

Coulomb problem for vector bosons

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The Coulomb problem for vector bosons W^\pm incorporates a well-known difficulty; the charge of the boson localized in a close vicinity of the attractive Coulomb center proves to be infinite. The paradox is shown to be resolved by the QED vacuum polarization, which brings in a strong effective repulsion that eradicates the infinite charge of the boson on the Coulomb center. This property allows one to define the Coulomb problem for vector bosons properly.

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I. INTRODUCTION

Consider a charged vector boson, which propagates in the Coulomb field created by a heavy pointlike charge $Z|e|$ assuming that the boson is massive, its mass being produced via the Higgs mechanism; the W^\pm -bosons give an example. We study relativistic effects in this Coulomb problem. A situation where they can be important arises, for example, for small primordial charged black holes since an impact of their Coulomb field on a W -boson prevails over the gravitational field.

It has “always” been known that there is a difficulty in the Coulomb problem for vector bosons. Soon after Proca formulated a theory for vector particles [1] it became clear that it produces inadequate results for the Coulomb problem [2–4]. This fact inspired Corben and Schwinger [5] to modify the Proca theory, tuning the Lagrangian and equations of motion in such a way as to force the gyromagnetic ratio of the vector boson to acquire a favorable value $g = 2$. Later on the formalism of [5] was found to have a connection with the non-Abelian gauge theory [6], which makes it relevant for the present day studies. In particular, $g = 2$ is the gyromagnetic ratio of the W -boson in the standard model. A role of the identity $g = 2$ was thoroughly discussed in literature, see e. g. Refs. [7,8].

Reference [5] found a realistic discrete energy spectrum for the Coulomb problem for vector bosons. However, it discovered also a fundamental flaw in the problem. For two series of quantum states the charge of the vector boson located on the Coulomb center turns infinite, which indicates the fall of the boson on the center. One of these series has the total angular momentum zero, $j = 0$, another one has $j = 1$ (being further specified by a label “ $\gamma - 3/2$,” see Sec. IV D). This effect takes place for arbitrary small value of the Coulomb charge Z , which is physically unacceptable. All this indicates that the Coulomb problem is poorly defined. Thus, the Coulomb problem derived from

the standard model produces results, which challenge the model itself.

This difficulty was inspirational for several lines of research. Early efforts are summarized in Ref. [9]. More recent Refs. [10–12] suggested a new, refined modification of the formalism for vector bosons. Reference [13] claimed that it complies with results of Corben and Schwinger. Some authors considered other forms of the equation governing vector bosons [14–16], which produce more acceptable results for the Coulomb problem, but this advantage is partially undermined by the fact that it does not rely on a renormalizable theory.

The Coulomb corrections for a threshold $e^+e^- \rightarrow W^+W^-$ cross section were calculated in Ref. [17]. Comparing our results with these works one should keep in mind that the Corben-Schwinger problem, which we consider in this work, is distinguished by the following properties.

- (1) It manifests itself only for spin $S = 1$ particles, being absent for particles with $S = 0$ and $1/2$.
- (2) It is relevant in a close vicinity of the Coulomb center, when the ultrarelativistic limit is applicable.
- (3) The problem persists for the charge density, being absent in the energy spectrum (presuming that the magnetic g -factor is $g = 2$).

The papers Ref. [17] addressed mainly the Coulomb corrections for W^+W^- production for the *nonrelativistic* W -bosons. The questions we consider do not show up in this problem. Also, the authors of Ref. [17] were interested in corrections important for the experiment. In particular, they take into account the well-known Coulomb factor (which also exists for $S = 0$ and $S = 1/2$ particle production) and the effects due to the width of the W -boson, which prove to be important mostly in the near-threshold region.

In contrast, for understanding of the Corben-Schwinger problem the width of the W -boson is irrelevant. The reason stems from a simple fact that this problem arises close to the Coulomb center, where even the energy of the W -boson is negligible in the wave equation. Having this fact in mind, we will neglect the width in the following discussion,

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finding a solution for the Corben-Schwinger problem at small distances.

In order to apply our results to the full-scale Coulomb problem, when the events at large distances, where the width is important, are also accounted for, one can adopt one of the following attitudes. First, one can restrict consideration to sufficiently large Coulomb charges, presuming that the Coulomb binding energy $B \approx \alpha^2 Z^2 m/2$, where $m \approx 80.419$ GeV is the mass of the W -boson, is larger than its width $\Gamma = 2.12$ GeV, data from Ref. [18]. Condition $B \gg \Gamma$, which makes the effects related to the width negligible, is satisfied provided $Z > 30$. Alternatively, one can be interested in theoretical modifications of the standard model, in which the width is small (by taking, for example, the mass of a vector particle smaller than 2 fermion masses). At last, the most accurate treatment would require that our solution of the Corben-Schwinger problem is combined with the solution of the wave equation valid at large distances (which include the effects related to the width) by sawing the two solutions in the region of intermediate distances, but we will not dwell on this subject in this paper any longer.

In this work we suggest a clear way to formulate the Coulomb problem for vector particles properly, within the frames of the standard model. Our main observation is that the polarization of the QED vacuum has a profound impact in the problem forcing the density of charge of a vector boson to decrease at the origin, thus making the Coulomb problem stable, well defined. This decrease has an exponential character for the $j = 0$ state. For the $j = 1, \gamma - 3/2$ state the suppression is of a power-type. In both these states the suppression eradicates the difficulty of the Coulomb problem.

From the first glance this result looks surprising. Presumably, the vacuum polarization is meant to make the attractive Coulomb field only stronger at small distances, which should result in an increase of the charge density at the origin. Additionally, the vacuum polarization for spinor and scalar particles in the Coulomb field is known to produce only small, perturbative effects. In contrast, we claim a strong *reduction* of the charge density for the vector particle. To grasp a physical mechanism involved it is necessary to notice that the equation of motion for vector particles incorporates a particular term, which explicitly depends on the external current and has no counterparts for scalars and spinors, see the last term in Eq. (2.10). Precisely this term brings in a strong effective repulsion, which stems from the vacuum polarization and makes the Coulomb problem stable, well defined.

The renormalizability of the standard model means that if all essential processes are taken care of, then the results must be finite (at least within the perturbation theory after the renormalization). The bound state problem is, strictly speaking, not a perturbative problem. However, in the spirit of the standard model the amplitude of the photon ex-

change between leptons or/and quarks at high transferred momenta should be considered alongside exchange by the Higgs and Z -bosons. From this perspective the catastrophic behavior of the charge density of a vector boson at small distance, i.e. at large transferred momenta, in the Coulomb problem could have been considered as an indication that the Coulomb problem for vector bosons should include the processes related to the Higgs and Z -bosons exchange from the very beginning. In contrast to this widely spread presumption we find a way to formulate the Coulomb problem for vector bosons entirely in terms of the W and electromagnetic fields, as a pure QED problem.

A complete standard model calculation, where all possible processes are accounted for accurately, would require specific information on the nature of a heavy particle that creates the Coulomb field, i.e. on all its quantum numbers related to the standard model. This information is not necessarily feasible. A simple example gives primordial black holes; it is not easy to assert with certainty whether they have, or have not the weak charge, and what are their other quantum numbers in the standard model. The same questions arise in relation to other possible candidates for the heavy Coulomb center. As a result, a presumption that the exchange of the Higgs and Z -boson should play a basic role in the Coulomb problem leads to complications. It is fortunate therefore that the detailed information on properties of the heavy particle proves to be redundant, that the Coulomb problem can be properly defined using the only physical parameter of the heavy particle, its electric charge.

This point of view, which is advocated in the present work keeps the Coulomb problem simple and transparent. On the other hand, it also allows one to include all other processes, which are left outside the scope of the Coulomb problem, by means of perturbation theory. Our preliminary calculations indicate that the exchange by the Higgs and Z -bosons, as well as possible processes with lepton or quark exchange, give only small corrections. The reason stems from the fact that the found wave functions for vector bosons are suppressed at small distances. Consequently, the small-distance processes with the exchange by Higgs and Z -bosons are also suppressed (the exchange by a lepton or quark contains the vanishing at the Coulomb center fields, which describe the W -boson).

In Sec. II the Corben-Schwinger formalism for charged vector bosons is derived directly from the standard model. The pure Coulomb problem is discussed in Secs. III, IV, and V, and several Appendixes. This analysis follows Ref. [5], but some important details, including the non-relativistic limit (Sec. III) and the eigenvalue problem for $j = 0$ states (Sec. V) are discussed in more detail. Sections VI and VII present the main result of the paper. They show that the QED vacuum polarization plays a defining role in the problem, as was first noticed in our previous work [19]. The units $\hbar = c = 1$, $e^2 = 4\pi\alpha$, where $e < 0$, are used below.

II. W-MESONS IN ELECTROMAGNETIC FIELD

A. W-bosons in standard model

Consider boson fields in the electroweak part of the Lagrangian of the standard model, see e.g. Ref. [20],

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu + g A_\nu \times A_\mu)^2, \\ -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{1}{2}D_\mu \Phi^\dagger D^\mu \Phi. \quad (2.1)$$

Here A_ν and B_μ are the triplet of $SU(2)$ and the $U(1)$ gauge potentials, respectively, (abridged notation is used here). The covariant derivative $D_\mu \Phi$ takes into account that the Higgs field Φ has a hypercharge $Y = 2$, which describes its interaction with the $U(1)$ field, and is transformed as a doublet under the $SU(2)$ gauge transformations. Taking the unitary gauge one can present it via one real component

$$\Phi = \begin{pmatrix} 0 \\ \phi \end{pmatrix}, \quad \phi = \phi^+. \quad (2.2)$$

Assuming that the scalar field develops the vacuum expectation value $\phi = \phi_0$ and the Higgs mechanism takes place, one finds that the gauge field can be presented as a new $U(1)$ field A_μ , and a triplet of massive fields W_μ^\pm, Z_μ

$$A_\mu = \sin\theta A_\mu^3 + \cos\theta B_\mu, \quad (2.3)$$

$$Z_\mu = \cos\theta A_\mu^3 - \sin\theta B_\mu, \quad (2.4)$$

$$W_\mu = (A_\mu^1 - iA_\mu^2)/\sqrt{2}. \quad (2.5)$$

Here $W_\mu \equiv W_\mu^-$ represents the W -boson with charge $e = -|e|$, and θ is the Weinberg angle.

Expanding the Lagrangian equation (2.1) in the vicinity of $\phi = \phi_0$ and retaining only bilinear in the fields W_μ, W_μ^+ terms, including their interaction with the electromagnetic field, one derives an effective Lagrangian

$$\mathcal{L}^W = -\frac{1}{2}(\nabla_\mu W_\nu - \nabla_\nu W_\mu)^\dagger (\nabla^\mu W^\nu - \nabla^\nu W^\mu) \\ + ieF^{\mu\nu} W_\mu^+ W_\nu + m^2 W_\mu^+ W^\mu, \quad (2.6)$$

which describes the propagation of W -bosons in an external electromagnetic field. Here m is the mass of W . The external field is accounted for in Eq. (2.6) in the derivative $\nabla_\mu = \partial_\mu + ieA_\mu$ and by the term with the field $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The first and the last terms in Eq. (2.6) are present in the Proca formalism [1], while the second one was introduced by Corben and Schwinger [5].

From Eq. (2.6) one derives the classical Lagrange equation of motion for vector bosons

$$(\nabla^2 + m^2)W^\mu + 2ieF^{\mu\nu}W_\nu - \nabla^\mu \nabla^\nu W_\nu = 0. \quad (2.7)$$

Here an identity $[\nabla_\mu, \nabla_\nu] = ieF_{\mu\nu}$ was used. Taking a covariant derivative in Eq. (2.7) one finds

$$m^2 \nabla_\mu W^\mu + ie j_\mu W^\mu = 0, \quad (2.8)$$

where

$$j^\mu = \partial_\nu F^{\nu\mu}, \quad (2.9)$$

is the external current, which creates the external field $F^{\nu\mu}$. Evaluating $\nabla_\mu W^\mu$ from Eq. (2.8) and substituting the result back into Eq. (2.7) one rewrites the latter one in a more transparent form

$$(\nabla^2 + m^2)W^\mu + 2ieF^{\mu\nu}W_\nu + \frac{ie}{m^2} \nabla^\mu (j_\nu W^\nu) = 0. \quad (2.10)$$

This equation of motion for vector bosons was suggested in Ref. [5]. The coefficient 2 in front of the second term ensures that the g -factor of the boson takes the value $g = 2$, see Eq. (2.25) below.

The derivation outlined shows that Eq. (2.10) represents the classical equation of motion for W -bosons in the external electromagnetic field, which is valid within the frames of the standard model. This equation has similarities with the Klein-Gordon and Dirac equations (if the latter one is written as the second-order differential equation), but there is also an important distinction. It is produced by the last term in Eq. (2.10), which explicitly contains the external current; there is no similar term for scalars and spinors. We will see how important this term is, when we discuss the vacuum polarization.

We will use below a current of vector bosons j_μ^W , which can be obtained by considering a variation of the Lagrangian equation (2.6) under variation of A_μ , which yields

$$j_\mu^W = j_\mu^{(1)} + j_\mu^{(2)} + j_\mu^{(3)}, \quad (2.11)$$

$$j_\mu^{(1)} = -ie(W_\nu^+ \nabla_\mu W^\nu - \nabla_\mu W_\nu^+ W^\nu), \quad (2.12)$$

$$j_\mu^{(2)} = -ie(\nabla_\nu W_\mu^+ W^\nu - W_\nu^+ \nabla^\nu W_\mu), \quad (2.13)$$

$$j_\mu^{(3)} = -ie\partial^\nu (W_\mu^+ W_\nu - W_\nu^+ W_\mu). \quad (2.14)$$

Differentiating in Eq. (2.14) term by term and taking into account Eq. (2.8) one verifies that

$$j_\mu^{(3)} = j_\mu^{(2)} - ie(W_\mu^+ \nabla_\nu W^\nu - \nabla^\nu W_\nu^+ W_\mu) \\ = j_\mu^{(2)} - \frac{e^2}{m^2} (W_\mu^+ W_\nu + W_\nu^+ W_\mu) j^\nu. \quad (2.15)$$

Using this result, the current Eq. (2.11) can be written in a compact form

$$j_\mu^W = -ie(W_\nu^+ \nabla_\mu W^\nu + 2\nabla_\nu W_\mu^+ W^\nu - \text{c.c.}) \\ - \frac{e^2}{m^2} (W_\mu^+ W_\nu + W_\nu^+ W_\mu) j^\nu. \quad (2.16)$$

Here c.c. refers to two complex conjugated terms.

B. Static electric field

Consider a static electric field described by the electric potential $A_0 = A_0(\mathbf{r})$ and charge density $\rho = \rho(\mathbf{r}) = -\Delta A_0$. For a stationary state of the W -boson one can presume that

$$\nabla_0^2 W_\mu = -(\varepsilon - U)^2 W_\mu, \quad (2.17)$$

where ε is the energy of the stationary state, and $U = U(\mathbf{r}) = eA_0$ is the potential energy of the W -boson in the electric field. Equation (2.8) in this case gives

$$\mathbf{w} = (\varepsilon - U - Y)^{-1} \nabla \cdot \mathbf{W}. \quad (2.18)$$

The four-vector $W^\mu = (W_0, \mathbf{W})$ is presented here via the three-vector \mathbf{W} and the modifies zeroth-component $\mathbf{w} = iW_0$. In order to simplify notation we introduce also a very important for us quantity $Y = Y(\mathbf{r})$,

$$Y = \frac{e\rho}{m^2} = -\frac{\Delta U}{m^2}. \quad (2.19)$$

Equations (2.8) and (2.17) show that this definition complies with (2.18). The quantity Y appears in the equations of motion alongside the initial potential U , see e.g. Eq. (2.18). In this sense it plays a role of an effective potential energy, which is specific for vector bosons. We will call it the Y -term, or Y -potential. In this notation Eq. (2.10) reads

$$((\varepsilon - U)^2 - m^2)\mathbf{W} = -\Delta \mathbf{W} - 2\nabla U \mathbf{w} - \nabla(Y\mathbf{w}), \quad (2.20)$$

$$((\varepsilon - U)^2 - m^2)\mathbf{w} = -\Delta \mathbf{w} + 2\nabla U \cdot \mathbf{W} + (\varepsilon - U)Y\mathbf{w}. \quad (2.21)$$

A relation between \mathbf{w} and \mathbf{W} given by Eq. (2.18) shows that among four equations of motion Eqs. (2.20) and (2.21) only three are independent, precisely what one expects for massive vector particles.

It will be useful to present Eqs. (2.20) in a slightly different form, which can be derived by combining it with Eq. (2.18) and using an identity $\Delta \mathbf{W} = \nabla \times (\nabla \times \mathbf{W}) - \nabla(\nabla \cdot \mathbf{W})$, which gives

$$((\varepsilon - U)^2 - m^2)\mathbf{W} = \nabla \times (\nabla \times \mathbf{W}) - (\varepsilon - U)\nabla \mathbf{w} - \nabla U \mathbf{w}. \quad (2.22)$$

From the expression for the current of vector bosons Eq. (2.16) one derives the charge density

$$\rho^W = 2e((\varepsilon - U)(\mathbf{W}^+ \cdot \mathbf{W} + \mathbf{w}^+ \mathbf{w}) + \mathbf{W}^+ \cdot \nabla \mathbf{w} + \mathbf{W} \cdot \nabla \mathbf{w}^+ - Y\mathbf{w}^+ \mathbf{w}). \quad (2.23)$$

C. G-factor

The behavior of vector bosons in the homogeneous magnetic fields was studied in detail, see e.g. [9] and references therein. The spectrum of this problem reads, see Sec. I,

$$\varepsilon^2 = m^2 + p_z^2 + 2|e|B(n + 1/2 + \sigma). \quad (2.24)$$

Here $n = 0, 1, \dots$ specifies the Landau levels, and $\sigma = -1, 0, 1$ gives a projection of spin $S = 1$ of the vector boson. Equation (2.24) shows that vector bosons possess the magnetic moment

$$\boldsymbol{\mu} = eS/m, \quad (2.25)$$

which means that the magnetic g -factor is $g = 2$.

III. NONRELATIVISTIC LIMIT

Consider a vector boson in a static electric field with the potential energy $U = eA_0(\mathbf{r})$. If we presume that the non-relativistic approach is valid, which needs that $|U| \ll m$, then in the lowest order of the perturbation theory in powers of U/m one immediately finds from Eqs. (2.18) and (2.20)

$$E\mathbf{W} = -\frac{1}{2m}\Delta \mathbf{W} + U\mathbf{W}. \quad (3.1)$$

Here $E \simeq \varepsilon - m$ is the energy, the vector \mathbf{W} plays a role of the wave function for the vector boson, and the nonrelativistic Hamiltonian on the right-hand side has a usual form for a massive charged particle.

Let us find corrections to Eq. (3.1) induced by relativistic effects. The wave function of the massive vector particle $\boldsymbol{\Phi}$ is well defined in the rest frame. Therefore the vector \mathbf{W} , which describes the moving vector particle, inevitably deviates from the wave function $\boldsymbol{\Phi}$. A relation between \mathbf{W} and $\boldsymbol{\Phi}$ is easy to articulate for the free motion, when it is given by the Lorentz boost, see e.g. a book [21],

$$\mathbf{W} = \boldsymbol{\Phi} + \frac{\mathbf{p}(\mathbf{p} \cdot \boldsymbol{\Phi})}{m(m + \varepsilon)}. \quad (3.2)$$

Generically, the potential energy brings in complications, but within the necessary accuracy we can neglect them, presuming also that $\varepsilon \simeq m$. Then Eq. (3.2) gives

$$\mathbf{W} \simeq \boldsymbol{\Phi} + \frac{\mathbf{p}(\mathbf{p} \cdot \boldsymbol{\Phi})}{2m^2}, \quad (3.3)$$

where $\mathbf{p} = -i\nabla$. This relation plays a role similar to the Foldy-Wouthuysen transformation [22] for fermions.

Substituting Eq. (3.3) in Eqs. (2.18) and (2.20) and expanding the latter ones in powers of U/m one finds the following Schrödinger-type equation for the wave function $\boldsymbol{\Phi}$ of the vector boson

$$E\boldsymbol{\Phi}_i = H_{ij}\boldsymbol{\Phi}_j, \quad (3.4)$$

$$H_{ij} = \left(\frac{\mathbf{p}^2}{2m} + U\right)\delta_{ij} + \delta H_{ij}, \quad (3.5)$$

$$\delta H_{ij} = -\frac{\mathbf{p}^4}{8m^3} \delta_{ij} - \frac{\mathbf{F} \cdot (\mathbf{p} \times \mathbf{S}_{ij})}{2m^2} + \frac{\Delta U}{6m^2} \delta_{ij} + \frac{1}{6m^2} \left(3 \frac{\partial^2 U}{\partial r_i \partial r_j} - \Delta U \delta_{ij} \right). \quad (3.6)$$

Here $i, j = 1, 2, 3$ label components of three-vectors, \mathbf{S} is the spin, which operates on a vector \mathbf{V} according to $\mathbf{S}_{ij} V_j = -i \epsilon_{ijk} V_k$.

The relativistic correction to the Hamiltonian δH of vector particles is given in Eq. (3.6). It is instructive to compare this correction with the known Darwin Hamiltonian δH_D , which accounts for relativistic effects for spinor particles

$$\delta H_D = -\frac{\mathbf{p}^4}{8m^3} - \frac{\mathbf{F} \cdot (\mathbf{p} \times \mathbf{s})}{2m^2} + \frac{\Delta U}{8m^2}. \quad (3.7)$$

Here $\mathbf{s} = \boldsymbol{\sigma}/2$ is the operator of spin for spinor particles. The three terms in the first line of Eq. (3.6) resemble their counterparts in Eq. (3.7), the only distinction is the numerical coefficient in front of the term with ΔU . We conclude that these three terms have conventional meaning, describing the relativistic correction to the kinetic energy, the spin-orbit interaction, and the contact correction to the potential. The coefficient in front of the term responsible for the spin-orbit interaction in Eq. (3.5) complies with the gyromagnetic ratio $g = 2$ of the vector boson, if one presumes that the Thomas ‘‘one-half rule’’ is applicable for vector particles the same way is for spinors.

The last, fourth term in Eq. (3.6) finds no counterpart in the Darwin Hamiltonian. It is instructive to write a contribution of this term to the energy shift

$$\delta E_Q = \frac{1}{6} \int Q_{ij} \frac{\partial^2 A_0}{\partial r_j \partial r_i} d^3 r, \quad (3.8)$$

$$Q_{ij} = \frac{e}{m^2} (3\Phi_i^* \Phi_j - \delta_{ij} |\Phi|^2). \quad (3.9)$$

Equations (3.8) and (3.9) show that Q_{ij} plays a role of the density of the quadrupole moment for vector bosons. We conclude that the last, fourth term in Eq. (3.6) indicates that vector bosons have a quadrupole moment.

From the first glance the contact and the quadrupole terms in the Eq. (3.6) have similarity with the Y-term in Eq. (2.19). However this resemblance is coincidental, since the Y-term does not contribute to (3.6), which takes into account corrections of the order of $(Z\alpha)^2$. Equation (2.19) allows to estimate the Y-potential as $Y \sim (mr_0)^{-2} U \sim (Z\alpha)^2 U$, where $r_0 = (Zam)^{-1}$ is the Bohr radius. The Y-potential comes into the equation of motion with the factor \mathbf{w} , see the last term in Eq. (2.20). Equation (2.18) gives an estimate $|\mathbf{w}| \sim (mr_0)^{-1} |\mathbf{W}| \sim Z\alpha |\mathbf{W}|$. Overall, an estimate for the correction produced by the Y-term in Eq. (2.20) is $\sim (Z\alpha)^3$, which means that the Y-term is too small to contribute to Eq. (3.6). Thus, the contact and

quadrupole interactions in Eq. (3.6) have no direct connection with the Y-term. This fact makes a difference in coefficients in front of the contact term in Eq. (3.6) and the Y-term in Eq. (2.19) acceptable. In particular, the fact that they have opposite signs produces no contradiction.

IV. COULOMB PROBLEM

Consider the pure Coulomb field, presuming that it is created by a pointlike heavy object with charge $Z|e| > 0$. Then for $r > 0$ one has

$$U = -\frac{Z\alpha}{r}, \quad Y = 0. \quad (4.1)$$

The second identity here follows from Eq. (2.19).

A. Perturbation theory

Let us treat the Coulomb problem using the nonrelativistic perturbation theory. Take the nonrelativistic equation (3.1) as a starting point, and consider the Hamiltonian equation (3.6) as a perturbation. Conventional calculations, see Appendix B, lead to the following result for the shift of the energy level characterized by the main quantum number n , orbital momentum l , and total angular momentum $j = l, l \pm 1$

$$\delta E_{nlj} = \frac{m(Z\alpha)^4}{n^3} \left(\frac{3}{8n} - \frac{1}{2j+1} \right). \quad (4.2)$$

This formula is similar to the one that describes the energy shifts for spinor particles; the only distinction comes from values of j in Eq. (4.2), which are integers for vector particles and half-integers for spinors. The order of several lowest levels shows the following pattern

$$\begin{aligned} n = 1 & \quad 1s_1; & n = 2 & \quad 2p_0, \quad \{2s_1, 2p_1\}, \quad 2p_2; \\ n = 3 & \quad 3p_0, \quad \{3s_1, 3p_1, 3d_1\}, & \{3p_2, 3d_2\}, & \quad 3d_3. \end{aligned} \quad (4.3)$$

Here the atomiclike notation nl_j is adopted, the brackets combine together the degenerate energy levels.

B. Central field

Consider the static central electric field (the Coulomb problem gives an important example). The conservation of the total angular momentum j in this field allows one to separate the angular variables. We will use for this purpose the electric, longitudinal, and magnetic spherical vectors, $\mathbf{Y}_{jm}^{(e)}$, $\mathbf{Y}_{jm}^{(l)}$, $\mathbf{Y}_{jm}^{(m)}$ defined conventionally, see [21] and Appendix C. Generically, one can present the vector \mathbf{W} as a linear combination of three spherical vectors with the given value of j . It is convenient to refer to the three terms in this combination as the electric, longitudinal, and magnetic modes (or polarizations) of a vector boson. The parity conservation simplifies the problem further on. The state with the magnetic polarization, which parity is different

from the parity of the other two modes, is not coupled with these modes. Therefore the magnetically polarized mode can be written in a simple form

$$\mathbf{W} = f\mathbf{Y}_{jm}^{(m)}, \quad (4.4)$$

where $f = f(r)$ is the radial function. The two modes related to electric and longitudinal polarizations have same parity, which makes coupling between these modes possible. One needs therefore to consider them on the same footing assuming that

$$\mathbf{W} = u\mathbf{Y}_{jm}^{(e)} + v\mathbf{Y}_{jm}^{(l)}, \quad (4.5)$$

where $u = u(r)$, $v = v(r)$. We will refer to them as electro-longitudinal modes, or polarizations.

C. Magnetic polarization, $j \geq 1$

For the magnetic mode the angular momentum is restricted $j \geq 1$ (the magnetic spherical vector is not defined for $j = 0$, see Eq. (C1)). Substituting Eq. (4.4) into Eq. (2.20) one finds the following equation for the radial function f

$$(\Delta_j + (\varepsilon + Z\alpha/r)^2 - m^2)f = 0. \quad (4.6)$$

Here Δ_j is

$$\Delta_j = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - \frac{j(j+1)}{r^2}. \quad (4.7)$$

The form of Eq. (4.6) coincides with the Klein-Gordon equation. Therefore the spectrum of the magnetic mode replicates the spectrum of scalar particles, which is given by the Sommerfeld formula

$$\varepsilon = m \left(1 + \frac{(Z\alpha)^2}{(\gamma + n - j - 1/2)^2} \right)^{-1/2}. \quad (4.8)$$

Here

$$\gamma = ((j + 1/2)^2 - (Z\alpha)^2)^{1/2}. \quad (4.9)$$

In Eq. (4.8) $n = 1, 2, \dots$ plays a role of the main quantum number. In the nonrelativistic limit the magnetic mode corresponds to the states $2p_1, 3d_2, 4f_3, \dots$

D. Electrolongitudinal polarizations, $j \geq 1$

Consider electrolongitudinal polarizations, when the vector \mathbf{W} is given by Eq. (4.5). Substituting it into Eqs. (2.18) and (2.20) and using the properties of the spherical vectors from Appendix C one finds a system of coupled equations for radial functions u , v

$$(\Delta_j + (\varepsilon + Z\alpha/r)^2 - m^2)u = -2\sqrt{j(j+1)} \frac{v}{r^2}, \quad (4.10)$$

$$(\Delta_j + (\varepsilon + Z\alpha/r)^2 - m^2)v = -2\sqrt{j(j+1)} \frac{u}{r^2} + \frac{2v}{r^2} - \frac{2Z\alpha w}{r^2}. \quad (4.11)$$

Here $w = w(r)$ denotes the radial part of \mathbf{w} . Using Eq. (2.18) one finds for it

$$\mathbf{w} = w\mathbf{Y}_{jm}, \quad (4.12)$$

$$w = \frac{1}{\varepsilon + Z\alpha/r} \left(-\sqrt{j(j+1)} \frac{u}{r} + \frac{dv}{dr} + \frac{2v}{r} \right). \quad (4.13)$$

Equations (4.10) and (4.11) are sufficient to define the functions u , v , but it is convenient to compliment them by the radial form of Eq. (2.21), which reads

$$(\Delta_j + (\varepsilon + Z\alpha/r)^2 - m^2)w = \frac{2Z\alpha v}{r^2}. \quad (4.14)$$

Let us verify first that Eqs. (4.10) and (4.11) describe two different modes. Consider with this purpose distances so small that $m \ll Z\alpha/r$, where the potential energy dominates over mass. In this region Eqs. (4.10) and (4.11) reduce to

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{(Z\alpha)^2 - j(j+1)}{r^2} \right) u + 2(j(j+1))^{1/2} \frac{v}{r^2} = 0, \quad (4.15)$$

$$\left(\frac{d^2}{dr^2} + \frac{4}{r} \frac{d}{dr} + \frac{(Z\alpha)^2 - j(j+1) + 2}{r^2} \right) v = 0. \quad (4.16)$$

One derives from Eqs. (4.15) and (4.16) that there exists a mode, in which at small distances v is small, $|v| \ll |u|$, which means that in this region the polarization is predominantly electric. From Eq. (4.15) one finds that this mode satisfies the following asymptotic conditions at $r \rightarrow 0$

$$u \rightarrow ar^{\gamma-1/2}, \quad |v| \ll |u|. \quad (4.17)$$

We will call it the “ $\gamma - 1/2$ ” mode below.

In order to find the second mode let us assume the following asymptotic behavior for $r \rightarrow 0$

$$u \rightarrow br^\nu, \quad (4.18)$$

$$v \rightarrow cr^\nu. \quad (4.19)$$

Substituting Eqs. (4.18) and (4.19) in Eqs. (4.15) and (4.16) one finds a system of two homogeneous linear equations, in which ν plays a role of the eigenvalue. Solving this system one finds ν and the ratio c/b , deriving

$$u \rightarrow br^{\gamma-3/2}, \quad (4.20)$$

$$v \rightarrow b \frac{\gamma - 1/2}{\sqrt{j(j+1)}} r^{\gamma-3/2}. \quad (4.21)$$

This mode will be referred to as the $\gamma - 3/2$ mode [23].

Let us find now the discrete energy spectrum. Introduce a function $g = g(r)$

$$\begin{aligned} g &= Z\alpha u + \sqrt{j(j+1)}v \\ &= Z\alpha u + \frac{\sqrt{j(j+1)}}{\varepsilon + Z\alpha/r} \left(-\sqrt{j(j+1)}\frac{u}{r} + \frac{dv}{dr} + \frac{2v}{r} \right). \end{aligned} \quad (4.22)$$

Here Eq. (4.13) was used in the second identity. Taking the corresponding linear combination of Eqs. (4.10) and (4.14) one finds that g satisfies the Klein-Gordon equation

$$(\Delta_j + (\varepsilon + Z\alpha/r)^2 - m^2)g = 0. \quad (4.23)$$

This result leaves only two options; either g equals zero identically, or, alternatively, the spectrum of electrolongitudinal modes can be found from Eq. (4.23). The first alternative takes place for $j = 0$, when only the longitudinal mode is present. The function u in this case should be taken as zero, which makes zero also the function g in Eq. (4.22). Thus, Eq. (4.23) provides no help for $j = 0$ states.

For $j \geq 1$ the function g is nonzero, for both $\gamma - 1/2$ and $\gamma - 3/2$ modes, see Appendix D. Equation (4.23) defines the spectrum, which therefore satisfies the Sommerfeld formula Eq. (4.23). In the nonrelativistic limit the mixed electric-longitudinal modes correspond to the following states with $j \geq 1$: $1s_1, 2p_2, 3d_1, 3d_3, 4f_2 \dots$

E. Longitudinal polarization, $j = 0$

Consider zero angular momentum $j = 0$, which corresponds to purely longitudinal polarization, see Eq. (C1). The state with $j = 0$ is described by one radial function $v = v(r)$,

$$\mathbf{W} = v\mathbf{n}, \quad \mathbf{n} = \mathbf{r}/r. \quad (4.24)$$

The radial function v satisfies Eqs. (4.10) and (4.11) in which the function u is to be put to zero (electric polarization for $j = 0$ is impossible). These equations therefore yield

$$\begin{aligned} \frac{d^2v}{dr^2} + \frac{2}{r} \frac{dv}{dr} + ((\varepsilon + Z\alpha/r)^2 - m^2)v \\ = \frac{2v}{r^2} - \frac{2Z\alpha}{r^2} \frac{1}{\varepsilon + Z\alpha/r} \left(\frac{dv}{dr} + \frac{2v}{r} \right). \end{aligned} \quad (4.25)$$

In order to make the physical meaning of this equation more transparent let us eliminate the first derivative by means of a substitution $v \rightarrow \varphi$

$$v = \frac{Z\alpha}{\varepsilon^2} \left(\varepsilon + \frac{Z\alpha}{r} \right) \frac{\varphi}{r} = \frac{1+x}{x^2} \varphi, \quad (4.26)$$

where it is convenient also to scale the radial variable $r \rightarrow x$

$$r = \frac{Z\alpha}{\varepsilon} x, \quad (4.27)$$

assuming $\varphi = \varphi(x)$. In this notation Eq. (4.25) can be rewritten as a conventional Schrödinger-type eigenvalue problem

$$H\varphi = -\kappa^2\varphi, \quad (4.28)$$

$$H = -\frac{d^2}{dx^2} - \frac{2(Z\alpha)^2}{x} - \frac{(Z\alpha)^2}{x^2} + \frac{2}{(x+1)^2}, \quad (4.29)$$

where $-\kappa^2$, which plays a role of an eigenvalue, is related to the energy of the discrete level

$$\kappa^2 = (Z\alpha)^2 \frac{m^2 - \varepsilon^2}{\varepsilon^2} > 0. \quad (4.30)$$

The operator H in Eq. (4.29) possesses three singular points, $x = 0$, $x = \infty$, and $x = -1$. The last one lies in the nonphysical region, but it presents an obstacle for an analytical study anyway. One can overcome this difficulty using a substitution $\varphi \rightarrow \tilde{\varphi}$

$$\varphi = \left(\frac{d}{dx} + (\gamma + 1/2) \frac{x+1}{x} - \frac{1}{x+1} \right) \tilde{\varphi}. \quad (4.31)$$

It can be shown that $\tilde{\varphi}$ satisfies an eigenvalue problem

$$\tilde{H}\tilde{\varphi} = -\kappa^2\tilde{\varphi}, \quad (4.32)$$

$$\tilde{H} = -\frac{d^2}{dx^2} - 2\frac{(Z\alpha)^2}{x} + \left(\gamma + \frac{1}{2} \right) \left(\gamma + \frac{3}{2} \right) \frac{1}{x^2}. \quad (4.33)$$

The main result of the transformation Eq. (4.31) is that the operator \tilde{H} has only two singular points, $x = 0$ and $x = \infty$. An interesting method, which allows one to ‘‘invent’’ the substitution Eq. (4.31) and derive then Eq. (4.32) is presented in Appendix E. It takes its origins in an elegant treatment of quantum mechanics developed by the Göttingen School and is known as matrix mechanics.

A regular at $r = 0$ solution of the eigenvalue problem (4.32) reads

$$\tilde{\varphi} = e^{-\kappa x} x^{L+1} F\left(L+1 - \frac{(Z\alpha)^2}{\kappa}, 2L+2, 2\kappa x \right). \quad (4.34)$$

Here $F(\alpha, \beta, z)$ is the confluent hypergeometric function and L is defined by

$$L = \gamma + 1/2. \quad (4.35)$$

To make the solution given by Eq. (4.34) regular at infinity one should assume that

$$\kappa = \frac{(Z\alpha)^2}{L+n-1} = \frac{(Z\alpha)^2}{\gamma+n-1/2}, \quad n = 2, 3 \dots \quad (4.36)$$

The corresponding eigenfunctions are given by Eq. (4.34), in which the hypergeometric function is reduced to a polynomial

$$\tilde{\varphi} = e^{-\kappa x} x^{\gamma+3/2} F(2-n, 2\gamma+3, 2\kappa x). \quad (4.37)$$

Equations (4.31) and (4.37) give then the function φ , while

(4.24) and (4.26) transform it into v and W . The function φ exhibits the following behavior at the boundaries

$$\varphi \propto \exp(-\kappa x), \quad x \rightarrow \infty, \quad (4.38)$$

$$\varphi \propto x^{\gamma+1/2}, \quad x \rightarrow 0. \quad (4.39)$$

Equation (4.36) gives the spectrum

$$\varepsilon = m \left(1 + \frac{(Z\alpha)^2}{(\gamma + n - 1/2)^2} \right)^{-1/2}, \quad n = 2, 3, \dots \quad (4.40)$$

which complies with the Sommerfeld formula Eq. (4.8). In the nonrelativistic limit the longitudinal mode corresponds to the following states with $j = 0$: $2p_0, 3p_0, 4p_0, \dots$

F. Summary for Coulomb problem

Our discussion of the Coulomb problem for vector particles confirms that for all polarizations and all angular momenta j the discrete energy spectrum is described by the Sommerfeld formula Eq. (4.8), as was first found by Corben and Schwinger [5].

For $j \geq 1$ there exists three modes. One of them is purely magnetic, it has $l = j$, while two others are constructed from the electric and longitudinal polarizations, each one of these two modes has an admixture of $l = j + 1$ and $l = j - 1$ states. These two modes coexist for $j \geq 1$, while for $j = 0$ only one of them, which in this case has a purely longitudinal polarization and $l = 1$ is present.

From Eq. (4.8) one derives that the spectrum of the Coulomb problem is degenerate; it is triply degenerate provided $n \geq j + 2$, $j \geq 1$, doubly degenerate for levels with $n = j + 1$, $j \geq 1$, while the states which have either $n = j$ or $j = 0$ remain nondegenerate. This conclusion agrees with the nonrelativistic expansion, see Eq. (4.3).

Interestingly, one and the same Sommerfeld formula Eq. (4.8) describes the discrete energy spectrum in the Coulomb problem for scalar, Dirac, and vector particles. The only distinction is related to the angular momentum j , which takes the integer values $j = 0, 1, \dots$ for bosons and half-integer $j = 1/2, 3/2, \dots$ for fermions.

V. CATASTROPHE WITH CHARGE

Consider the charge density of a vector boson for a state with $j = 0$. Equations (2.23) and (4.24) give

$$\rho^W = 2e \left[\left(\varepsilon + \frac{Z\alpha}{r} \right) (v^2 + w^2) + 2v \frac{dw}{dr} - Y w^2 \right], \quad (5.1)$$

where w defined by Eqs. (2.18), (4.12), and (4.13) reads

$$w = \frac{1}{\varepsilon + Z\alpha/r} \left(\frac{dv}{dr} + \frac{2v}{r} \right). \quad (5.2)$$

In the region of small distances $r \ll Z\alpha/m$ Eq. (4.39) shows that $\varphi \propto r^{\gamma+1/2}$. Consequently, from Eqs. (4.26)

and (5.2) we find the following estimates for v and w

$$v \sim r^{\gamma-3/2}, \quad (5.3)$$

$$w \sim \frac{\gamma + 1/2}{Z\alpha} r^{\gamma-3/2}. \quad (5.4)$$

From here one derives an estimate for the charge density (5.1) of the vector boson

$$\rho^W \sim -2e \frac{(1-\gamma)(1+2\gamma)}{Z\alpha} r^{2\gamma-4}, \quad r > 0. \quad (5.5)$$

It diverges at the origin so badly that the total charge $Q^W = \int \rho^W d^3r$ localized in any small sphere surrounding the origin is infinite.

The trouble does not stop here. Remember the density $\rho = Z|e|\delta(\mathbf{r})$ of the Coulomb charge, which is located at the origin. This density results in the Y -term defined by Eq. (2.19)

$$Y = \frac{e\rho}{m^2} = -\frac{4\pi Z\alpha}{m^2} \delta(\mathbf{r}). \quad (5.6)$$

We did not consider it previously because the functions we dealt with were regular at the origin, allowing one to hope that their regular behavior makes the Y -term irrelevant. Since the charge density does not follow this pattern, we need to take the term given by Eq. (5.6) into account. The contribution of the δ -function in Eq. (5.6) to the boson charge density is given by the last term in Eq. (5.1), which reads

$$(\rho^W)_{Y\text{-term}} = e \frac{8\pi Z\alpha}{m^2} w^2(0) \delta(\mathbf{r}). \quad (5.7)$$

Equation (5.4) shows that $w(0) = \infty$, which makes the density Eq. (5.7) infinite as well.

We see that there are two closely located, though different regions, which contribute to an infinite charge of the W -boson in the $j = 0$ state. One region is $r > 0$, where the density of charge Eq. (5.5) behaves singularly as $r \rightarrow 0$. Another region is located strictly at the origin $r = 0$, where an infinite coefficient $w^2(0) = \infty$ in front of the δ -function in Eq. (5.7) makes the charge infinite as well.

The origin of Eq. (5.7) can be traced down to the last term in Eq. (2.23). It contributes therefore to the charge density for all states. There is one more state, in which the coefficient in Eq. (5.7) turns infinite, signaling a catastrophic behavior of the charge. This is the $\gamma - 3/2$ state with $j = 1$, see Eqs. (4.20) and (4.21). To justify this statement, note that Eqs. (4.13), (4.20), and (4.21) imply that $w \propto r^{\gamma-3/2}$. For $j = 1$ the inequality $\gamma < 3/2$ holds. Therefore for the state $\gamma - 3/2$, $j = 1$ one finds $w(0) = \infty$, which makes the charge of the W -boson located strictly at the origin infinite. (There is no problem in that case with the charge in the region $r > 0$).

The catastrophic behavior of the charge of the W -boson in $j = 0$ and $j = 1$, $\gamma - 3/2$ states was discovered in [5],

forcing the authors of this work to conclude that the pure Coulomb problem for W -bosons is poorly defined.

VI. VACUUM POLARIZATION

Consider the conventional QED vacuum polarization. The potential energy of the W -boson propagating in the Coulomb field acquires an additional term, let us call it $S(r)$, which describes the polarization

$$U(r) = -(1 + S(r)) \frac{Z\alpha}{r}. \quad (6.1)$$

It suffices to consider the polarization effect in the lowest-order approximation, when it is described by the known Uehling potential. Its small-distance asymptotic behavior is given by a simple logarithmic function, see e.g. [21],

$$S(r) \simeq -\alpha\beta \ln(m_Z r), \quad r \rightarrow 0. \quad (6.2)$$

This function is related to the logarithm responsible for the scaling of the QED coupling constant

$$\alpha^{-1}(\mu) = \alpha^{-1}(\mu_0) - \beta \ln(\mu/\mu_0). \quad (6.3)$$

The relation between Eqs. (6.2) and (6.3) is well known, see e.g. book [21], which presents it for one generation of leptons. The factor β , which governs the scaling of the coupling constant and the potential in Eq. (6.2) equals the lowest coefficient of the Gell-Mann–Low β -function. It is normalized here in such a way that for one generation of leptons $\beta = \beta_e = 2/3\pi$.

It is important for us that $\alpha(\mu)$ rises with the mass parameter μ , i.e. β is positive, $\beta > 0$; theoretical and experimental data agree on this fact, for a brief review see e.g. Ref. [18], the experimental data are provided by Refs. [24–26]. An estimation of β can be found from two reliable reference points $\alpha^{-1}(m_\tau) = 133.498 \pm 0.017$ and $\alpha^{-1}(m_Z) = 127.918 \pm 0.018$ provided in Ref. [18]. Using them and taking the masses $m_\tau = 1776.99 + 0.29 - 0.26$ MeV and $m_Z = 91.1876 \pm 0.0021$ GeV recommended in [18] one derives from Eq. (6.3) that

$$\beta \simeq 1.42(1). \quad (6.4)$$

A more simple estimation of β can be done if one takes into account a contribution of all known charged fermions “naively” (neglecting complications, related to the QCD vacuum as well as possible contribution of scalars). This estimate yields

$$\beta_{\text{est}} \simeq \frac{2}{3\pi} \sum_i \frac{q_i^2}{e^2} = \frac{2}{3\pi} \left(1 + \frac{5}{3}\right) 3 \simeq 1.70. \quad (6.5)$$

Here summation runs over all charged fermions, q_i is the charge of the fermion, the terms 1 and $3/5$ in the bracket are due to the lepton and quark contribution for one generation, and the factor 3 after the bracket accounts for three generations. A discrepancy between “simple-minded” Eq. (6.5) and more solid-based Eq. (6.4) is below 20%.

The normalization of the logarithmic function on the mass of the Z -boson m_Z adopted in Eq. (6.2) presumes that the fine-structure constant α is taken at precisely this scale, $\alpha \equiv \alpha(m_Z) \simeq 1/128$.

We are interested in high-momenta behavior in Eqs. (6.2) and (6.3), where $\mu \sim 1/r \gg m$. An accuracy of Eqs. (6.4) and (6.5), as well as any other feasible estimation, is limited in this region by a contribution of unknown heavy charged fermions and scalars. However, this uncertainty does not affect our final conclusions. For our purposes it suffices to stick to a widely accepted hypothesis that β is a positive constant (or a slow-varying function up to the grand unification limit).

Substituting Eqs. (6.1) and (6.2) into Eq. (2.19) one derives

$$Y(r) \simeq \frac{Z\alpha^2\beta}{m^2 r^3}, \quad r \rightarrow 0, \quad (6.6)$$

where the lowest term of the α -expansion is retained. It is vital that for small distances, when $r \ll \sqrt{\alpha}/m$, $Y(r)$ is positive and large,

$$Y(r) \gg |U(r)| \gg m. \quad (6.7)$$

Note that the direct contribution of the vacuum polarization given by the term $S(r)$ in Eq. (6.1) is not pronounced. In contrast, the Y -term Eq. (6.6) becomes dominant at small distances, making the effects related to the QED vacuum polarization very important. Since this term plays a crucial role below, let us verify its sign again. Consider a positive Coulomb center, $Z > 0$. Then the vacuum polarization produces negative charge density, $\rho < 0$. Since the charge of the W^- meson is negative, $e < 0$, we find from Eq. (2.19) that $Y = e\rho/m^2 > 0$. We see that indeed, the Y -term is positive, in accord with Eq. (6.6).

A. Longitudinal polarization, $j = 0$

Equation (4.24) shows that a longitudinal state with $j = 0$ is described by the single radial function $v = v(r)$. Equation (2.18) allows one to express the function w via v

$$w = (\varepsilon - U - Y)^{-1}(v' + 2v/r). \quad (6.8)$$

We need now to write the classical equation of motion for v , in which the term Y is taken into account. The simplest way is to substitute \mathbf{W} and w from Eqs. (4.24) and (6.8) into Eq. (2.22), which yields

$$\begin{aligned} ((\varepsilon - U)^2 - m^2)v = & -(\varepsilon - U) \frac{d}{dr} \left(\frac{v' + 2v/r}{\varepsilon - U - Y} \right) \\ & - U' \frac{v' + 2v/r}{\varepsilon - U - Y}. \end{aligned} \quad (6.9)$$

It is taken into account here that Eq. (4.24) ensures that $\nabla \times \mathbf{W} = 0$. For a purely Coulomb case, when $Y = 0$ for $r > 0$, Eq. (6.9) reduces to Eq. (4.25). Equation (6.9) can be rewritten in a more compact form

$$v'' + Gv' + Hv = 0, \quad (6.10)$$

where the coefficients $G = G(r)$ and $H = H(r)$ are

$$G = \frac{2}{r} + \frac{U'}{\varepsilon - U} + \frac{U' + Y'}{\varepsilon - U - Y}, \quad (6.11)$$

$$H = -\frac{2}{r^2} + \frac{2}{r} \left(\frac{U'}{\varepsilon - U} + \frac{U' + Y'}{\varepsilon - U - Y} \right) + \frac{\varepsilon - U - Y}{\varepsilon - U} ((\varepsilon - U)^2 - m^2). \quad (6.12)$$

For a qualitative analyses it is convenient to eliminate the term with the first derivative by scaling the radial function $v \rightarrow \varphi = \varphi(r)$

$$v = \frac{1}{r} [(\varepsilon - U)(\varepsilon - U - Y)]^{1/2} \varphi. \quad (6.13)$$

(This definition reduces to Eq. (4.26) when $Y = 0$). The classical equation of motion for W -bosons takes a simple form

$$-\varphi'' + \mathcal{U}\varphi = 0, \quad (6.14)$$

$$\mathcal{U} = -H + G^2/4 + G'/2, \quad (6.15)$$

where G , H are defined in Eqs. (6.11) and (6.12). Equation (6.14) can be looked at as a Schrödinger-type equation, in which $\mathcal{U} = \mathcal{U}(r)$ plays the role of an effective potential energy.

According to Eqs. (6.6) and (6.7) the Y -term is large and positive at small distances. This fact makes the effective potential $\mathcal{U}(r)$ in Eq. (6.15) also large and positive when $r \rightarrow 0$

$$\mathcal{U}(r) \simeq -H(r) \simeq -U(r)Y(r) \simeq \frac{Z^2 \alpha^3 \beta}{m^2 r^4}. \quad (6.16)$$

Compare this result with the effective potential $[\mathcal{U}(r)]_C$ for the pure Coulomb field. The latter one is a part of the Hamiltonian in Eq. (4.29). For $r \rightarrow 0$ one finds from Eq. (4.28) that

$$[\mathcal{U}(r)]_C \simeq -(Z\alpha)^2/r^2. \quad (6.17)$$

It is taken into account here that the variables x and r in Eq. (4.29) are proportional, see Eq. (4.27). Equation (6.16) shows that the vacuum polarization produces a strong repulsion in the effective potential $\mathcal{U}(r)$, in contrast with a mild attraction, which exhibits $[\mathcal{U}(r)]_C$ in Eq. (6.17) for the pure Coulomb case.

When the estimate equation (6.16) is applicable, Eq. (6.14) allows an analytical solution

$$\varphi(r) \propto r \exp\left(-\frac{Z\alpha(\alpha\beta)^{1/2}}{mr}\right). \quad (6.18)$$

It shows that $\varphi(r)$ exponentially decreases at small distances. According to Eqs. (6.8) and (6.13) the functions $v(r)$, $w(r)$ also decrease exponentially here; correspond-

ingly, the charge density of the W -boson Eq. (5.1) decreases exponentially at the origin as well

$$v \rightarrow \frac{a}{m} \frac{1}{r^2} \exp\left(-\frac{Z\alpha(\alpha\beta)^{1/2}}{mr}\right), \quad (6.19)$$

$$w \rightarrow -\frac{a}{(\alpha\beta)^{1/2}} \frac{1}{r} \exp\left(-\frac{Z\alpha(\alpha\beta)^{1/2}}{mr}\right), \quad (6.20)$$

$$\rho^W \rightarrow -\frac{4a^2 Z\alpha e}{m^2} \frac{1}{r^5} \exp\left(-\frac{2Z\alpha(\alpha\beta)^{1/2}}{mr}\right). \quad (6.21)$$

Here a constant a depends on the normalization of v , which is specified in Eq. (7.3) below. Equation (6.21) shows that for $r > 0$ in the vicinity of the origin the charge density is finite and small, which makes the charge located in this region finite as well. Equation (6.20) shows that $w(0) = 0$, which eradicates the contribution of the δ -function in Eq. (5.7). Thus, the charge located strictly at origin $r = 0$ is zero.

We verified that an account of the QED vacuum polarization erases the infinite charge of a vector boson for $j = 0$ state.

B. Electrolongitudinal polarizations, $j \geq 1$

Equation (6.7) shows that in the region of small distances $r \ll \alpha/m$ the Y -term, which is related to the vacuum polarization, is large. This fact makes the function w in Eq. (2.18) small, $|w| \ll |W|$. As a result the asymptotic form of the equation of motion (2.22) at small distances reads

$$\frac{(Z\alpha)^2}{r^2} \mathbf{W} = \nabla \times (\nabla \times \mathbf{W}). \quad (6.22)$$

Equation (4.5) ensures that $\nabla \times \mathbf{W}$ is not zero identically provided $j \geq 1$, which makes Eq. (6.22) meaningful.

Using Eq. (4.5) to represent the electrolongitudinal modes and identities Eqs. (C3) for the spherical vectors one rewrites Eq. (6.22) in terms of the radial functions u , v

$$(Z\alpha)^2 u = -r^2 u'' - ru' - u + \sqrt{j(j+1)}v, \quad (6.23)$$

$$(Z\alpha)^2 v = -\sqrt{j(j+1)}(ru' + u - \sqrt{j(j+1)}v). \quad (6.24)$$

Their solution is straightforward

$$u = br^\lambda, \quad (6.25)$$

$$v = b\sqrt{j(j+1)} \frac{\lambda + 1}{\gamma^2 - 1/4} r^\lambda. \quad (6.26)$$

Here b is a constant, and λ can take one of the two possible values,

$$\lambda = \lambda_\pm = \frac{1}{2} \frac{j(j+1) \pm k}{\gamma^2 - 1/4}, \quad (6.27)$$

where k satisfies

$$k^2 = j^2(j+1)^2 - 4(Z\alpha)^2(\gamma^2 - \frac{5}{4})(\gamma^2 - \frac{1}{4}), \quad (6.28)$$

with γ defined in Eq. (4.9). The two available values of λ_{\pm} should be attributed to the two electrolongitudinal modes.

Comparing Eqs. (6.25) and (6.26), which are valid when the vacuum polarization is taken into account, with Eqs. (4.17), (4.20), and (4.21), which describe the purely Coulomb case, we see that the polarization changes drastically the behavior of the wave functions. One finds from Eqs. (6.27) and (6.28) that λ_{\pm} are positive for all j , $j \geq 1$, provided Z is not very large, $Z\alpha \leq 1/2$.

From Eqs. (6.25) and (6.26) one deduces therefore that $u(r)$, $v(r) \rightarrow 0$, when $r \rightarrow 0$. Equation (2.18) guarantees then that $w(0) = 0$. As a result the contribution of the δ -function in Eq. (5.7) to the charge density turns zero for all electrolongitudinal modes. This differs qualitatively from the pure Coulomb case, which gives an infinite charge located at the origin for $j = 1$, " l " = 0 state.

We conclude that the QED vacuum polarization suppresses the wave functions of a vector boson at the origin, eradicating thus the infinite charge of the boson, which plagues the problem for the pure Coulomb field.

VII. NUMERICAL EXAMPLE

To be more informative on the behavior of vector bosons in the Coulomb field let us solve the corresponding equations of motion numerically. Consider the $j = 0$ state, describing it with the help of Eqs. (4.24) and (6.9). We need to specify the factor $S(r)$, which describes the vacuum polarization in the potential in Eq. (6.1). For small r , $r \ll Z\alpha/m$ this factor plays a major role, while for larger it is less important. Let us construct a simple model, which gives a correct asymptotic behavior Eq. (6.2) as $r \rightarrow 0$, and is physically reasonable, though not perfect, at larger r . Take with this purpose the Uehling potential, see e.g. [21], assuming that only charged leptons and quarks contribute to it

$$S(r) = \frac{2\alpha}{3\pi} \sum_i \frac{q_i^2}{e^2} F(m_i r). \quad (7.1)$$

Here

$$F(x) = \int_1^{\infty} \exp(-2x\zeta) \left(1 + \frac{1}{2\zeta^2}\right) \frac{\sqrt{\zeta^2 - 1}}{\zeta^2} d\zeta. \quad (7.2)$$

Summation in Eq. (7.1) runs over all quarks and charged leptons, their charges q_i and masses m_i are taken from Ref. [18]. The model presented by Eq. (7.1) neglects complications related to the QCD vacuum, which may be substantial at large distances, but the role of the polarization is insignificant in this region anyway. For small distances $r \rightarrow 0$ the model Eq. (7.1) reproduces the correct asymptotic formula Eq. (6.2), in which the coefficient β is given by Eq. (6.5). A comparison of Eqs. (6.4) and (6.5)

shows that at small distances the accuracy of the model potential can be estimated as $\sim 20\%$, which is sufficient for us. Using $S(r)$ from Eq. (7.1) one defines the potential $U(r)$ in Eq. (6.1) and the Y -potential in Eq. (2.19). After that the solution of Eq. (6.9) is straightforward. This solution should be normalized on the total charge e of the W -boson

$$e = \int \rho^W(r) d^3r, \quad (7.3)$$

where the charge density is defined by Eq. (5.1).

Figure 1 shows the radial function $v(r)$ for the $2p_1$ state ($n = 2$, $j = 0$) for $Z = 1$. For large distances $mr \gg 1$ the function $v(r)$ is close to the conventional nonrelativistic wave function of the $2p$ state, as it should be, see Sec. III. In the ultrarelativistic region $mr \ll 1$ the function $v(r)$ changes sign, then shows an extremum, and decreases exponentially when $r \rightarrow 0$, in accord with an analytical estimate Eq. (6.19), which is shown by the dashed line. The fact that the function $v(r)$ has a node in the relativistic region produces no controversy since $v(r)$ is not a proper wave function and the conventional theorem, which counts

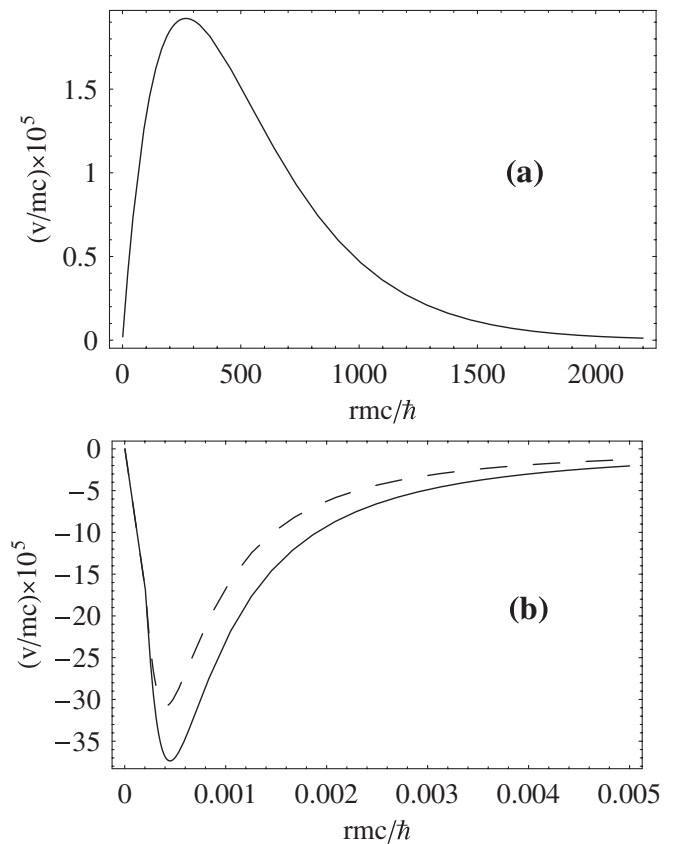


FIG. 1. The $2p_1$ radial function $v(r)$, which describes the $n = 2$, $j = 0$ discrete state of a W -boson in the Coulomb field of a charge $Z = 1$. (a) Large distances $r \gg \hbar/mc$, $v(r)$ is close to conventional nonrelativistic wave function $2p$; (b) ultrarelativistic region $r \ll \hbar/mc$. Solid line—numerical solution, dashed line—analytical prediction of Eq. (6.19).

the nodes of the wave functions for discrete levels is not applicable.

In our discussion we did not try to construct proper wave functions, being content with a possibility to calculate the current. As an example, Fig. 2 shows the charge density for the $2p_1$ state in the Coulomb field of $Z = 1$. In the non-relativistic region $mr \gg 1$ it behaves conventionally. For the ultrarelativistic case $mr \ll 1$ the density changes sign, exhibits an extremum, and then decreases exponentially when $r \rightarrow 0$ in agreement with Eq. (6.21). Note that the “wrong” sign of the charge density, i.e. the positive charge density for the negatively charged W -boson (Fig. 2 shows $\rho^W/e = -\rho^W/|e|$) produces no contradiction with general principles. The Pauli’s theorem, see e.g. Ref. [21], implies that the energy of a boson field is positively defined, but the sign of the charge density of the boson field remains not unequivocally determined [27].

The total positive charge located at small distances proves to be very small; for $2p_1$, $Z = 1$ state it is

$$Q_+^W = 4\pi \int_0^{r_0} \rho^W(r)r^2 dr \approx 0.676 \times 10^{-14}|e|. \quad (7.4)$$

Here $r_0 \approx 1.01 \times 10^{-3}m^{-1}$ is a node of $\rho^W(r)$.

It is instructive to compare the found charge density with the one in the pure Coulomb problem, which is shown in Fig. 2(b) by the dashed-dotted line that reproduces Eq. (5.5). One should be content in this case with an arbitrary normalization, since for the pure Coulomb field the normalization integral in Eq. (7.3) is divergent. Figure 2(b) illustrates the fact that the vacuum polarization reduces the charge density at the origin.

The energy shift $\delta\varepsilon$ of the level $2p_1$ ($\delta\varepsilon$ is a deviation of energy from the Sommerfeld formula Eq. (4.8)) due to the vacuum polarization is found to be $\delta\varepsilon/m = -1.90 \times 10^{-7}$. In relative units it is much bigger than the Lamb shift in atoms $\delta\varepsilon_{LS}/m_e \sim Z^4\alpha^5$. The reason is obvious. The Lamb shift in atoms originates mostly from within the Compton distances $r \sim r_c = 1/m_e$, which are smaller than the Bohr radius for the electron $r_B \sim 1/(Z\alpha m_e)$. For W -bosons the situation is different. Light fermions, which contribute to Eq. (7.1), allow the polarization potential to spread to large distances, as far as the Bohr radius of the W -boson, $r \sim 1/m_i \sim 1/(Z\alpha m)$. Therefore the energy shift due to the vacuum polarization gains substantial contribution from the nonrelativistic region, where the wave function is large, which makes $\delta\varepsilon$ large as well (large compared to the Lamb shift in atoms). The accuracy of the energy shift calculations is limited by an accuracy of our model at large distances. The contribution of the QCD vacuum, which could be substantial here, is not described properly by the model based on Eq. (7.1). Consequently, the presented above value for the energy shift should be considered only as an estimate.

Nevertheless, one can derive an important lesson from this estimate. The found energy shift is small on the absolute scale, being lower than the nonrelativistic binding energy by a factor of $|\delta\varepsilon| \times 4/(Z^2\alpha^2 m) \sim 1.4 \times 10^{-2}$. Thus, the dramatic variation of the function $v(r)$ at the origin, which is produced by the vacuum polarization, makes only a small impact on the spectrum. This is in contrast to a strong influence, which the vacuum polarization exercises on the charge distribution of vector bosons. The fact that the energy shift is small makes the Sommerfeld formula Eq. (4.8) a good approximation for discrete energy levels.

VIII. DISCUSSION

We demonstrated that the conventional QED vacuum polarization plays a very important, defining role in the Coulomb problem for vector bosons. Let us summarize the reasons leading to this conclusion. The Uehling potential, which describes the vacuum polarization in the simplest approximation is known to be a weakly attractive and slowly varying function. For spinor particles it produces

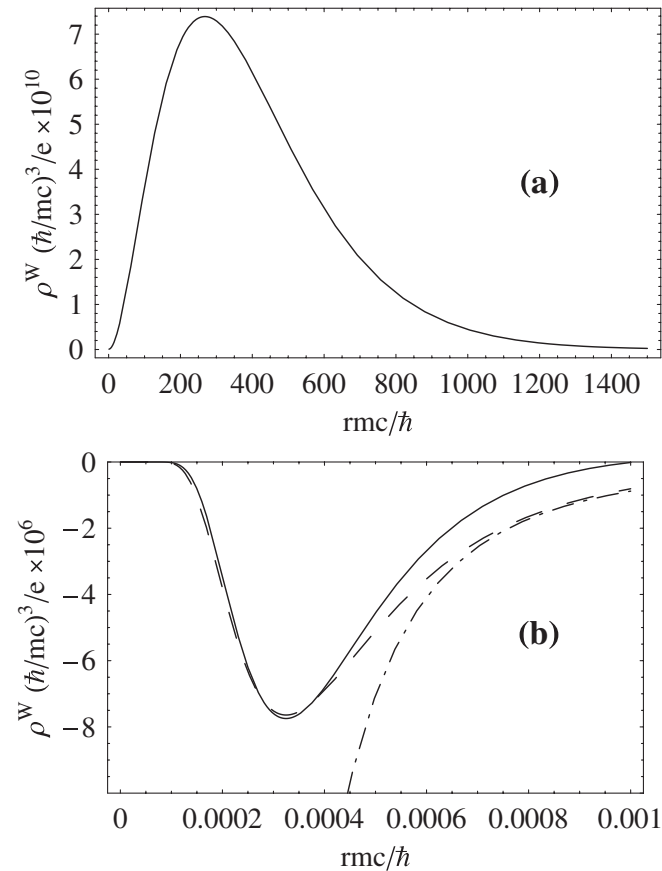


FIG. 2. The charge distribution $\rho^W(r)$ for the $2p_1$ state of a W -boson in the Coulomb field of a charge $Z = 1$. (a) Nonrelativistic region of large distances $r \gg \hbar/mc$; (b) ultrarelativistic region $r \ll \hbar/mc$. Solid line—numerical solution, dashed line—analytical prediction of Eq. (6.21), dashed-dotted line—the pure Coulomb case Eq. (5.5), when the charge density diverges at the origin (shown with arbitrary normalization).

a small enhancement of the fermion wave functions on the Coulomb center. For vector bosons the situation is different because the equations of motion for vector particles explicitly incorporate the external current. As a result, the density of the polarized charge ρ comes into the equations of motion for vector bosons. The corresponding term in the equations was called the Y -potential, $Y = e\rho/m^2$. The charge density ρ is negative for an attractive Coulomb center, $\rho < 0$ when $Z > 0$, being singular on the Coulomb center, $|\rho| \sim 1/r^3$. One derives from this that the vacuum polarization produces a repulsive Y -potential, $Y = e\rho/m^2 \propto 1/r^3 > 0$ (remember, $e < 0$). Since the Y -term is singular at the origin, it plays a dominant role at small distances.

Strong effective repulsion produced by the Y -potential reduces the fields, which describe W -bosons on the Coulomb center. For $j = 0$ this reduction is dramatic, exponential. For $j = 1$, $\gamma - 3/2$ the suppression is of a more moderate power-type nature, but in both cases it is strong enough to eliminate the infinite charge, which is located at the origin in the pure Coulomb approximation.

The above comments appeal to a chain of calculations. It is interesting to look at the obtained result from a more general perspective. The renormalizability of the standard model implies that by renormalizing relevant physical quantities one is *bound* to obtain sensible physical results. The relevant quantity in question is the charge density of a vector boson. It follows from this that the important physical quantity, which should be renormalized, is the coupling constant. Its renormalization is effectively fulfilled when the vacuum polarization is taken into account. Thus, it makes sense that the account of the vacuum polarization results in acceptable physical results.

A proposed approach is very straightforward, which makes the Coulomb problem for vector bosons as simple and reliable as it is for scalars and spinors. All discrete energy levels can be easily evaluated, all relevant fields can be calculated and normalized properly. Presumably, all scattering data can also be evaluated, though the scattering problem was not discussed in detail in the present work. All these quantities include the Coulomb charge Z accurately, not relying on perturbation theory. Starting from this base, one can consider all other processes left outside the scope of the Coulomb problem by treating them as perturbations. This includes the conventional QED processes, such as the radiative decay, photoionization, and the radiative corrections. This includes also processes related to possible exchange of Higgs and Z -bosons.

Previous attempts to formulate the Coulomb problem for vector bosons within the framework of the standard model have been facing a difficulty related to an infinite charge of the boson located near an attractive Coulomb center. This work finds that the polarization of the QED vacuum eradicates the problem. Usually the QED radiative corrections produce only small perturbations. It is interesting that in

the case discussed the radiative correction plays a major, defining role.

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APPENDIX A: HOMOGENEOUS MAGNETIC FIELD

For a static homogeneous magnetic field Eq. (2.10) reads

$$(\varepsilon^2 - m^2)\mathbf{W} = -(\nabla - ie\mathbf{A})^2\mathbf{W} - 2ie\mathbf{B} \times \mathbf{W}. \quad (\text{A1})$$

Assuming that the magnetic field is directed along the z -axis and introducing the new variables w_σ , $\sigma = 0, \pm 1$, $w_{\pm 1} = (\mathbf{W}_x \pm i\mathbf{W}_y)/\sqrt{2}$, $w_0 = \mathbf{W}_z$ one rewrites Eq. (A1) in a simple form

$$(\varepsilon^2 - m^2)w_\sigma = -(\nabla - ie\mathbf{A})^2w_\sigma + 2eB\sigma w_\sigma, \quad (\text{A2})$$

which looks similar to the nonrelativistic Schrödinger equation for a particle in the homogeneous magnetic field. This similarity allows one to write the spectrum Eq. (2.24).

APPENDIX B: RELATIVISTIC CORRECTIONS TO ENERGY LEVELS

Here we present separate expectation values for four relativistic corrections in the same order as they appear in Eq. (3.6). For $l = 0$

$$\delta E_{n0j} = \frac{m(Z\alpha)^4}{n^3} \left[\left(\frac{3}{8n} - 1 \right) + 0 + \frac{2}{3} + 0 \right]. \quad (\text{B1})$$

Here and below we specify the terms having zero expectation values by writing the corresponding zeros explicitly. For $l \geq 1$

$$\delta E_{nlj} = \frac{m(Z\alpha)^4}{n^3} \left[\left(\frac{3}{8n} - \frac{1}{2l+1} \right) + \frac{\langle ls \rangle}{l(l+1)(2l+1)} + 0 - \frac{6\langle ls \rangle^2 + 3\langle ls \rangle - 4l(l+1)}{l(l+1)(2l-1)(2l+1)(2l+3)} \right], \quad (\text{B2})$$

where

$$\langle ls \rangle = \frac{1}{2}[j(j+1) - l(l+1) - 2]. \quad (\text{B3})$$

Both Eq. (B1) and (B2) lead to Eq. (4.2). The total relativistic correction would look very complicated and show no degeneracy if magnetic dipole or electric quadrupole moments of a vector particle are different from those values, which follow from the gauge theory.

APPENDIX C: SPHERICAL VECTORS

The conventional definition of spherical vectors, see e.g. [21], reads

$$\begin{aligned} \mathbf{Y}_{jm}^{(e)} &= \nabla_n Y_{jm} / \sqrt{j(j+1)}, & \mathbf{Y}_{jm}^{(l)} &= \mathbf{n} Y_{jm}, \\ \mathbf{Y}_{jm}^{(m)} &= \mathbf{n} \times \mathbf{Y}_{jm}^{(e)}. \end{aligned} \quad (\text{C1})$$

Here $Y_{jm} \equiv Y_{jm}(\theta, \varphi)$ is the spherical function, $\mathbf{Y}_{jm}^{(e)}$, $\mathbf{Y}_{jm}^{(l)}$, $\mathbf{Y}_{jm}^{(m)}$ are the electric, longitudinal, and magnetic vectors. The symbol ∇_n in Eq. (C1) indicates the angular part of the gradient, $\nabla F(\theta, \phi) = \nabla_n F(\theta, \phi)/r$, and $\mathbf{n} = \mathbf{r}/r$ is a unit vector along the radius vector.

Definitions Eqs. (C1) imply the following properties of the spherical vectors

$$\begin{aligned} \nabla_n \cdot \mathbf{Y}_{jm}^{(e)} &= -\sqrt{j(j+1)} Y_{jm}, & \nabla_n \cdot \mathbf{Y}_{jm}^{(l)} &= 2Y_{jm}, \\ \nabla_n \cdot \mathbf{Y}_{jm}^{(m)} &= 0, \end{aligned} \quad (\text{C2})$$

$$\begin{aligned} \nabla \times \mathbf{Y}_{jm}^{(e)} &= \mathbf{Y}_{jm}^{(m)}, & \nabla \times \mathbf{Y}_{jm}^{(l)} &= -\sqrt{j(j+1)} \mathbf{Y}_{jm}^{(m)}, \\ \nabla \times \mathbf{Y}_{jm}^{(m)} &= -\mathbf{Y}_{jm}^{(e)} - \sqrt{j(j+1)} \mathbf{Y}_{jm}^{(l)}. \end{aligned} \quad (\text{C3})$$

The formulas for the Laplace operator read

$$\begin{aligned} \Delta_n \mathbf{Y}_{jn}^{(e)} &= -j(j+1) \mathbf{Y}_{jn}^{(e)} + 2\sqrt{j(j+1)} \mathbf{Y}_{jn}^{(l)}, \\ \Delta_n \mathbf{Y}_{jn}^{(l)} &= 2\sqrt{j(j+1)} \mathbf{Y}_{jn}^{(e)} - (j(j+1) + 2) \mathbf{Y}_{jn}^{(l)}, \\ \Delta_n \mathbf{Y}_{jn}^{(m)} &= -j(j+1) \mathbf{Y}_{jn}^{(m)}. \end{aligned} \quad (\text{C4})$$

Here Δ_n describes the angular part of the Laplacian, i.e. $\Delta F(\theta, \phi) = \Delta_n F/r^2$. The parity for electric and longitudinal polarizations equals $P = (-1)^j$, for magnetic polarization the parity is $P = (-1)^{j+1}$. The orbital moment l takes the value $l = j$ for the magnetic polarization, in agreement with the parity for this state. The electric and longitudinal polarizations are constructed as linear combinations of the two states with $l = j \pm 1$. For $j = 0$ there exists only one spherical vector, which is purely longitudinal and has $l = 1$.

APPENDIX D: SPECTRUM OF ELECTROLONGITUDINAL MODES FOR $j \geq 1$

Let us verify that for $j \geq 1$ the function g introduced in Eq. (4.22) is nonzero. Consider first the $\gamma - 1/2$ mode. Substituting Eq. (4.17) into Eq. (4.22) one finds

$$g \rightarrow a \frac{1}{Z\alpha} (1/4 - \gamma^2) r^{\gamma-1/2}, \quad (\text{D1})$$

which indicates that in this mode g is not zero.

Consider now the $\gamma - 3/2$ mode, which incorporates both possible polarizations at small distances. We need here the expressions for u and v at small distances that are more accurate, then the ones in Eq. (4.20). They can be derived by using $mr \ll 1$ as a perturbation in Eqs. (4.15) and (4.16), and pushing calculations one step beyond the simplest approximation given by Eqs. (4.20) and (4.21).

The result reads

$$u \rightarrow b \left(1 - \frac{2 + (Z\alpha)^2}{\gamma + 1/2} \frac{\varepsilon r}{Z\alpha} \right) r^{\gamma-3/2}, \quad (\text{D2})$$

$$v \rightarrow \frac{b}{\sqrt{j(j+1)}} \left(\gamma - \frac{1}{2} - Z\alpha\varepsilon r \right) r^{\gamma-3/2}. \quad (\text{D3})$$

Substituting Eqs. (D2) and (D3) into Eq. (4.22) one finds that the main term $\propto r^{\gamma-3/2}$ cancels out in g , but the next one survives, giving

$$g \rightarrow b \frac{\varepsilon}{Z\alpha} (2\gamma - 1 - (Z\alpha)^2) r^{\gamma-1/2}. \quad (\text{D4})$$

We verified that for $j \geq 1$ the function g is not an identical zero for both electrolongitudinal modes.

APPENDIX E: LONGITUDINAL MODE $j = 0$ AND MATRIX MECHANICS

In order to find the spectrum of the operator H in Eq. (4.29) let us employ a method, which finds its inspiration in an elegant approach to quantum mechanics developed by the Göttingen School and often called the *matrix mechanics*; the book Ref. [28] gives its systematic presentation. We modify it for our purposes as follows. Assume that one needs to find discrete spectrum of some Hermitian operator \mathcal{H} (in our case it is the operator \mathcal{H} in Eq. (4.29)). Let us presume that one is able to find the operator θ , which satisfies

$$\mathcal{H} = \theta^\dagger \theta + \lambda_0, \quad (\text{E1})$$

where λ_0 is a number. Define then a new operator $\tilde{\mathcal{H}}$,

$$\tilde{\mathcal{H}} = \theta \theta^\dagger + \lambda_0. \quad (\text{E2})$$

Let us verify that the two operators \mathcal{H} , $\tilde{\mathcal{H}}$ have very similar sets of eigenvalues. Consider an eigenfunction φ of \mathcal{H} , with the eigenvalue λ

$$\mathcal{H} \varphi = \lambda \varphi. \quad (\text{E3})$$

Taking

$$\tilde{\varphi} = \theta \varphi, \quad (\text{E4})$$

one verifies that

$$\begin{aligned} \tilde{\mathcal{H}} \tilde{\varphi} &= (\theta \theta^\dagger + \lambda_0) \theta \varphi = \theta (\theta^\dagger \theta + \lambda_0) \varphi = \theta \mathcal{H} \varphi \\ &= \lambda \theta \varphi = \lambda \tilde{\varphi}. \end{aligned} \quad (\text{E5})$$

This shows that either the function $\tilde{\varphi}$ is an eigenfunction of $\tilde{\mathcal{H}}$ with the eigenvalue λ , or $\tilde{\varphi} = 0$. The first options makes λ an eigenvalue of both operators \mathcal{H} , $\tilde{\mathcal{H}}$. The second one implies that

$$\theta \varphi = 0, \quad (\text{E6})$$

which indicates that $\lambda = \lambda_0$ is a candidate for an eigen-

value of \mathcal{H} because Eq. (E6) implies $\mathcal{H}\varphi = \lambda_0\varphi$. Equation (E6) provides a convenient way to derive the corresponding eigenfunction. There is a subtlety here though. The one found from Eq. (E6) φ may, or may not satisfy the boundary conditions. If it does, then it represents the eigenfunction and $\lambda = \lambda_0$ is an eigenvalue. Otherwise, λ_0 does not belong to the discrete spectrum, as would be the case in an example discussed. One should also verify that an action of the operator θ in Eq. (E4) (as well as the operator θ^\dagger in Eq. (E8) below) does not spoil the boundary conditions. We presume here that this is the case, and verify later on that this assumption holds for a particular example discussed, see Eq. (4.38) and (4.39).

We conclude that any discrete eigenvalue of \mathcal{H} is an eigenvalue of $\tilde{\mathcal{H}}$ as well, with one possible exception of λ_0 . By reversing the argument, one derives that if $\tilde{\varphi}$ is an eigenfunction of $\tilde{\mathcal{H}}$ with the eigenvalue $\tilde{\lambda}$,

$$\tilde{\mathcal{H}}\tilde{\varphi} = \tilde{\lambda}\tilde{\varphi} \quad (\text{E7})$$

then

$$\varphi = \theta^\dagger\tilde{\varphi} \quad (\text{E8})$$

satisfies Eq. (E3) with $\lambda = \tilde{\lambda}$. We see that the two sets of discrete eigenvalues of the two operators \mathcal{H} , $\tilde{\mathcal{H}}$ are the same, except for λ_0 , which may, or may not be present in one, or both sets of spectra. The crucial point for us is that the operator $\tilde{\mathcal{H}}$ can be more simple for analyses than the initial operator \mathcal{H} .

Taking the operator \mathcal{H} from Eq. (4.29), we construct the operators θ , θ^\dagger in the form

$$\theta = -\frac{d}{dx} + a + \frac{b}{x} + \frac{c}{x+1}, \quad (\text{E9})$$

$$\theta^\dagger = \frac{d}{dx} + a + \frac{b}{x} + \frac{c}{x+1}, \quad (\text{E10})$$

where a, b, c are real numbers. From Eqs. (E9) and (E10) it follows that

$$\begin{aligned} \theta^\dagger\theta = & -\frac{d^2}{dx^2} + a^2 + \frac{2b(a+c)}{x} + \frac{b(b-1)}{x^2} \\ & + \frac{2c(a-b)}{x+1} + \frac{c(c-1)}{(x+1)^2}. \end{aligned} \quad (\text{E11})$$

There are four x -dependent rational terms in Eq. (E11), while only three coefficients a, b, c are available for tuning to make them identical to similar terms present in the

operator \mathcal{H} . However, the coefficients in Eq. (4.29) prove to be “user-friendly,” making this procedure possible. Taking

$$a = b = \gamma + 1/2, \quad c = -1, \quad (\text{E12})$$

$$\lambda_0 = -(\gamma + 1/2)^2, \quad (\text{E13})$$

one satisfies Eq. (E1). Taking θ , θ^\dagger defined in Eqs. (E9), (E10), and (E12) one constructs $\tilde{\mathcal{H}}$ Eq. (E5), with the result given in Eq. (4.33). The “nasty” singular at $x = -1$ term disappears from $\tilde{\mathcal{H}}$. The latter operator describes a conventional Coulomb-type problem with $L = \gamma + 1/2$ playing a role of an effective (noninteger) angular momentum. From Eq. (4.32) one finds that the regular at $x = 0$ solution of the eigenvalue problem $\tilde{\mathcal{H}}\tilde{\varphi} = -\kappa^2\tilde{\varphi}$, satisfies Eq. (4.34). Equation (4.36), which ensures that this solution is regular at infinity, completely defines a set of discrete eigenvalues of $\tilde{\mathcal{H}}$.

The set of eigenvalues of $\tilde{\mathcal{H}}$ gives the eigenvalues of the original operator \mathcal{H} , except for possibly one additional eigenvalue λ_0 , which is discussed below. The eigenfunctions of \mathcal{H} can be found from Eq. (E8). Using Eqs. (E10) and (E12) one presents them in a form of Eq. (4.31).

In order to verify whether λ_0 is an eigenvalue of \mathcal{H} one needs to find φ from Eq. (E6). Equation (E9) gives

$$\left(-\frac{d}{dx} + (\gamma + 1/2)\frac{x+1}{x} - \frac{1}{x+1}\right)\varphi = 0, \quad (\text{E14})$$

which leads to

$$\varphi = (x+1)^{-1}x^{\gamma+1/2}\exp[(\gamma+1/2)x]. \quad (\text{E15})$$

Since this function is singular at $x = \infty$, it cannot be an eigenfunction. Consequently λ_0 is not an eigenvalue.

The function φ defined by Eq. (4.31) exhibits regular behavior at both boundaries Eqs. (4.38) and (4.39). This ensures that φ is an eigenfunction. Note that specifying the operators θ , θ^\dagger one had an additional option. One could have chosen in Eqs. (E12) and all the following relevant formulas $-\gamma$ instead of γ . In this case, however, instead of Eq. (4.39) one obtains $\varphi \propto x^{-\gamma+1/2}$ for $x \rightarrow 0$, which indicates a singular, unacceptable for an eigenfunction behavior.

We conclude that the full set of all discrete eigenvalues of \mathcal{H} is specified by Eq. (4.36). The corresponding eigenfunctions are given by Eqs. (4.31) and (4.37).

- [1] A. Proca, *Comptes Rendus* **202**, 1490 (1936).
 [2] H. F. W. Massey and H. C. Corben, *Camb. Phil. Soc.* **35**, 463 (1939).

- [3] J. R. Oppenheimer, H. Snyder, and R. Serber, *Phys. Rev.* **57**, 75 (1940).
 [4] I. E. Tamm, *Phys. Rev.* **58**, 952 (1940); *Dokl. Akad. Nauk*

- SSSR **9**, 551 (1940) (in Russian).
- [5] H. C. Corben and J. Schwinger, Phys. Rev. **58**, 953 (1940).
- [6] J. Schwinger, Rev. Mod. Phys. **36**, 609 (1964).
- [7] H. Cheng and T. T. Wu, Phys. Rev. D **5**, 3247 (1972).
- [8] K. Huang, *Quarks, Leptons and Gauge Fields* (World Scientific, Singapore, 1992), 2nd ed.
- [9] B. Vijayalakshmi, M. Seetharaman, and P. M. Mathews, J. Phys. A **12**, 665 (1979).
- [10] A. A. Pomeransky and I. B. Khriplovich, JETP **86**, 839 (1998).
- [11] A. A. Pomeransky and R. A. Sen'kov, Phys. Lett. B **468**, 251 (1999).
- [12] A. A. Pomeransky, R. A. Sen'kov, and I. B. Khriplovich, Usp. Fiz. Nauk **43**, 1129 (2000) [Sov. Phys. Usp. **43**, 1055 (2000)].
- [13] A. J. Silenko, hep-th/0404074.
- [14] W. I. Fushchych, A. G. Nikitin, and V. M. Susloparov, Nuovo Cimento **87**, 415 (1985).
- [15] W. I. Fushchych and A. G. Nikitin, *Symmetries of Equations of Quantum Mechanics* (N. Y. Allerton Press, New York, 1994).
- [16] A. G. Sergheyev, Ukr. J. Phys. **42**, 1171 (1997).
- [17] V. S. Fadin, V. A. Khoze, A. D. Martin, and W. J. Stirling, Phys. Lett. B **363**, 112 (1995); V. S. Fadin, V. A. Khoze, A. D. Martin, and A. Chapovsky, Phys. Rev. D **52**, 1377 (1995); Phys. Lett. B **311**, 311 (1993).
- [18] S. Eidelman *et al.*, Phys. Lett. B **592**, 1 (2004).
- [19] M. Yu. Kuchiev and V. V. Flambaum, Mod. Phys. Lett. A **21**, 781 (2006); (unpublished).
- [20] S. Weinberg, *Modern Applications, The Quantum Theory of Fields Vol. II* (Cambridge University Press, Cambridge, England, 2001).
- [21] V. B. Berestetsky, E. M. Lifshits, and L. P. Pitaevsky, *Quantum Electrodynamics* (Pergamon, New York, 1982).
- [22] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
- [23] Reference [5] uses notation " $l = j - 1$ " and " $l = j + 1$ " for the modes, which we call " $\gamma - 1/2$ " and " $\gamma - 3/2$."
- [24] I. Levine *et al.*, Phys. Rev. Lett. **78**, 424 (1997).
- [25] S. Okada *et al.*, Phys. Rev. Lett. **81**, 2428 (1998).
- [26] G. Abbiendi *et al.*, Eur. Phys. J. C **13**, 553 (2000).
- [27] The scalar field $\phi(r)$ propagating in a static potential $U(r)$ provides a simple example. The charge density $\rho_\phi = 2e(\epsilon - U(r))\phi^*(r)\phi(r)$, where ϵ is the energy, shows the wrong sign in a strongly repulsive region $\epsilon < U(r)$. It is necessarily small because the field is suppressed there; compare Eq. (7.4), which indicates that the wrong density of charge of the W -boson in the $2p_1$ state is very small.
- [28] H. S. Green, *Matrix Mechanics* (P. Noordhoff Ltd-Groninger, The Netherlands, 1965).