The $AdS_5 \times S^5$ superstring worldsheet S matrix and crossing symmetry

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An S matrix satisfying the Yang-Baxter equation with symmetries relevant to the $AdS_5 \times S^5$ superstring recently has been determined up to an unknown scalar factor. Such scalar factors are typically fixed using crossing relations; however, due to the lack of conventional relativistic invariance, in this case its determination remained an open problem. In this paper we propose an algebraic way to implement crossing relations for the $AdS_5 \times S^5$ superstring worldsheet S matrix. We base our construction on a Hopf-algebraic formulation of crossing in terms of the antipode and introduce generalized rapidities living on the universal cover of the parameter space which is constructed through an auxillary, coupling-constant dependent, elliptic curve. We determine the crossing transformation and write functional equations for the scalar factor of the S matrix in the generalized rapidity plane.

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I. INTRODUCTION

One of the most fascinating discoveries in recent years was the unravelling of integrable structures in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory [1–4] and in superstring theory on $AdS_5 \times S^5$ space [5–11]. In view of the AdS/CFT correspondence [12] which links the two theories and identifies anomalous dimensions of the gauge theory with energies of string excitations in $AdS_5 \times S^5$, these discoveries have thus opened up the possibility of detailed testing of the proposed correspondence. Even more interestingly, the techniques of integrability allow in principle for an exact quantization of the superstring in $AdS_5 \times S^5$ and/or the determination of anomalous dimensions in the $\mathcal{N} = 4$ SYM.

However it turned out that at three loop order there is a set of discrepancies between gauge-theory calculations and string theory results [13–16]. This does not signify a contradiction since the domains of applicability are nonoverlapping and in order to perform a comparison an extrapolation is needed—in fact an order of limits problem was suggested to be a possible explanation. One source of the discrepancy, the so-called "wrapping interactions" have apparently been ruled out [17,18] so the problem remains open. A key ingredient which enters the calculation of the anomalous dimensions or string energies is the S matrix. "Phenomenologically" one can quantify the disagreement by a *scalar* dressing factor between the "string" S matrix and the asymptotic "gauge-theory" S matrix [19]. Initially this has been proposed to hold in subsectors of the full theory [20], but subsequently it has been extended to the whole S matrix. The structure of the dressing factor on the string theory side appears to be quite complicated with $1/\sqrt{\lambda}$ deviations [21–23] from the strong coupling expression [20].

In fact, in a remarkable series of papers, [24,25], the S matrix in various sectors was uncovered. This culminated with [26] where the S matrix with $su(2|2) \times su(2|2) \subset psu(2, 2|4)$ symmetry was determined up to an unknown scalar function $S_0(p_1, p_2)$:

$$S(p_1, p_2) = S_0(p_1, p_2) \cdot [\hat{S}_{su(2|2)}(p_1, p_2) \otimes \hat{S}_{su(2|2)}(p_1, p_2)], \qquad (1)$$

with $\hat{S}_{su(2|2)}(p_1, p_2)$ being the S matrix determined uniquely by $su(2|2)$ symmetry in [26]. A key remaining problem is the determination of the scalar "dressing factor" $S_0(p_1, p_2)$. The aim of this paper is to propose functional equations which $S_0(p_1, p_2)$ has to satisfy.

A similar situation exists in *relativistic* integrable quantum field theories, where symmetries (including the nonlocal Yangian or affine quantum algebra) determine completely the matrix form of the S matrix up to an overall scalar factor. This scalar factor is in turn determined by requiring unitarity and crossing symmetry of the S matrix. Only these two conditions together determine uniquely a scalar factor with the minimal number of poles/zeroes in the physical region. The only remaining ambiguities are the Castillejo-Dalitz-Dyson factors which serve to introduce further poles, if needed on physical grounds, but generically the minimal solution suffices (see [27]).

It is important to emphasize that the two conditions—unitarity and crossing—have quite a different status. Unitarity is, as is the Yang-Baxter equation, a consistency condition for the Faddeev-Zamolodchikov algebra and does not involve any dynamical assumptions about the theory in question. The matrix form of the S matrix from [26] indeed directly satisfies the unitarity condition. On the other hand the crossing condition involves a link between the scattering of particles and the same process with a particle changed into an antiparticle. The definite form of the resulting equation involves in an essential way explicit relativistic kinematics as formulated in the rapidity plane. In this form it depends crucially on the relativistic invariance of the theory.

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It is this last property which makes it very difficult to adopt a similar strategy in the case of the worldsheet theory of the $AdS_5 \times S^5$ superstring. There the light-cone quantization of the worldsheet theory, which seems necessary in order to deal only with physical excitations, is *inconsistent* with Minkowski metric on the worldsheet [9,13,15,28]. Therefore, the guiding principle of relativistic invariance is lost, making it unclear not only how to implement crossing but even whether such a property should hold at all.

The aim of this paper is to propose how to implement crossing properties directly in the context of the S matrix (1). We believe that on general grounds it is very probable that some form of crossing symmetry should hold for the $AdS_5 \times S^5$ superstring. First, if one would attempt quantization without using light-cone gauge, but at the cost of introducing ghosts, it is probable that one could get an explicitly relativistic quantum field theory (QFT), which would thus have to obey crossing conditions. Second, there are subsectors of the full $AdS_5 \times S^5$ worldsheet theory, which can be formulated in an explicitly relativistic manner [29]. There one also expects crossing to hold. For these reasons we believe that a form of the crossing condition should exist for (1). In order to find its concrete form we will use an algebraic formulation of this condition which appears in the language of Hopf algebras, the natural mathematical framework for incorporating nonlocal symmetries in integrable quantum field theories and which includes as notable examples the Yangian and quantum affine algebras (see e.g. [30–32]). Applying this procedure in the context of the S matrix (1) we find that the form of crossing transformation is uniquely fixed. We then find an analog of the rapidity parametrization in relativistic QFT involving a coupling-constant dependent elliptic curve and use this to derive equations for the scalar factor $S_0(z_1, z_2)$, where z_i are the "generalized" rapidities.

The plan of this paper is as follows: In Sec. II we recall the status of crossing symmetry in relativistic integrable quantum field theories, and we emphasize its Hopf-algebraic reformulation which will be the basis of our construction. In Sec. III we describe in some detail the construction of the $su(2|2)$ S matrix of [26] in the form which will be convenient for our purposes. In Sec. IV we derive the form of the crossing transformation and exhibit crossing properties of the S matrix. In Sec. V we proceed to derive a generalized rapidity parametrization in the context of (1) which directly generalizes the rapidity variable θ of a relativistic QFT. We use that to write, in Sec. VI, the unitarity and crossing equations for the scalar factor $S_0(z_1, z_2)$ in the generalized rapidity plane. We close the paper with some conclusions and two appendices.

II. CROSSING SYMMETRY IN RELATIVISTIC INTEGRABLE QFT

In a relativistic integrable QFT with some symmetry group the S matrix has the form

$$S(\theta_1, \theta_2) = S_0(\theta_1 - \theta_2) \cdot \hat{S}(\theta_1 - \theta_2), \tag{2}$$

where the θ_i are the *rapidities* which parametrize the energies and momenta of the particles through

$$E_i = m \cosh\theta_i, \qquad p_i = m \sinh\theta_i. \tag{3}$$

Relativistic invariance ensures that the S matrix is a function only of the difference $\theta = \theta_1 - \theta_2$. The matrix $\hat{S}(\theta)$ is typically uniquely fixed, up to multiplication by a scalar function $S_0(\theta)$, using either the Yang-Baxter equation or nonlocal symmetries from Yangian or quantum affine algebras.

Subsequently the scalar factor is fixed by requiring unitarity

$$S_{ij}^{nm}(\theta) S_{nm}^{kl}(-\theta) = \delta_i^k \delta_j^l \tag{4}$$

and crossing invariance of the S matrix

$$S_{\bar{k}j}^{l\bar{i}}(i\pi - \theta) = S_{ij}^{kl}(\theta), \tag{5}$$

where the bars over indices indicate a change from a particle to an antiparticle and may typically involve some nontrivial action of a charge conjugation matrix.

For the case of the $su(2|2) \times su(2|2)$ symmetric S matrix relevant for the $AdS_5 \times S^5$ superstring it is unclear how to generalize the crossing relation written in the form (5). First, the S matrix depends nontrivially on two variables and cannot be written as a function of a single variable. Second, one does not know how to implement charge conjugation and even more importantly what is the analogue of the $i\pi - \theta$ in (5). In order to overcome these difficulties we use a reformulation of the crossing condition in terms of an underlying symmetry algebra which is a Hopf algebra. This is in fact a natural setting for the symmetry algebras of integrable relativistic QFT as *nonlocal* symmetry charges are naturally incorporated in this framework through a nontrivial coproduct (i.e. a prescription of how the nonlocal charge acts on a two-particle state). The relativistic crossing symmetry requirement has been translated already into this framework (see [33–35] for the supersymmetric case) using another ingredient of a Hopf algebra—the antipode.

In order to motivate this formulation let us rewrite the relation (5) reintroducing two separate rapidities, θ_1 and θ_2, and keeping in mind that crossing involves changing the first particle into an antiparticle. Equation (5) can then be rewritten as

$$S(i\pi - \theta_1 + \theta_2)^{\text{cross}} = S(\theta_1 - \theta_2), \tag{6}$$

where the superscript cross stands for the relevant transformation of the indices. Now we reverse the signs of θ_1 and θ_2 to get

$$S(\theta_1 + i\pi - \theta_2)^{\text{cross}} = S(-(\theta_1 - \theta_2)) = S(\theta_1 - \theta_2)^{-1}, \tag{7}$$

where in the last equality we used unitarity. We see that the particle to antiparticle transformation involves a shift of the rapidity $\theta \rightarrow \theta + i\pi$ which reverses the signs of the energy and momentum.

It is this last form which has a direct Hopf-algebraic interpretation. To see that let us introduce the R matrix $R(\theta_1, \theta_2)$ related to the S matrix through a (graded) permutation P:

$$S(\theta_1, \theta_2) = PR(\theta_1, \theta_2). \tag{8}$$

The R matrix in a Hopf algebra satisfies the direct counterpart of (7) (see e.g. proposition 4.2.7 in [30])

$$(\mathcal{S} \otimes \mathrm{id})R = R^{-1}, \qquad (\mathrm{id} \otimes \mathcal{S}^{-1})R = R^{-1}, \tag{9}$$

where $\mathcal{S}$ is the antipode mapping which has the physical interpretation of a particle to antiparticle transformation. Directly from the Yangian or quantum affine algebra viewpoint one shows that the antipode involves a shift of θ (since in this framework the rapidity labels the representations of the Yangian) and also possibly some charge conjugation matrices when the above equations are considered in some definite representation of a Lie (super) algebra.

We would like to emphasize that (9) does not involve any assumptions on the underlying relativistic invariance of the theory. The action of the antipode follows just purely algebraically from the relevant Hopf algebra. Because of this property, we propose to use (9) as a basis for generalizing crossing to the case of the worldsheet $AdS_5 \times S^5$ theory. We will perform this construction in Sec. IV.

Before we end this section let us discuss the rapidity parametrization (3). It was introduced in order for the S matrix to be a meromorphic (single-valued) function without any cuts which would appear if the S matrix was considered as a function of physical momenta. Since we do not have much intuition about the analytical structure of the S matrix for the nonstandard dispersion relations characteristic of the $AdS_5 \times S^5$ worldsheet theory, we would like to abstract the above mentioned characteristic of the rapidity parametrization and use it as a guiding principle in our case.

The physical relativistic energies and momenta are linked by the mass-shell condition

$$E^2 - p^2 = m^2. \tag{10}$$

The rapidity parametrization can be thought of as a uniformization of the above curve (roughly as a universal covering space[1])—namely, a mapping by single-valued functions from the rapidity plane of complex θ

$$E = m\cosh\theta, \qquad p = m\sinh\theta. \tag{11}$$

The particle-antiparticle interchange $(E, p) \rightarrow (-E, -p)$ then becomes a translation $\theta \rightarrow \theta + i\pi$ which is no longer an involution, a fact that will be important for us later. We propose to implement the crossing conditions (9) for the $AdS_5 \times S^5$ worldsheet S matrix on the appropriate universal covering space which would play the role of the space of generalized rapidities, and by its very definition would avoid the appearance of cuts. We will construct the universal covering space in this context in Sec. V and write the resulting crossing equations for the scalar factor in Sec. VI.

III. THE $su(2|2)$ S MATRIX

In [26] the S matrix has been considered mainly with explicit (centrally extended) $su(2|2)$ symmetry. This was at the cost of introducing explicit length-changing operators which have nontrivial braiding (commutation) relations with the excitations. A consequence of that was the fact that these braiding factors had to be incorporated when verifying the Yang-Baxter equation. This nonstandard modification of the Yang-Baxter equation would make the Hopf-algebraic interpretation advocated in the present paper quite problematic. However, as pointed out in Appendix B of [26], one can formulate the scattering matrix with only a manifest $su(1|2)$ symmetry, but with the length-changing operators eliminated. Then the Yang-Baxter equation is just the ordinary Yang-Baxter equation without any braiding factors. For these reasons we will adopt here the $su(1|2)$ symmetric formulation. For completeness, we will now recapitulate in some detail the derivation of the $su(2|2)$ symmetric S matrix in this formulation as we will use some of the explicit constructions presented here in the derivation of the crossing relations.

A word of caution is necessary here concerning the identification of the $su(2|2) \times su(2|2)$ S matrix of [26] with the S matrix of the worldsheet $AdS_5 \times S^5$ superstring. In the latter paper the S matrix was formulated from the spin-chain point of view, but in fact all arguments could be recast as implementing the symmetries in a Zamolodchikov-Faddeev algebra for the worldsheet theory extended by the length-changing operators. One cannot be completely certain that this is the only way of implementing these symmetries but the high degree of uniqueness of the resulting S matrix makes it quite probable that this is indeed so. In this paper we will therefore adopt this hypothesis for the S matrix of the worldsheet theory.

We will denote, as in [26], the basis states as $|\phi\rangle \equiv |\phi^1\rangle$, $|\chi\rangle \equiv |\phi^2 Z^+\rangle$ and the fermionic $|\psi^{1,2}\rangle$. The operators which will form the $su(1|2)$ subalgebra have the bosonic index restricted to 1. For completeness let us quote the action of the generators of $su(1|2)$ on this basis states. The bosonic rotation generators are canonical:

$$\mathfrak{R} = \tfrac{1}{2}|\phi\rangle\langle\phi| - \tfrac{1}{2}|\chi\rangle\langle\chi|, \tag{12}$$

$$\mathfrak{L}^{\alpha}_{\ \beta} = \delta_{\gamma\beta}|\psi^{\alpha}\rangle\langle\psi^{\gamma}| - \tfrac{1}{2}\delta^{\alpha}_{\beta}|\psi^{\gamma}\rangle\langle\psi^{\gamma}|. \tag{13}$$

The fermionic supercharges act on the basis states as

[1]Note, however, that the universal cover here is the sphere and the rapidity parametrization can be considered as a mapping onto that sphere from the plane (albeit with an essential singularity at infinity).

follows:

$$\begin{aligned}\mathfrak{Q}^1 &= a|\psi^1\rangle\langle\phi| + b|\chi\rangle\langle\psi^2|,\\ \mathfrak{Q}^2 &= a|\psi^2\rangle\langle\phi| - b|\chi\rangle\langle\psi^1|,\end{aligned} \tag{14}$$

$$\begin{aligned}\mathfrak{S}_1 &= c|\psi^2\rangle\langle\chi| + d|\phi\rangle\langle\psi^1|,\\ \mathfrak{S}_2 &= -c|\psi^1\rangle\langle\chi| + d|\phi\rangle\langle\psi^2|.\end{aligned} \tag{15}$$

The (complex) parameters a, b, c, d parametrize the allowed representations and encode the energy and momenta of the states. They are not completely unconstrained. The commutation relation

$$\{\mathfrak{Q}^\alpha, \mathfrak{S}_\beta\} = \mathfrak{L}^\alpha_\beta + \delta^\alpha_\beta \mathfrak{R} - \delta^\alpha_\beta \mathfrak{H}, \tag{16}$$

leads to the relation

$$ad - bc = 1, \tag{17}$$

while the central charge $\mathfrak{H}$ is fixed to be

$$\mathfrak{H} = (\tfrac{1}{2} + bc)\cdot \mathrm{id}. \tag{18}$$

From the point of view of the full $psu(2,2|4)$ symmetry of the $AdS_5 \times S^5$ superstring, $\mathfrak{H}$ is related to the anomalous dimension Δ through

$$\mathfrak{H} = \tfrac{1}{2}(\Delta - J). \tag{19}$$

Moreover, a can be absorbed into relative normalization of the fermionic and bosonic states. We will set it usually to 1.

A. The $su(1|2)$ invariant S matrix

The $su(1|2)$ invariant S matrix acts in the tensor product of two representations as

$$S: V_1 \otimes V_2 \to V_2 \otimes V_1. \tag{20}$$

It will turn out to be more convenient to consider the R matrix acting as

$$R: V_1 \otimes V_2 \to V_1 \otimes V_2, \tag{21}$$

related to the S matrix through the graded permutation operator P:

$$S = PR. \tag{22}$$

From $su(1|2)$ invariance the R matrix is then a linear combination of projectors onto the three irreducible $su(1|2)$ representations appearing in the tensor product $V_1 \otimes V_2$:

$$R = S_1 \cdot \mathrm{proj}_1 + S_2 \cdot \mathrm{proj}_2 + S_3 \cdot \mathrm{proj}_3. \tag{23}$$

For our purposes we will need an explicit construction of these projectors. To this end we may first construct the $su(1|2)$ invariant Casimir operator in $V_1 \otimes V_2$:

$$\mathcal{C}_{12} = \tfrac{1}{2}(\mathfrak{Q}^\alpha \mathfrak{S}_\alpha - \mathfrak{S}_\alpha \mathfrak{Q}^\alpha) + (\mathfrak{R} + \mathfrak{H})^2 + \tfrac{1}{2}\mathfrak{L}^\alpha_\beta \mathfrak{L}^\beta_\alpha. \tag{24}$$

This operator has three distinct eigenvalues in $V_1 \otimes V_2$ corresponding to the three irreducible representations. These are

$$\lambda_1 = (2 + b_1c_1 + b_2c_2)^2 - (b_1c_1 + b_2c_2) - 2, \tag{25}$$

$$\lambda_2 = (1 + b_1c_1 + b_2c_2)^2 - 1, \tag{26}$$

$$\lambda_3 = (b_1c_1 + b_2c_2)(1 + b_1c_1 + b_2c_2). \tag{27}$$

The projectors can then be constructed explicitly as e.g.

$$\mathrm{proj}_1 = \frac{(\mathcal{C}_{12} - \lambda_2)(\mathcal{C}_{12} - \lambda_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \tag{28}$$

and similar formulas for proj_2 and proj_3.

B. Implementation of the $su(2|2)$ symmetry

It turns out [26] that it is necessary to extend $su(2|2)$ by additional central charges which, however, act in such a way that they vanish on states with vanishing *total* momentum i.e. exactly the states that satisfy "level matching" for the closed string (or cyclic traces on the gauge-theory/spin-chain side). The two conditions which arise link the a, b, c, d parameters with the physical momentum p:

$$ab = \alpha(e^{-ip} - 1), \qquad cd = \beta(e^{ip} - 1), \tag{29}$$

where α and β are constants *common* for all excitations. Let us note that the above two equations lead to an additional constraint linking a, b, c, d. Namely, calculating e^{ip} from one of these equations and inserting into the other one leads to a quartic constraint

$$abcd + \beta ab + \alpha cd = 0. \tag{30}$$

The coupling constant is linked to α and β through $\alpha\beta = g^2/2$. This condition restricts further the allowed parameters labeling the explicit $su(1|2)$ representations.

Let us also quote a convenient parametrization of a, b, c, d in terms of the $x^\pm$ parameters [26] (with $a = 1$):

$$\begin{gathered} a = 1, \qquad b = -\alpha\left(1 - \frac{x^-}{x^+}\right), \qquad c = \frac{i\beta}{x^-},\\ d = -i(x^+ - x^-), \end{gathered} \tag{31}$$

where the parameters $x^\pm$ satisfy

$$x^+ + \frac{\alpha\beta}{x^+} - x^- - \frac{\alpha\beta}{x^-} = i. \tag{32}$$

The inverse mapping is given by

$$x^- = \frac{i\beta}{c}, \qquad x^+ = \frac{\alpha x^-}{\alpha + b}. \tag{33}$$

In order to completely determine the scalar coefficients S_i in (23) it is enough to assume symmetry under one of the other $su(2|2)$ supercharges (see [26]). In the explicit $su(1|2)$ setup these supercharges add the length-changing operators $\mathcal{Z}$ or $\mathcal{Z}^+$ to *all* states in the multiplet. Therefore invoking these symmetries will necessarily involve braiding factors. Explicitly the S matrix has to satisfy

$$(B_1\tilde{\mathfrak{Q}}_2 \otimes \mathrm{id} + (-1)^F \otimes \tilde{\mathfrak{Q}}_1) \cdot S - S \cdot (B_2\tilde{\mathfrak{Q}}_1 \otimes \mathrm{id} + (-1)^F \otimes \tilde{\mathfrak{Q}}_2) = 0, \quad (34)$$

where B_i are braiding factors coming from the commutation of length-changing operators. It turns out that the above equation for just one of the supercharges

$$\tilde{\mathfrak{Q}}_i = a_i|\psi^1\rangle\langle\chi| - b_i|\phi\rangle\langle\psi^2| \quad (35)$$

fixes uniquely both the braiding factors B_i and the S matrix coefficients S_i up to a common scalar factor. In terms of the a, b, c, d the expressions are quite lengthy but become rather simple in terms of $x^\pm$. We recover thus the result of Appendix B of [26]:

$$S_1 = \frac{x_2^+ - x_1^-}{x_2^- - x_1^+}, \quad (36)$$

$$S_2 = 1, \quad (37)$$

$$S_3 = \frac{x_2^-}{x_2^+}\frac{x_1^+}{x_1^-}\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}, \quad (38)$$

$$B_i = \frac{x_i^-}{x_i^+}, \quad (39)$$

where $S_2 = 1$ was chosen as in [26] to normalize the S matrix to the gauge-theory asymptotic S matrix.

C. The full $su(2|2) \times su(2|2)$ S matrix

The full $su(2|2) \times su(2|2)$ S matrix is uniquely fixed by imposing $su(2|2)$ symmetry in each factor. The two algebras are linked by sharing the same $\mathfrak{H}$ operator. Therefore, one is led to use the same parameters $x^\pm$ (or a, b, c, d) for both factors. One thus has essentially

$$S(p_1, p_2) = S_0(p_1, p_2) \cdot [\hat{S}_{su(2|2)}(p_1, p_2) \otimes \hat{S}_{su(2|2)}(p_1, p_2)]. \quad (40)$$

However, care has to be taken to implement correctly all anticommutation relations which necessitates the introduction of various factors of $(-1)^F$—therefore, the tensor product notation used above has to be understood somewhat symbolically. Let us note that one may also use a different explicit algebra for one of the $su(2|2)$ factors which would also eliminate length-changing processes i.e. $su(2|1)$ instead of $su(1|2)$. We checked that this does not modify the crossing properties derived in the following section apart from a trivial "braiding factor" so we will not consider this possibility further.

IV. THE ANTIPODE AND CROSSING PROPERTIES

In this section we will implement the formulation of the crossing property of the R matrix using the Hopf-algebraic conditions

$$(\mathcal{S} \otimes \mathrm{id})R = R^{-1}, \qquad (\mathrm{id} \otimes \mathcal{S}^{-1})R = R^{-1}, \quad (41)$$

where $\mathcal{S}$ is the antipode. It is an antihomomorphism of the Hopf (super) algebra i.e.

$$\mathcal{S}(AB) = (-1)^{d(A)d(B)}\mathcal{S}(B)\mathcal{S}(A). \quad (42)$$

The above equations descend to equations in specific representations $V_1 \otimes V_2$ through the introduction of a charge conjugation matrix $\mathcal{C}$:

$$\pi(\mathcal{S}(A)) = \mathcal{C}^{-1}\bar{\pi}(A)^{\mathrm{st}}\mathcal{C}, \quad (43)$$

where $\bar{\pi}$ is the representation for the antiparticles which can be distinct from π, and st stands for the supertranspose defined as

$$M_{ij}^{\mathrm{st}} = (-1)^{d(i)d(j)+d(j)}M_{ji}. \quad (44)$$

In order to apply the above framework to the case of the $su(2|2)$ (and consequently the full $su(2|2) \times su(2|2)$) S matrix we encounter some difficulties. First we do not have a complete description of the Hopf algebra of (nonlocal) symmetries of the S matrix. There are strong indications that it is not in fact a Yangian. However, we will assume that the explicit $su(1|2)$ algebra is part of the full Hopf algebra, and implementing (43) for $A \in su(1|2)$ will determine $\mathcal{C}$ and partially the representation $\bar{\pi}$. Then we will find that in order for (41) to have any chance of having a solution will uniquely determine the remaining freedom in $\bar{\pi}$ and give equations for the scalar factor S_0.

For algebra elements belonging to a Lie superalgebra the antipode acts very simply as $\mathcal{S}(A) = -A$. Then the Eq. (43) takes the form

$$\mathcal{C} \cdot \pi(A) + \bar{\pi}(A)^{\mathrm{st}}\mathcal{C} = 0. \quad (45)$$

Let us assume that the representation π is defined through the parameters a_1, b_1, c_1, and d_1. Inserting the generators of $su(1|2)$ into (45), we obtain one constraint on the representation $\bar{\pi}$

$$\bar{c} = -\frac{1 + b_1c_1}{\bar{b}}. \quad (46)$$

The charge conjugation matrix $\mathcal{C}$ is then seen to act as follows on the basis states

$$\mathcal{C}|\phi\rangle = \frac{a_1b_1}{\bar{a}\,\bar{b}}|\chi\rangle, \qquad \mathcal{C}|\chi\rangle = |\phi\rangle, \quad (47)$$

$$\mathcal{C}|\psi^1\rangle = -\frac{b_1}{\bar{a}}|\psi^2\rangle, \qquad \mathcal{C}|\psi^2\rangle = \frac{b_1}{\bar{a}}|\psi^1\rangle, \quad (48)$$

where an overall factor is arbitrary but in any case it cancels out from all subsequent equations. The Eq. (45) does not lead to any constraints on $\bar{b}$ (as $\bar{a}$ is again just an arbitrary normalization). We believe that if we would know the full Hopf algebra and therefore the action of the antipode on the nonlocal generators, we could also determine $\bar{b}$

directly from (43). We will determine it, however, using other reasoning.

Let us consider the first equation in (41) rewritten using (43) and denoting the R matrix with the proper scalar factor by $R_{\rm final}$:

$$(\mathcal{C}^{-1}\otimes {\rm id})R_{\rm final}(\bar{1},2)^{{\rm st}_1}(\mathcal{C}\otimes {\rm id})R_{\rm final}(1,2)={\rm id}, \quad (49)$$

where the superscript st_1 denotes the supertranspose in the first entry of $R_{\rm final}(\bar{1},2)$ defined explicitly as

$$(R^{{\rm st}_1})^{b_1b_2}_{a_1a_2}=(-1)^{d(a_1)d(b_1)+d(a_1)}R^{a_1b_2}_{b_1a_2}. \quad (50)$$

Typically, given a solution of the Yang-Baxter equation $R(1,2)$ which is invariant under all relevant symmetries fixes $R_{\rm final}(1,2)$ up to scalar multiplication by a function $S_0(1,2)$:

$$R_{\rm final}(1,2)\equiv S_0(1,2)R(1,2). \quad (51)$$

In our case $R(1,2)$ is the solution (23). Inserting (51) into (49) we find that in order for the S matrix to be crossing symmetric the expression

$$(\mathcal{C}^{-1}\otimes {\rm id})R(\bar{1},2)^{{\rm st}_1}(\mathcal{C}\otimes {\rm id})R(1,2) \quad (52)$$

has to be a multiple of the identity. This is a very nontrivial equation which *a priori* does not need to hold at all.

In order to analyze it in detail we first determine how do the individual representations transform under crossing:

$$(\mathcal{C}^{-1}\otimes {\rm id}){\rm proj}_i(\bar{1},2)^{{\rm st}_1}(\mathcal{C}\otimes {\rm id})=M_{ik}{\rm proj}_k(1,2). \quad (53)$$

It turns out that the matrix M_{ik} is quite nontrivial and does not have any vanishing entries (see the explicit formulas in Appendix B). Now the requirement that (52) equals[2] $1/f(1,2)\cdot {\rm id}$ is equivalent to the system of three scalar equations:

$$S_i(\bar{1},2)M_{i1}S_1(1,2)=1/f(1,2), \quad (54)$$

$$S_i(\bar{1},2)M_{i2}S_2(1,2)=1/f(1,2), \quad (55)$$

$$S_i(\bar{1},2)M_{i3}S_3(1,2)=1/f(1,2). \quad (56)$$

Equating the left-hand sides of the first two equations gives two solutions for $\bar{b}$. Equating subsequently the left-hand side of the third equation picks a unique choice for $\bar{b}$:

$$\bar{b}=\frac{a_2b_1b_2(a_1c_2(1+b_2c_2)-a_2c_1(1+b_1c_1))}{\bar{a}c_2(a_1b_1-a_2b_2)(1+b_2c_2)}. \quad (57)$$

Now this expression should be a function only of a_1, b_1, c_1 and should *not* depend on the second particle. Quite remarkably, one can show using (30) that the dependence on the second particle cancels out and $\bar{b}$ becomes

$$\bar{b}=\frac{-\alpha a_1b_1}{\bar{a}(\alpha+a_1b_1)}, \quad (58)$$

which fixes uniquely (up to a trivial rescaling of $\bar{a}$ which we will set to 1) the representation $\bar{\pi}$. We believe that the fact that such a solution depending only on a_1, b_1, c_1 exists at all is a very strong indication that crossing symmetry should hold for the $\mathrm{AdS}_5\times S^5$ worldsheet theory.

A. The crossing transformation

Before we present the result for the scalar factor $f(1,2)$, let us examine more closely the interpretation of the particle-antiparticle transformation (46) and (58). Expressing the transformation of the $x^\pm$ induced by these formulas we get the very simple result

$$x^+\longrightarrow\frac{\alpha\beta}{x^+}, \qquad x^-\longrightarrow\frac{\alpha\beta}{x^-}. \quad (59)$$

Using $e^{ip}=x^+/x^-$ we see that the momentum changes sign. The same also holds for the energy. This is reassuring since the analogous transformation for a relativistic theory $\theta\longrightarrow\theta+i\pi$ also reverses the signs of both the momentum and energy. Let us note, however, that we did not assume the form of transformation (59) but obtained it purely algebraically.

We may now obtain the function $f(1,2)$ from any of the Eqs. (54)–(56). Again the expression in terms of the a_i, b_i, and c_i is quite complicated but it simplifies considerably when expressed in terms of the $x_i^\pm$ variables:

$$f(1,2)=\frac{(\frac{\alpha\beta}{x_1^+}-x_2^-)(x_1^+-x_2^+)}{(\frac{\alpha\beta}{x_1^-}-x_2^-)(x_1^--x_2^+)}. \quad (60)$$

Let us first note that the above function is not a constant, so a nontrivial scalar factor is needed in order to form a crossing symmetric S matrix. If we would be interested only in a $su(2|2)$ symmetric S matrix we would thus be led to the equation

$$S_0(\bar{1},2)S_0(1,2)=f(1,2). \quad (61)$$

On the other hand, if we consider the case of our main interest i.e. $su(2|2)\times su(2|2)$ symmetry the relevant equation would be

$$S_0(\bar{1},2)S_0(1,2)=f(1,2)^2. \quad (62)$$

We have verified the above by explicitly constructing the $su(2|2)\times su(2|2)$ symmetric S matrix taking into account various $(-1)^F$ factors and also the various signs appearing in the supertranspose. We found that the relevant "crossing" function is indeed just the square of $f(1,2)$.

The above Eq. (62) has to be supplemented by the unitarity equation

$$S_0(1,2)S_0(2,1)=1 \quad (63)$$

and an analogous crossing relation for the second particle

[2] We write the scalar function in this form for later convenience.

(i.e. $2 \to \bar{2}$) which is obtained using

$$(\mathrm{id} \otimes \mathcal{C}^{-1}) R_{\mathrm{final}}(1, \bar{2})^{\mathrm{st}_2} (\mathrm{id} \otimes \mathcal{C}) R_{\mathrm{final}}(1, 2) = \mathrm{id}, \tag{64}$$

where the superscript st_2 is defined as

$$(R^{\mathrm{st}_2})^{b_1 b_2}_{a_1 a_2} = (-1)^{d(a_2)d(b_2)+d(b_2)} R^{b_1 a_2}_{a_1 b_2}. \tag{65}$$

The result is

$$S_0(1, \bar{2}) S_0(1, 2) = f(1, 2)^2. \tag{66}$$

Let us note one slightly troubling property of (62). The right-hand side is *not* symmetric under the interchange (59), while the left-hand side apparently is symmetric. This would lead to an apparent contradiction. In order to avoid that conclusion we have to keep in mind that $x^\pm$ are not independent variables but are linked through the constraint (32). So such an expression as (60) has cuts. As advocated in Sec. II, in order to deal with meromorphic functions we would have to pass to the universal covering space of (32) or Eqs. (17) and (30) and only there one would have to consider the Eqs. (62). On this covering space the transformation $1 \to \bar{1}$ would no longer necessarily be an involution.

Such behavior in fact holds for the conventional case of the rapidity parametrization in relativistic QFT. There the $1 \to \bar{1}$ transformation corresponds to $\theta \to \theta + i\pi$ which does not square to the identity [while the original transformation $(E, p) \to (-E, -p)$ on the curve $E^2 - p^2 = m^2$ is an involution]. Also, for such generic Hopf algebras involving nonlocal symmetry charges like Yangians, the square of the antipode is not equal to the identity $\mathcal{S}^2 \neq \mathrm{id}$. Therefore, we expect the universal cover to be the natural algebraic scene for considering the S matrices. Of course we have to resort to such algebraic arguments since we lack a deeper physical understanding of the structure of the worldsheet theory.

In the next section we will explicitly construct the universal covering space of the parameter space given by (17) and (30), and in Sec. VI we will write the crossing and unitarity equations directly on that generalized "rapidity plane."

V. THE GENERALIZED RAPIDITY PLANE

In this section we would like to introduce analogues of the rapidity variable θ in relativistic quantum field theory. Let us recall that, as described in Sec. II, the rapidity parameter space can be understood to be the universal cover (with the caveat of an additional mapping from the plane as mentioned above) of the (E, p) variables subject to the constraint of the relativistic mass-shell condition $E^2 - p^2 = m^2$. Intuitively this means that the rapidity variable θ allows to get rid of the cuts and to deal with purely meromorphic functions.

The $su(2|2)$ S matrix is expressed in terms of the complex parameters a, b, c, d of the $\mathfrak{Q}^\alpha$ and $\mathfrak{S}_\alpha$ operators subject to two constraints[3]:

$$ad - bc = 1, \tag{67}$$

$$abcd + \beta ab + \alpha cd = 0, \tag{68}$$

where the first constraint is necessary for $sl(1|2)$ symmetry, while the second quartic constant follows from the structure of the central charges in order to have $su(2|2)$ symmetry at zero total momentum. The a parameter can be absorbed in the relative normalization of the states and so we will set it to 1. Consequently, one can express d as $d = 1 + bc$ and the remaining *nonlinear* constraint is

$$(bc^2 + c)(b + \alpha) + \beta b = 0. \tag{69}$$

The above algebraic curve has degree 4, and so if it would be nonsingular it would have genus $g = (4 - 1) \times (4 - 2)/2 = 3$. It turns out, however, that it contains singularities: a double point and a cusp which brings down the genus to 1. This means that it can be uniformized by elliptic functions defined on the complex plane, which is the universal cover of (69).

We will now find the explicit form of these mappings. First let us express c as

$$c = \frac{-1 + v}{2b}, \tag{70}$$

where v satisfies the equation

$$v^2 = 1 - 4\frac{\beta b^2}{b + \alpha}. \tag{71}$$

Multiplying both sides by $\beta^2(b + \alpha)^2$, and introducing $y \equiv \beta(b + \alpha)v$, we obtain

$$y^2 = -4\beta^3 b^3 + (1 - 4\alpha\beta)\beta^2 b^2 + 2\alpha\beta^2 b + \alpha^2\beta^2. \tag{72}$$

Now a final coordinate transformation $b = -(x + (4\alpha\beta - 1)/12)/\beta$ reduces this elliptic curve to the standard Weierstrass form

$$y^2 = 4x^3 - g_2 x - g_3, \tag{73}$$

where

$$g_2 = \tfrac{1}{12}(1 + 16\alpha\beta + 16\alpha^2\beta^2), \tag{74}$$

$$g_3 = \tfrac{1}{216}(1 + 8\alpha\beta)(-1 - 16\alpha\beta + 8\alpha^2\beta^2). \tag{75}$$

The resulting parametrization is $y = \mathcal{P}'(z)$ and $x = \mathcal{P}(z)$. So finally

$$b(z) = -\frac{1}{\beta}\left(\mathcal{P}(z) + \frac{4\alpha\beta - 1}{12}\right), \tag{76}$$

$$c(z) = \frac{\mathcal{P}'(z) - \beta b(z) - \alpha\beta}{2\beta b(z)(b(z) + \alpha)}. \tag{77}$$

Using formulas (33) we may obtain the functions[4] $x^+(z)$ and $x^-(z)$. Before we proceed let us describe in more detail

[3] A similar analysis directly in terms of the x^+ and x^- parameters will be performed in Appendix A.

[4] An alternate but essentially equivalent parametrization is derived in Appendix A.

the elliptic curve (73). It depends explicitly on the gauge-theory coupling-constant $g^2 = 2\alpha\beta$. For nonzero (real) coupling it is nonsingular; its discriminant is

$$\Delta = g_2^3 - 27g_3^2 = \alpha^4\beta^4(1 + 16\alpha\beta). \tag{78}$$

It has two half-periods ω_1 and ω_2 which, together with $\omega_3 \equiv -\omega_1 - \omega_2$ get mapped by the Weierstrass function $\mathcal{P}(z)$ to the zeroes e_1, e_2, e_3 of the polynomial

$$4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3). \tag{79}$$

Using the explicit forms of g_2 and g_3 we find that one of the zeroes has the following simple form:

$$e_1 = \frac{1 + 8\alpha\beta}{12}. \tag{80}$$

We will denote the corresponding half-period by ω_1 i.e.

$$\mathcal{P}(\omega_1) = e_1. \tag{81}$$

We also have another identity which will be useful later,

$$(e_1 - e_2)(e_1 - e_3) = \alpha^2\beta^2. \tag{82}$$

A. Crossing transformations in the z plane

Now we have to see how the particle-antiparticle transformation

$$x^{\pm} \to \frac{\alpha\beta}{x^{\pm}} \tag{83}$$

is represented on the generalized rapidity z plane. It turns out that this transformation has a very simple representation similar to the transformation $\theta \to \theta + i\pi$ in the relativistic case.

Using the addition laws[5] for $\mathcal{P}(z)$ and $\mathcal{P}'(z)$

$$\mathcal{P}(z + \omega_1) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\mathcal{P}(z) - e_1}, \tag{84}$$

and

$$\mathcal{P}'(z + \omega_1) = -(e_1 - e_2)(e_1 - e_3)\frac{\mathcal{P}'(z)}{(\mathcal{P}(z) - e_1)^2}, \tag{85}$$

after straightforward but slightly tedious calculations we obtain

$$b(z + \omega_1) = \bar{b}(z), \qquad c(z + \omega_1) = \bar{c}(z). \tag{86}$$

Consequently we have that

$$x^{\pm}(z + \omega_1) = \frac{\alpha\beta}{x^{\pm}(z)}. \tag{87}$$

Therefore, the $1 \to \bar{1}$ transformation on the universal cover is represented by a translation by the half-period ω_1 defined through (81):

$$z \to z + \omega_1. \tag{88}$$

There is another natural transformation $x^+ \to -x^-$, $x^- \to -x^+$ which interchanges the sign of momentum while keeping the energy unchanged. This is represented also very simply by $z \to -z$.

[5] See e.g. Sec. 20.33 of [36].

VI. CROSSING EQUATIONS FOR THE $su(2|2) \times su(2|2)$ S MATRIX

We are now ready to write the final form of the functional equations for the scalar factor of the $su(2|2) \times su(2|2)$ invariant S matrix. We consider it to be defined as a meromorphic function on two copies of the complex plane which represent the "generalized rapidities" of the two particles. It has to satisfy the unitarity equation

$$S_0(z_1, z_2)S_0(z_2, z_1) = 1 \tag{89}$$

and crossing with respect to the first and second particle

$$S_0(z_1 + \omega_1, z_2)S_0(z_1, z_2) = f(z_1, z_2)^2, \tag{90}$$

$$S_0(z_1, z_2 - \omega_1)S_0(z_1, z_2) = f(z_1, z_2)^2. \tag{91}$$

In the above we used the fact that performing crossing with respect to the second particle leads to the same scalar function $f(1, 2)$. The transformation $z_2 \to z_2 - \omega_1$ arises from the Hopf-algebraic crossing equation for the second particle [see (41)] which involves the *inverse* of the antipode $\mathcal{S}^{-1}$. Let us note that in order for the two crossing conditions (90) and (91) to be consistent with unitarity (89), the function $f(z_1, z_2)$ has to satisfy a nontrivial consistency relation

$$f(z_1 - \omega_1, z_2) = \frac{1}{f(z_2, z_1)}. \tag{92}$$

We find that indeed (60) satisfies the above condition. We believe that this is another argument for the relevance of such a crossing condition, formulated on the universal cover, to the $AdS_5 \times S^5$ superstring worldsheet S matrix. In fact if we would have chosen the translation $z_2 \to z_2 + \omega_1$ the resulting equation would be $f(z_1, z_2) = 1/f(z_2, z_1)$ which does *not* hold.

The natural question is now to determine a minimal solution $S_0(z_1, z_2)$ and the corresponding form of CDD factors. The equations are, however, quite complicated and the standard iterative technique for solving the coupled crossing and unitarity relations (see e.g. [35,37]) does not work here. We postpone the study of this issue to a separate publication [38].

VII. CONCLUSIONS

In this paper we have proposed how to implement crossing relations for the $su(2|2) \times su(2|2)$ symmetric S matrix relevant for the $AdS_5 \times S^5$ superstring worldsheet theory. Once constructed, these relations provide functional equations for the overall scalar factor (the so-called "dressing factor") of the S matrix.

Our proposal involves two basic steps. First, a Hopf-algebraic reformulation of the relativistic crossing relation allows us to address the problem in a purely algebraic manner. The lack of knowledge of the full Hopf algebra structure of the nonlocal symmetries allows us to determine only a part of the relations directly from properties of the antipode; however, using the structure of the full S matrix allowed us to fix uniquely the remaining ambiguity in the crossing transformation. We found that the original S matrix (normalized to the asymptotic gauge-theory result) transforms nontrivially under crossing thus necessitating a nonconstant scalar "dressing factor."

In a second step, in order to eliminate cuts and to deal only with meromorphic functions, we proposed to introduce the "generalized rapidity plane" which is a universal covering space of the space of parameters appearing in the S matrix. This space is constructed through a coupling-constant dependent elliptic curve. On this space the crossing transformation acts very simply as a translation by a specific half-period.

Finally, we derive functional equations for the scalar "dressing factor" on the universal covering space. We propose to investigate its solutions in a forthcoming paper [38]. Apart from that, there are numerous interesting directions for further study. It would be interesting to understand more directly the geometric structure of the parameter space and perhaps link it more directly to the properties of the worldsheet theory. Another more mathematical question would be to try to find the whole structure of the Hopf algebra relevant in this case and, in particular, to understand more intrinsically the mathematical origin of the quartic constraint (30). Finally, it would be also very interesting to make contact with near-BMN quantization of the $\mathrm{AdS}_5 \times S^5$ superstring (like the very recent work [39]) especially if formulated in an analogous explicit $su(1|2)$ picture.

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APPENDIX A: UNIFORMIZATION USING $x^{\pm}$ VARIABLES

It is interesting to derive directly the uniformization of the parameters $x^{\pm}$ satisfying the defining equations

$$x^+ + \frac{\alpha\beta}{x^+} - x^- - \frac{\alpha\beta}{x^-} = i. \tag{A1}$$

In particular we should obtain the same elliptic curve as in Sec. V. Equation (A1) can be rewritten as

$$x^+ - x^- = \frac{ix^+x^-}{x^+x^- - \alpha\beta}. \tag{A2}$$

Denoting $w = -x^+x^-$ and $x^+ + x^- = iy/(w + \alpha\beta)$ we may derive the equation linking y and w:

$$y^2 - w^2 = 4w(w + \alpha\beta)^2. \tag{A3}$$

Performing a final substitution $w = x - (1 + 8\alpha\beta)/12$ we obtain finally the curve in Weierstrass form

$$y^2 = 4x^3 - g_2x - g_3, \tag{A4}$$

which exactly coincides with (73). Putting all the above together we obtain finally a parametrization for $x^{\pm}$:

$$x^+(z) = \frac{i}{2}\frac{\mathcal{P}'(z) + w(z)}{w(z) + \alpha\beta}, \tag{A5}$$

$$x^-(z) = \frac{i}{2}\frac{\mathcal{P}'(z) - w(z)}{w(z) + \alpha\beta}, \tag{A6}$$

where

$$w(z) = \mathcal{P}(z) - \frac{1 + 8\alpha\beta}{12}. \tag{A7}$$

Let us note the amusing fact that if we compare the above parametrization and the similar one derived in Sec. V, the $x^-(z)$ functions coincide, while $x^+(z)$ here corresponds to $\alpha\beta/x^+(z)$ there. In fact this can be implemented by a linear transformation of the z variable[6]:

$$x^{\pm}_{\text{Sec. 5}}(z) = x^{\pm}_{\text{here}}\left(\frac{\omega_1}{2} + \omega_2 - z\right), \tag{A8}$$

showing that the construction presented here and the one in Sec. V are essentially equivalent.

APPENDIX B: CROSSING PROPERTIES OF THE ELEMENTARY PROJECTORS

In this appendix, for completeness we write the elements of the matrix M_{ij} encoding the transformation properties of the projectors proj_i under crossing [see Eq. (53)]:

$$M_{11} = \frac{b_1c_1(1 + b_1c_1)}{(-1 + b_1c_1 - b_2c_2)(b_1c_1 - b_2c_2)}, \tag{B1}$$

$$M_{12} = -\frac{b_1c_1(1 + b_2c_2)}{(-1 + b_1c_1 - b_2c_2)(b_1c_1 - b_2c_2)}, \tag{B2}$$

$$M_{13} = \frac{b_2c_2(1 + b_2c_2)}{(-1 + b_1c_1 - b_2c_2)(b_1c_1 - b_2c_2)}, \tag{B3}$$

$$M_{21} = -\frac{2(1 + b_1c_1)(1 + b_2c_2)}{(-1 + b_1c_1 - b_2c_2)(1 + b_1c_1 - b_2c_2)}, \tag{B4}$$

$$M_{22} = \frac{b_1c_1(1 + b_1c_1) + b_2c_2(1 + b_2c_2)}{(-1 + b_1c_1 - b_2c_2)(1 + b_1c_1 - b_2c_2)}, \tag{B5}$$

[6] We checked this numerically without, however, carrying out an analytic proof.

$$M_{23} = -\frac{2b_1 b_2 c_1 c_2}{(-1 + b_1 c_1 - b_2 c_2)(1 + b_1 c_1 - b_2 c_2)}, \quad \text{(B6)}$$

$$M_{31} = \frac{b_2 c_2 (1 + b_2 c_2)}{(b_1 c_1 - b_2 c_2)(1 + b_1 c_1 - b_2 c_2)}, \quad \text{(B7)}$$

$$M_{32} = -\frac{b_2 c_2 (1 + b_1 c_1)}{(b_1 c_1 - b_2 c_2)(1 + b_1 c_1 - b_2 c_2)}, \quad \text{(B8)}$$

$$M_{33} = \frac{b_1 c_1 (1 + b_1 c_1)}{(b_1 c_1 - b_2 c_2)(1 + b_1 c_1 - b_2 c_2)}. \quad \text{(B9)}$$

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