Modified dispersion relations and black hole physics

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A modified formulation of the energy-momentum relation is proposed in the context of doubly special relativity. We investigate its impact on black hole physics. It turns out that such a modification will give corrections to both the temperature and the entropy of black holes. In particular, this modified dispersion relation also changes the picture of Hawking radiation greatly when the size of black holes approaches the Planck scale. It can prevent black holes from total evaporation, as a result providing a plausible mechanism to treat the remnant of black holes as a candidate for dark matter.

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I. INTRODUCTION

One generally believed feature of quantum gravity is the existence of a minimal observable length [1,2]. Recent development of loop quantum gravity has also greatly strengthened such beliefs by manifestly showing the discreteness of area and volume spectra [3]. At the same time, such effects have also invoked many investigations and debates on the fate of Lorentz symmetry at Planck scale [4]. One reason is that such effects seemingly lead to a paradox due to the apparent confliction between the existence of a minimum length and Lorentz symmetry, which in principle may contract any object to arbitrarily small size by Lorentz boost. Nowadays, one intriguing approach dubbed as doubly special relativity (DSR) is proposed to solve this paradox (for details see the recent review [5] and references therein). In particular, a general formalism of modifying special relativity has been proposed in [6] to preserve the relativity of inertial frames, while at the same time keeping a physical energy such as Planck energy as an invariant. This is accomplished by a nonlinear action of the Lorentz transformation in momentum space. This formalism also points to the possibility that the usual relation between the energy and momentum in special relativity may be modified at Planck scale, conventionally named as modified dispersion relations (MDR). Such relations can also be derived in the study of the semiclassical limit of loop quantum gravity [7,8].

The modification of energy-momentum relations and its implications have been greatly investigated by many theorists [9–13]. It may be responsible for some peculiar phenomena in experiments and astronomic observations, such as the threshold anomalies of ultrahigh energy cosmic ray and gamma ray burst. Such modifications may further lead to some predictions which can be falsified in planned experiments. Among these are the energy dependence of the speed of light and the helicity independence of dispersion relations, observable in the coming AUGER and GLAST experiments [8]. Moreover, a modified dispersion relation may present alternatives to inflationary cosmology [14], and this is testable in the future measurement of CMB spectrum.

In this paper we intend to study the impact of modified dispersion relations on black hole physics. We first present a modified dispersion relation advocated by doubly special relativity, and then show that this modified relation may contribute corrections to the temperature of black holes as well as the entropy. We find the entropy has a logarithmic correction while the temperature is bounded with a finite value as the mass of black holes approaches to the Planck scale such that black holes will finally stop radiating, in contrast to the ordinary picture where the temperature can be arbitrarily high as the mass approaches to zero and finally divergent when black holes fully evaporate. A comparison with effects due to the generalized uncertainty principle is also discussed.

II. MODIFIED DISPERSION RELATIONS

As pointed out in [6], in a DSR framework the modified dispersion relation may be written as

$$E^2 f_1^2(E;\eta) - P^2 f_2^2(E;\eta) = m_0^2, \tag{1}$$

where f_1 and f_2 are two functions of energy from which a specific formulation of boost generator can be defined. In this paper we adopt a modified dispersion relation (MDR) by taking $f_1^2 = [1 - \eta(l_p E)^n]$ and $f_2^2 = 1$, such that

$$E^2 = \frac{p^2 + m_0^2}{[1 - \eta(l_p E)^n]}, \tag{2}$$

where Planck length $l_p \equiv \sqrt{8\pi G} \equiv 1/M_p$ and η is a dimensionless parameter. If $l_p E \ll 1$, this modified dispersion relation goes back to the ordinary one,

$$E^2 = p^2 + m_0^2 + \eta(l_p E)^n(p^2 + m_0^2 + \cdots). \tag{3}$$

However, when $l_p E \sim 1$ the relation changes greatly and we need to treat it nonperturbatively. For convenience, we

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take $n = 2$ and $\eta = 1$,

$$l_p^2 E^4 - E^2 + (p^2 + m_0^2) = 0. \tag{4}$$

This gives a relation as

$$E^2 = \frac{1}{2l_p^2}[1 - \sqrt{1 - 4l_p^2(p^2 + m_0^2)}]. \tag{5}$$

Because of the appearance of square root in the above equation, we notice that both the momentum and the static mass of a single particle are bounded by $m_0 \leq M_p/2$ and $p^2 \leq (M_p^2/4 - m_0^2)$; in particular, for massless particles the limiting momentum is $M_p/2$. The existence of a maximum momentum reflects the feature that there is a minimal observable length in quantum gravity, and more delicate analysis can be found in [1,2,15], where the generalized uncertainty principle is employed, while in the context of DSR our argument here can be considered as a result of the nonlinear Lorentz transformation in momentum space [5,6]. From (5) it is also easy to see that a single particle has a maximum energy $E_{\max} = M_p/\sqrt{2}$.

III. LINKING MODIFIED DISPERSION RELATIONS TO BLACK HOLE PHYSICS

Now we consider the impact of this MDR on Schwarzschild black holes. The effect that the existence of a minimum length can prevent black holes from total evaporation has been investigated in [15], where the generalized uncertainty principle (GUP) plays an essential role. Here we will consider a modified dispersion relation rather than GUP. First, we identify above quantities E and p as the energy and momentum of photons emitted from the black hole, respectively, of course for photons m_0 is set to zero. Then we adopt the argument presented in [15] that the characteristic temperature of this black hole is supposed to be proportional to the photon energy E, namely $E = T$. On the other hand, we apply the ordinary uncertainty relation to photons in the vicinity of black hole horizons. As pointed out in [15,16], for these photons there is an intrinsic uncertainty about the Schwarzschild radius R:

$$p \sim \delta p \sim \frac{1}{\delta x} \sim \frac{1}{4\pi R}, \tag{6}$$

where a "calibration factor" 4π is introduced. Using the fact that $R = 2MG \sim M/4\pi M_p^2$ [17], we obtain the temperature of Schwarzschild black holes has the form

$$T = \left[\frac{M_p^2}{2}\left(1 - \sqrt{1 - \frac{4M_p^2}{M^2}}\right)\right]^{1/2}. \tag{7}$$

This requires that the mass of black holes $M \geq 2M_p$, and correspondingly the temperature $T \leq M_p/\sqrt{2}$ [18]. For large black holes with $M \gg 2M_p$, it goes back to the ordinary form $T = M_p^2/M$.

Next we consider the possible correction to the entropy of black holes due to the modification of the temperature, assuming the first thermodynamical law still exactly holds even for small black holes where the quantum effect of gravity may play an essential role [19]. Plugging the temperature into $dM = TdS$, we have

$$dM = \left[\frac{M_p^2}{2}\left(1 - \sqrt{1 - \frac{4M_p^2}{M^2}}\right)\right]^{1/2} dS. \tag{8}$$

Thus, the entropy can be calculated from the integration,

$$S = \frac{1}{2\sqrt{G}} \int_{A_{\min}}^{A} (A - \sqrt{A^2 - 8GA})^{-1/2} dA, \tag{9}$$

where $A_{\min} = 8G \sim l_p^2/\pi$ is the cutoff corresponding to a black hole with minimum mass $M = 2M_p$. Define $t = \sqrt{1 - 8G/A}$, (9) can be integrated out,

$$S = \frac{1}{\sqrt{2}}\left[2(1+t)^{-1/2}\Big|_{t_{\min}}^{t} + \frac{\sqrt{1+t}}{1-t}\Bigg|_{t_{\min}}^{t} - \frac{1}{\sqrt{2}}\ln\left(\frac{1+t}{1-t}\right)\Bigg|_{\sqrt{1+t_{\min}}/\sqrt{2}}^{\sqrt{1+t}/\sqrt{2}}\right] + S_{\min}, \tag{10}$$

where $S_{\min} = A_{\min}/4G$ is a constant term such that for minimum black holes the familiar Bekenstein-Hawking entropy formula still holds. When $A \gg 8G$, it becomes

$$S = \frac{1}{\sqrt{2}}\left[\frac{A}{8G}(1+t)^{3/2} - \frac{1}{\sqrt{2}}\ln\left[\frac{A}{8G}(1+t)\right]\right]. \tag{11}$$

It is obvious that for large black holes, namely $t \longrightarrow 1$, this gives the familiar formula

$$S = \frac{A}{4G} - \frac{1}{2}\ln\frac{A}{4G} + \cdots. \tag{12}$$

Therefore, in this case we find modified dispersion relations contribute a logarithmic correction to black hole entropy [20].

The entropy correction can also be evaluated using the scheme proposed in [21], where the Bekenstein entropy assumption is applied to determine the minimum increase of horizon area when a black hole absorbs a classical particle with energy ϵ and size δx. Explicitly, the assumption is

$$\frac{\delta A_{\min}}{4G} \geq 2\pi\epsilon\delta x. \tag{13}$$

On the other hand, from (5) we may obtain a uncertainty relation between the energy and momentum of a single particle as

$$\delta E \sim \frac{\delta p}{(1 - 2l_p^2 p^2)}. \tag{14}$$

Identifying $\epsilon \sim \delta E$ and $\delta p \sim 1/\delta x \sim 1/(4\pi R)$, we may have

$$\delta A_{\min} \cong 16\pi G \ln 2\epsilon\delta x = \frac{4G\ln 2}{1 - l_p^2/2\pi A}, \tag{15}$$

where a calibration factor 2 ln2 is introduced. Thus

$$\frac{dS}{dA} \cong \frac{\delta S}{\delta A_{\min}} = \frac{1}{4G}\left(1 - \frac{l_p^2}{2\pi A}\right), \tag{16}$$

and consequently

$$S = \frac{A}{4G} - \ln\left(\frac{A}{4G}\right). \tag{17}$$

Comparing with (12) we find the answer is almost the same but the factor in logarithmic term is different, which results from the fact that some approximations for large black holes have been taken into account during the derivation of (17), for instance in (14). In this sense we argue that we find a more precise way to obtain corrections to black hole entropy. This can also be understood from the fact that in our case the exact expression for entropy corresponding to the temperature (7) can be obtained, as shown in (10), in contrast to [21] where such exact expressions are not available. Moreover, it is worthwhile to point out that in those papers the entropy is obtained approximately at first, and then the temperature is derived with the use of the first thermodynamical law. With no surprise, such a logic gives rise to a different result for temperature from (7), but approximately equal at the large black hole limit. From this point of view, our results are more general. It is also this advantage that makes them applicable to investigate the fate of black holes at the late stage of radiation. We briefly present our analysis below, as a similar discussion has appeared in [15].

Thanks to the Stefan-Boltzmann law [22], the evaporation rate of black holes can be estimated by

$$\frac{dx}{dt} = \frac{-1}{t_f}(x - \sqrt{x^2 - 4})^2, \tag{18}$$

where $x = M/M_p$ and $t_f = 16\pi/(\sigma M_p)$. The solution to this equation reads as

$$t = t_c - \frac{t_f}{24}[x^3 + (x^2 - 4)^{3/2} - 6x], \tag{19}$$

where t_c is an integral constant. From the above equation, we find that $dx/dt = -4/t_f$ is a finite number at the end $x = 2$ rather than infinity in the ordinary case. As a matter of fact when the size of black holes approaches the Planck scale, they will cease radiation although the temperature reaches a maximum. This can be seen from the behavior of the heat capacity. From Eq. (7), we obtain

$$C = \frac{dM}{dT} = -\frac{M^3 T}{M_p^4}\left(1 - \frac{4M_p^2}{M^2}\right)^{1/2}. \tag{20}$$

It is interesting that the heat capacity becomes vanishing when the black hole mass approaches a nonzero scale, $M = 2M_p$. This maybe implies the ground state of the black hole. As an analogy, let us look at a system consisting of the harmonic oscillators, the heat capacity is vanishing when the system is in the ground state. This is because the zero energy is independent of the temperature, $\partial E_0/\partial T = 0$. This phenomenon provides a mechanism to take black hole remnants as a natural candidate for cold dark matter due to their weakly interacting features [23].

At the end of this section we point out that the effect of generalized uncertainty principle will not change our conclusions but provide modifications. The well-known uncertainty relation can be generalized as

$$\delta x \delta p \geq 1 + l_p^2 \delta p^2. \tag{21}$$

Thus

$$\delta p = \frac{\delta x}{2l_p^2}(1 \mp \sqrt{1 - 4l_p^2/\delta x^2}). \tag{22}$$

Set $p \sim \delta p$ and plug it into Eq. (5); we obtain

$$T \sim \left[\frac{M_p^2}{2}(1 - \sqrt{5 - 2x(x - \sqrt{x^2 - 4})})\right]^{1/2}, \tag{23}$$

where $x = M/M_p \geq 5/2$. Thus the black hole ceases radiation as it reaches this minimum value at Planck scale.

IV. CONCLUDING REMARKS

In this paper we have shown that the modified dispersion relations have important impacts on black hole physics at the high energy level. First, MDR contributes a correction to the temperature of black holes and provides an effective cutoff such that a upper limit will be reached as the area of black hole horizon takes the minimum value. Correspondingly, the black hole entropy is corrected with a logarithmic term. Second, MDR provides a plausible mechanism to prevent black holes from fully evaporating and the remnant can be treated as a candidate for cold dark matter.

Through the paper we only investigate a special case in doubly special relativity with specified functions of $f_1(E;\eta)$ and $f_2(E;\eta)$, but it is obviously possible to extend our discussion to other general cases. For instance, we may take $n = 1$ such that $f_1^2 = [1 - \eta(l_p E)]$, a parallel analysis can be done and the entropy of black holes is expected to receive a correction proportional to the square root of the area. Among all the possible modified dispersion relations, which is the proper one should await further tests in experiments.

It is interesting to notice that both generalized uncertainty relations and modified dispersion relations may be rooted at the algebraic structure of commutators among position and momentum variables. More deep relations between them and implications to quantum gravity phenomenology are under investigation.

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[17] We assume that the modified dispersion relation will not change this relation which is consistent with the result in [24], where the impact of doubly special relativity on gravity is investigated. also see [25] for recent work on the thermodynamics of modified black holes from gravity's rainbow.
[18] It seems possible to relate this formalism to the Immirzi parameter γ in loop quantum gravity where $A_{\min} = \sqrt{3}/2\gamma l_p^2$, since in dispersion relations we can freely add some parameter η in the term $(l_pE)^2$ in (4) as we pointed out. Then setting these two minimum areas equal leads to a relation $\gamma = 2\eta/(\sqrt{3}\pi)$.
[19] We make this assumption based on the belief that the thermodynamical laws of black holes can still be captured even in the quantum theory of gravity. This belief has been supported by recent work in isolated horizon programs as well as the stretched horizon program, where the first law of black hole thermodynamics can still be established in a quasilocal fashion. In the context of DSR, which may be viewed as the semiclassical effect of quantum gravity, we assume the modification of dispersion relations of particle would not change this picture. However, before proceeding, we cautiously point out that this exact form of the first law may not be generally true but receives a small modification, or even fails for instance in the Einstein-Aether theory as investigated in [13].
[20] The factor $-1/2$ in the logarithmic term happens to be the same as the one appearing in [26] where this factor is rigorously fixed in the context of loop quantum gravity. However, we stress that, at the semiclassical level as discussed in our paper, this factor cannot be fixed uniquely, but depends on the value of η which has been set as a unit in our paper. A straightforward calculation shows that in general the factor would be $-\eta/2$. If we insist that the minimum area in our paper is the same as that in loop quantum gravity as we suggested in above reference, then the factor turns out to be $-\sqrt{3}\pi\gamma/4$, rather than $-1/2$. Reversely, if we insist on setting the factor exactly to be $-1/2$, namely $\eta = 1$, then from this approach we have an Immirzi parameter $\gamma \simeq 0.37$, different from the result in [26]. This discrepancy can be understood as this specific MDR we proposed is only a coarse grained model at the semiclassical limit of quantum gravity.
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