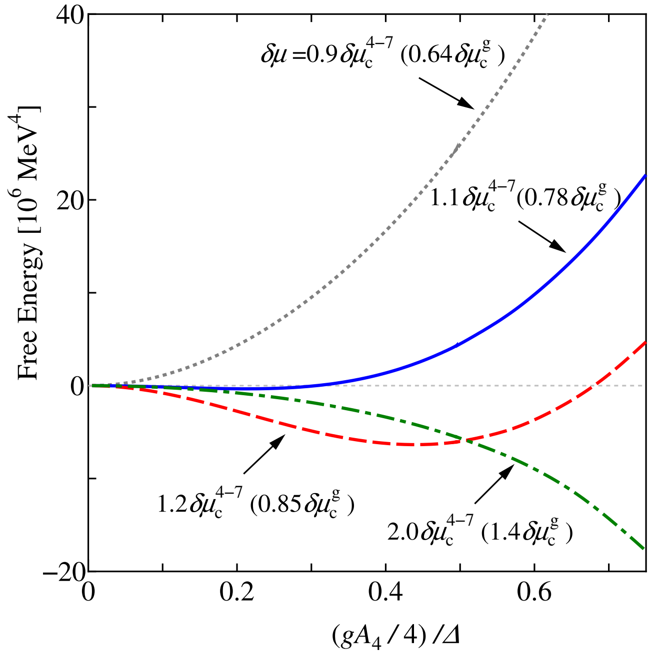

FIG. 2 (color online). Free energy difference from the energy without vector potentials in two-flavor quark matter as a function of $\boldsymbol{A}_4$. The critical $\delta\mu$ for $\boldsymbol{A}_4$ is $\delta\mu_c^{4-7} = \delta\mu_c^{\mathrm{g}}/\sqrt{2}$ and the g2SC phase occurs at $\delta\mu = \delta\mu_c^{\mathrm{g}}$.

The same observation is apparent also in Fig. 3. The Meissner mass squared for $\boldsymbol{A}_8$ is negative divergent at $\delta\mu = \delta\mu_c^{\mathrm{g}}$, meaning that the potential has a cusp at $\boldsymbol{A}_8 = 0$ then, which is confirmed in our results as seen at $\delta\mu = 1.01\delta\mu_c^{\mathrm{g}}$ (short-dashed curve). When $\delta\mu$ pulls away from the onset value, the negative Meissner mass squared becomes smaller, and at the same time, $\boldsymbol{A}_8$ acquires a larger expectation value of order $\delta\mu$ again.

We would emphasize that these findings are unexpected results; if the Ginzburg-Landau expansion of (4) works with a positive definite quartic term in $\boldsymbol{A}_\alpha$, an infinitesimal negative κ (potential curvature) simply leads to an infinitesimal $\boldsymbol{A}_\alpha$. Therefore, our results imply that not only the quadratic term but also quartic and even higher-order terms are significantly affected by gapless quarks when $\delta\mu > \delta\mu_c^{\mathrm{g}}$. The reason why the Ginzburg-Landau expansion breaks down can be understood in a diagrammatic way.

Figure 4 shows an example of the diagrammatic expansion of the free energy in terms of $\boldsymbol{A}_\alpha$. The dimensionless expansion parameter is obviously $g\langle \boldsymbol{A}_\alpha\rangle/\epsilon(\boldsymbol{p})$ where $\epsilon(\boldsymbol{p})$ is the quark energy stemming from the propagator. Therefore such an expansion is no longer legitimate once gapless quarks whose $\epsilon(\boldsymbol{p})$ can become vanishingly small enter the loop. In other words, in the gapless phase, the Meissner mass squared is far from informative on the true ground state; the smallness of the Meissner mass squared does not mean the weakness of the instability.

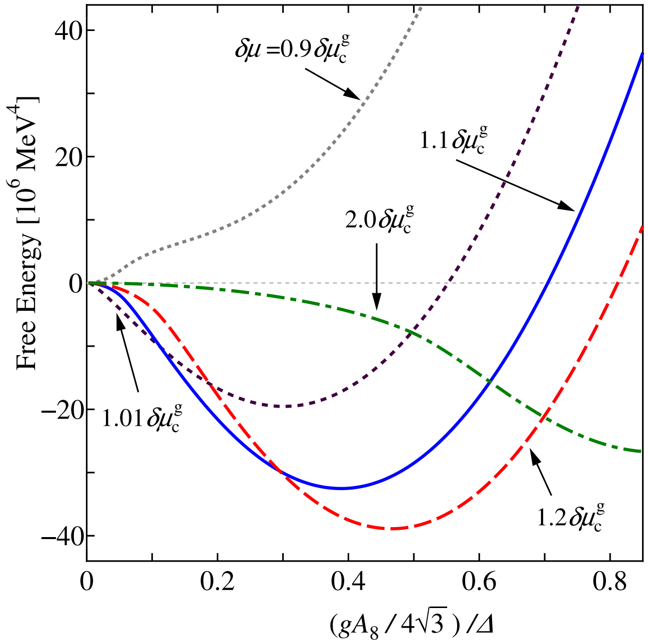

FIG. 3 (color online). Free energy difference from the energy without vector potentials in two-flavor quark matter as a function of $\boldsymbol{A}_8$ in the unit of Δ.

It should be noted that our potential analysis neither preserves neutrality nor solves the equations of motion. As long as $\boldsymbol{A}_\alpha$ is small, the coupling between $\boldsymbol{A}_\alpha$ and the other parameters such as the chemical potentials and the gap parameters is small due to approximate rotational symmetry. We can thus expect that the free energy we estimated would not be modified significantly by neutrality and the condensation energy in the region where $\boldsymbol{A}_\alpha$ is small. This is not the case, however, once $\boldsymbol{A}_\alpha \sim \delta\mu$ develops. Hence, strictly speaking, we cannot say anything about the exact location of the potential minimum, but we can at least insist that there is no stable LOFF-like state with small $\boldsymbol{q}$ directly resulting from the chromomagnetic instability in the gapless phase even when the negative Meissner mass squared is tiny.

Our results in the two-flavor case are not direct evidence but suggestive; we can anticipate that the situation in Fig. 1(a) is realized for the off-diagonal gluons at $\delta\mu_c^{4-7} < \delta\mu \ll \delta\mu_c^g$, while the situation in Fig. 1(b) is likely to be the case for $\delta\mu \gtrsim \delta\mu_c^g$. In the three-flavor case we conjecture that the instability picture is close to Fig. 1(b) since the chromomagnetic instability then occurs only in the gapless region of $\delta\mu$. These are our central conclusions derived from the numerical results.

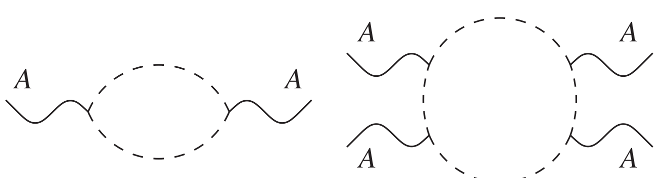

FIG. 4. Diagrammatic expansion of the free energy in terms of $g\langle \boldsymbol{A}_\alpha\rangle/\epsilon(\boldsymbol{p})$ where the quark energy $\epsilon(\boldsymbol{p})$ comes from the quark propagator.

Finally let us comment on the possibility of coexistence of both $\boldsymbol{A}_4$ and $\boldsymbol{A}_8$ in the two-flavor case. We shall call such a state the *gluonic phase* [25]. One should be careful about the terminology not to fall in a mere interpretation; we would use the nomenclature, the gluonic phase, differently from the original usage in Ref. [25], but to mean a state in which all $\boldsymbol{A}_\alpha$ in the covariant derivative cannot be simultaneously removed by any gauge rotation of Δ_η. Thus one can uniquely define the gluonic phase in a way distinct from the LOFF and LOFF-like states.

If $\boldsymbol{A}_4$ and $\boldsymbol{A}_8$ are not parallel, A_4^1 and A_8^2 for instance, then we cannot find an appropriate gauge rotation $\Delta \to V\Delta$ to eliminate them simultaneously. Namely, the gauge rotation matrix V satisfying $V^\dagger \partial^1 V = igA_4^1 t^4$ and $V^\dagger \partial^2 V = igA_8^2 t^8$ does not exist if V is assumed not to have any singularity.

The gluonic phase has one more significant difference from the LOFF state besides the covariant derivative; it has nonvanishing chromomagnetic field. For our example A_4^1 and A_8^2 produce a nonzero field strength tensor,

$$B_3^a = F^{a12} = -\frac{\sqrt{3}}{2}\delta^{a5} g A_4^1 A_8^2, \tag{11}$$

which means that the system has a uniform chromomagnetic field in it. However, such a state would never be realized, otherwise the field energy diverges. To put it in another way, the vector potentials A_4^1 and A_8^2 do not solve the Yang-Mills equations of motion, $D_\mu F^{\mu\nu a} = 0$ [26]. Therefore, we do not think that the gluonic phase results from the chromomagnetic instability.

In fact, one can reduce the field energy by making $\boldsymbol{A}_4$ and $\boldsymbol{A}_8$ be parallel to each other. Then the Yang-Mills action simply vanishes. This argument can be easily extended to more generic $\boldsymbol{A}_\alpha$ in the three-flavor case. We would thus reach a conclusion that all nonvanishing $\boldsymbol{A}_\alpha$ as a result of the chromomagnetic instability are aligned to the same direction energetically. Then such $\boldsymbol{A}_\alpha$ can be eliminated by a gauge rotation of Δ_η. That is, the likely destination is a state characterized by the gap parameters $\exp[-igt^\alpha \boldsymbol{A}_\alpha \cdot \boldsymbol{x}] \cdot \Delta$ where the summation over α is taken. This is what is called the *colored crystalline phase* in Ref. [15,22] and, as we have seen in (9), characterized by a LOFF ansatz beyond the single plane-wave one. Although the difference from the single plane-wave ansatz $\exp[-2i\boldsymbol{q}_\eta \cdot \boldsymbol{x}]\Delta_\eta$ is just a color rotation, it changes the physics because η of Δ_η has the information of flavor as we have already discussed in $\kappa_{\eta'\sigma'}^{\eta\sigma}$.

In summary, based on our numerical results for the free energy as a function of the color vector potentials, we have reached a speculation that the chromomagnetic instability in the gapless color superconducting region leads to the

LOFF phase, meaning that the LOFF phase is not an alternative but a destination of the instability. In contrast, the instability found in the 2SC (not g2SC) phase drives the system toward a LOFF-like state which is qualitatively distinct from the LOFF phase.

I would like to thank Kei Iida for discussions and Krishna Rajagopal for critical comments. I am also grateful to Larry McLerran for many things I learned through valuable conversations. This work was supported by the RIKEN BNL Research Center.

[1] For a review, see, K. Rajagopal and F. Wilczek, hep-ph/0011333.
[2] M. G. Alford, K. Rajagopal, and F. Wilczek, Nucl. Phys. **B537**, 443 (1999).
[3] M. G. Alford, J. A. Bowers, and K. Rajagopal, Phys. Rev. D **63**, 074016 (2001).
[4] A. I. Larkin and Yu. N. Ovchinnikov, Sov. Phys. JETP **20**, 762 (1965); P. Fulde and R. A. Ferrell, Phys. Rev. **135**, A550 (1964).
[5] J. A. Bowers and K. Rajagopal, Phys. Rev. D **66**, 065002 (2002).
[6] E. Gubankova, W. V. Liu, and F. Wilczek, Phys. Rev. Lett. **91**, 032001 (2003).
[7] M. Huang and I. Shovkovy, Phys. Lett. B **564**, 205 (2003); Nucl. Phys. A **729**, 835 (2003).
[8] M. Alford, C. Kouvaris, and K. Rajagopal, Phys. Rev. Lett. **92**, 222001 (2004); M. Alford, C. Kouvaris, and K. Rajagopal, Phys. Rev. D **71**, 054009 (2005).
[9] G. Sarma, J. Phys. Chem. Solids **24**, 1029 (1963).
[10] R. Casalbuoni, R. Gatto, N. Ippolito, G. Nardulli, and M. Ruggieri, Phys. Lett. B **627**, 89 (2005); **634**, 565(E) (2006).
[11] M. Mannarelli, K. Rajagopal, and R. Sharma, hep-ph/0603076.
[12] M. Huang and I. A. Shovkovy, Phys. Rev. D **70**, 051501 (2004); **70**, 094030 (2004).
[13] R. Casalbuoni, R. Gatto, M. Mannarelli, G. Nardulli, and M. Ruggieri, Phys. Lett. B **605**, 362 (2005); **615**, 297(E) (2005).
[14] M. Alford and Q. h. Wang, J. Phys. G **31**, 719 (2005).
[15] K. Fukushima, Phys. Rev. D **72**, 074002 (2005).
[16] I. Giannakis and H. C. Ren, Phys. Lett. B **611**, 137 (2005).
[17] I. Giannakis and H. C. Ren, Nucl. Phys. **B723**, 255 (2005).
[18] M. Ciminale, G. Nardulli, M. Ruggieri, and R. Gatto, hep-ph/0602180.
[19] E. V. Gorbar, M. Hashimoto, and V. A. Miransky, Phys. Rev. Lett. **96**, 022005 (2006).
[20] D. H. Rischke, Phys. Rev. D **62**, 034007 (2000); **62**, 054017 (2000).
[21] K. Iida and K. Fukushima, hep-ph/0603179.
[22] K. Fukushima, hep-ph/0510299.
[23] E. V. Gorbar, M. Hashimoto, V. A. Miransky, and I. A. Shovkovy, hep-ph/0602251.
[24] M. Huang, Phys. Rev. D **73**, 045007 (2006).
[25] E. V. Gorbar, M. Hashimoto, and V. A. Miransky, Phys. Lett. B **632**, 305 (2006).
[26] L. McLerran (private communication).