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Towards the string dual of $\mathcal{N} = 1$ supersymmetric QCD-like theories

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We construct supergravity plus branes solutions, which we argue to be related to 4d $\mathcal{N} = 1$ SQCD with a quartic superpotential. The geometries depend on the ratio N_f/N_c which can be kept of order one, present a good singularity at the origin, and are weakly curved elsewhere. We support our field theory interpretation by studying a variety of features like R-symmetry breaking, instantons, Seiberg duality, Wilson loops and pair creation, running of couplings, and domain walls. In a second part of this paper, we address a different problem: the analysis of the interesting physics of different members of a family of supergravity solutions dual to (unflavored) $\mathcal{N} = 1$ SYM plus some UV completion.

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I. INTRODUCTION

The AdS/CFT conjecture originally proposed by Maldacena [1] and refined in [2,3] is one of the most powerful analytic tools for studying strong-coupling effects in gauge theories. There are many examples that go beyond the initially conjectured duality and first steps in generalizing it to nonconformal models were taken in [4]. Later, very interesting developments led to the construction of the gauge-string duality in phenomenologically more relevant theories i.e. minimally or nonsupersymmetric gauge theories [5].

Conceptually, the clearer setup for duals to less symmetric theories is obtained by breaking conformality and (partially) supersymmetry by deforming $\mathcal{N} = 4$ SYM with relevant operators or vacuum expectation values (VEV)'s. The models put forward in [5] and (even though there are some important technical differences) the Klebanov-Tseytlin and Klebanov-Strassler model(s) [6] belong to this class.

On the other hand, a different set of models, that are less conventional regarding the UV completion of the field theory have been developed. The idea here is to start from a set of Dp-branes (usually with $p > 3$), that wrap a q-dimensional compact manifold in a way such that two conditions are satisfied: first, one imposes that the low-energy description of the system is $(p - q)$-dimensional, that is, the size of the q-manifold is small and is not observable at low energies. Secondly, one also requires that a minimal amount of supersymmetry (SUSY) is preserved, in order to have technical control over the theory (for example, the resolution of the Einstein equations is eased). According to intuition, in this second class of phenomenologically interesting dualities, the UV completion of the (usually four-dimensional) field theory of interest is a higher-dimensional field (or string) theory. There are several models that belong to this latter class. In this paper we will concentrate on the model dual to $\mathcal{N} = 1$ SYM [7], that builds on a geometry originally found in 4d gauged supergravity in [8]. It must be noted that all of the models in this category (and also those in the class described in the previous paragraph [5,6]) are afflicted by the fact that they are not dual to the ''pure'' field theory of interest, but instead, the field theory degrees of freedom are entangled with the KK modes (on the q-manifold or with the modes in the latest steps in the cascade) in a way that depends on the energy scale of the field theory. The KK modes enter the theory at an energy scale which is inversely proportional to the size of the q-manifold and the main problem is that this size is comparable to the scale at which one wants to study nonperturbative phenomena such as confinement, spontaneous breaking of chiral symmetry, etc. Nevertheless, this limitation can be seen as an artifact of the supergravity approximation and will hopefully be avoided once the formulation of the string sigma model on these Ramond-Ramond (RR) backgrounds becomes available. Many articles have studied different aspects of these models. Instead of revisiting the main results here, we refer the interested reader to the review articles [9].

An important characteristic of the different models mentioned above, is that these supergravity backgrounds are conjectured to be dual to field theories with adjoint matter (also some quiver gauge theories are in this class). A problem that remains basically unsolved is how to deal with field theories containing matter fields transforming in the fundamental representation. It is a very natural problem to tackle if one is interested in making contact with phenomenological theories, like QCD.

There were some attempts to study this problem. To begin with, the ''quenched'' approximation (when the number of flavors is negligible compared to the number of colors, hence using a probe brane approximation in the dual picture) was studied in detail and in the different models, leading to a set of interesting results, regarding breaking of symmetries, meson spectrum, and interactions. This line was developed in detail; for a list of references,

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see the citations to the paper that initiated this approach [10].

The need to go beyond this quenched approximation motivated some authors to search for more complicated backgrounds including the effects of a large number of flavors [11]. These new backgrounds, that typically depend on two coordinates, are afflicted (in the case of four-dimensional field theories), by the usual problems caused by the addition of D7-branes: a codimension-two object with an associated conical singularity, a maximum number of D7-branes (or O_7 planes) that can be added in a compact space, etc. There is another line of research that used noncritical string theory [12,13] to construct duals to interesting field theories. The use of noncritical strings cures the problem of the existence of extra massive states (the KK modes pointed out above), but it typically relies on backgrounds which have large curvature, with the consequent problem that solutions need to be stringy-corrected (see [14] for some work in this direction). The results obtained are nevertheless indicating that this approach has some potential.

Perhaps being conservative, one could say that the results regarding the duality with flavored field theories using the backgrounds of [11,12], are not as spectacular and clear as the ones obtained with duals to field theories with adjoint fields only. All this indicates that a different approach is necessary, which might try to combine the successes of the previous ones and to avoid their inconveniences as much as possible.

In this paper we propose a way of finding a dual to $\mathcal{N} = 1$ SQCD-like theories using critical strings, focusing our attention here on type IIB string theory. The basic idea will be to add flavors to the background in [7], by using different technical points developed in [15] and in noncritical string approaches [12,16]. Below, we describe our approach and give a guide to read this paper.

A second part of the paper, namely, Section VIII, deals with backgrounds dual to different UV completions of minimally SUSY Super-Yang-Mills. For these geometries, unlike in the model [7], the dilaton does not diverge in the UV. We study relevant aspects of the dual gauge theory.

A. General idea

Let us describe here the procedure we will adopt to add a large number of flavors N_f to a given supergravity background, more particularly to the one constructed with wrapped D5-branes and conjectured to be dual to $\mathcal{N} = 1$ SYM plus massive adjoint matter [7]. Even though we will concentrate on this particular case, the procedure described below could be used to add many flavors in different backgrounds dual to $\mathcal{N} = 1$, $\mathcal{N} = 2$ SYM in 2 + 1 and 3 + 1 dimensions, etc.

We want to preserve SUSY in order to solve Bogomol'nyi-Prasad-Sommerfield (BPS) equations rather than Einstein equations; since the original background preserves four supercharges, we cannot break more SUSY in duals to four-dimensional field theories. It is then necessary to find a noncompact and holomorphic two-cycle (Σ_2), where we can place N_f D5-branes that share the 3 + 1 gauge theory dimensions with those N_c D5-branes that generated the original background [7]. These surfaces must be holomorphic to preserve SUSY, and noncompact so that the symmetry added by the flavor branes is global (in other words, the coupling of the effective four-dimensional field theory on the flavor branes $g_4^2 = g_6^2/\mathrm{Vol}(\Sigma_2)$, vanishes) [17].

The problem of finding noncompact SUSY preserving two-cycles in the geometry of [7] was solved in [15]. We will use a particular solution found there in order to place *many* D5-branes. This new stack of N_f branes is heavy, so it will backreact and the original background will be modified. We can think of an action describing the dynamics of the backreacted system that reads[1]

$$S = S_{\mathrm{IIB}} + S_{\mathrm{flavor}}. \tag{1.1}$$

Basically, we have added the open string sector in S_{flavor}. The procedure might be thought of as consisting of two steps; first we take the background [7], where the N_c color branes have been replaced by a flux, that we represent by S_{IIB} in (1.1). On top of this, we add a bunch of N_f flavor branes and find a new background solving new BPS (and Einstein) equations that encode the presence of the flavor branes. The part of the action (1.1) corresponding to the flavors will be the Dirac + WZ action of the N_f five-branes. In Einstein frame it reads

$$S_{\mathrm{flavor}} = T_5 \sum^{N_f} \Big(- \int_{\mathcal{M}_6} d^6 x e^{\phi/2} \sqrt{-\hat{g}_{(6)}} + \int_{\mathcal{M}_6} P[C_6] \Big), \tag{1.2}$$

where the integrals are taken over the six-dimensional worldvolume of the flavor branes $\mathcal{M}_6$ and $\hat{g}_{(6)}$ stands for the determinant of the pullback of the metric on such worldvolume. We will take these D5's extended along six coordinates that we call $x_0, x_1, x_2, x_3, r, \psi$ at constant θ, φ, $\tilde{\theta}$, $\tilde{\varphi}$. These supersymmetric embeddings were called *cylinder solutions* in [15].

The action S_{IIB} is ten-dimensional in contrast with S_{flavor}, that is six-dimensional since the "flavor branes" are localized in the directions θ, φ, $\tilde{\theta}$, $\tilde{\varphi}$. So, trying to find solutions to the equations of motion derived from (1.1), will involve writing Einstein equations with Dirac delta functions on the coordinates where the branes are localized. Thus, the solution will depend on $(\theta, \tilde{\theta}, \varphi, \tilde{\varphi})$ and in

[1] It is known that it is not possible to write a polynomial action for IIB supergravity due to the self-duality condition, we nevertheless will deal with solutions that have $F_5 = 0$, so, what we have in mind in this case is an action of the form $S_{\mathrm{IIB}} = \int d^{10} x \sqrt{g} (R - \frac{1}{2} (\partial \phi)^2 - e^{-\phi} \frac{F_3^2}{12})$.

this respect, this solution would be similar to those found in [11]. This is a very hard problem in principle and it is nice to notice that the (six-dimensional) noncritical string approach of [12] to $\mathcal{N} = 1$ flavored solutions is not afflicted by this technical difficulty, since S_{flavor} is basically a cosmological term.[2] We will circumvent this difficult technical obstacle by following a procedure first proposed in [16] in the context of flavor branes. In this paper, an eight-dimensional noncritical string action was used and an ingenious trick developed. Indeed, it was found in [16] that one could solve Einstein equations without Dirac delta functions if one *smears* the very many N_f branes in the extra directions (in our case θ, $\tilde{\theta}$, φ, $\tilde{\varphi}$).

Since N_f and N_c are very large numbers of the same order, interchanging the sum in (1.2) by an integral over the extra directions will produce a fully ten-dimensional action, erasing the explicit dependence on the extra coordinates. This gives rise to some new global symmetries. We will comment on the interpretation of this procedure from the dual field theory perspective in Section V A.

We proceed in the rest of the paper by first describing in detail the background in [7] and a singular version of it (that is easier to deal with and illustrates the technical details). We will then propose a new deformed background, where the fingerprint of the flavor branes will be the explicit violation of the Bianchi identity (a sort of Dirac-string like singularity). We will find BPS equations describing this situation and in Appendix A will develop a superpotential approach and compare with a purely IIB SUSY approach to this problem. The outcome being that the BPS equations obtained from the superpotential coincide with those coming from imposing vanishing of the variations of the gravitino and dilatino, $\delta\psi_\mu$, $\delta\lambda$, in ten dimensions.

Then, we apply a similar approach to the nonsingular case, obtain a set of BPS equations, find some interesting asymptotic solutions with numerical interpolation, and study their (strongly-coupled) dual gauge theory predictions.

The content of the paper is the following: in Section II, we present the dual to $\mathcal{N} = 1$ SYM plus adjoint matter that will be the arena on which we will construct flavored solutions. The presentation is detailed enough to make the paper self-contained and the reader familiar with these results may skip it. In Section III, we will provide a derivation of the flavored-BPS equations for the singular background, that solve the Einstein equations derived from (1.1). This derivation is complemented in Appendix A, using a superpotential approach. The presentation in this section will be very detailed, and it was written in order to describe and explain in a simpler context the procedure we will follow in the physically interesting non-Abelian case.

In Section IV we will write BPS equations for the flavoring of the nonsingular background and, as an interesting particular case, we will also deal with the case $N_f = 2N_c$ that presents unique features. Then, we study in detail the asymptotics of the solutions and provide a careful numerical treatment to the equations derived in this section. In Section V we first provide arguments to describe the field theory dual to our backgrounds, that indicate that (at low energies) we are dealing with $\mathcal{N} = 1$ SQCD plus a quartic superpotential for the quark superfield. Then, we will initiate the study of the field theory properties of these flavored solutions, most notably, we will analyze Wilson loops and SQCD-string breaking, instanton action, theta angle, beta function, $U(1)_R$ symmetry breaking, domain walls, and Seiberg duality.

In Section VI we will come back to the interesting case $N_f = 2N_c$. Here, we will analyze distinctive gauge theory features; most notably (non)-confinement (screening of quarks), $U(1)_R$ symmetry preservation and Seiberg duality, that becomes particularly interesting in this case. We will also comment on finite temperature aspects of this field theory.

In Section VII, a different approach to flavored backgrounds will be introduced. We argue that it might be possible to account for the violation of the Bianchi identity induced by the addition of the flavor branes, by turning on nontrivial fluxes which are not present in the unflavored solution of [7].

In the second part of the paper (as a byproduct of our results above), in Section VIII we will consider the particular case $N_f = 0$ (which in the following we will call "unflavored case") and find a one-parameter family of deformed (nonsingular) solutions that correspond to different UV completions of $\mathcal{N} = 1$ SYM. We present a gauge theory interpretation of this family of solutions and study many field theory aspects as seen from it; most notably, confinement, k-string tensions, PP-waves, rotating strings, and beta function, pointing out in each case the similarities and differences between different members of the family and the solution in [7].

Section IX is left for conclusions and possible future directions. With the aim of making the main text more readable, we wrote many appendixes, where we have relegated lots of technical details.

Reader's guide.—Given that this is a considerably long paper, but that different parts may be read almost independently, we feel it is useful to write a guide in order to help the reader find the results of her/his interest. Readers interested in the construction of the flavored backreacted solutions can concentrate on Sections III and IV supplemented with Appendices A and B and also read Section VII. Readers interested in the field theory features reproduced by the flavored solutions can go directly to

[2] But as we mentioned, the noncritical string approach is seriously afflicted by string corrections to the gravity approximation; nevertheless, the AdS solutions in [12] are likely to persist after these corrections.

Sections V and VI and just look at the explicit expressions for the backgrounds (Sections IV B, IV C, IV D, and IV E) when needed. Finally, who is interested in the deformed unflavored solutions can read, in a self-contained way, Sections IVA and VIII, supplemented with Appendices B and E.

II. THE DUAL TO $\mathcal{N} = 1$ SYM

We work with the model presented in [7] (the solution was first found in a 4d context in [8]). Let us briefly describe the main points of this supergravity dual to $\mathcal{N} = 1$ SYM and its UV completion. We start with N_c D5-branes. The field theory that lives on them is 6D SYM with 16 supercharges. Then, suppose that we wrap two directions of the D5-branes on a curved two-manifold that can be chosen to be a sphere. In order to preserve some fraction of SUSY, a twisting procedure has to be implemented [18] and actually, there are two ways of doing it. The one we will be interested in this paper deals with a twisting that preserves four supercharges. In this case the two-cycle mentioned above lives inside a Calabi-Yau 3-fold. The corresponding supergravity solution can be argued to be dual to a four-dimensional field theory only for low energies (small values of the radial coordinate). Indeed, at high energies, the modes of the gauge theory explore also the two-cycle and as the energy is increased further, the theory first becomes six-dimensional and then, blowing up of the dilaton forces one to S-dualize. Therefore the UV completion of the model is given by a little string theory.

The supergravity solution that interests us, preserves four supercharges and has the topology $\mathbb{R}^{1,3} \times \mathbb{R} \times S^2 \times S^3$. There is a fibration of the two spheres in such a way that $\mathcal{N} = 1$ supersymmetry is preserved. By going near $r = 0$ it can be seen that the topology is $\mathbb{R}^{1,6} \times S^3$. The full solution and Killing spinors are written in detail in [15]. Let us write some aspects for future reference and to set conventions. The metric in the Einstein frame reads,

$$ds_{10}^2 = \alpha' g_s N_c e^{\phi/2}\bigg[\frac{1}{\alpha' g_s N_c} dx_{1,3}^2 + dr^2 + e^{2h}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{4}(\tilde{\omega}_i - A^i)^2\bigg], \tag{2.1}$$

where ϕ is the dilaton. The angles $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$ parametrize a two-sphere. This sphere is fibered in the ten-dimensional metric by the one-forms A^i ($i = 1, 2, 3$). They are given in terms of a function $a(r)$ and the angles (θ, φ) as follows:

$$A^1 = -a(r)d\theta, \qquad A^2 = a(r)\sin\theta d\varphi, \qquad A^3 = -\cos\theta d\varphi. \tag{2.2}$$

The $\tilde{\omega}_i$ one-forms are defined as

$$\begin{aligned} \tilde{\omega}_1 &= \cos\psi d\tilde{\theta} + \sin\psi \sin\tilde{\theta} d\tilde{\varphi}, \\ \tilde{\omega}_2 &= -\sin\psi d\tilde{\theta} + \cos\psi \sin\tilde{\theta} d\tilde{\varphi}, \\ \tilde{\omega}_3 &= d\psi + \cos\tilde{\theta} d\tilde{\varphi}. \end{aligned} \tag{2.3}$$

The geometry in (2.1) preserves four supercharges and is nonsingular when the functions $a(r)$, $h(r)$ and the dilaton $\phi(r)$ are:

$$a(r) = \frac{2r}{\sinh 2r}, \qquad e^{2h} = r\coth 2r - \frac{r^2}{\sinh^2 2r} - \frac{1}{4}, \qquad e^{-2\phi} = e^{-2\phi_0}\frac{2e^h}{\sinh 2r}, \tag{2.4}$$

where ϕ_0 is the value of the dilaton at $r = 0$. The solution of type IIB supergravity includes a RR threeform $F_{(3)}$ that is given by

$$\frac{1}{g_s\alpha' N_c}F_{(3)} = -\frac{1}{4}(\tilde{\omega}_1 - A^1)\wedge(\tilde{\omega}_2 - A^2)\wedge(\tilde{\omega}_3 - A^3) + \frac{1}{4}\sum_a F^a \wedge (\tilde{\omega}_a - A^a), \tag{2.5}$$

where F^a is the field strength of the $SU(2)$ gauge field A^a, defined as:

$$F^a = dA^a + \tfrac{1}{2}\epsilon_{abc}A^b \wedge A^c. \tag{2.6}$$

The different components of F^a read

$$F^1 = -a' dr \wedge d\theta, \qquad F^2 = a' \sin\theta dr \wedge d\varphi, \qquad F^3 = (1 - a^2)\sin\theta d\theta \wedge d\varphi, \tag{2.7}$$

where the prime denotes derivative with respect to r. Since $dF_{(3)} = 0$, one can represent $F_{(3)}$ in terms of a twoform potential $C_{(2)}$ as $F_{(3)} = dC_{(2)}$. Actually, it is not difficult to verify that $C_{(2)}$ can be taken as:

$$\begin{aligned} \frac{C_{(2)}}{g_s\alpha' N_c} = \frac{1}{4}[&\psi(\sin\theta d\theta \wedge d\varphi - \sin\tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi}) \\ &- \cos\theta\cos\tilde{\theta} d\varphi \wedge d\tilde{\varphi} \\ &- a(d\theta \wedge \tilde{\omega}_1 - \sin\theta d\varphi \wedge \tilde{\omega}_2)]. \end{aligned} \tag{2.8}$$

The equation of motion of $F_{(3)}$ in the Einstein frame is $d(e^{\phi *}F_{(3)}) = 0$, where $*$ denotes Hodge duality. Let us stress that the configuration presented above is nonsingular.

For future reference, let us quote here the asymptotic expansions of the functions $a(r)$, $e^{2h(r)}$, $e^{2\phi(r)}$ near $r = 0$,

$$a(r) \sim 1 - \frac{2}{3}r^2 + \ldots, \qquad e^{2h} \sim r^2 + \ldots, \qquad e^{2\phi} \sim e^{2\phi_0}(1 + \ldots), \tag{2.9}$$

and for large values of the radial coordinate,

$$a(r) \sim 4re^{-2r} + \ldots, \qquad e^{2h} \sim r + \ldots, \qquad e^{2\phi} \sim e^{2\phi_0} \frac{e^{2r}}{4\sqrt{r}}. \tag{2.10}$$

The BPS equations that the configuration in (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8) solves (for a complete derivation from the spinor transformations of IIB sugra, see Appendix A in [15]) read

$$\begin{aligned} \phi' &= \frac{1}{Q}\left[e^{2h} - \frac{e^{-2h}}{16}(a^2-1)^2\right], \\ h' &= \frac{1}{2Q}\left[a^2 + 1 + \frac{e^{-2h}}{4}(a^2-1)^2\right], \\ a' &= -\frac{2a}{Q}\left[e^{2h} + \frac{1}{4}(a^2-1)\right], \end{aligned} \tag{2.11}$$

with

$$Q \equiv \sqrt{e^{4h} + \tfrac{1}{2}e^{2h}(a^2+1) + \tfrac{1}{16}(a^2-1)^2}. \tag{2.12}$$

The Einstein equations read

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \tfrac{1}{2}(\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi) + \tfrac{1}{12}e^{\phi}(3F_{\mu\lambda_2\lambda_3}F_\nu^{\lambda_2\lambda_3} - \tfrac{1}{2}g_{\mu\nu}F_{(3)}^2). \tag{2.13}$$

The Maxwell equation as quoted above reads $d(e^{\phi*}F_{(3)}) = 0$ and the Bianchi identity is $dF_3 = 0$. Both of them are solved by (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8).

Finally, in the next section, as a warm up example, we will add many flavor branes to a particular singular solution of the system (2.11), which is characterized by $a(r) = 0$, $e^{2h} = r$, $e^{2\phi - 2\phi_0} = \frac{e^{2r}}{4\sqrt{r}}$,

$$\begin{aligned} ds_{10}^2 = \alpha' g_s N_c e^{\phi/2}\Big[&\frac{1}{\alpha' g_s N_c}dx_{1,3}^2 + dr^2 \\ &+ e^{2h}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{4}(\tilde{\omega}_1^2 + \tilde{\omega}_2^2) \\ &+ \frac{1}{4}(\tilde{\omega}^3 + \cos\theta d\varphi)^2\Big], \end{aligned} \tag{2.14}$$

and a RR potential

$$\frac{C_{(2)}}{g_s\alpha' N_c} = \frac{1}{4}[\psi(\sin\theta d\theta \wedge d\varphi - \sin\tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi}) - \cos\theta\cos\tilde{\theta} d\varphi \wedge d\tilde{\varphi}]. \tag{2.15}$$

Notice that at $r = 0$ this background presents a (bad) singularity that is solved by the turning on of the function $a(r)$ in the full solution (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8). On the field theory side, this way of resolving the singularity boils into the phenomena of confinement and breaking of the R-symmetry.

Since they will be useful in the remainder of this paper, let us summarize some aspects of the dual field theory.

A. Dual field theory

In [7] the solution presented in (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8) was argued to be dual to $\mathcal{N} = 1$ SYM plus some KK massive adjoint matter. The 4D field theory is obtained by reduction of N_c D5-branes on S^2 with a twist that we explain below. Therefore, as the energy scale of the 4D field theory becomes comparable to the inverse volume of S^2, the KK modes begin to enter the spectrum.

To analyze the spectrum in more detail, we briefly review the twisting procedure. In order to have a supersymmetric theory on a curved manifold like the S^2 here, one needs globally defined spinors. In our case the argument goes as follows. As D5-branes wrap the two-sphere, the Lorentz symmetry along the branes decomposes as $SO(1,3) \times SO(2)$. There is also an $SU(2)_L \times SU(2)_R$ symmetry that rotates the transverse coordinates and corresponds to the R-symmetry of the supercharges of the field theory on the D5-branes. One can properly define $\mathcal{N} = 1$ supersymmetry on the curved space that is obtained by wrapping the D5-branes on the two-cycle, by identifying a $U(1)$ subgroup of either $SU(2)_L$ or $SU(2)_R$ R-symmetry with the $SO(2)$ of the two-sphere. To fix notations, let us choose the $U(1)$ in $SU(2)_L$. Having done the identification with $SO(2)$ of the sphere, we denote this twisted $U(1)$ as $U(1)_T$.

After this twisting procedure is performed, the fields in the theory are labeled by the quantum numbers of $SO(1,3) \times U(1)_T \times SU(2)_R$. The bosonic fields are (the a indicates an adjoint index)

$$A_\mu^a = (4, 0, 1), \qquad \Phi^a = (1, \pm, 1), \qquad \xi^a = (1, \pm, 2). \tag{2.16}$$

Respectively they are the gluon, two massive scalars that are coming from the reduction of the original 6D gauge field on S^2 (explicitly from the A_φ and A_θ components), and finally four other massive scalars (that originally represented the positions of the D5-branes in the transverse $\mathbb{R}^4$). As a general rule, all the fields that transform under $U(1)_T$, the second entry in the above charge designation, are massive. For the fermions one has,

$$\lambda^a = (2, 0, 1), (\bar{2}, 0, 1), \qquad \Psi^a = (2, ++, 1), (\bar{2}, --, 1), \qquad \psi^a = (2, +, 2), (\bar{2}, -, 2). \tag{2.17}$$

These fields are the gluino plus some massive fermions whose $U(1)_T$ quantum number is nonzero. The KK modes in the 4D theory are obtained by the harmonic decomposition of the massive modes, Φ, ξ, Ψ, and ψ that are shown above. Their mass is of the order of $M_{KK}^2 = (\mathrm{Vol}_{S^2})^{-1} \propto \frac{1}{g_s\alpha' N_c}$. A very important point to notice here is that these KK modes are charged under $U(1)_T \times U(1)_R$ where the second $U(1)$ is a subgroup of the $SU(2)_R$ that is left untouched in the twisting procedure. On the other hand, the gluonic gauge field and the gluino are not charged under either of the $U(1)$'s.

The dynamics of these KK modes mixes with the dynamics of confinement in this model because the strong-coupling scale of the theory is of the order of the KK mass. One way to evade the mixing problem would be to work instead with the full string solution, namely, the world sheet sigma model on this background (or in the S-dual NS5 background) which would give us control over the duality to all orders in α', hence we would be able to decouple the dynamics of KK modes from the gauge dynamics. This direction is unfortunately not (yet) available. Meanwhile, in [19] a procedure was developed to determine when a field theory observable computed from the supergravity solution is affected by the presence of the massive KK modes or is purely an effect of the massless fields A_μ, λ.

In order to get a better intuition of the dynamics, one can schematically write a Lagrangian for these fields as follows:

$$\begin{aligned} L = -\mathrm{Tr}[&\tfrac{1}{4}F_{\mu\nu}^2 + i\lambda D\lambda - (D_\mu \Phi_i)^2 - (D_\mu \xi_k)^2 \\ &+ \Psi(iD - M_{KK})\Psi + M_{KK}^2(\xi_k^2 + \Phi_i^2) \\ &+ V[\xi, \Phi, \Psi]]. \end{aligned} \tag{2.18}$$

The potential typically contains the scalar potential for the bosons, Yukawa type interactions, and more. This expression is schematic because of (at least) two reasons. First of all, the potential presumably contains very complicated interactions involving the KK and massless fields. Secondly, there is mixing between the infinite tower of spherical harmonics that are obtained by reduction on S^2 and S^3, see [20,21] for a careful treatment.

III. ADDING FLAVOR BRANES TO THE SINGULAR SOLUTION

In this section we will add flavors to the particular solution (2.14) and (2.15). Even though there is little physical significance for this solution, the point of this section is to illustrate in detail the type of formalism we use and its remarkable consistency. Readers more interested in physically more relevant aspects of this paper should perhaps skip this section in a first reading, but since the formalism and technical subtleties are quite involved, we decided to spell them out explicitly here in a simpler context.

A. Deforming the space

The first step will be to deform the background (2.14) and (2.15). First, we will study the deformation of the backgrounds without the addition of flavors and then we will treat a deformation due to the presence of the N_f flavor branes. For this we propose a set of vielbeins given by,

$$\begin{aligned} &e^{xi} = e^f dx_i, \qquad e^r = e^f dr, \qquad e^\theta = e^{f+h} d\theta, \\ &e^\varphi = e^{f+h}\sin\theta d\varphi, \qquad e^1 = \frac{e^{f+g}\tilde{\omega}_1}{2}, \\ &e^2 = \frac{e^{f+g}\tilde{\omega}_2}{2}, \qquad e^3 = \frac{e^{f+k}(\tilde{\omega}_3 + \cos\theta d\varphi)}{2}, \end{aligned} \tag{3.1}$$

that leads to the metric (Einstein frame):

$$\begin{aligned} ds^2 = e^{2f(r)}\Big[& dx_{1,3}^2 + dr^2 + e^{2h(r)}(d\theta^2 + \sin^2\theta d\varphi^2) \\ &+ \frac{e^{2g(r)}}{4}(\tilde{\omega}_1^2 + \tilde{\omega}_2^2) + \frac{e^{2k(r)}}{4}(\tilde{\omega}_3 + \cos\theta d\varphi)^2\Big]. \end{aligned} \tag{3.2}$$

Notice that compared to (2.14), we took $\alpha' g_s = 1$, while N_c has been absorbed in e^{2h}, e^{2g}, e^{2k}. In our configuration, we will also have a dilaton and a RR threeform

$$\begin{aligned} &\phi(r), \\ &F_{(3)} = -2N_c e^{-3f-2g-k} e^1 \wedge e^2 \wedge e^3 \\ &\qquad + \frac{N_c}{2} e^{-3f-2h-k} e^\theta \wedge e^\varphi \wedge e^3, \end{aligned} \tag{3.3}$$

where N_c, the number of color D5-branes, comes from the quantization condition:

$$\frac{1}{2\kappa_{(10)}} \int_{S^3} F_{(3)} = N_c T_5. \tag{3.4}$$

The S^3 on which we integrate is parametrized by $\tilde{\theta}$, $\tilde{\varphi}$, ψ, so only the first term in (3.3) contributes.

We plug this configuration into the IIB SUSY transformations (see Eq. (B3)). The projections for the Killing spinor are:

$$\Gamma_{r123}\epsilon = \epsilon, \qquad \Gamma_{r\theta\varphi 3}\epsilon = \epsilon, \qquad \epsilon = i\epsilon^*. \tag{3.5}$$

(The sign of the two first expressions can be changed, but we fix it in order to match the full non-Abelian case). After some algebra, the following equations arise,

$$4f = \phi, \tag{3.6}$$

$$h' = \tfrac{1}{4}N_c e^{-2h-k} + \tfrac{1}{4}e^{-2h+k}, \tag{3.7}$$

$$g' = -N_c e^{-2g-k} + e^{-2g+k}, \tag{3.8}$$

$$k' = \tfrac{1}{4}N_c e^{-2h-k} - N_c e^{-2g-k} - \tfrac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k}, \tag{3.9}$$

$$\phi' = -\tfrac{1}{4}N_c e^{-2h-k} + N_c e^{-2g-k}. \tag{3.10}$$

B. Flavoring the space

We want to backreact the space with D5-flavor branes. Let us try to do this following the procedure introduced in [12], i.e. adding an open string sector to the gravity action. It is important to remark that the construction of [12] involved noncritical strings and thus curvatures of order of the string scale. This fact hindered the possibility of finding reliable quantitative results from a supergravity approximation. On the other hand, we are using ten-dimensional string theory and weak curvature can be obtained by taking $g_s N_c \gg 1$ as usual. The action of the system is:

$$S = S_{\rm grav} + S_{\rm flavor}, \tag{3.11}$$

where $S_{\rm grav}$ (Einstein frame) is given by:

$$S_{\rm grav} = \frac{1}{2\kappa^2_{(10)}}\int d^{10}x\sqrt{-g}\Big[R - \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{12}e^{\phi}F^2_{(3)}\Big], \tag{3.12}$$

whereas $S_{\rm flavor}$ is the Dirac + WZ action for the N_f D5-flavor branes (we take the worldvolume gauge field to be zero). In Einstein frame:

$$S_{\rm flavor} = T_5\sum^{N_f}\Big(-\int_{\mathcal{M}_6} d^6x e^{\frac{\phi}{2}}\sqrt{-\hat{g}_{(6)}} + \int_{\mathcal{M}_6} P[C_6]\Big), \tag{3.13}$$

where the integrals are taken over the six-dimensional worldvolume of the flavor branes $\mathcal{M}_6$ and $\hat{g}_{(6)}$ stands for the determinant of the pullback of the metric in such worldvolume. We will take these D5's extended along $x_0, \ldots, x_3, r, \psi$ at constant θ, φ, $\tilde{\theta}$, $\tilde{\varphi}$. Notice that this configuration preserves the $U(1)_R$ symmetry associated to shifts in ψ. This is one of the embeddings which were called *cylinder solutions* in [15]. Moreover, notice that this brane configuration makes clear the need of the deformed ansatz (3.2) where $k(r) \neq g(r)$ (compare to (2.1)).

Following a procedure analogous to [16] we think of the $N_f \to \infty$ branes as being homogeneously smeared along the two transverse S^2's parametrized by θ, φ and $\tilde{\theta}$, $\tilde{\varphi}$. We will elaborate on how the smearing may affect the dual field theory in Section VA. The smearing erases the dependence on the angular coordinates and makes it possible to consider an ansatz with functions only depending on r, enormously simplifying computations. We use the same ansatz (3.2) for the metric and also fix $\phi = 4f$. We have:

$$-T_5\sum^{N_f}\int_{\mathcal{M}_6} d^6x e^{\phi/2}\sqrt{-\hat{g}_{(6)}} \to -\frac{T_5 N_f}{(4\pi)^2}\int d^{10}x \sin\theta\sin\tilde{\theta} e^{\phi/2}\sqrt{-\hat{g}_{(6)}}, \tag{3.14}$$

$$T_5\sum^{N_f}\int_{\mathcal{M}_6} P[C_6] \to \frac{T_5 N_f}{(4\pi)^2}\int {\rm Vol}(\mathcal{Y}_4)\wedge C_{(6)}, \tag{3.15}$$

where we have defined ${\rm Vol}(\mathcal{Y}_4) = \sin\theta\sin\tilde{\theta} d\theta\wedge d\varphi\wedge d\tilde{\theta}\wedge d\tilde{\varphi}$ and the new integrals span the full space-time. We will need the expressions (we choose $\alpha' = g_s = 1$)

$$T_5 = \frac{1}{(2\pi)^5}, \qquad \frac{1}{2\kappa^2_{(10)}} = \frac{1}{(2\pi)^7}. \tag{3.16}$$

Let us turn our attention to the effect of the Wess-Zumino term (WZ) of the flavor brane action (3.15). Since it does not depend on the metric nor on the dilaton, it does not enter the Einstein equations. However, it alters the equation of motion for the 6-form $C_{(6)}$, which before was $d * F_{(7)} \equiv dF_{(3)} = 0$. Now we have a source term so[3]:

$$dF_{(3)} = \tfrac{1}{4}N_f{\rm Vol}(\mathcal{Y}_4) = \tfrac{1}{4}N_f \sin\theta\sin\tilde{\theta} d\theta\wedge d\varphi\wedge d\tilde{\theta}\wedge d\tilde{\varphi}. \tag{3.17}$$

[3]In general, if for a form $F_{(n)} = dA_{(n-1)}$ there is an action $-\frac{1}{2n!}\int\sqrt{|g|}F^2 + \int G\wedge A$, the equation of motion for the form reads $d * F = {\rm sign}(g)(-1)^{D-n+1}G$. In this case, the relevant part of the action (go to string frame for this computation) reads: $-\frac{1}{2\kappa^2_{(10)}}\frac{1}{2\cdot 7!}\int\sqrt{|g|}F^2_{(7)} + \frac{T_5N_f}{(4\pi)^2}\int{\rm Vol}(\mathcal{Y}_4)\wedge C_{(6)}$ so the equation of motion is $\frac{1}{2\kappa^2_{(10)}} d * F_{(7)} = -\frac{T_5N_f}{(4\pi)^2}{\rm Vol}(\mathcal{Y}_4)$. Taking into account $F_{(3)} = - * F_{(7)}$ and (3.16) we arrive at (3.17).

Notice that this particular form for the violation of the Bianchi identity is an effect of the smearing: we replace the sum of delta functions on the position of each of the N_f branes by a constant density, so there is a continuous distribution of charge which acts as a source for the RR form. We can slightly modify (3.3) to solve (3.17)[4]:

$$F_{(3)} = -\frac{N_c}{4}\sin\tilde{\theta}d\tilde{\theta}\wedge d\tilde{\varphi}\wedge(d\psi+\cos\theta d\varphi) - \frac{N_f-N_c}{4}\sin\theta d\theta\wedge d\varphi\wedge(d\psi+\cos\tilde{\theta}d\tilde{\varphi}). \quad (3.18)$$

On the other hand, we still have $dF_{(7)} = 0$. In order to find the new system of BPS equations, we recompute (B3) using the *modified* RR threeform field strength (3.18). It is quite straightforward in this way to get:

$$h' = \tfrac{1}{4}(N_c - N_f)e^{-2h-k} + \tfrac{1}{4}e^{-2h+k}, \quad (3.19)$$

$$g' = -N_c e^{-2g-k} + e^{-2g+k}, \quad (3.20)$$

$$k' = \tfrac{1}{4}(N_c - N_f)e^{-2h-k} - N_c e^{-2g-k} - \tfrac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k}, \quad (3.21)$$

$$\phi' = -\tfrac{1}{4}(N_c - N_f)e^{-2h-k} + N_c e^{-2g-k}. \quad (3.22)$$

Nevertheless, it is important to point out that, apart from the IIB sugra action, we also have the flavor branes action, so, in order to preserve supersymmetry, it is necessary that their action preserves κ-symmetry. As stated above, this is indeed the case in this construction. In Appendix A, we reobtain this system using the so-called "superpotential" approach and in Section III C we check that this first order system automatically implies that the set of second order equations of motion are satisfied.

The system (3.19), (3.20), (3.21), and (3.22) has a very simple solution when $N_f = 2N_c$ that we present in Section IV E and discuss in Section VI. In this work, we will not undertake the study of the system for $N_f \neq 2N_c$.

[4]More generally, we could consider:

$$F_{(3)} = -\frac{N_c + N_c'}{4}\sin\tilde{\theta}d\tilde{\theta}\wedge d\tilde{\varphi}\wedge(d\psi+\cos\theta d\varphi) - \frac{N_f - N_c - N_c'}{4}\sin\theta d\theta\wedge d\varphi\wedge(d\psi+\cos\tilde{\theta}d\tilde{\varphi})$$

but N_c' is just a shift in N_c. As explained below (3.4), it is the coefficient of the first term the one identified with the number of colors.

Using this toy example, we have clarified our proposal. In order to add backreacting flavors, we will consider a deformed background, solution of the equations of motion derived from the sugra action plus an action like (3.14) and (3.15). These last terms come from the action of the (SUSY preserving) flavor branes. We modify the RR form so that the failure of the Bianchi identity indicates the presence of the smeared flavor branes. After that, we impose vanishing of the IIB SUSY variations. This will produce a set of BPS equations that satisfy the Einstein and Maxwell equations and ensure the SUSY of the whole construction.

C. The Einstein equations

In order to check for good the consistency of the approach, in this section we will check that the solutions found above indeed satisfy the full set of equations of motion.

First of all, we have already guaranteed the equation for the RR form (see (3.17) and (3.18)). The equations for the worldvolume fields on the flavor branes are also satisfied since κ-symmetry is preserved. We still have to check the equations for the dilaton and the metric. The relevant action is given by the sum of (3.12) and (3.14), since (3.15) does not depend on the dilaton nor on the metric. The equation for the dilaton is:

$$\frac{1}{\sqrt{-g_{(10)}}}\partial_\mu(g^{\mu\nu}\sqrt{-g_{(10)}}\partial_\nu\phi) - \frac{1}{12}e^{\phi}F_{(3)}^2 - \frac{N_f}{8}e^{\phi/2}\frac{\sqrt{-\hat{g}_{(6)}}}{\sqrt{-g_{(10)}}}\sin\theta\sin\tilde{\theta} = 0, \quad (3.23)$$

whereas the equations coming from variations of the metric are (compare with (2.13)):

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \tfrac{1}{2}(\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi) + \tfrac{1}{12}e^{\phi}(3F_{\mu\lambda_2\lambda_3}F_\nu^{\lambda_2\lambda_3} - \tfrac{1}{2}g_{\mu\nu}F_{(3)}^2) + T_{\mu\nu}^{\text{flavor}}, \quad (3.24)$$

where the energy-momentum tensor generated by the flavor branes comes from the variation of the Lagrangian of the flavor branes (3.14):

$$\mathcal{L}_{\text{flavor}} = -\frac{T_5 N_f}{(4\pi)^2}\sin\theta\sin\tilde{\theta}e^{\phi/2}\sqrt{-\hat{g}_{(6)}} \quad (3.25)$$

and therefore is:

$$T_{\text{flavor}}^{\mu\nu} = \frac{2\kappa_{(10)}^2}{\sqrt{-g_{(10)}}}\frac{\delta\mathcal{L}_{\text{flavor}}}{\delta g_{\mu\nu}} = -\frac{N_f}{8}\sin\theta\sin\tilde{\theta}e^{\phi/2}\delta_\alpha^\mu\delta_\beta^\nu\hat{g}_{(6)}^{\alpha\beta}\frac{\sqrt{-\hat{g}_{(6)}}}{\sqrt{-g_{(10)}}}, \quad (3.26)$$

where α, β span the brane worldvolume. In components, after lowering the indices:

$$
\begin{aligned}
T^{\rm flavor}_{x_i x_j} &= -\frac{N_f}{2}\eta_{ij}e^{-2h-2g},\\
T^{\rm flavor}_{rr} &= -\frac{N_f}{2}e^{-2h-2g},\\
T^{\rm flavor}_{\psi\psi} &= -\frac{N_f}{8}e^{2k-2h-2g},\\
T^{\rm flavor}_{\varphi\psi} &= -\frac{N_f}{8}e^{2k-2h-2g}\cos\theta,\\
T^{\rm flavor}_{\varphi\varphi} &= -\frac{N_f}{8}e^{2k-2h-2g}\cos^2\theta,\\
T^{\rm flavor}_{\tilde\varphi\psi} &= -\frac{N_f}{8}e^{2k-2h-2g}\cos\tilde\theta,\\
T^{\rm flavor}_{\tilde\varphi\tilde\varphi} &= -\frac{N_f}{8}e^{2k-2h-2g}\cos^2\tilde\theta,\\
T^{\rm flavor}_{\varphi\tilde\varphi} &= -\frac{N_f}{8}e^{2k-2h-2g}\cos\theta\cos\tilde\theta.
\end{aligned} \tag{3.27}
$$

Now it is straightforward to check (using *Mathematica*) that the Eqs. (3.19), (3.20), (3.21), and (3.22) imply that (3.23) and (3.24) are satisfied.

To summarize, we have explicitly showed in this section that a consistent procedure to add flavor branes to a given background is to consider the BPS equations obtained by imposing the vanishing of SUSY transformations for the gravitino and dilatino in a system of a "deformed space-time" (like (3.2) with respect to (2.14)) and modified RR forms so that the failure of the Bianchi identity is indicating the presence of the smeared flavor branes. In the next section, we will apply this proposal to add flavor branes to the background dual to $\mathcal{N} = 1$ SYM (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8).

IV. ADDING FLAVOR BRANES TO THE NONSINGULAR SOLUTION

In this section we will solve the important problem of adding N_f flavor branes to the background dual to $\mathcal{N} = 1$ SYM (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8).[5] As pointed out in the introduction, the resolution of this problem is instrumental in the duality between string theory and $\mathcal{N} = 1$ SQCD-like theories. We will be more sketchy here than in Section III and just present the ansatze and corresponding BPS equations for different cases. Many technical details are left for Appendix B. Our procedure (as explained in detail in Section III), will be first to propose a deformation of the background dual to $\mathcal{N} = 1$ SYM (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8). Then, we will obtain the BPS equations that describe the deformation due to the presence of flavor branes, that will be extended along the directions x_0, x_1, x_2, x_3, r, ψ and smeared over the directions $(\theta, \tilde\theta, \varphi, \tilde\varphi)$. As before, these flavor branes will be sources for RR forms that will flux the deformed background and the mark of their existence as singular sources will be the violation of the Bianchi identity.

A. Deforming the space

We start by proposing a deformation to the background (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8). The ansatz we use below is a subcase (with no fractional branes $H_{(3)} = F_{(5)} = 0$) of that first proposed in [23] and further analyzed in [24]. We borrow some results and notation from [24]. Although it is not the main motivation of this work, this unflavored setup encodes some interesting physics which we will analyze in Section VIII. We consider the Einstein-frame metric:

$$
\begin{aligned}
ds^2 = e^{2f(r)}\Big[& dx_{1,3}^2 + dr^2 + e^{2h(r)}(d\theta^2 + \sin^2\theta d\varphi^2)\\
&+ \frac{e^{2g(r)}}{4}((\tilde\omega_1 + a(r)d\theta)^2 + (\tilde\omega_2 - a(r)\sin\theta d\varphi)^2)\\
&+ \frac{e^{2k(r)}}{4}(\tilde\omega_3 + \cos\theta d\varphi)^2\Big].
\end{aligned} \tag{4.1}
$$

The vielbein we consider for this metric is the straightforward generalization of (3.1) by the inclusion of the $a(r)$ dependence in e^1 and e^2. Apart from the dilaton

$$\phi = 4f, \tag{4.2}$$

we also excite the RR 3-form field strength which we take to be of the same form as (2.5):

$$
\begin{aligned}
F_{(3)} = \frac{N_c}{4}\big[&-(\tilde\omega_1 + b(r)d\theta)\wedge(\tilde\omega_2 - b(r)\sin\theta d\varphi)\\
&\wedge(\tilde\omega_3 + \cos\theta d\varphi) + b' dr\wedge(-d\theta\wedge\tilde\omega_1 + \sin\theta d\varphi\\
&\wedge\tilde\omega_2) + (1 - b(r)^2)\sin\theta d\theta\wedge d\varphi\wedge\tilde\omega_3\big].
\end{aligned} \tag{4.3}
$$

The condition $dF_{(3)} = 0$ is automatically ensured by this ansatz. It is useful to rewrite this expression in terms of the vielbein forms:

$$
\begin{aligned}
F_{(3)} = &-2N_c e^{-3f-2g-k}e^1\wedge e^2\wedge e^3 + \frac{N_c}{2}b' e^{-3f-g-h}e^r\\
&\wedge(-e^\theta\wedge e^1 + e^\varphi\wedge e^2)\\
&+ \frac{N_c}{2}e^{-3f-2h-k}(a^2 - 2ab + 1)e^\theta\wedge e^\varphi\wedge e^3\\
&+ N_c e^{-3f-h-g-k}(b - a)(-e^\theta\wedge e^2 + e^1\wedge e^\varphi)\wedge e^3
\end{aligned} \tag{4.4}
$$

We now analyze the dilatino and gravitino transformations (B3) (in this case, the superpotential approach is far more complicated because, as we will see, there are algebraic constraints). After very lengthy algebra (that is explicitly reported in Appendix B), we obtain a system of BPS equations and constraints, some of which can be solved, leaving us with two differential equations for a and k. We define:

[5] In the probe approximation also the paper [22] discussed the problem.

$$e^{-k(r)}dr \equiv d\rho. \tag{4.5}$$

The differential equations read:

$$\begin{aligned}\partial_\rho a &= \frac{-2}{-1+2\rho\coth 2\rho}\left[\frac{e^{2k}}{N_c}\frac{(a\cosh 2\rho-1)^2}{\sinh 2\rho} + a(2\rho - a\sinh 2\rho)\right],\\ \partial_\rho k &= \frac{2(1+a^2-2a\cosh 2\rho)^{-1}}{(-1+2\rho\coth 2\rho)}\left[\frac{e^{2k}}{N_c}a\sinh 2\rho(a\cosh 2\rho - 1) + (2\rho - 4a\rho\cosh 2\rho + \frac{a^2}{2}\sinh 4\rho)\right].\end{aligned} \tag{4.6}$$

The equation for $f = \frac{\phi}{4}$ reads

$$\partial_\rho f = -\frac{(-1+a\cosh 2\rho)^2\sinh^{-2}(2\rho)(-4\rho+\sinh 4\rho)}{4(1+a^2-2a\cosh 2\rho)(-1+2\rho\coth 2\rho)}, \tag{4.7}$$

and can be integrated once $a(\rho)$ is known. The functions $b(\rho)$, $g(\rho)$, $h(\rho)$ can be solved to be:

$$b(\rho) = \frac{2\rho}{\sinh(2\rho)}, \qquad e^{2g} = N_c\frac{b\cosh(2\rho)-1}{a\cosh(2\rho)-1}, \qquad e^{2h} = \frac{e^{2g}}{4}(2a\cosh(2\rho) - 1 - a^2). \tag{4.8}$$

B. Flavoring the space

We consider the same ansatz for the metric:

$$\begin{aligned}ds^2 = e^{2f(r)}\Big[&dx_{1,3}^2 + dr^2 + e^{2h(r)}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{e^{2g(r)}}{4}((\tilde\omega_1 + a(r)d\theta)^2 + (\tilde\omega_2 - a(r)\sin\theta d\varphi)^2)\\ &+ \frac{e^{2k(r)}}{4}(\tilde\omega_3 + \cos\theta d\varphi)^2\Big]\end{aligned} \tag{4.9}$$

and the dilaton is still (4.2).

One should now think of introducing backreacting flavor branes along the lines of Section III. We incorporate smeared flavored D5-branes on the nonsingular solution and the analysis of the equations is quite similar. We again consider D5's extended along $x_0, \ldots, x_3, r, \psi$, each at constant θ, φ, $\tilde\theta$, $\tilde\varphi$. The analysis in [15] shows that these branes preserve the same supersymmetry as the background for any value of the angles θ, φ, $\tilde\theta$, $\tilde\varphi$. One can therefore think of smearing along this space.[6] As in Section III B, the Bianchi identity gets modified to (3.17):

$$dF_{(3)} = \tfrac{1}{4}N_f \mathrm{Vol}(\mathcal{Y}_4) = \tfrac{1}{4}N_f\sin\theta\sin\tilde\theta d\theta\wedge d\varphi\wedge d\tilde\theta\wedge d\tilde\varphi. \tag{4.10}$$

We solve this by adding to the 3-form written in (4.3) the same as in (3.18):

$$F_{(3)}^{\mathrm{flavor}} = -\frac{N_f}{4}\sin\theta d\theta\wedge d\varphi\wedge(d\psi + \cos\tilde\theta d\tilde\varphi). \tag{4.11}$$

Written in flat indices, the 3-form now reads:

$$\begin{aligned}F_{(3)} = &-2N_c e^{-3f-2g-k}e^1\wedge e^2\wedge e^3 + \frac{N_c}{2}b' e^{-3f-g-h}e^r\\ &\wedge(-e^\theta\wedge e^1 + e^\varphi\wedge e^2)\\ &+\frac{N_c}{2}e^{-3f-2h-k}(a^2-2ab+1-x)e^\theta\wedge e^\varphi\wedge e^3\\ &+N_c e^{-3f-h-g-k}(b-a)(-e^\theta\wedge e^2 + e^1\wedge e^\varphi)\wedge e^3,\end{aligned} \tag{4.12}$$

where we have defined

$$x \equiv \frac{N_f}{N_c}. \tag{4.13}$$

We now insert the expression (4.12) for $F_{(3)}$ in the transformation of the spinors equations (B3). We find a set of BPS equations: seven first order equations and two algebraic constraints (for details, the reader is referred to Appendix B). This system[7] can be partially solved, leaving us with (here we use the definition (4.5)):

$$\begin{aligned}b &= \frac{(2-x)\rho}{\sinh(2\rho)}, \qquad e^{2g} = \frac{N_c}{2}\frac{2b\cosh 2\rho - 2 + x}{a\cosh 2\rho - 1},\\ e^{2h} &= \frac{e^{2g}}{4}(2a\cosh(2\rho) - 1 - a^2)\end{aligned} \tag{4.14}$$

[6] Notice that in the unflavored construction the spheres parametrized by θ, φ, $\tilde\theta$, $\tilde\varphi$ do not play an important role from the point of view of the IR $\mathcal{N}=1$ SYM theory. Since we want to add flavor to this theory, we can expect that the effect of the smearing will not alter dramatically the resulting IR $\mathcal{N}=1$ SQCD theory. We will comment more about this in Section V A.

[7] These expressions are clearly not valid when $N_f = 2N_c$. We will deal with this special case in Section IV E and Appendix D.

plus two coupled first order equations for $a(\rho)$, $k(\rho)$ and a differential equation for $f = \frac{\phi}{4}$ that can be integrated in terms of $a(\rho)$. These equations read ($\tilde{k} \equiv k - \frac{1}{2}\log N_c$):

$$\partial_\rho a = \frac{2}{2\rho\coth 2\rho - 1}\Big(-\frac{2e^{2\tilde{k}} - x}{2-x}\frac{(a\cosh 2\rho - 1)^2}{\sinh 2\rho} + a^2\sinh 2\rho - 2a\rho\Big), \tag{4.15}$$

$$\partial_\rho \tilde{k} = \frac{2}{(2\rho\coth 2\rho - 1)(1 - 2a\cosh 2\rho + a^2)} \times\Big(\frac{2e^{2\tilde{k}} + x}{2-x}a\sinh 2\rho(a\cosh 2\rho - 1) + 2\rho\Big(a^2\frac{\sinh 2\rho}{2\rho}\cosh 2\rho - 2a\cosh 2\rho + 1\Big)\Big), \tag{4.16}$$

and

$$\partial_\rho f = \frac{(-1 + a\cosh 2\rho)\sinh^{-2}(2\rho)}{4(1 + a^2 - 2a\cosh 2\rho)(-1 + 2\rho\coth 2\rho)} \times\Big[-4\rho + \sinh 4\rho + 4a\rho\cosh 2\rho - 2a\sinh 2\rho - \frac{4}{(2-x)}a(\sinh 2\rho)^3\Big]. \tag{4.17}$$

The Eqs. (4.14), (4.15), (4.16), and (4.17) constitute an important result of this paper. In order to summarize the flavored setup, the metric is given by (4.9), the dilaton by (4.2), and the RR threeform is the sum of (4.3) and (4.11) (or, in flat indices, it is (4.12)), while all the functions in the ansatz are determined by (4.14), (4.15), (4.16), and (4.17). One can check (using *Mathematica*) that these conditions solve the full system of second order equations given by (3.23) and (3.24) and $d * F_{(3)} = 0$. Notice that the expressions (3.27) are still valid in this case. So, if we can find solutions to (4.15), (4.16), and (4.17), we will have the string dual to a family of SQCD-like theories with N_c colors and N_f flavors; we now turn into this.

C. Asymptotic solutions ($N_f \neq 2N_c$)

Unfortunately, it was not possible for us to find exact explicit solutions to the system (4.15), (4.16), and (4.17). We study the problem in the following way, we first find asymptotic solutions near $\rho \to \infty$ (the UV of the dual theory) and near $\rho = 0$ (the IR of the dual gauge theory). Then, we study a numerical interpolation showing that both expansions can be joined smoothly; for many purposes, this is as good as finding an exact solution.

1. Expansions for $\rho \to \infty$

Let us disregard exponentially suppressed terms, and assume that $a = e^{-2\rho}j$ where j is some function that can be Laurent expanded in ρ^{-1}. Then, when $x < 2$, the large ρ expansion reads:

$$a = e^{-2\rho}\Big((4 - 2x)\rho + \frac{x}{2} + \frac{x(4-x)}{8(x-2)}\rho^{-1} + \mathcal{O}(\rho^{-2})\Big) + \dots \tag{4.18}$$

When $x = 0$ we have $a = 4\rho e^{-2\rho} + \dots$ which agrees with the usual solution (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8), see (2.10) for comparisons. The other functions read:

$$\begin{aligned} e^{2k} &= N_c\Big(1 - \frac{x}{8(2-x)}\rho^{-2} + \dots\Big),\\ e^{2g} &= N_c\Big(1 + \frac{x}{4(2-x)}\rho^{-1} + \dots\Big),\\ e^{2h} &= \frac{N_c}{2}\Big((2-x)\rho + \frac{x-1}{2} + \dots\Big),\\ \partial_\rho f &= \frac{1}{4} - \frac{1}{16}\rho^{-1} + \dots \end{aligned} \tag{4.19}$$

Equation (4.19) is clearly not valid for $x > 2$. In this case we find a different expansion:

$$\begin{aligned} a &= e^{-2\rho}\Big(1 + \frac{x-1}{2(x-2)}\rho^{-1} + \frac{x-1}{8(x-2)}\rho^{-2}\Big) + \dots,\\ e^{2k} &= N_c\Big(x - 1 - \frac{x(x-1)}{8(x-2)}\rho^{-2} + \dots\Big),\\ e^{2g} &= N_c\Big(2(x-2)\rho + 1 + \frac{x(x-1)}{4(x-2)}\rho^{-1} + \dots\Big),\\ e^{2h} &= N_c\Big(\frac{x-1}{4} + \frac{x(x-1)}{16(x-2)}\rho^{-1} + \dots\Big),\\ \partial_\rho f &= \frac{1}{4} - \frac{1}{16}\rho^{-1} + \dots \end{aligned} \tag{4.20}$$

The other possible behavior at $\rho \to \infty$ is a geometry asymptoting to Minkowski times the deformed conifold. Since we will not use such solutions in our physical interpretation, we relegate their description to Appendix C.

2. Expanding around $\rho \to 0$

A series expansion can be found for the functions $a(\rho)$, $k(\rho)$ that solve the BPS equations near $\rho = 0$. This is actually a two-parameter family of solutions labeled by two free numbers which we denote c_1 and c_2. The solution for the functions reads in this case

$$a = 1 - 2\rho^2 + c_1\rho^3 + \frac{(80 - 40x + 9xc_1^2)}{12(2-x)}\rho^4 + \ldots,$$
$$e^{2k} = N_c\bigg(c_2\rho^2 + \frac{3c_1c_2x}{2(x-2)}\rho^3 + \frac{c_2(256 - 256x + x^2(64 + 27c_1^2))}{48(2-x)^2}\rho^4 + \ldots\bigg),$$
$$e^{2g} = N_c\bigg(\frac{2(2-x)}{3c_1}\frac{1}{\rho} - \frac{x}{2} + \frac{8(2-x)}{9c_1}\rho + \ldots\bigg),$$
$$e^{2h} = N_c\bigg(\frac{2(2-x)}{3c_1}\rho - \frac{x}{2}\rho^2 - \frac{8(2-x)}{9c_1}\rho^3 + \ldots\bigg),$$
$$e^{2\phi-2\phi_0} = 1 + \frac{3c_1x}{2(2-x)}\rho + \frac{27x^2c_1^2}{16(2-x)^2}\rho^2 + \ldots \qquad (4.21)$$

Some comments about this expansion are in order. First, we should notice that (4.21) above does not reduce to (2.9) when $x = 0$. This is not very surprising, since the behavior of gauge theories in the IR changes radically in the presence of massless flavors. Second and perhaps more important, we notice that this solution is *singular*. Indeed, when computing the Ricci scalar when $\rho \to 0$, one gets $R \sim \rho^{-2}$ and there does not seem to exist a choice of the constants (c_1, c_2) that avoids this.

The presence of a singularity might be source of concern and cast doubts on the reliability of the solutions. An important point that we would like to emphasize, though, is that there is an important and significant difference between the singularity in (4.21) and others present in the literature, as the one in (2.14) for example. There is one criterium developed in [25] (see also [26] for other criteria) that suggests a way of deciding when an (IR) singularity in a supergravity solution should be accepted or rejected as unphysical. This criterium is quite easy to apply and coincides with other different criteria developed in previous literature. It consists in analyzing the (Einstein-frame) component of the metric g_{tt}. If this is bounded, the singularity should be accepted as a good background. The idea in this criteria is that an excitation travelling towards the origin (that is a low-energy object on the gauge theory dual) will have an energy as measured by an inertial gauge theory observer given by $E = \sqrt{|g_{tt}|}E_0$ (with E_0 the proper energy of the object). So, if g_{tt} diverges (the singularity is repulsive), the object gains unbounded energy from the field theory inertial observer point of view and this is to be considered as unphysical.

According to this criterium, the singularity encoded in the solution (4.21) above should be accepted as being physically relevant; indeed $g_{tt} = e^{\phi/2} = e^{\phi_0/2}(1 + \ldots)$, see (4.21). This means that even though the (good) singularity we find might be a signal of some IR dynamics that we are overlooking, we can still perform computations to try to answer nonperturbative field theory questions. As we will see in Section V, the background matches several gauge theory expectations even when we are performing computations near $\rho = 0$.

To complete the study of these solutions, it is worthwhile to do a detailed numerical analysis to which we turn now.

D. Numerical study ($N_f < 2N_c$)

In order to get a qualitative idea of the behavior of the solutions, we have studied numerically the Eqs. (4.14), (4.15), (4.16), and (4.17) with the initial conditions (4.21) near $\rho = 0$. We also impose the large ρ asymptotic behavior (4.19), what gives a relation between the parameters c_1 and c_2. We also fix $c_2 = 9c_1^2/16$. This amounts to requiring that the domain wall tension scales with $N_c - \frac{N_f}{2}$ (see Section V F). However, the qualitative behavior of the plots does not depend on this constraint. With these conditions, all the functions are uniquely determined for a given value of $x \equiv \frac{N_f}{N_c}$, up to the constant ϕ_0 which we will set to zero. Some results are reported in Fig. 1.

Notice that the $x \to 0$ limit is not continuous (at least in the small ρ region) as it is apparent from the expansion (4.21). In order to show it graphically, we have included in Fig. 1 a plot of the function a near $\rho = 0$.

E. The simple $N_f = 2N_c$ solution

From the expressions (4.14), (4.15), (4.16), and (4.17), one can readily see that $N_f = 2N_c$ is a special point and, in fact, the manipulations leading to some of those expres-

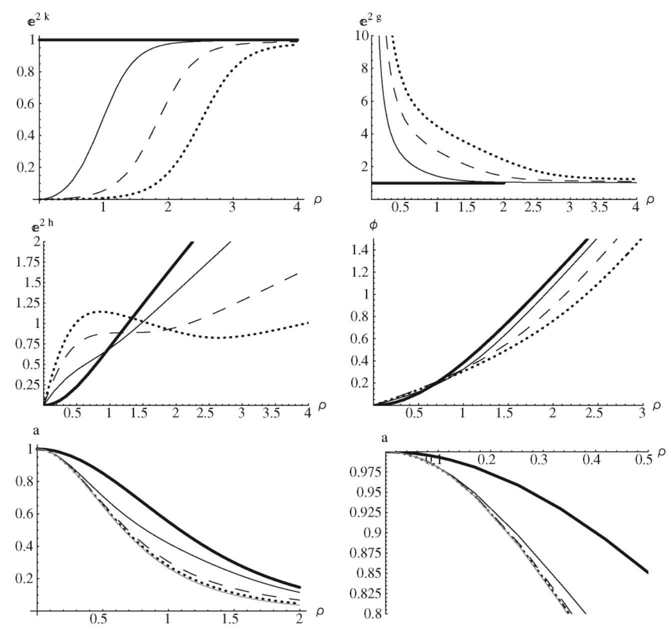

FIG. 1. Some functions of the flavored solutions for $N_f = 0.5N_c$ (thin solid lines), $N_f = 1.2N_c$ (dashed lines), $N_f = 1.6N_c$ (dotted lines). For comparison, we also plot (thick solid lines) the usual unflavored solution. The graphs are e^{2k}, e^{2g}, e^{2h}, ϕ, a, and a zoom of the plot for a near $\rho = 0$. In the figures for a we have also plotted with a light solid line $a = \frac{1}{\cosh(2\rho)}$.

sions are ill-defined in this case. Nevertheless, from the first equation in (4.14) we expect to have $b = 0$. Then, considering the system (B11)–(B15) and (B17), it is easy to see that it can be solved taking $a = b = 0$ and $\mathcal{A} = 1$, $\mathcal{B} = 0$ (however, this is not the only possibility, see Appendix D). In fact, $a = b = 0$ leads us back to the construction of Section III B. For the particular $N_f = 2N_c$ case, there is an extremely simple solution, in which the other functions satisfy:

$$e^{2h} = \frac{N_c}{\xi}, \qquad e^{2g} = \frac{4N_c}{4-\xi}, \qquad e^{2k} = N_c, \qquad \phi = 4f = \phi_0 + r, \tag{4.22}$$

where we have introduced the constant $0 < \xi < 4$. The resulting Einstein-frame metric, dilaton, and RR threeform read:

$$\begin{aligned} ds^2 &= e^{\phi/2}\Big[dx_{1,3}^2 + N_c\Big(dr^2 + \frac{1}{\xi}(d\theta^2 + \sin^2\theta d\varphi^2) \\ &\quad + \frac{1}{4-\xi}(d\tilde\theta^2 + \sin^2\tilde\theta d\tilde\varphi^2) \\ &\quad + \frac{1}{4}(d\psi + \cos\theta d\varphi + \cos\tilde\theta d\tilde\varphi)^2\Big)\Big], \\ e^{\phi} &= e^{\phi_0}e^r, \\ F_{(3)} &= -\frac{N_c}{4}(\sin\tilde\theta d\tilde\theta \wedge d\tilde\varphi + \sin\theta d\theta \wedge d\varphi) \\ &\quad \wedge (d\psi + \cos\theta d\varphi + \cos\tilde\theta d\tilde\varphi). \end{aligned} \tag{4.23}$$

We will discuss the gauge theory associated to this background in Section VI.

1. A Flavored black hole

In Section IV C 2, we discussed the fact that the $0 < N_f < 2N_c$ solutions present a curvature singularity, which is good according to the criterium of having bounded g_{tt}. The $N_f = 2N_c$ case presents the same kind of singularity at $r \to -\infty$. However, the simplicity of this metric allows us to check that this singularity also satisfies an alternative criterium, that basically states that a gravity singularity is good if it can be smoothed out by turning on some temperature in the dual field theory and thus finding a black hole solution [26]. So, in order to construct the black hole, let us first make a change of variable,

$$r = 2\log z, \tag{4.24}$$

and define:

$$\mathcal{F} = 1 - \left(\frac{z_0}{z}\right)^4. \tag{4.25}$$

Keeping the same dilaton and threeform but deforming the metric to:

$$\begin{aligned} ds^2 = e^{\phi_0/2}z\Big[&-\mathcal{F}dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \\ &+ N_c\Big(\frac{4}{z^2}\mathcal{F}^{-1}dz^2 + \frac{1}{\xi}(d\theta^2 + \sin^2\theta d\varphi^2) \\ &+ \frac{1}{4-\xi}(d\tilde\theta^2 + \sin^2\tilde\theta d\tilde\varphi^2) \\ &+ \frac{1}{4}(d\psi + \cos\theta d\varphi + \cos\tilde\theta d\tilde\varphi)^2\Big)\Big] \end{aligned} \tag{4.26}$$

provides a solution of the Einstein equations (3.23) and (3.24) for which the singularity is cloaked behind a horizon. As a technical comment, notice that, due to the change in the metric, the expressions for T_{tt}^{flavor} and T_{rr}^{flavor} get multiplied, with respect to those in (3.27) by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively. The rest of (3.27) remains unchanged.

Let us stress that this solution has its own interest since it is, to our knowledge, the first solution dual to a field theory in four dimensions with adjoints and fundamentals at finite temperature.[8] We will explore some of the gauge theory aspects of this solution (4.26) in Section VI.

Finally, to close this section, we mention a numerical solution described in detail in Appendix D. This solution for the case $N_f = 2N_c$, appears after a careful analysis of the BPS equations and has remarkable features. We comment more on it at the end of Section VI.

V. GAUGE THEORY ASPECTS AND PREDICTIONS OF THE SOLUTION ($N_f \neq 2N_c$)

In this section we will study the family of gauge theories dual to the solution(s) we have presented in the previous section. We will first try to give arguments towards a definite Lagrangian dual to the background (4.9), (4.10), (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), and (4.17) and then we will study different strongly coupled aspects of these field theories to support our proposal. These aspects include R-symmetry breaking, pair creation and Wilson loops, instantons, Seiberg duality, domain walls, and asymptotic beta functions.

A. General aspects of the gauge theory

Below we will present some arguments to motivate a Lagrangian description for the field theory dual to our backgrounds (4.9), (4.10), (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), and (4.17). We will argue that the dual field theory is related in the IR to $\mathcal{N} = 1$ SQCD plus a quartic superpotential for the quark superfields. Some aspects related to the smearing of the flavor branes and to the six-dimensional little string UV completion of the theory are not completely under control, and, in particular, we could expect that some operator we are overlooking might

[8] In an interesting work [27], the addition of finite temperature to a three-dimensional gauge theory with adjoint and fundamental fields was studied.

still be present and slightly deform the IR dynamics of the theory. Nonetheless we are confident that the result to which we arrive is robust and we present many tests in the following, which support our interpretation.

Our argument starts by reminding the discussion in Section II A, that the field theory dual to the unflavored solution of [7] is six-dimensional $\mathcal{N} = 1$ $SU(N_c)$ SYM compactified on a twisted two-sphere in such a way that the low-energy theory is four-dimensional and $\mathcal{N} = 1$ supersymmetric. The spectrum of the corresponding $U(1)$ theory is related to that of the four-dimensional $U(M)$ $\mathcal{N} = 1^*$ theory, that is $\mathcal{N} = 4$ SYM with equal masses for the adjoint chiral multiplets, in its Higgs vacuum [20]. Let us briefly review the main features of this four-dimensional theory before arguing how to extend this remarkable result to the large N_c case. The superpotential of the $\mathcal{N} = 1^*$ theory can be written (in $\mathcal{N} = 1$ notations) as

$$W = \mathrm{Tr}_{U(M)}\left(i\Phi_1[\Phi_2, \Phi_3] + \frac{\mu}{2}\sum_i \Phi_i^2\right) \qquad (5.1)$$

and therefore the F-term equations of motion read

$$[\Phi_i, \Phi_j] = i\mu\epsilon_{ijk}\Phi_k. \qquad (5.2)$$

It is immediate to recognize an $su(2)$ algebra in this equation. This allows for nontrivial solutions to (5.2) where the adjoint fields Φ_i are taken to be proportional to the generators $J_i^{(n)}$ of any irreducible representation of $su(2)$. For any dimension n there is only one such representation and therefore the classical vacua of this theory correspond to the partition of M into positive integer numbers [28]

$$\sum_{n=1}^{M} nk_n = M, \qquad (5.3)$$

where k_n is the number of times the n-dimensional irreducible representation appears in the expectation value of the adjoint fields. Any specific such vacuum has residual gauge group $\prod_n U(k_n)$. To obtain a vacuum with gauge group $U(1)$, one has to take $k_M = 1$ and all other k_n's to zero, whereas to obtain a vacuum which has exactly $SU(N_c)$ residual gauge group, one needs to start from $SU(M)$ $\mathcal{N} = 1^*$ with M an integer multiple of N_c, $M = mN_c$, and to take $k_m = N_c$ in (5.3). The expectation values of the adjoint fields in this Higgs vacuum satisfy the relation

$$\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = \mu^2\frac{M^2 - 1}{4}, \qquad (5.4)$$

that is they describe a (fuzzy) two-sphere.

In the particular case $U(1)$, it was shown in [20] that the matching of the spectrum to that of the compactified six-dimensional theory we started with, is exact in the $M \rightarrow \infty$ limit. Because of the ways the $U(1)$ and the $SU(N_c)$ vacua are built, it is natural to argue that the results of [20,21] extend to the more general $SU(N_c)$ case, that is the spectrum and Lagrangian of the $k_m = N_c$ Higgs vacuum of $\mathcal{N} = 1^*$ is exactly the same as that of the six-dimensional $SU(N_c)$ SYM wrapped on a twisted two-sphere when $m \rightarrow \infty$.

Let us consider now the case we are interested in, that is the addition of flavors to the compactified six-dimensional theory. When we add N_f flavor branes into the background, we are effectively adding a set of massless chiral multiplets transforming in the fundamental and antifundamental of the $SU(N_c)$ gauge group. Let us call these multiplets $Q = (S, q)$ and $\tilde{Q} = (\tilde{S}, \tilde{q})$. These fields are massless because the flavor branes are at zero distance from the color branes that generated the background.

A generic Lagrangian for this new system could then be written as

$$L = \mathrm{Tr}\left[\int d^4\theta(\Phi_{KK}^\dagger e^V \Phi_{KK} + Q^\dagger e^V Q + \tilde{Q}^\dagger e^V \tilde{Q}) + \int d^2\theta(W_\alpha^2 + \mathcal{W})\right], \qquad (5.5)$$

where $\mathcal{W}$ is a chiral superpotential describing the mass terms and self-interaction of the adjoint Kaluza-Klein states, and their coupling to the fundamental fields. Because of the argument above, we can think of the Kaluza-Klein states as the three massive adjoint fields of the $\mathcal{N} = 1^*$ theory. The most natural way to couple the fundamental fields to an adjoint state is mediated from the only allowed term in $\mathcal{N} = 2$ SQCD, which is the one typically appearing in intersecting branes setups. It reads

$$\mathcal{W} = \kappa\tilde{Q}\Phi Q \qquad (5.6)$$

with κ a coupling constant whose value will not affect our discussion. In a theory like the $\mathcal{N} = 1^*$ in a Higgs vacuum described above, if the fundamentals couple to a single adjoint field through (5.6), the $SU(2)$ global symmetry that rotates the three adjoints breaks down to a $U(1)$ subgroup. The term (5.6) breaks explicitly also the $SU(N_f) \times SU(N_f)$ flavor group to its diagonal $SU(N_f)$ subgroup.

The explicit flavor group breaking is exactly matched in our dual string picture by the presence of flavor D5-branes alone, rather than of flavor D5 and anti-D5 branes. This feature is in contrast with other models of theories with fundamental degrees of freedom where the chiral symmetry breaking is spontaneous and flavors are added to the picture via both D-branes and anti D-branes, which reconnect in the IR to account for the spontaneous breaking [29]. Since in our case the breaking of the global symmetry is explicit rather than spontaneous, we do not expect to find Goldstone bosons in the spectrum.

Regarding the breaking of the $SU(2)$ to $U(1)$, a little more care is required since the smearing of the flavor branes restores the $SU(2)$ symmetry in our model. Let us start, therefore, from a different distribution of the flavor branes to try to understand what is happening. We could smear the flavor branes on the (θ, φ) two-sphere, while putting all of them on a single point on the $(\tilde{\theta}, \tilde{\varphi})$ directions, for simplicity let us say at $\tilde{\theta} = 0$. Since the exact unflavored solution of [7] is invariant under an $SU(2)$ acting on the ψ, $\tilde{\theta}$, and $\tilde{\varphi}$ angles, this new configuration breaks this $SU(2)$ background isometry to $U(1)$. We claim that this is the configuration dual to the fundamental-adjoint coupling (5.6). From the field theory point of view this would probably be the most natural brane configuration to consider, but as we already mentioned above, finding a solution to the Einstein and flux equations for this less symmetric configuration would probably turn out to be technically impossible. This is the reason why we considered a smeared configuration for the flavor branes. Let us try, therefore, to understand what this means in field theory terms. We know from (5.4) that the three massive adjoint fields describe a two-sphere (in the limit $M \to \infty$ it is a smooth sphere). Therefore, we can take them to be parametrized by two angles ($\tilde{\theta}$ and $\tilde{\varphi}$), by the usual projections over the coordinate planes of $\mathbb{R}^3$ of a two-sphere of radius $R^2 = \mu^2(M^2 - 1)/4$ centered at the origin. We will denote these adjoints parametrizing the $S^2(\tilde{\theta}, \tilde{\varphi})$ as $\Phi^{(i)}_{(\tilde{\theta},\tilde{\varphi})}$.

Then, we propose that the fundamental-adjoint coupling corresponding to our model is, schematically:

$$\mathcal{W} \sim \kappa \int_{S^2} d\tilde{\theta} d\tilde{\varphi} \sin\tilde{\theta}\, \tilde{Q}(\Phi^{(1)}_{(\tilde{\theta},\tilde{\varphi})} + \Phi^{(2)}_{(\tilde{\theta},\tilde{\varphi})} + \Phi^{(3)}_{(\tilde{\theta},\tilde{\varphi})})Q. \tag{5.7}$$

Let us quickly go through now the low-energy effective Lagrangian corresponding to the superpotential (5.7). In general, if the parameter μ is much bigger than the scale of strong coupling $\Lambda_{\rm sqcd}$ we can integrate out the KK fields from (5.5) to end up with an $\mathcal{N} = 1$ SQCD-like theory that looks

$$L = \mathrm{Tr}\bigg[\int d^4\theta(Q^\dagger e^V Q + \tilde{Q}^\dagger e^V \tilde{Q}) + \int d^2\theta W_\alpha^2 + \mathcal{W}'\bigg], \tag{5.8}$$

where the effective superpotential $\mathcal{W}'$ will depend on the way the adjoints couple to the fundamentals and to themselves in (5.5). For our case (5.7), after integrating out the massive KK fields, we might expect to obtain some combination of the quartic operators (5.6):

$$\mathcal{W}' \sim \frac{\kappa^2}{\mu}(\mathrm{Tr}(\tilde{Q}Q\tilde{Q}Q) + (\mathrm{Tr}\tilde{Q}Q)^2) \tag{5.9}$$

and possibly some more complicated operator due to the cubic coupling of the adjoint KK fields.

Could we take the KK masses to be very large ($\mu \to \infty$), we would end up with pure SQCD. Unfortunately, there is no known way to sensibly separate μ from $\Lambda_{\rm sqcd}$ without requiring the full string dynamics.[9] In fact, one can expect that taking $\mu \to \infty$ requires reducing the "external" space (that is topologically $S^2 \times S^3$) to a stringy scale. In this context, it is very suggestive that our construction has some similarity with the noncritical string model of Klebanov and Maldacena [12], which, in fact, should be the limit of our construction when continuously increasing $\mu \to \infty$. Let us emphasize once again that, in the process, the curvature of space-time reaches the string scale and therefore one cannot expect to get reliable results from supergravity (although it has been shown that it is possible to get some qualitative information from that kind of setup using a gravity action as a toy model [12,13,16]).

We can now complete the comparison of the isometries of the field theory and of our background. We have already analyzed the matching of the flavor symmetry and the $SU(2)$ Kaluza-Klein symmetry. The field theory at weak coupling also has a $U(1)_B \times U(1)_R$ symmetry, where $U(1)_B$ is an exact baryonic symmetry, while $U(1)_R$ is the anomalous R-symmetry. The unbroken baryonic symmetry could be associated with rotations along φ, whereas the anomalous R-symmetry corresponds to shifts of the coordinate ψ. As we will analyze in detail in Section V D, the supergravity background reproduces faithfully the breaking pattern $U(1)_R \to Z_{2N_c-N_f} \to Z_2$.

Finally, let us mention that in the special case $N_f = 2N_c$ we can reasonably claim that all the extra uncontrolled terms appearing along with (5.9) in the effective superpotential $\mathcal{W}'$ can be turned off, giving on the string side our simple solution (4.23). As we show in Section VI, this background has very special features which we relate to the scale invariance of the dual field theory. On the other hand, for $N_f < 2N_c$, we could not find a way on the gravity side to turn off these extra terms, as the comparison between the solutions of Section IV D and of Appendix D to the simple solution (4.23) seems to confirm. Nonetheless, as all the rest of this section shows, the string dual perspective of many strong-coupling effects, remarkably agrees with expectations from the field theory described by (5.8) and (5.9), and allows therefore to make interesting predictions for its nonperturbative dynamics.

As a concluding remark, it would certainly prove very interesting to try to match with our approach, the results of the careful analysis done in [32] on the vacuum structure of $\mathcal{N} = 2$ SQCD with supersymmetry broken to $\mathcal{N} = 1$ by a mass term for the adjoint superfield, which is closely related to the theory (5.8) and (5.9).

[9] This is the typical problem that afflicts all the supergravity duals to confining field theories. This will be cleanly solved when a world sheet CFT is found for these models lifting the limitation $\alpha' \to 0$. Meanwhile, from a supergravity perspective, the KK modes can be disentangled form the gauge theory dynamics following [19,30,31].

B. Wilson loop and pair creation

One natural question is what happens to the gauge theory Wilson loop, now that we have massless flavors. In principle, as in QCD we should not observe an area law, but the SQCD-string should elongate until its tension is equal to the mass of the lightest meson, and then break. So, if we find that the (very massive) quarks that we introduce in the system feel each other up to a maximal distance only, this will be indication that we are seeing a phenomenon like pair creation.[10] In order to study this, we will follow a very careful treatment to compute the Wilson loop in gravity duals [34] developed in [35] (for a good summary of the results, see pages 19–25 in [36]). As usual, we propose that the Wilson loop for two nondynamical quarks (strings stretching up to $\rho \to \infty$) separated a distance L in the gauge theory coordinates, should be computed as the action of a fundamental string that is parametrized by $t = \tau$, $x_1 = \sigma$, $\rho = \rho(\sigma)$ [34]. By solving the Nambu-Goto action, one obtains a one-parameter family of solutions depending on an integration constant which we will define to be ρ_0 (the minimal ρ reached by the string). We convert to string frame the metric (4.9) and use (4.2) and (4.5). Then, the length and energy (renormalized by subtracting the infinite masses of the nondynamical quarks) read:

$$\begin{aligned} L(\rho_0) &= 2\int_{\rho_0}^{\rho_1} e^{k(\rho)}\frac{e^{\phi(\rho_0)}}{\sqrt{e^{2\phi(\rho)}-e^{2\phi(\rho_0)}}}d\rho, \\ E(\rho_0) &= \frac{1}{2\pi\alpha'}\bigg[2\int_{\rho_0}^{\rho_1}\frac{e^{2\phi(\rho)+k(\rho)}}{\sqrt{e^{2\phi(\rho)}-e^{2\phi(\rho_0)}}}d\rho \\ &\quad - 2\int_{0}^{\rho_1} e^{\phi(\rho)+k(\rho)}d\rho\bigg], \end{aligned} \tag{5.10}$$

where ρ_1 is a cutoff related to the mass of the heavy quarks that can be taken to infinity smoothly. These integrals can be performed numerically. In order to do so, we have to fix the parameters c_1 and c_2 of the expansion (4.21) in order to give initial conditions for the Eqs. (4.15) and (4.16). Imposing the large ρ asymptotic behavior (4.19) gives a relation between them. Moreover, we also fix $c_2 = 9c_1^2/16$ so that the domain wall tension scales with $N_c - \frac{N_f}{2}$ (see Section V F). However, this condition is not essential for the qualitative behavior. The numerical result is reported in Fig. 2. There is a finite value of ρ_0, say ρ_0^*, for which the solution reaches a maximum $L_{\max}$ (at this point, $\frac{dL}{d\rho_0} = \frac{dE}{d\rho_0} = 0$). There is also a minimum value of the length $L_{\min} = \pi\sqrt{g_s N_c \alpha'}$. Our interpretation is that $L_{\min}$ is related to the KK scale at which the UV completion sets in, and, therefore, the background loses predictivity. For a range of $L_{\min} < L < L_{\max}$, there are two solutions of the Nambu-Goto action. The physical one (the one of lower energy), corresponds to $\rho_0 > \rho_0^*$ which, in particular, implies that the string never approaches $\rho = 0$ and therefore it does not approach the curvature singularity.

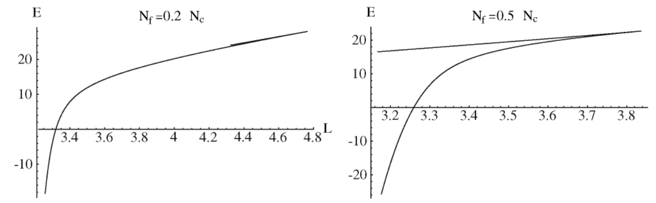

FIG. 2. Energy vs length of the Wilson loop for $\frac{N_f}{N_c} = 0.2$ and $\frac{N_f}{N_c} = 0.5$. Energies are in units of $\frac{e^{\phi_0}\sqrt{g_s N_c}}{\sqrt{\alpha\prime}}$, whereas lengths are in units of $\sqrt{g_s N_c \alpha'}$. There is a value of the integration constant ρ_0 for which the string reaches its maximum L and E. Longer strings are not solutions of the Nambu-Goto action. We interpret it as string breaking due to quark-antiquark pair creation.

As explained above, when one stretches a flux tube in a confining theory with dynamical quarks, the tube must break at some finite length when there is enough energy to create a light meson which screens the color charge. The geometry can reproduce this behavior (the breaking happens at $L_{\max}$) because the backreaction of the flavors has been taken into account. The graphs qualitatively match those found on the lattice in similar (nonsupersymmetric) setups [37]. Thus, our solution is showing that there is pair creation and giving a prediction for the lightest meson in terms of the parameters of the model. The fact that $L_{\min}$ and $L_{\max}$ are roughly of the same order reflects the known fact that, when constructing the gravity dual of a confining theory, the KK scale and $\Lambda_{\rm qcd}$ are of the same order and cannot be separated within a weakly curved framework.

The discussion above should be compared with what happens in unflavored backgrounds, like (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8), the KS background [6], or the ones we present in Section VIII, where there is no possibility of extremizing the function $L(\rho_0)$. However, notice that the behavior found above is very similar to the one observed in the paper [38] for a gravity dual of YM* without flavors. Such a background, like ours, presents a good singularity in the IR and the authors of [33] considered the Wilson loop behavior unphysical and related to the fact that the string wanted actually to fall in the singularity. Nevertheless, there is an important difference between the two models since g_{tt} (and therefore the QCD string tension) never vanishes in our setup.

C. Instanton action

Let us compute the action of an instanton in the field theory. We will propose that a gauge theory instanton should be thought of as a Euclidean D1-brane wrapping some contractible two-cycle in the geometry (4.1). The action of the D1-brane is:

[10] In [33], screening due to backreacting flavors was found as a change in the slope of Regge trajectories.

$$S = T_1 \int d^2 z e^{-\phi}\sqrt{\det g} + T_1 i \int C_{(2)} = \frac{8\pi^2}{g^2_{\text{sqcd}}} + i\theta_{\text{sqcd}}, \tag{5.11}$$

where we have compared with the action of the instanton in order to get an expression for the four-dimensional gauge coupling and theta angle. Since $dF_{(3)} \neq 0$, it is not possible to define a twoform potential such that $F_{(3)} = dC_{(2)}$. But if we consider the subspace given by $\tilde{\theta} = \theta$, $\tilde{\varphi} = 2\pi - \varphi$ (and take also large constant ρ) we can write, from (4.12):

$$F_{(3)\text{eff}} = \frac{N_c}{4}(2-x)\sin\theta d\theta \wedge d\varphi \wedge d\psi \tag{5.12}$$

and therefore:

$$C_{(2)\text{eff}} = (\psi - \psi_0)\frac{N_c}{4}(2-x)\sin\theta d\theta \wedge d\varphi. \tag{5.13}$$

We now pick the two-cycle defined by [39] :

$$\theta = \tilde{\theta}, \qquad \tilde{\varphi} = 2\pi - \varphi, \qquad \psi = (2n+1)\pi. \tag{5.14}$$

So we can identify[11]:

$$\theta_{\text{sqcd}} = \tfrac{1}{2}(2N_c - N_f)(\psi - \psi_0), \tag{5.15}$$

$$\frac{4\pi^2}{g^2_{\text{sqcd}}} = e^{2h} + \frac{e^{2g}}{4}(a-1)^2. \tag{5.16}$$

It is quite nice to see that the coupling constant satisfies the confining behavior we expect for $N_f < 2N_c$. Indeed, plugging in the expansions for the functions near the origin (4.21), we see that even though the solution is singular, this singularity is not causing problems and the coupling grows at low energies. Conversely, in the UV, using the expansion in (4.19), we observe that the coupling is asymptotically small. We will further discuss the quantities written in (5.15) and (5.16) in Sections V D and V E respectively.

D. $U(1)_R$ breaking

In the previous subsection, we have identified a contractible two-cycle on which we wrapped a Euclidean D1-brane that we identified as an instanton. Since this cycle contracts to zero near the origin of the coordinates, the partition function for this Euclidean brane should be equal to one, up to a phase. For the original presentation of this argument see [40]. The partition function of this instantonic string is

$$Z = (\ldots)\exp\left[\frac{i}{2\pi}\int_{\Sigma_2} C_{(2)\text{eff}}\right], \tag{5.17}$$

where Σ_2 is the cycle (5.14) and $(\ldots)$ represents the Born-Infeld part of the action, see (5.11). Under a shift in the R-symmetry $\psi \to \psi + 2\epsilon$ (we write the factor of 2 since $\psi \in [0, 4\pi)$ in order to have $\epsilon \in [0, 2\pi)$) we will have

$$Z \to Z[0]\exp[i\epsilon(2N_c - N_f)], \tag{5.18}$$

so, asking for invariance of Z imposes that (this is equivalent to requiring that the θ_{sqcd} defined in (5.15) is unchanged modulo 2π under shifts of ψ):

$$\epsilon = \frac{2\kappa\pi}{2N_c - N_f} \qquad \kappa = 1, \ldots, 2N_c - N_f. \tag{5.19}$$

So we see that quantum effects select a discrete subset $\mathbb{Z}_{2N_c - N_f}$ out of the possible rotations of the angle ψ associated with the R-symmetry, that is consequently broken according to $U(1)_R \to \mathbb{Z}_{2N_c-N_f} \to \mathbb{Z}_2$, with the last step in the chain of breakings due to the fact that the full background is only invariant under $\psi \to \psi + 2\pi$. This last step in breaking is understood as the formation of a condensate, which the functions $a(\rho)$, $b(\rho)$ should be dual to [41].

Notice that when $N_f = 2N_c$ the breaking above does not happen and the $U(1)_R$ symmetry preservation indicates a theory with higher invariance, in this case, scale invariance: the $U(1)_R$ anomaly is in the same anomaly multiplet as the beta function so, when $N_f = 2N_c$ the coupling in (5.16) should not run. We study this point in more detail in Section VI.

E. Beta function

Let us start by summarizing the result of the Wilsonian beta function for SQCD

$$\beta_g = -\frac{g^3_{\text{sqcd}}}{16\pi^2}(3N_c - N_f(1-\gamma_0)), \tag{5.20}$$

where γ_0 is the anomalous dimension of the quark (and antiquark) superfields, while for the coupling of the quartic superpotential we have

$$\beta_h \sim (1 + 2\gamma_0). \tag{5.21}$$

An approach to the calculation of the beta function of SYM from a gravity dual is presented in [39]. Without entering into the many possible discussions about the validity of this computation (more on this in Section VIII H), let us just repeat the steps in [39] for our case. The coupling is defined in (5.16) and we define the radius-energy relation near the UV of the field theory (at large values of ρ) as

$$\left(\frac{\Lambda}{\mu}\right)^3 \sim a(\rho) \sim b(\rho) \sim e^{-2\phi}, \tag{5.22}$$

where with the symbol $\sim$ above we mean that all these definitions, when taken at large values of the radial coordinate, give the same result. The idea of the first two definitions is that these functions should be dual to the gaugino condensate or other condensate as proposed in [41]. On the other hand, one might consider that the

[11] We remind the reader that we have used that the tension of the D1-brane is $T_1 = \frac{1}{2\pi\alpha' g_s}$ and we are using $\alpha' = g_s = 1$ along our computations.

exponential of the dilaton is related to the strong-coupling scale as proposed in [7,42]. Considering only the large ρ expansion of the coupling defined in (5.16) and the functions in (5.22), we have

$$\frac{4\pi^2}{g^2_{\rm sqcd}} \sim N_c\left(1-\frac{x}{2}\right)\rho + \ldots, \qquad \log\frac{\mu}{\Lambda} \sim \frac{2}{3}\rho. \tag{5.23}$$

So, a straightforward computation leads us to

$$\beta = \frac{dg_{\rm sqcd}}{d\log\frac{\mu}{\Lambda}} = -\frac{3g^3_{\rm sqcd}}{32\pi^2}(2N_c - N_f) + \ldots, \tag{5.24}$$

where the last equality should be understood as valid only in an expansion for large values of ρ. This shows that in the regime described by the large ρ region of the background, the anomalous dimension of the quark superfield is $\gamma_0 = -1/2$ (and therefore $\beta_h = 0$). Once again, this computation should be severely criticized regarding the approximations and assumptions that are done (see Section VIII for a different sort of criticism to this approach), but we just wanted to spell it here to show that the coefficient $2N_c - N_f$ appears. This also reinforces the fact that for the case in which $N_f = 2N_c$ something special is happening (and we know the solution changes qualitatively for that case). It would be interesting to understand what happens when $N_f > 2N_c$.

F. Domain walls

The domain walls of $\mathcal{N} = 1$ SYM associated to the IR spontaneous breaking of the R-symmetry $\mathbb{Z}_{2\rm N_c} \to \mathbb{Z}_2$ correspond to D5-branes wrapping the finite S^3 at $\rho = 0$ [7]. Their associated worldvolume κ-symmetry matrix is:

$$i\Gamma_{x_0x_1x_2123}\epsilon^* = \epsilon. \tag{5.25}$$

As pointed out in [43], this projection commutes with those in (B4) if and only if $\mathcal{A} = 0$, i.e., when the brane is placed at $\rho = 0$. So from the brane point of view, one finds, consistently, that these objects are one-half BPS.

In the flavored theory we are dealing with, we have argued that there is a also a spontaneous IR breaking $\mathbb{Z}_{2\rm N_c - N_f} \to \mathbb{Z}_2$ and therefore one would also expect to have domain walls. It is easy to see, using the expansion (4.21) and the expressions in Appendix B that it is also true in our flavored case that $\lim_{\rho\to 0}\mathcal{A} = 0$, so the same kind of embedding at $\rho = 0$ preserves again half of the supersymmetry. Notice also that the action for the domain wall D5-brane wrapping the three-cycle parametrized by $\tilde\theta$, $\tilde\varphi$, ψ at constant ρ is:

$$E_{D5} \propto T_5 e^{2\phi+2g+k} \propto (g_sN_c)^{1/2}N_c e^{2\phi_0}\frac{2(2-x)\sqrt{c_2}}{3c_1}(1+2\rho^2+\ldots) \tag{5.26}$$

and it has indeed a minimum at $\rho = 0$ (we have used Eq. (4.21) and restored the dependence on g_s). Also, despite e^{2g} and e^{2k} tend to infinity and zero, respectively, the relevant combination (5.26) goes to a constant at $\rho = 0$, rendering a finite tension for the domain wall (its precise value would depend on the integration constants c_1, c_2 and could be used to fix them). Thus, we would like to stress that, even if at $\rho = 0$ we are precisely on top of the curvature singularity, it is appealing to find sensible results from these simple computations.

G. Seiberg duality

A remarkable feature of many $\mathcal{N} = 1$ gauge theories that include matter transforming in the fundamental representation of the gauge group (or even in the bifundamental representation for a quiver gauge theory) is Seiberg duality [44]. In its simplest form, this can be stated as saying that an $\mathcal{N} = 1$ $SU(N_c)$ gauge theory with N_f quarks and an $\mathcal{N} = 1$ $SU(N_f - N_c)$ gauge theory with N_f quarks and a gauge-singlet meson have the same IR physics. When the theory contains a quartic superpotential for the fundamental fields, the duality is a bit more complicated [45] but nonetheless it retains the same relations between the number of colors and flavors of the two Seiberg-dual theories as in the simpler case.

In this section we will argue that Seiberg duality appears in our construction in a remarkably simple and elegant way: it corresponds to switching between two alternative ways of writing the background.[12]

The central point on which the realization of Seiberg duality in our model is based, is that the internal space of our background has $S^2 \times S^3$ topology, but the choice to use either the space spanned by the pair (θ, φ) or by $(\tilde\theta, \tilde\varphi)$ as the S^2 base of the S^3 is (at least at first sight) arbitrary. In fact, let us take the metric and $F_{(3)}$ flux to be written as, from (4.9),

$$ds^2 = e^{\phi/2}\Big(dx^2_{1,3} + dr^2 + e^{2h}(\mathrm{e}_1^2 + \mathrm{e}_2^2) + \frac{e^{2g}}{4}((\tilde\omega_1 + a(r)\mathrm{e}_1)^2 + (\tilde\omega_2 + a(r)\mathrm{e}_2)^2) + \frac{e^{2k}}{4}\hat\omega_3^2\Big) \tag{5.27}$$

and from (4.3) and (4.11)

$$F_{(3)} = \frac{N_c}{4}\{-\tilde\omega_1\wedge\tilde\omega_2\wedge\hat\omega_3 - b(r)(\mathrm{e}_1\wedge\tilde\omega_2 - \mathrm{e}_2\wedge\tilde\omega_1)\wedge\hat\omega_3 - b'(r)dr\wedge(\mathrm{e}_1\wedge\tilde\omega_1 + \mathrm{e}_2\wedge\tilde\omega_2) - (1-x)\mathrm{e}_1\wedge\mathrm{e}_2\wedge\hat\omega_3\}, \tag{5.28}$$

[12] A first hint of what we will develop for the non-Abelian background can be easily gained in the simpler Abelian case by looking at the way the value of the integral of $F_{(3)}$ (3.18) changes when it is evaluated over the 3-sphere (ψ, θ, φ) or $(\psi, \tilde\theta, \tilde\varphi)$.

where the functions ϕ, a, g, h, and k are numerically evaluated in Section IV D, b is given by (4.14)

$$b(\rho) = (2 - x)\frac{\rho}{\sinh 2\rho} \tag{5.29}$$

and for convenience we have defined the following one-forms

$$\begin{aligned}
&\mathrm{e}_1 = d\theta, \qquad \tilde{\mathrm{e}}_1 = d\tilde{\theta},\\
&\mathrm{e}_2 = -\sin\theta d\varphi, \qquad \tilde{\mathrm{e}}_2 = -\sin\tilde{\theta} d\tilde{\varphi},\\
&\omega_1 = \cos\psi \mathrm{e}_1 - \sin\psi \mathrm{e}_2, \qquad \tilde{\omega}_1 = \cos\psi\tilde{\mathrm{e}}_1 - \sin\psi\tilde{\mathrm{e}}_2,\\
&\omega_2 = -\sin\psi \mathrm{e}_1 - \cos\psi \mathrm{e}_2, \qquad \tilde{\omega}_2 = -\sin\psi\tilde{\mathrm{e}}_1 - \cos\psi\tilde{\mathrm{e}}_2,\\
&\omega_3 = d\psi + \cos\theta d\varphi, \qquad \tilde{\omega}_3 = d\psi + \cos\tilde{\theta} d\tilde{\varphi},\\
&\hat{\omega}_3 = \omega_3 + \cos\tilde{\theta} d\tilde{\varphi} = \tilde{\omega}_3 + \cos\theta d\varphi.
\end{aligned} \tag{5.30}$$

We know that the number of colors of the dual theory can be evaluated by integrating the threeform flux over the internal $\tilde{S}^3$ spanned by the angles $\tilde{\theta}$, $\tilde{\varphi}$, ψ, as in (3.4)

$$N_c T_5 = \frac{1}{2\kappa_{(10)}} \int_{\tilde{S}^3} F_{(3)}, \tag{5.31}$$

whereas the number N_f of flavors in the gauge theory is determined by the failing of the Bianchi identity for $F_{(3)}$, Eq. (4.10).

It is easy to show that the oneforms we defined in (5.30) satisfy the following identities[13]

$$\begin{aligned}
&\omega_1^2 + \omega_2^2 = \mathrm{e}_1^2 + \mathrm{e}_2^2, \qquad \tilde{\omega}_1^2 + \tilde{\omega}_2^2 = \tilde{\mathrm{e}}_1^2 + \tilde{\mathrm{e}}_2^2,\\
&\mathrm{e}_1\tilde{\omega}_1 + \mathrm{e}_2\tilde{\omega}_2 = \omega_1\tilde{\mathrm{e}}_1 + \omega_2\tilde{\mathrm{e}}_2, \qquad \omega_1 \wedge \omega_2 = -\mathrm{e}_1 \wedge \mathrm{e}_2,\\
&\tilde{\omega}_1 \wedge \tilde{\omega}_2 = -\tilde{\mathrm{e}}_1 \wedge \tilde{\mathrm{e}}_2,\\
&\tilde{\mathrm{e}}_1 \wedge \omega_2 - \tilde{\mathrm{e}}_2 \wedge \omega_1 = -(\mathrm{e}_1 \wedge \tilde{\omega}_2 - \mathrm{e}_2 \wedge \tilde{\omega}_1),\\
&\tilde{\mathrm{e}}_1 \wedge \omega_1 + \tilde{\mathrm{e}}_2 \wedge \omega_2 = -(\mathrm{e}_1 \wedge \tilde{\omega}_1 + \mathrm{e}_2 \wedge \tilde{\omega}_2).
\end{aligned} \tag{5.32}$$

We can use these expressions to rewrite the metric and $F_{(3)}$ in a new form. It is very important to notice that even though these two quantities will look quite different from above, what we do is only a rewriting of (5.27) and (5.28). For the metric we obtain then

$$\begin{aligned}
ds^2 = e^{\phi/2}\Big(&dx_{1,3}^2 + dr^2 + e^{2\bar{h}}(\tilde{\mathrm{e}}_1^2 + \tilde{\mathrm{e}}_2^2)\\
&+ \frac{e^{2\bar{g}}}{4}((\omega_1 + \bar{a}(r)\tilde{\mathrm{e}}_1)^2 + (\omega_2 + \bar{a}(r)\tilde{\mathrm{e}}_2)^2) + \frac{e^{2k}}{4}\hat{\omega}_3^2\Big)
\end{aligned} \tag{5.33}$$

and for the threeform flux

$$\begin{aligned}
F_{(3)} = \frac{\bar{N}_c}{4}\{&-\omega_1 \wedge \omega_2 \wedge \hat{\omega}_3 - \bar{b}(r)(\tilde{\mathrm{e}}_1 \wedge \omega_2 - \tilde{\mathrm{e}}_2 \wedge \omega_1)\\
&\wedge \hat{\omega}_3 - \bar{b}'(r)dr \wedge (\tilde{\mathrm{e}}_1 \wedge \omega_1 + \tilde{\mathrm{e}}_2 \wedge \omega_2)\\
&- (1 - \bar{x})\tilde{\mathrm{e}}_1 \wedge \tilde{\mathrm{e}}_2 \wedge \hat{\omega}_3\},
\end{aligned} \tag{5.34}$$

where in (5.33) we have defined

$$\begin{aligned}
&e^{2\bar{h}} = \frac{e^{2g}}{4}\left(1 - \frac{a^2 e^{2g}}{4e^{2h} + a^2 e^{2g}}\right), \qquad e^{2\bar{g}} = 4e^{2h} + a^2 e^{2g},\\
&\bar{a} = a\frac{e^{2g}}{4e^{2h} + a^2 e^{2g}},
\end{aligned} \tag{5.35}$$

and in (5.34) we have set

$$\bar{N}_c = N_f - N_c, \qquad \bar{x} = \frac{x}{x - 1}, \qquad \bar{b}(r) = \frac{b(r)}{1 - x} \tag{5.36}$$

in order for the new expressions of the metric and the threeform flux to have again the same apparent form as in the original versions (5.27) and (5.28). By using the definition of $x = N_f/N_c$ and the expression for $\bar{N}_c$, it is immediate to show that $\bar{N}_f = N_f$. In the same way, it follows from (5.36) and (5.29) that

$$\bar{b}(\rho) = (2 - \bar{x})\frac{\rho}{\sinh 2\rho}, \tag{5.37}$$

that is, exactly the same form as in (5.29) with the new value of the N_f/N_c ratio.

Because of the way the metric looks now (5.33), it seems natural to identify as the three-sphere of the internal manifold, the one spanned by θ, φ, and ψ. Therefore evaluating the integral of $F_{(3)}$ over this S^3, we find that the number of colors of the field theory dual to the background (5.33) and (5.34) is $\bar{N}_c = N_f - N_c$, whereas the number of flavors is still N_f. This is very interesting because this is exactly the way the rank of the gauge group changes under Seiberg duality.

Let us now compare the geometries in the IR, where Seiberg duality tells us that the two field theories are the same. By using the small ρ expansions of Section IV C, we can derive the behavior of $\bar{h}$, $\bar{g}$, and $\bar{a}$ near the origin

$$\begin{aligned}
&e^{2\bar{h}} = e^{2h} - \frac{4}{3}(2 - x)\rho^4 + \ldots,\\
&e^{2\bar{g}} = e^{2g} + \frac{4}{3}(2 - x)\rho^2 + \ldots,\\
&\bar{a} = a - 2c_1\rho^3 + \ldots,
\end{aligned} \tag{5.38}$$

and remarkably we see that the newly defined functions correspond to their original homonyms up to the third term in the small ρ series expansions. This is the holographic dual of the statement that the Seiberg pair theories have the same IR behavior.

This result might look a little puzzling at first sight: in (5.33) and (5.34) we have only written in a different way

[13] We thank Agostino Butti for bringing these identities to our attention while describing to us the realization of the $SU(2) \times SU(2)$ isometry of the baryonic branch backgrounds of [24].

the metric and threeform flux but at the same time claim that the background written in this way corresponds to a different theory from the one that is dual to the (same) original solution (5.27) and (5.28). We believe the right way to look at this puzzle is in terms of the dictionary we use to translate the geometry data into field theory predictions. So in a sense, we have two different dictionaries we can use on the same background, corresponding to the two ways of identifying the S^3 inside the internal space.

This is a remarkable result: just by rewriting the flavored background in a different form, we can capture the most relevant features of Seiberg duality for theories with flavors. In fact in a simple and elegant way we can reproduce the change in the rank of the color gauge group while leaving the number of flavors untouched, and we can show that the IR dynamics of the two theories is exactly the same thanks to (5.38). It would be very interesting to study how information about the Seiberg-dual field theory is encoded into string theory in this setup, and which predictions this allows us to make.

VI. FEATURES OF THE $N_f = 2N_c$ SOLUTION

We now discuss the gauge theory interpretation of the particularly simple $N_f = 2N_c$ solution (or $x = 2$ in the notation introduced above), presented in Section IV E, see Eq. (4.23). For this solution (and only for this one), the effective four-dimensional Yang-Mills coupling, see (5.16), is constant (notice that the coupling is constant regardless of the cycle used to define it)

$$\frac{4\pi^2}{g_{\rm sqcd}^2} = e^{2h} + \frac{e^{2g}}{4}(a-1)^2 = N_c \frac{4}{\xi(4-\xi)}. \tag{6.1}$$

It is also the only solution for which the $U(1)_R$ symmetry is preserved. There is not anomalous UV breaking (we can apply the discussion of Section V D, Eq. (5.18)), while there is not spontaneous IR breaking since $a = b = 0$. We would therefore not expect to have domain walls and this is in fact the case since its tension as defined in (5.26) would vanish ($e^{2\phi} \to 0$ when $r \to -\infty$). We think that these two features are signaling an IR conformal fixed point in the effective four-dimensional theory.

That this happens only for $N_f = 2N_c$ is in precise agreement with the field theory interpretation of Section V A. Notice that, unlike in pure $\mathcal{N} = 1$ SQCD, there should not be a conformal window for $N_f \leq 2N_c$. In fact, having an IR effective quartic superpotential for the quarks (5.9) fixes their R-charge to $\frac{1}{2}$. The Novikov Shifman Vainshtein Zakharov (NSVZ) beta function, which can be written in terms of the (IR) R-charges of the fields can consequently only vanish when $N_f = 2N_c$. This can also be seen using the Wilsonian beta functions for the gauge coupling $\beta_g \sim (3N_c - N_f(1-\gamma_0))$ and the quartic coupling $\beta_h \sim 1 + 2\gamma_0$ which vanish simultaneously if and only if $\gamma_0 = -\frac{1}{2}$ and $N_f = 2N_c$ (γ_0 is the anomalous dimension of the quark superfield).

Nevertheless, we cannot claim that the solution (4.23) is a faithful dual to a conformal four-dimensional theory. Since there is no AdS_5 space, the conformal group $SO(2, 4)$ does not show up in the gravity side and also there is no superconformal enhancement of the supersymmetry. We believe that the explanation resides on the fact that we start with a six-dimensional theory living on the D5-branes compactified with a twist on an S^2 and then add flavor branes. The four-dimensional theory never completely decouples from the Kaluza-Klein degrees of freedom and this spoils the would-be conformal symmetry. Indeed, even though the parameters and couplings might be tuned so that the massless modes are in a "conformal phase," the KK modes spoil this symmetry and the background reflects it by not allowing an AdS_5 to appear. To add to this interpretation we can, for instance, make a dilatation of the four-dimensional space-time $x_i \to \lambda x_i$, that can be compensated (we take string frame) by a shift $r \to r - 2\log\lambda$, which should be regarded as a rescaling of the energy scale (see (5.22)). This is not an isometry of the metric since the internal space gets rescaled, thus rescaling the KK scale. However, we find appealing that the geometry is not deformed in any other way, since the functions h, g, k, a do not depend on r. Notice that this happens only for this simple $N_f = 2N_c$ solution.

In any case, one could not expect to find a weakly curved AdS_5 since this would imply that the coefficients of the conformal anomaly are equal $a = c$ at leading order in N_c [46]. Since our theory has $N_f \sim N_c$, one has $a \neq c$ at leading order [47], ruling out the possibility of having a faithful supergravity dual. Nevertheless, we think it is very nice that some features can be matched in this setup.

Finally, let us conjecture on the gauge theory meaning of the parameter ξ. $\mathcal{N} = 1$ SQCD with $N_f = 2N_c$ and a quartic superpotential has a one complex-dimensional family of marginal deformations in the two complex-dimensional parameter space of the gauge coupling and of the coupling of the quartic term (see [45]). The parameter ξ changes the volume of the spheres (4.22) so it is natural to think that it would change the masses of the adjoints coming from the KK modes and, indeed, modify the value of the coupling at the "conformal fixed point" (6.1). Thus, it may be a parameter signaling this conformal line.

Let us see this more precisely. In this geometry, we could define couplings (not necessarily the SQCD coupling in (6.1)), to be proportional to the inverse volume of two-cycles

$$\frac{1}{g_4^2} \sim \frac{{\rm Vol}\Sigma_2}{g_6^2}. \tag{6.2}$$

Here $g_6^2 = \alpha' g_s N$ is the coupling of the little string theory. So, different definitions would imply the choice of different two-cycles in the geometry, like

$$\frac{1}{\hat{g}^2} = \frac{\mathrm{Vol}\hat{S}^2(\theta, \varphi)}{g_6^2}, \qquad \frac{1}{\tilde{g}^2} = \frac{\mathrm{Vol}\tilde{S}^2(\tilde{\theta}, \tilde{\varphi})}{g_6^2}, \qquad \frac{1}{g_{\mathrm{sqcd}}^2} = \frac{\mathrm{Vol}S^2(\theta, \tilde{\theta}, \varphi, \tilde{\varphi})}{g_6^2}. \tag{6.3}$$

Let us be more precise and define the cycles above. The first two-cycles are given by the effective geometry obtained from (4.23) when imposing that all the coordinates are constant except (θ, φ), or $(\tilde{\theta}, \tilde{\varphi})$ in the case of $\hat{S}^2(\theta, \varphi)$ and $\tilde{S}^2(\tilde{\theta}, \tilde{\varphi})$, respectively. The last cycle is defined like in (5.14) and gives the coupling in (6.1).

So, computing these "couplings" one gets

$$\frac{1}{\hat{g}^2 N_c} \sim \frac{1}{\sqrt{\xi}} E\left[\pi, \frac{\xi - 4}{\xi}\right], \qquad \frac{1}{\tilde{g}^2 N_c} \sim \frac{1}{\sqrt{4-\xi}} E\left[\pi, \frac{\xi}{\xi - 4}\right], \qquad \frac{1}{g_{\mathrm{sqcd}}^2 N_c} \sim \frac{4}{\xi(4-\xi)}, \tag{6.4}$$

where $E[\pi, k]$ is the second kind Elliptic function. Now, let us observe something quite nice. If we interchange the cycles $\hat{S}^2(\theta, \varphi) \leftrightarrow \tilde{S}^2(\tilde{\theta}, \tilde{\varphi})$, this is equivalent to the interchange $\xi \leftrightarrow (4 - \xi)$, or $\hat{g} \leftrightarrow \tilde{g}$. The Seiberg-like duality defined in Section V G does precisely this job. So, we have a line of fixed points, parametrized by ξ (solutions where the two-spheres have constant radius), which is reflected under Seiberg duality. This is what is known to happen in a field theory like the one discussed in Section V, that is $\mathcal{N} = 1$ SQCD plus a quartic potential, when the theory is at its conformal point ($N_f = 2N_c$, and the anomalous dimension of the quark superfields is $\gamma_0 = -\frac{1}{2}$). For a review of these results, see Strassler's beautiful lecture notes [45]. Indeed, following [45], we can see that the line of fixed points has a self-dual point (under Seiberg duality), that in our case is $\xi = 2$. Then the duality, in our case the interchange $\xi \leftrightarrow (4 - \xi)$ or the interchange of two-cycles, interchanges points on the fixed line that lie on different sides of the self-dual point. We can see that this is happening in Eqs. (6.4), for a holomorphic coupling that should be a combination of g_{sqcd} and the quartic coupling.

This suggests that as for the coupling (6.1), it might be possible to find another two-cycle whose volume is inversely proportional to the quartic coupling. This is a very nice check that points to the consistency of our approach and interpretation.

Finally, we would like to stress two very interesting facts. First, let us compute the Wilson loop (see Section V B, Eq. (5.10)) for this particular case. It can be seen that for any value of ρ_0, one obtains that the quark-antiquark pair has energy $E_{q\bar{q}} = 0$ and that the separation of the pair is always $L = \pi\sqrt{N_c g_s \alpha'}$. Following the interpretation of Section V B, below this length this computation is unphysical due to the UV completion. For $L > \pi\sqrt{N_c g_s \alpha'}$, the configuration in which the $q\bar{q}$ are joined by a string is never preferred to having two independent particles (notice that in this case, the QCD string tension vanishes $g_{tt}(r = -\infty) = 0$). We interpret this observation as the existence of total screening, the "mesons" have zero energy, in agreement with the fact that this theory does not dynamically generate a scale.

The second observation concerns the black hole solution in (4.26). We can compute the viscosity of the dual quark-gluon plasma following the approach developed in [48]. Since our background (4.26) satisfies the hypothesis of the theorem in [49], we can see that the relation between the shear viscosity and the temperature is of the predicted form $\mathcal{D} = \frac{1}{4\pi T}$. It may be interesting to study other hydrodynamical quantities using the dual solution (4.26). This is a very simple background dual to a field theory with adjoints, fundamentals, and at finite temperature.

Finally, to close this section, we would like to call the reader's attention to the solution presented in Appendix D. There we described a numerical solution for the case $N_f = 2N_c$, that appears after a careful analysis of the BPS equations. Interesting features of this solution are that it has a behavior similar to the flavored case (with $N_f \neq 2N_c$), asymptotes in the UV to the simple solution presented in Eq. (4.23), and the fact that the solution is Seiberg-duality invariant. We do not know how to interpret this new solution from a dual field theory viewpoint. It seems to behave as a flow from a fixed point where a relevant operator is being inserted, breaks "conformality", and goes down to an IR theory similar to the one described by the solutions with $N_f < 2N_c$. We leave this interesting problem for future study.

VII. AN ALTERNATIVE APPROACH: FLAVORS AND FLUXES

As we have seen in Section IV B, the addition of many flavor branes to the background dual to D5-branes wrapped over an S^2 [7] is encoded in the nonclosure of the threeform RR flux $F_{(3)}$. When the flavor branes are smeared over their transverse directions, the failing of the Bianchi identity is proportional to the volume form of the transverse $S^2 \times S^2$

$$dF_3 = \frac{N_f}{4} \sin\theta \sin\tilde{\theta} d\theta \wedge d\varphi \wedge d\tilde{\theta} \wedge d\tilde{\varphi}. \tag{7.1}$$

In Sections III and IV we show that this effect gives rise to a background which has a uniform distribution of $F_{(3)}$ charge along the two two-spheres. As we have shown in the preceding sections this procedure works very nicely and gives rise to a remarkable matching between gauge theory expectations and string predictions.

Nonetheless, we think it could also be very interesting to approach the same problem from a complementary approach. In this section we sketch the general idea and some basic features of this possible new scenario, but setting up and solving the whole problem would require a much deeper development and thorough discussion, which we leave as a possibility for the future.

In this new approach, the failing of the Bianchi identity for the threeform $F_{(3)}$ is achieved through nontrivial fluxes only. Indeed, we know from supergravity that the presence of an NS threeform background deforms the definition of the field strengths of the RR fluxes. Let us suppose to turn on some axion $C_{(0)}$ and NS flux $H = dB_{(2)}$, then the field strength $F_{(3)}$ is defined by

$$F_3 = dC_{(2)} - C_{(0)}H \tag{7.2}$$

and we see that if the axion is nonconstant we can have a nontrivial Bianchi identity for $F_{(3)}$

$$dF_3 = -dC_{(0)} \wedge H = -F_{(1)} \wedge H. \tag{7.3}$$

Notice that in this case the decomposition of $F_{(3)}$ in an exact and nonexact part is natural, and therefore the definition of $C_{(2)}$ is unambiguous (up to gauge transformations, obviously).

In the preceding sections we have always assumed that the SUSY transformations of the flavored background are exactly the same as those of the unflavored solution [7,15]. This is due to the fact that we are adding to the this solution an (infinite) set of BPS branes, which by definition preserve the same supersymmetries as the original background. But when we look at the flavoring of the background from the flux side, this could fail, and in fact, this is what one usually expects when turning on new fluxes in a background. The only condition we think it is still sensible to impose, is that the background can accommodate supersymmetrically embedded branes.

Regarding the geometrical properties of the six-dimensional internal manifold. There are two possibilities to build backgrounds with fluxes that preserve four-dimensional $\mathcal{N} = 1$ supersymmetry: $SU(3)$-structure manifolds [50], and a particular class of $SU(2)$-structure manifolds [51]. In the unflavored background of [7], the internal manifold has $SU(3)$-structure [50] and since we can most likely think of the flavored background as a deformation of the unflavored solution, it seems reasonable to assume that its internal part is an $SU(3)$-structure manifold as well.

Starting from these two assumptions, we can write the spinor ansatz for the supersymmetry transformations. As usual for solutions with four-dimensional Lorentz invariance, we split the IIB ten-dimensional spinors in their four-dimensional and internal six-dimensional parts

$$\begin{aligned} \epsilon_1 &= \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_-, \\ \epsilon_2 &= \zeta_+ \otimes \eta^2_+ + \zeta_- \otimes \eta^2_-, \end{aligned} \tag{7.4}$$

where η^1 and η^2 are spinors of the internal manifold. Notice that $\eta^i_- = (\eta^i_+)^*$. Since $SU(3)$-structures allow only for one globally defined spinor η, we need to take η^1 and η^2 to be proportional

$$\eta^1_+ = \mathfrak{a}\eta_+, \qquad \eta^2_+ = \mathfrak{b}\eta_+. \tag{7.5}$$

A necessary condition to have consistent supersymmetric embeddings of branes in an $SU(3)$-structure manifold, is that the two spinors η^1 and η^2 have the same norm, $\| \eta^1 \|^2 = \| \eta^2 \|^2$ [52], which requires that the functions $\mathfrak{a}$ and $\mathfrak{b}$ in (7.5) satisfy

$$\frac{\mathfrak{a}}{\mathfrak{b}} = e^{i\tilde{\phi}}, \qquad \frac{\mathfrak{a}}{\mathfrak{b}^*} = e^{i\tau} \tag{7.6}$$

with $\tilde{\phi}$ and τ real functions of the internal coordinates. For the smeared configuration we consider in this paper, symmetry imposes that they only depend on the radius of the internal manifold. Notice that the spinor η is defined up to a phase, and therefore τ in (7.6) can be arbitrarily gauge fixed by this symmetry [50]. On the contrary $\tilde{\phi}$ does not depend on the gauge choice.

Instead of $\mathfrak{a}$ and $\mathfrak{b}$ we can define α and β

$$\alpha = \mathfrak{a} + i\mathfrak{b}, \qquad \beta = \mathfrak{a} - i\mathfrak{b} \tag{7.7}$$

in such a way that when we define $\epsilon = \epsilon_1 + i\epsilon_2$, we get

$$\begin{aligned} \epsilon &= \alpha\zeta_+ \otimes \eta_+ + \bar{\beta}\zeta_- \otimes \eta_-, \\ \epsilon^* &= \beta\zeta_+ \otimes \eta_+ + \bar{\alpha}\zeta_- \otimes \eta_-. \end{aligned} \tag{7.8}$$

In terms of α and β the first condition in (7.6) reads

$$\frac{\alpha}{\beta} = -i \cot\frac{\tilde{\phi}}{2}. \tag{7.9}$$

At this point, one should write a smart ansatz for the metric and background fluxes, and then the $SU(3)$-structure techniques developed in [50,53] would allow to derive first order differential equations describing the solution. This is beyond the scope of the present work.

VIII. THE UNFLAVORED CASE $N_f = 0$: DEFORMED SOLUTIONS

This section might be read almost independently of the rest of the paper and addresses the problem of finding gravity duals to different UV completions to $\mathcal{N} = 1$ SYM from the one described in Section II. Concretely, we will deal with the setup presented in Section IVA. These solutions have appeared, in the context of this work, because in the procedure we devised to add flavors, we first deformed the original (unflavored) background (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8), see the details in Section IV. As explained in Section IVA, these backgrounds are an interesting subcase of the ansatz studied in [24].

Below, we will study a little more the interesting solutions of the unflavored system (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), and (4.8). In Section VIII A, we describe the backgrounds arising from the study of these equations and in the rest of Section VIII we discuss their gauge theory interpretation.

A. Description of the solutions

Equations (4.6) have a one-parameter family of regular solutions. In fact, for small values of the radial coordinate, we find a very well controlled expansion in terms of a parameter μ taking values in the interval $(-2, -2/3]$:

$$a(\rho) = 1 + \mu\rho^2 + \ldots,$$
$$e^{2k} = N_c\left(\frac{4}{6+3\mu} - \frac{20+36\mu+9\mu^2}{15(2+\mu)}\rho^2 + \ldots\right). \tag{8.1}$$

By inserting it in (4.7) and (4.8) we find, for the other functions:

$$e^{2g} = N_c\frac{4}{6+3\mu} + \ldots, \qquad e^{2h} = N_c\frac{4\rho^2}{6+3\mu} + \ldots,$$
$$e^{2f-2f_0} = 1 + \frac{(2+\mu)^2}{8}\rho^2 + \ldots. \tag{8.2}$$

In the case in which the parameter $\mu = -2/3$ we reobtain the known solution (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8), that is:

$$a = \frac{2\rho}{\sinh 2\rho}, \qquad e^{2k} = N_c \tag{8.3}$$

and in the case $\mu = -2$ (and $N_c = 0$) we get four-dimensional Minkowski times the deformed conifold (see Appendix C). For $\rho \to \infty$ (except in the particular $\mu = -\frac{2}{3}$ case), the solutions asymptote to the deformed conifold metric (for details, see Appendix C), i.e. very far away from the branes their effect becomes negligible and the background asymptotes to a Ricci-flat geometry.

In the Fig. 3 we present numerical plots of several of the functions defining the geometry for different values of μ. Notice, in particular, that, unlike the usual solution (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8), here the dilaton does not diverge.[14]

B. Gauge theory analysis

All the deformed solutions presented in Section VIII A are nonsingular duals to field theories with field content being a non-Abelian gauge field, a Majorana fermion, and a set of massive adjoint scalar multiplets. These field theories in principle confine since the warp factor in the metric (4.1) goes to a constant value near $\rho = 0$. This will be analyzed in more detail in Section VIII G. Regarding the string theory picture, this is a compactification of N_c D5-branes on a two-sphere, that preserves $\mathcal{N} = 1$ SUSY. As studied in many places, and recently reviewed in detail in [20], this leads to a field theory whose Lagrangian is the one of $\mathcal{N} = 1$ SYM coupled to a set of massive scalar multiplets. We will assume this picture below, and will argue that the dynamics of the massive modes is related to the parameter that labels the family of solutions.

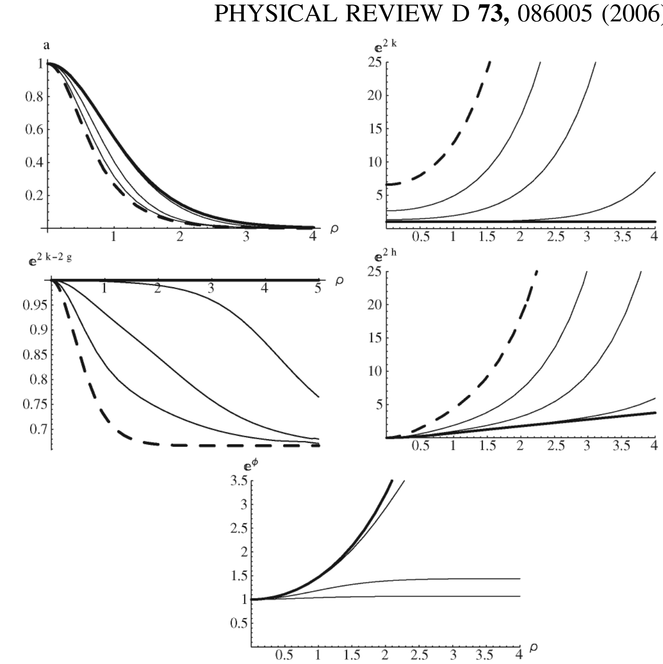

FIG. 3. We plot the different functions for different values of the parameter μ. The thick solid line corresponds to $\mu = -\frac{2}{3}$ (the usual case [7]), the solid thin lines correspond, respectively, to $\mu = -.68, -1, -1.5$, and the dashed line is a deformed conifold.

Contrary to what one might think, this is not the "braneside" of the solution (2.1) (for the case of D6-branes this "braneside" solution was found in [54]). If these new solutions were the "braneside," one should see a resolved conifold and the metric structure that characterizes the D5-branes, near the origin of the space. One might want to interpret this new solution as the "non-near-horizon" version of the solution of D5-branes wrapping a two-cycle inside the conifold. This must be at least partially correct, since at $\rho \to \infty$ one goes to Ricci-flat geometry and the effect of the fluxes disappears (notice also that a metric like (4.1) cannot be obtained by uplifting from 7d gauged sugra in the same way as (2.1)). However, it cannot be the whole story since, as we will see, the IR physics is also modified within the family of solutions.

The proposal we want to put forward is that different members of this family (that can be thought of as different values of μ in the window $(-2, -2/3]$) differ in the superpotential that affects the dynamics and masses of the KK modes. At the same time, it is also possible that different members of this family differ in VEV's for some operator. Then, the UV completion of the theory changes, modifying

[14]A usual criticism to solutions like (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8) is that since the dilaton diverges for large values of the radial coordinate, one must S-dualize and we should be dealing with NS5 branes, hence, the UV completion is a little string theory, which is an unconventional completion for a field theory.

the large ρ behavior of the background (this is in analogy with turning on an irrelevant operator in $AdS_5 \times S^5$). But this change in the UV dynamics also alters the infrared (in analogy with what would be a dangerously irrelevant operator).

Indeed, we know that all these infinite solutions preserve $\mathcal{N} = 1$ SUSY by virtue of a twisting procedure in the gauge theory. It should be quite interesting to analyze how the developments in [20,21] apply to this new set of configurations. It is also interesting to notice that any expansion that starts out of the window $(-2, -2/3]$ for the parameter μ mentioned above generates pathological solutions. This suggests that the dynamics of the KK modes is such that it shows some instability or similar sick behavior for solutions that are outside this window (as the supergravity solution becomes singular).[15] Besides, we stress that the values $\mu = -2$, $\mu = -2/3$ correspond to conifolds with maximal and zero deformation parameter. So, from the gravity side the existence of this window is clear, see Appendix C.

A first check of our proposal comes from constructing new backgrounds doing solution-generating transformations on these μ-family of solutions, that have a $U(1) \times U(1)$ isometry. These isometries are global symmetries of the KK part of the spectrum, hence, constructing a new background by $SL(3, R)$ transformations based on those $U(1)$'s (following, for example, the techniques developed in [55]), will only affect the dynamics of the KK modes. This idea was proposed in [19,30] to be the way to distinguish between a pure SYM effect and the dynamics of the KK modes influencing some observable. When we perform the transformation on this family of backgrounds (we will not report here the formulas, since they are quite lengthy), we observe a mixing between the parameter of deformation γ and the parameter μ (changing the dynamics of the KK modes).

In the following, we will study a sample of gauge theory observables that range from the possible existence of massless axionic glueballs, k-string tensions, annulons and PP-waves spectrum, Wilson loops, and beta function. Technical details can be found in Appendix E. These examples will support the validity of the proposal that we have made above.

C. Axionic massless glueballs

In the paper [56], the authors studied the possibility of finding a massless glueball in the exact solution (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8). They found that this glueball was not normalizable, which led them not to consider it to be a physical excitation. This is in contrast to what happens for the Klebanov-Strassler solution or a nonsupersymmetric deformation of it in which there are, in fact, physical massless glueballs [56,57].

In the exact (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8) solution ($\mu = -\frac{2}{3}$), the dilaton diverges at large values of the radial coordinate and this is the root of the non-normalizability of this mode. In this new set of solutions, we have that for $\mu \neq -2/3$ the dilaton is bounded at infinity, see Fig. 3. It is natural, therefore, to wonder whether this family of solutions might have a massless glueball. Let us analyze this in some detail. In principle, we do not expect this massless glueball to exist, since there is not in this model any apparent $U(1)_B$ symmetry that is spontaneously broken, leading to a massless supermultiplet of composites.

In the paper [58] the fluctuated equations of motion were studied in detail. Let us assume that the background fields vary according to

$$g_{\mu\nu} \to g_{\mu\nu} + \epsilon h_{\mu\nu}, \qquad \phi \to \phi + \epsilon\delta\phi, \qquad F_{\mu\nu\rho} \to F_{\mu\nu\rho} + \epsilon\delta F_{\mu\nu\rho}. \tag{8.4}$$

So, keeping only linear order in the parameter ϵ, we get equations for the fluctuated fields that we report in detail in Appendix E. One can easily see that a solution to these equations is the one found by the authors of [56]

$$\delta F_3 = *da, \qquad h_{\mu\nu} = \delta\phi = 0; \qquad F_3.\delta F_3 = 0. \tag{8.5}$$

Once again, the point in the paper [56], is that this fluctuation, due to the diverging dilaton is not normalizable, so they reject the excitation as a physical state. We might wonder what will happen now that we have a new family of solutions with a bounded dilaton. The norm of the fluctuation is

$$|\delta F_3|^2 = (*_4 da)^2 \int_0^\infty dr e^{2k+2g+2h+5\phi}. \tag{8.6}$$

In this case the dilaton does not diverge, but the functions g, h, k are unbounded making again the mode under consideration not normalizable. Even in this family of solutions, therefore, we do not seem to have this massless glueball. There might be another combination of fields that could play that role, but this does not seem likely, due to the fact that this family of theories (if they differ from (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8) in the masses and dynamics for the KK modes and possibly condensates) should not have a $U(1)$ symmetry that is spontaneously broken leading to a massless excitation.

D. Confining k-strings

One very interesting problem in $SU(N)$-QCD and similar theories is the study of flux tubes induced by color sources in higher representations. If the source has k fundamental indexes, the flux tube is called the k-string. The

[15]One might speculate with the existence of a potential term in the KK Lagrangian of the form (Φ denotes a generic KK mode) $V = \mu^2/2\Phi^2 - (\mu + 2)(\mu + 2/3)\Phi^{2n} + \ldots$ that captures this behavior.

law for the tensions of these k-strings has been subject of interest, from the lattice viewpoint and also from a more formal side.

In the very nice paper [59], Herzog and Klebanov tackled this problem using AdS/CFT-like dualities for confining theories. They considered a confining string (connecting quarks in the antisymmetric representation) as a D3-brane wrapping a cycle near the IR of the geometry. They found the tensions of these strings. Let us briefly summarize their approach and apply it to our cases of interest. This can be done easily with the expansion written in (8.1) and (8.2), since we have to look at the geometry near $\rho = 0$. It is easy to see that our geometry (after the choice of the cycle $\theta = \tilde{\theta}$, $\varphi = 2\pi - \tilde{\varphi}$, ψ and a rescaling $\psi \to \psi/2$) turns exactly into the geometry in Eq. (12) of [59]. So, near $\rho = 0$ the metric induced on these D3-branes that wrap the cycle above and extend in (t, x) forming a confining string, reads

$$ds^2 = e^{2f_0}[dx_{1,1}^2 + e^{2k(0)}(d\psi^2 + \cos^2\psi(d\theta^2 + \sin^2\theta d\varphi^2))]. \tag{8.7}$$

Following the approach of Herzog and Klebanov, we can compute the tension of the q-confining string as the tension of a D3-brane wrapping the finite three-cycle above

$$T \sim \left[b^2\sin^4\psi + \left(\psi - \frac{\sin 2\psi}{2} - \frac{\pi q}{N_c}\right)^2\right]^{1/2}. \tag{8.8}$$

The coefficient is $b = \frac{e^{2k(0)}}{N_c} = \frac{4}{6+3\mu}$. Only for $\mu = -2/3$, that is for the exact solution (8.3), we have $b = 1$. It is precisely this coefficient the one that propagates into formulas (12)–(18) of [59]. Then, when minimizing the action of this confining string, our formulas will be exactly like the ones in [59]. The tension of the confining string is minimal when

$$\psi = \frac{1-b^2}{2}\sin\psi + \frac{\pi q}{N_c}, \qquad T_q = b\sin\psi\sqrt{1 + (b^2-1)\cos^2\psi}. \tag{8.9}$$

The law that is conjectured to work for $\mathcal{N} = 1$ SYM (see for example [60]) is

$$\frac{T_k}{T_k'} = \frac{\sin(\pi k/N_c)}{\sin(\pi k'/N_c)}, \tag{8.10}$$

whereas the string tension in this family of models is not obeying a sine law, and only for the exact solution ($\mu = -2/3$) this behavior is reproduced. It is interesting to notice that observables like this seem to diverge for $\mu = -2$, but we know that this point is out of our family.

We mentioned above that we could interpret the parameter μ as different masses, and most likely also different parameters in the superpotential, for the KK modes. This matches nicely with the findings of the paper [61]. The authors there have pointed out that if one considers $\mathcal{N} = 1$ SYM as broken $\mathcal{N} = 2$ SYM, the sine law is not universal. The set of models we have resembles quite much these $\mathcal{N} = 2 \to \mathcal{N} = 1$ models (it is a little more subtle since we have a twisting). In the analysis done in [59] for the Klebanov-Strassler (KS) model, it was found that confining strings there do not present a sine law behavior. In this aspect, the family of deformed solutions we are studying, behaves like the KS solution.

E. Rotating strings

We will study rotating strings in the same spirit as advocated by Gubser, Klebanov, and Polyakov in the paper [62]. These rotating strings should be dual to large operators in the gauge theory (composed of a large number of adjoint fields). We will see that these strings display characteristic energy-angular momentum relations and sample (for short strings) the different dynamics of the KK modes, that is, they depend on the particular value of μ. There are a number of solutions already found in [31,63] that show this behavior. So, consider, for example, a string that rotates on S^2 and is stretched in the radial direction parametrized by

$$t = \kappa\tau, \qquad \varphi = \omega\tau, \qquad \theta = \frac{\pi}{2}, \qquad \rho = \rho(\sigma). \tag{8.11}$$

After writing the Nambu-Goto action, the energy and angular momentum can be computed (for a string that stretches from the origin to a position ρ_0) one gets [31,63]

$$E \sim \int_0^{\rho_0} \frac{e^{2\phi+2k}}{\sqrt{e^{2\phi+2k}(\kappa^2 - \omega^2(e^{2h} + \frac{e^{2g}}{4}a^2))}}, \qquad J \sim \int_0^{\rho_0} \frac{e^{2\phi+2k}(e^{2h} + \frac{e^{2g}}{4}a^2)}{\sqrt{e^{2\phi+2k}(\kappa^2 - \omega^2(e^{2h} + \frac{e^{2g}}{4}a^2))}}. \tag{8.12}$$

By doing an expansion near $\rho_0 \sim 0$, that is by considering short strings, it is easy to see that the dispersion relations for this string depend on the parameter μ. This means that the operator dual to this string configuration contains a gauge invariant combination of fields, including the KK fields discussed above. In other words, these operators will have different energy-angular momentum relations depending on the member of the family of solutions we consider.

Now, we would like to study string configurations whose energy and angular momentum expressions *do not* depend on the μ parameter. The latter are more interesting than the solutions discussed above, since they should be dual to pure gauge theory operators made out of A_μ, λ, and derivatives; so we will be more detailed. For this, we consider a configuration for a bosonic string that rotates on "the gauge theory space" and on an internal direction (chosen so that it will give the string some R-charge) and also stretches along the radial direction. We consider the ansatz (see Appendix E for details)[16]:

$$t = \kappa\tau, \qquad x = R(\sigma)\cos(\omega_1\tau), \qquad y = R(\sigma)\sin(\omega_1\tau),$$
$$\rho = \rho(\sigma), \qquad \psi = 2\omega_2\tau. \tag{8.13}$$

Let us take a look at some particularly interesting solutions.

1. Rotating folded closed string

Let us consider constant ρ. Then, one has to take $\rho = \omega_2 = 0$ (notice that, at $\rho = 0$, we have $\partial_\rho k = \partial_\rho \phi = 0$) and:

$$R = \frac{1}{\omega_1}\cos\omega_1\sigma. \tag{8.14}$$

This is dual to a high spin glueball. Notice that the slope of the Regge trajectory will depend on the tension of the string at the bottom of the geometry and therefore on e^{ϕ_0}.

2. Pointlike strings rotating in ψ

Consider:

$$R = \text{const}, \qquad \rho = \text{const}. \tag{8.15}$$

Then, we need $\omega_1 = 0$ and:

$$\partial_\rho k = 0 \Rightarrow \rho = 0, \qquad e^{2k}\omega_2^2 = 1. \tag{8.16}$$

These strings are dual to (long) R-charged operators.

It is interesting to notice that when computing observables for these strings, like energy and angular momentum (R-charges), they do not depend on the parameter μ hence, they are the same for different members of the family. This can be interpreted as an indication[17] that these strings are a purely gauge theory effect and are dual to (large) operators that are gauge invariant combinations of A_μ, λ. On the contrary, moving strings like the ones considered in (8.11) and (8.12) (originally found in [31,63]), show dependence on the μ parameter when computed in these new backgrounds. This is because they are dual to large operators containing KK modes, as established in [31] (following the proposal of [19]). It would be interesting to recompute our solutions above in the deformed background proposed in [19].

F. Penrose limits and PP-waves

This subsection is constructed on the work done in the very good paper [66]. Let us summarize their findings. They studied a Penrose limit of the background (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8) on a geodesic that is mostly localized near $\rho = 0$. The limit gives a PP-wave geometry on which the string theory can be quantized. The authors of [66] associated this geodesic on the string side with an operator on the gravity side that looks like a large chain composed of KK modes, that they called annulon. The spectrum of excitations of the annulon was put in correspondence with the spectrum of the string on the PP-wave geometry [66]. We will follow their approach and take the same Penrose limit on our one-parameter family of solutions. The details of the computation are spelled out in Appendix E. After following the approach in [66] adapted to the present case, we get a plane wave background that looks

$$e^{-2k(0)-\phi(0)/2}L^2 ds^2 = -2dx^+dx^- + d\vec{x}_3^2 + d\vec{y}_2^2 + dz^2 + \frac{1}{4}d\vec{v}_2^2 - \left(\frac{\vec{v}_2^2}{4} + \frac{\mu^2}{4}\vec{y}_2^2\right)d\phi_+^2,$$
$$F_3 = \left(2dv_1 \wedge dv_2 + \frac{\mu}{2}dz_1 \wedge dz_2\right) \wedge dx^+. \tag{8.17}$$

So, we see that we have four massless directions ($\vec{x}_3$, z) and four massive ones; $\vec{v}_2$ with mass[18] $m^2 = 1$ and $\vec{y}_2$ with $m^2 = \mu^2/4$. The excitations of this PP-wave are dual to operators composed out of KK modes and one can clearly see that the parameter μ is related to the mass of the KK modes. As we mentioned before, it may be the case that μ is also influencing the dynamics of the KK's or condensates, but this is erased in the Penrose limit and we only see the effect in the masses. So, following the treatment in [66], we can compute the spectrum for strings on this family of PP-wave backgrounds and put the dynamics of this string in correspondence with the dynamics of the family of annulons.

G. Wilson loop

We would like to analyze here a subtle point related to Wilson loops. We will follow the treatment developed in [35] and carefully revised in [36]. We will consider a probe brane that is wrapping the same 2-cycle as the branes that originate the geometry, but this probe brane is very far away from the origin (this probe was proven to be SUSY preserving in [15]). This leads to very massive fields that we associate with quarks, and we compute the expectation value of the Wilson loop as

$$\langle W \rangle \sim e^{S_{\rm NG}}, \tag{8.18}$$

where $S_{\rm NG}$ is the Nambu-Goto action for a string in the background (4.1). After choosing a configuration for a string that extends in the coordinates x, ρ, t and is parametrized by the world sheet coordinates σ, τ according to $x = \sigma$, $\rho = \rho(\sigma)$, $t = \tau$, the NG action will read

[16]See also [64] for the discussion of a similar configuration in the backgrounds of [7,65].

[17]Notice that most probably μ-independence is a necessary but not sufficient condition to ensure that the operator does not contain KK modes.

[18]Notice that the mass of the two modes parametrized by $\vec{v}_2$ is not affected by the μ-deformation and is common a feature to all members of the family.

$$S_{\rm NG} = \frac{1}{2\pi\alpha'}\int d\sigma\sqrt{|g_{xx}g_{tt}| + |g_{tt}g_{\rho\rho}|\rho'^2} = \frac{1}{2\pi\alpha'}\int d\sigma\sqrt{F^2(\sigma) + G^2(\sigma)\rho'^2}. \qquad (8.19)$$

Notice that this action implies the existence of a conserved quantity, that we associate with $F(\rho_0)$, where ρ_0 is the point of maximal proximity of the probe string to the origin of space $\rho = 0$. The separation of the two very heavy quarks is given by $L = 2F(\rho_0)\int_{\rho_0}^{\rho_1}\frac{G}{F\sqrt{F^2 - F(\rho_0)^2}}d\rho$, where ρ_1 is the distance where we put the probe brane (in principle one should take $\rho_1 \to \infty$ to make the quarks very massive).

So, following the treatment reviewed in [36], one gets that the energy of the quark antiquark pair is given by (normalized after we subtract the infinite mass of the nondynamical quarks)

$$E_{q\bar{q}} = F(\rho_0)L + 2\int_{\rho_0}^{\rho_1}\frac{G}{F}\sqrt{F^2 - F(\rho_0)^2}d\rho - 2\int_0^{\rho_1}Gd\rho. \qquad (8.20)$$

The authors of [35], proved a very nice theorem stating under which conditions the energy of the quark antiquark leads to confinement. Some hypotheses are done on the behavior of the functions $F(\rho) = e^{\phi}$ and $G(\rho) = e^{\phi+k}$

(1) $F(\rho)$ is analytic and near $\rho = 0$, $F = F(0) + \sum_k a_k\rho^k$
(2) $G(\rho)$ is analytic and near $\rho = 0$, $G = b_j\rho^j$ $(j > -1)$
(3) $F(\rho)$, $G(\rho)$ are positive
(4) $F'(\rho)$ is positive
(5) The integral $\int^{\infty}\frac{G}{F^2}d\rho$ converges.

We can see that the functions for our family of backgrounds behave, for small values of the radial coordinate, as

$$F \sim e^{\phi_0}\Big(1 + \frac{(2+\mu)^2}{4}\rho^2 + \ldots\Big), \qquad G \sim \frac{e^{\phi_0}g_sN_c\alpha'}{\sqrt{6+3\mu}}\Big(2 + \frac{(20 + 4\mu + \mu^2)}{20}\rho^2 + \ldots\Big). \qquad (8.21)$$

So, one can check that hypothesis (1)–(4) above are satisfied but, unless $\mu = -2/3$, hypothesis (5) fails to be satisfied by the members of this family. This is quite remarkable, since it separates the exact solution occurring when $\mu = -2/3$ from the other members of the family (just like what happens when computing string tensions in Subsection VIII D).

We want to argue nevertheless that all this family is dual to confining theories, since the SYM string tension will behave like $T = F(0)$, but the correction to the leading behavior might be different from the one predicted in [35,36]. It would be interesting to evaluate the integrals in (8.20), to obtain the correction to the main confining behavior.

H. Beta function

There is a computation of the $\mathcal{N} = 1$ SYM beta function done in [39]. In spite of being quite controversial (because some of the assumptions are not clear to be correct and one might criticize the regime of validity in which the computation is done), it works in a quite remarkable way, not only because the final result is correct, but also because it keeps working even when deforming the background in quite dramatic ways (like a noncommutative or dipole deformation [19,67]). One way in which we might view this is that the authors of [39] have found a two-cycle (needed to define the coupling) that has very interesting and robust geometrical properties. What we will attempt in this subsection is to do again this calculation of the beta function. We will see that the result seems to change drastically from the one in [39], but we will also point out an interesting subtlety regarding this point.

Let us start by taking the expansion for large values of ρ given in Eqs. (C4) and (C1). We see that the coupling constant, that is defined as the inverse volume of the cycle $\tilde{\theta} = \theta$, $\varphi = 2\pi - \tilde{\varphi}$, $\psi = \pi$, is given by

$$\frac{4\pi^2}{g^2N} = e^{2h} + e^{2g}\frac{(a-1)^2}{4}. \qquad (8.22)$$

Using the fact that the gaugino condensate is related to the function $b(\rho)$ [41] allows us to write $\log(E/\Lambda) = -1/3\log(b(\rho))$. Then computing the beta function gives

$$\frac{dg}{d(\log(\mu/\Lambda))} = -\Big(\frac{g^3N}{2}\Big)\Big(\frac{d(g^2N)^{-1}}{d\rho}\Big)\Big(\frac{d\rho}{d\log(\mu/\Lambda)}\Big) \sim -g. \qquad (8.23)$$

We see that this is not a running that we recognize as the one of a SYM theory. So, since this computation can be done with a member of the family very close to $\mu = -2/3$ (for which the computation above reproduces the NSVZ result), we see that separating slightly from the exact solution (8.3) changes drastically the result. This may cast doubts on this computation being relevant for the SYM beta function.

Nevertheless, there is an interesting fact that should not be overlooked. In Fig. 4 one can see that a coupling defined

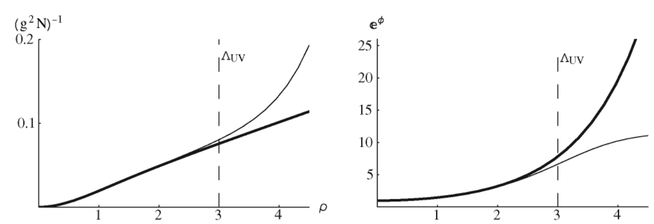

FIG. 4. We plot the inverse 't Hooft coupling and e^{ϕ} for $\mu = -.67$ (thin line) compared with the usual $\mu = -\frac{2}{3}$ case (thick line). Starting from some value of ρ ($\Lambda_{\rm UV}$), the UV completion sets in and the geometry asymptotes to a (Ricci-flat) conifold. $\Lambda_{\rm UV}$ decreases as μ decreases.

as in Eq. (8.22) will behave, near the UV of the field theory, like

$$\frac{1}{g^2 N} \sim \rho, \tag{8.24}$$

if we take the UV of the SYM field theory to be around the region bounded from above by the cutoff $\Lambda_{\rm UV}$, after which the stringy completion to the field theory should set in. With this caveat, following the procedure described in [39], one gets the typical NSVZ result. This addresses one of the criticisms that were raised to the papers [39]. We believe that this puts the whole result in a better context.

IX. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper we have studied two very interesting problems. The first was to find supergravity (string) backgrounds dual to minimally SUSY field theories in four dimensions containing fundamental matter with $N_f \sim N_c$. After writing the BPS equations that characterize the backgrounds, we solved them asymptotically and showed numerically the smooth behavior of these solutions. Then, we studied numerous gauge theory aspects of these solutions, checking known results and making interesting predictions. We believe the material presented in this part of the paper may be quite useful in future approaches to nonperturbative SQCD-like theories.

In a second part of the paper, we discussed a family of solutions dual to SYM plus some UV completion, that generalize the background of [7]. We studied the gauge theory aspects of this family of UV completions by analyzing the supergravity solutions. It is quite interesting to uncover the differences between gauge theory observables of different members of the family of solutions.

We would like to comment now on possible future directions that this paper opens. Indeed, the first thing that comes to mind is that the formalism developed in the first part of the paper could be immediately generalized to find string duals to field theories with flavor in different number of dimensions and with different number of SUSY's, using backgrounds like the ones in [68].

It would be interesting, following, for example, the technology developed in [65,69], to find nonsupersymmetric deformations of the solutions we have presented here. This might prove a way to get information on QCD-like theories with an arbitrary number of colors and flavors.

We left some open points, specially in Section V. Indeed, it would be very nice to get a more complete understanding of Seiberg duality, the tension of the domain walls, and the field theory living on them. Improving our understanding of the gauge theory, following, for example, the ideas in [21], would be desirable. It would be very nice to explore some hydrodynamical aspects of the solution dual to the field theory with $N_f = 2N_c$ at finite temperature. Also, finding a black hole in the solutions with arbitrary N_f and N_c is complicated, but might prove very useful. It would be interesting to follow the formalism in [70] for our background, in order to construct an effective 5d action with which the computation of any correlation function might be feasible (a lot of work should be done in this direction). Also, completing the approach we sketched in Section VII may have interesting consequences.

Finally, it would be of great interest to find a sigma model describing strings in our backgrounds. In this line, the background with $N_f = 2N_c$, due to its simplicity, is the most suitable to start studying these issues.

We believe that the solution to the problems presented above, among others, could significantly improve the understanding of gauge-strings duality for QCD-like theories.

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APPENDIX A: THE (ABELIAN) BPS EQUATIONS FROM A SUPERPOTENTIAL

It is instructive to obtain the BPS systems of Section III using an alternative approach, computing the so-called superpotential. Obtaining the non-Abelian system of Section IV with this technique would be much harder since it involves algebraic constraints.

Generically, for a Lagrangian of the form (η is the "time" variable, A, φ^a $a = 1, 2, \ldots$ are the fields):

$$L = e^{c_1 A}\Big[c_2(\partial_\eta A)^2 - \tfrac{1}{2}G_{ab}(\varphi)\partial_\eta\varphi^a\partial_\eta\varphi^b - V(\varphi)\Big] \tag{A1}$$

if one can find W such that:

$$V(\varphi) = \frac{c_3^2}{2}G^{ab}\frac{\partial W}{\partial\varphi^a}\frac{\partial W}{\partial\varphi^b} - \frac{c_1^2 c_3^2}{4c_2}W^2 \tag{A2}$$

then the equations of motion are automatically satisfied by the first order system:

$$\frac{dA}{d\eta} = \mp \frac{c_1 c_3}{2c_2} W, \qquad \frac{d\varphi^a}{d\eta} = \pm c_3 G^{ab} \frac{\partial W}{\partial \varphi^b}. \tag{A3}$$

Now, let us recover the system (3.19) and (3.22). The relevant action in Einstein frame is (3.11), i.e. (3.12)+(3.14), since (3.15) does not depend on the metric nor on the dilaton and therefore does not enter the Einstein equations (its effect was taken into account when writing (3.18)).

In order to obtain an effective Lagrangian, we substitute the ansatz (3.2) and (3.18). We also impose by hand the condition $\phi = 4f$. After integrating by parts the second derivatives arising from the Ricci scalar and defining $4Y = 8f + 2g + 2h + k$, we find the effective Lagrangian:

$$L = \frac{1}{2\kappa_{(10)}^2} \frac{1}{8} \sin\theta \sin\tilde{\theta} e^{4Y} [16Y'^2 - 2g'^2 - 2h'^2 - k'^2 - V] \tag{A4}$$

with:

$$V = -2e^{-2h} - 8e^{-2g} + 2e^{-4g+2k} + \frac{1}{8} e^{-4h+2k} + 2N_c^2 e^{-4g-2k} + \frac{(N_c - N_f)^2}{8} e^{-4h-2k} + N_f e^{-2g-2h}. \tag{A5}$$

The last term of this potential comes from the Dirac brane action (3.14) after using (3.16). The Lagrangian (A4) is of the form (A1) and, choosing $c_3 = 2$, Eq. (A2) reads,

$$V = \tfrac{1}{2}(\partial_g W)^2 + \tfrac{1}{2}(\partial_h W)^2 + (\partial_k W)^2 - W^2. \tag{A6}$$

Remarkably, this equation has a very simple solution:

$$W = -\tfrac{1}{4}(N_c - N_f) e^{-2h-k} + N_c e^{-2g-k} - \tfrac{1}{4} e^{-2h+k} - e^{-2g+k} - 2e^{-k}. \tag{A7}$$

In particular, it is amusing to see how simple is the modification of this W when the flavor branes are added. Inserting (A7) in (A3) (with the upper sign), we recover (3.19), (3.20), (3.21), and (3.22).

APPENDIX B: THE BPS EQUATIONS. A DETAILED DERIVATION

In this appendix we analyze in detail the computations that led us to the system of Eqs. (4.14), (4.15), (4.16), and (4.17) when flavoring the nonsingular solution (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8). The analysis of the generalized unflavored ansatz of Section IVA is incorporated in the same formalism just by taking $N_f = 0$.

It is important to notice that the ansatz for the metric we will use in the following is a subcase of that considered in [24] so we can profit from their analysis (although $N_f = 0$ in their case). For the sake of clarity, we rewrite here the ansatz of the main text (4.9) and (4.12). We take the Einstein-frame metric[19]:

$$ds^2 = e^{2f(r)} \Big[dx_{1,3}^2 + dr^2 + e^{2h(r)}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{e^{2g(r)}}{4}((\tilde{\omega}_1 + a(r)d\theta)^2 + (\tilde{\omega}_2 - a(r)\sin\theta d\varphi)^2) + \frac{e^{2k(r)}}{4}(\tilde{\omega}_3 + \cos\theta d\varphi)^2 \Big]. \tag{B1}$$

The vielbein we consider for this metric is the straightforward generalization of (3.1) by the inclusion of the $a(r)$ dependence in e^1 and e^2. Apart from the dilaton, there is also the RR threeform field strength:

$$F_{(3)} = -2N_c e^{-3f-2g-k} e^1 \wedge e^2 \wedge e^3 + \frac{N_c}{2} b' e^{-3f-g-h} e^r \wedge (-e^\theta \wedge e^1 + e^\varphi \wedge e^2) + \frac{1}{2} e^{-3f-2h-k}(N_c(a^2 - 2ab + 1) - N_f) e^\theta \wedge e^\varphi \wedge e^3 + e^{-3f-h-g-k} N_c (b - a)(-e^\theta \wedge e^2 + e^1 \wedge e^\varphi) \wedge e^3. \tag{B2}$$

Let us analyze the dilatino and gravitino transformations. For the functions we have in the ansatz, they read:

$$\delta\lambda = \frac{1}{2} i\Gamma^\mu \epsilon^* \partial_\mu \phi + \frac{1}{24} e^{\phi/2} \Gamma^{\mu_1\mu_2\mu_3} \epsilon F_{\mu_1\mu_2\mu_3},$$
$$\delta\psi_\mu = \partial_\mu \epsilon + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} \epsilon + \frac{e^{\phi/2}}{96} (\Gamma_\mu^{\mu_1\mu_2\mu_3} - 9\delta_\mu^{\mu_1} \Gamma^{\mu_2\mu_3}) i\epsilon^* F_{\mu_1\mu_2\mu_3} \tag{B3}$$

The projections one has to impose on the Killing spinor are:

$$\epsilon = i\epsilon^*, \qquad \Gamma_{\theta\varphi}\epsilon = \Gamma_{12}\epsilon, \qquad \Gamma_{r123}\epsilon = (\mathcal{A} + \mathcal{B}\Gamma_{\varphi 2})\epsilon. \tag{B4}$$

Notice that consistency implies:

$$\mathcal{A}^2 + \mathcal{B}^2 = 1 \tag{B5}$$

so one can parametrize:

$$\mathcal{A} = \cos\alpha, \qquad \mathcal{B} = \sin\alpha, \tag{B6}$$

and then the spinor reads:

[19]Comparing to [24] (we write a subindex B for quantities defined in that paper): $h_{1B} = h_{2B} = \chi_B = K_B = Q_B = 0$. Noticing that in [24] the metric is written in string frame: $A_B = 2f$, $dt_B = 2e^{-k}dr$, $e^{x_B} = \frac{1}{2}e^{4f+h+g}$, $e^{g_B} = 2e^{h-g}$, $e^{6p_B} = 8e^{-8f-2k-h-g}$. Afterwards, we will define $\tilde{g} = g_B$ and $\rho = t_B/2$. Notice also the definitions $du_B = e^{-4p_B}dt_B$, $v_B = e^{6p_B+2x_B}$ and that in [24] N_c has been set to 2. In order to use the most usual convention, we take $a = -a_B$, $b = -b_B$.

$$\epsilon = e^{-(\alpha/2)\Gamma_{\varphi 2}}\epsilon_0, \tag{B7}$$

where ϵ_0 satisfies the unrotated projections:

$$\epsilon_0 = i\epsilon_0^*, \qquad \Gamma_{\theta\varphi}\epsilon_0 = \Gamma_{12}\epsilon_0, \qquad \Gamma_{r123}\epsilon_0 = \epsilon_0. \tag{B8}$$

With this, it is straightforward to obtain the set of differential equations and constrains coming from (B3). The spin connection ($de^a + \omega^a_b \wedge e^b = 0$) is needed. We write it here for completeness:

$$\begin{aligned}
\omega^{x_i r} &= e^{-f} f' e^{x_i},\\
\omega^{3r} &= e^{-f}(f' + k')e^3,\\
\omega^{\theta 1} &= -\omega^{\varphi 2} = \tfrac{1}{4}a' e^{-f-h+g} e^r,\\
\omega^{1\varphi} &= \omega^{2\theta} = a\sinh(k-g)e^{-f-h}e^3,\\
\omega^{\theta r} &= e^{-f}(f' + h')e^\theta + \tfrac{1}{4}a' e^{-f-h+g}e^1,\\
\omega^{\varphi r} &= e^{-f}(f' + h')e^\varphi - \tfrac{1}{4}a' e^{-f-h+g}e^2,\\
\omega^{1r} &= e^{-f}(f' + g')e^1 + \tfrac{1}{4}a' e^{-f-h+g}e^\theta,\\
\omega^{2r} &= e^{-f}(f' + g')e^2 - \tfrac{1}{4}a' e^{-f-h+g}e^\varphi,\\
\omega^{23} &= -e^{-f+k-2g}e^1 + ae^{-f-h}\cosh(k-g)e^\theta,\\
\omega^{21} &= (-e^{-f+k-2g} + 2e^{-f-k})e^3 - e^{-h-f}\cot\theta e^\varphi,\\
\omega^{13} &= e^{-f+k-2g}e^2 + ae^{-f-h}\cosh(k-g)e^\varphi,\\
\omega^{\theta\varphi} &= -\tfrac{1}{4}e^{-2h-f+k}(a^2-1)e^3 - e^{-h-f}\cot\theta e^\varphi,\\
\omega^{3\varphi} &= -\tfrac{1}{4}e^{-2h-f+k}(a^2-1)e^\theta + ae^{-f-h}\sinh(k-g)e^1,\\
\omega^{3\theta} &= \tfrac{1}{4}e^{-2h-f+k}(a^2-1)e^\varphi + ae^{-f-h}\sinh(k-g)e^2.
\end{aligned} \tag{B9}$$

Comparing the equation for the variation of the dilatino and that of $\delta\psi_x$, we immediately find the condition

$$\phi = 4f. \tag{B10}$$

Moreover, from $\delta\psi_x = 0$ we get:

$$\begin{aligned}
f' = {}& \frac{N_c}{4}\mathcal{A}e^{-2g-k} - \frac{1}{16}\mathcal{A}e^{-2h-k}(N_c(a^2-2ab+1) - N_f)\\
&+ \frac{1}{4}\mathcal{B}e^{-h-g-k}N_c(b-a),
\end{aligned} \tag{B11}$$

$$\begin{aligned}
b' = {}& \frac{1}{N_c}\bigg[2N_c e^{h-g-k}\mathcal{B} - \frac{1}{2}e^{g-h-k}\mathcal{B}(N_c(a^2-2ab+1)\\
&- N_f) - 2\mathcal{A}e^{-k}N_c(b-a)\bigg].
\end{aligned} \tag{B12}$$

From $\delta\psi_\theta = 0$ (or equivalently, from $\delta\psi_\varphi = 0$) and using the previous expressions:

$$\begin{aligned}
h' = {}& -\mathcal{B}e^{-h}(a\cosh(k-g) + \tfrac{1}{2}e^{-k-g}N_c(b-a))\\
&- \tfrac{1}{4}\mathcal{A}e^{-2h+k}(a^2-1)\\
&+ \tfrac{1}{4}\mathcal{A}e^{-2h-k}(N_c(a^2-2ab+1) - N_f),
\end{aligned} \tag{B13}$$

$$\begin{aligned}
a' = {}& -4e^{-g}\mathcal{A}a\cosh(k-g) + \mathcal{B}e^{-g-h+k}(a^2-1)\\
&- 2N_c\mathcal{B}e^{h-3g-k} - \tfrac{1}{2}\mathcal{B}e^{-h-g-k}(N_c(a^2-2ab+1)\\
&- N_f).
\end{aligned} \tag{B14}$$

From $\delta\psi_1 = 0$ (or $\delta\psi_2 = 0$):

$$\begin{aligned}
g' = {}& \mathcal{A}e^{k-2g} - \mathcal{B}ae^{-h}\sinh(k-g)\\
&- \tfrac{1}{2}\mathcal{B}e^{-h-g-k}N_c(b-a) - N_c\mathcal{A}e^{-2g-k},\\
0 = {}& -4a\mathcal{A}e^{k-g-h} - 4\mathcal{B}e^{k-2g} + \mathcal{B}e^{-2h+k}(a^2-1).
\end{aligned} \tag{B15}$$

From $\delta\psi_3 = 0$ one finds:

$$\begin{aligned}
k' = {}& -\mathcal{A}e^{k-2g} + 2\mathcal{A}e^{-k} + \tfrac{1}{4}\mathcal{A}e^{-2h+k}(a^2-1)\\
&+ 2a\mathcal{B}e^{-h}\sinh(k-g) - N_c\mathcal{A}e^{-2g-k}\\
&+ \tfrac{1}{4}\mathcal{A}e^{-2h-k}(N_c(a^2-2ab+1) - N_f)\\
&- \mathcal{B}e^{-h-k-g}N_c(b-a),
\end{aligned} \tag{B16}$$

and the algebraic equation:

$$\begin{aligned}
0 = {}& \mathcal{B}e^{k-2g} - 2\mathcal{B}e^{-k} - \tfrac{1}{4}\mathcal{B}e^{-2h+k}(a^2-1)\\
&+ 2a\mathcal{A}e^{-h}\sinh(k-g) + N_c\mathcal{B}e^{-2g-k}\\
&- \tfrac{1}{4}\mathcal{B}e^{-2h-k}(N_c(a^2-2ab+1) - N_f)\\
&- \mathcal{A}e^{-h-g-k}N_c(b-a).
\end{aligned} \tag{B17}$$

Finally, $\delta\psi_r = 0$ yields:

$$\epsilon_0 = e^{\phi/8}\eta, \tag{B18}$$

where η is some constant spinor satisfying the same projections as ϵ_0 and:

$$\alpha' = -\tfrac{1}{2}e^{-h+g}(a' + N_c b' e^{-2g}). \tag{B19}$$

In order to simplify these equations, let us make some definitions:

$$e^{2\tilde{g}} = 4e^{2h-2g} \tag{B20}$$

and, following [24]:

$$\begin{aligned}
S &= \frac{1}{2a}\sqrt{a^4 + 2a^2(-1 + e^{2\tilde{g}}) + (1 + e^{2\tilde{g}})^2},\\
C &= \frac{1}{2a}(1 + a^2 + e^{2\tilde{g}}).
\end{aligned} \tag{B21}$$

Then, the set of algebraic constraints (B5), (B15), and (B17) can be written as:

$$\mathcal{A} = \frac{C-a}{S}, \qquad \mathcal{B} = -\frac{e^{\tilde{g}}}{S}, \tag{B22}$$

$$e^{2g} = \frac{N_c(bC-1) + \frac{1}{2}N_f}{aC-1}. \tag{B23}$$

Although the system seems overdetermined, it is not. By deriving in (B22) and using the rest of expressions, one arrives at (B19), which is therefore redundant. By deriving

(B23) and using all the differential equations and constraints, one finds an identity. Therefore, apart from (B19), one of the other differential equations (say the g' one) is also redundant. Let us count the number of functions. We have f, g, $\tilde{g}$, k, a, b. They are not independent because of (B23). But since the equation of g' plays no role, one can think of having five independent functions with a first order equation for each.

Remarkably, the system of first order equations and algebraic constraints (B11)–(B17) solves the set of Einstein second order equations. The equations are (3.23) and (3.24) along with the equation of motion for the RR form $\partial_\mu(e^{\phi}\sqrt{-g_{10}}F^{\mu\nu\rho}) = 0$. We have checked this using *Mathematica*.

We then would like to solve, after substituting the expressions (B22) and (B23), the Eqs. (B11), (B12), (B14), and (B16) and the equation for $\tilde{g}$ that can be obtained from the above:

$$\tilde{g}' = -\mathcal{B}ae^{-h+g-k} - \tfrac{1}{4}\mathcal{A}e^{-2h+k}(a^2-1) - \mathcal{A}e^{k-2g} + \tfrac{1}{4}\mathcal{A}e^{-2h-k}(N_c(a^2-2ab+1)-N_f) + N_c\mathcal{A}e^{-2g-k}. \tag{B24}$$

Let us define:

$$d\rho = e^{-k}dr, \tag{B25}$$

then, one can prove that:

$$\partial_\rho C = 2S, \qquad \partial_\rho S = 2C, \tag{B26}$$

and by also noticing the identity $C^2 - S^2 = 1$ and by fixing an integration constant (the origin of ρ) we can write:

$$C = \cosh(2\rho), \qquad S = \sinh(2\rho). \tag{B27}$$

From this one can obtain for instance $\tilde{g}$ in terms of a:

$$e^{2\tilde{g}} = 2a\cosh 2\rho - 1 - a^2. \tag{B28}$$

The equation for b is reduced to:

$$\partial_\rho b = \frac{1}{\sinh(2\rho)}\left(2(1 - b\cosh(2\rho)) - \frac{N_f}{N_c}\right) \tag{B29}$$

which is explicitly solved by:

$$b = \frac{(2 - \frac{N_f}{N_c})\rho + \text{const}}{\sinh(2\rho)}. \tag{B30}$$

We will fix this constant to zero. In the usual unflavored case, setting the analogous constant to zero is required in order to have regularity at the origin. As we have seen in Section IV, the flavored case presents a singularity anyway. However, setting the constant to zero seems to allow a milder singular behavior at the origin. Since f never appears in the right-hand sides of the differential equation, we are left with having to solve two coupled equations for a and k. These are the Eqs. (4.15) and (4.16) of the main text. It does not seem possible to find an explicit general solution of these equations. The equation for f is written in (4.17).

Finally, let us comment on three particular explicit solutions of this system: When $N_f = 0$, one has, of course, the usual unflavored solution of [7,8]:

$$e^{2k} = 1, \qquad a = \frac{2\rho}{\sinh 2\rho}, \qquad e^{4f} = e^{4f_0}\frac{\sinh^{1/2}2\rho}{2^{1/2}(\rho\coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4})^{1/4}}, \qquad (N_f = 0). \tag{B31}$$

When $N_c = N_f = 0$ one finds a Ricci-flat geometry which is nothing but the deformed conifold (see Appendix C for more details). And finally, when $N_f = 2N_c$, one has the solution presented in Section IV E. Notice that for these two last cases the manipulations leading to (B26)–(B30), (4.15), and (4.16) are ill-defined, but the solutions can be obtained from the system (B11)–(B17).

APPENDIX C: SOLUTIONS ASYMPTOTING TO THE CONIFOLD

When $N_f = N_c = 0$, the Ricci-flat solution compatible with the ansatz (4.1) is the deformed conifold. Actually, (4.1) is the natural way of writing the deformed conifold metric when it is obtained from gauged supergravity [71]. The different functions of the ansatz take the following values:

$$a(\rho) = \frac{1}{\cosh(2\rho)}, \qquad e^{2k} = \frac{2}{3}\epsilon^{4/3}\mathcal{K}(\rho)^{-2}, \qquad e^{2h} = \frac{1}{4}\epsilon^{4/3}\frac{\sinh^2(2\rho)}{\cosh(2\rho)}\mathcal{K}(\rho), \qquad e^{2g} = \epsilon^{4/3}\cosh(2\rho)\mathcal{K}(\rho), \tag{C1}$$

where ϵ is the deformation parameter, the dilaton is constant, and we have defined:

$$\mathcal{K}(\rho) = \frac{(\sinh(4\rho) - 4\rho)^{1/3}}{2^{1/3}\sinh(2\rho)}. \tag{C2}$$

We have argued, and it is apparent from Fig. 3, that the unflavored equations (4.6), (4.7), and (4.8) as well as the flavored equations (4.14), (4.15), (4.16), and (4.17) have large ρ solutions in which the effect of the branes becomes asymptotically negligible and they must therefore approach the Ricci-flat geometry (C1). In this limit, the only difference between flavored and unflavored is a rescaling of e^{2k}, e^{2h}, e^{2g} by $(1 - \frac{N_f}{2N_c})$.

In fact, this can be proved explicitly by discarding exponentially suppressed terms and considering an expansion starting with:

$$a\cosh 2\rho - 1 = \beta e^{-4\rho/3}(2\rho - 1) + \ldots \tag{C3}$$

We can write:

$$a = 2e^{-2\rho}(1 + \beta e^{-4\rho/3}(2\rho - 1)) + \ldots, \qquad e^{2\tilde{k}} = \frac{2}{3\beta} e^{4\rho/3} + \ldots, \tag{C4}$$

and similar expressions for the rest of functions which agree at leading order with (C1). By comparing (C4) to (C1), we see that β is a constant related to the deformation parameter of the conifold.

APPENDIX D: AN INTERESTING SOLUTION WITH $N_f = 2N_c$

The differential equations (4.15) and (4.16) are not valid for the special value $x = 2$. To study the solutions in this case start then from the unsimplified equations (B11)–(B17). In this appendix, we will show that there is another solution apart from the one presented in Section IV E. The manipulation of the differential equations and algebraic constraints goes along similar lines to the $x < 2$ case, with the important exception that now we use the constraint (B17) to solve for a and b rather than for e^{2g}. We impose that the two independent terms in the equation vanish simultaneously, which gives

$$a = \frac{1}{\cosh 2\rho} \quad \text{and} \quad b = 0. \tag{D1}$$

The choice of this solution to (B17) is suggested by the behavior of the $x < 2$ solutions for x that goes towards 2, see the plots at the bottom of Fig. 1.

At this point, it is easy to show that some of the remaining equations can be used to evaluate

$$\mathcal{A} = \tanh 2\rho, \qquad \mathcal{B} = -\frac{1}{\cosh 2\rho}, \qquad e^{h-g} = \frac{1}{2}\tanh 2\rho, \tag{D2}$$

and now, as before, we can reduce our system of BPS equations to a system of two coupled first order differential equations

$$\partial_\rho e^{h+g} = e^{2k} - N_c, \qquad \partial_\rho k = -(e^{2k} + N_c)e^{-h-g} + 2\coth 2\rho \tag{D3}$$

with the only remaining unknown function determined once we find the solution to (D3):

$$\partial_\rho f = \frac{N_c}{4} e^{-g-h}. \tag{D4}$$

To solve Eqs. (D3), we can obtain e^{2k} from the first one, and substitute it into the second one to obtain a second order differential equation containing only e^{g+h}

$$\tfrac{1}{2}e^{g+h}\partial_\rho^2 e^{g+h} + (\partial_\rho e^{g+h})^2 + \partial_\rho e^{g+h}(3N_c - 2e^{g+h}\coth 2\rho) + 2(N_c - e^{g+h}\coth 2\rho) = 0. \tag{D5}$$

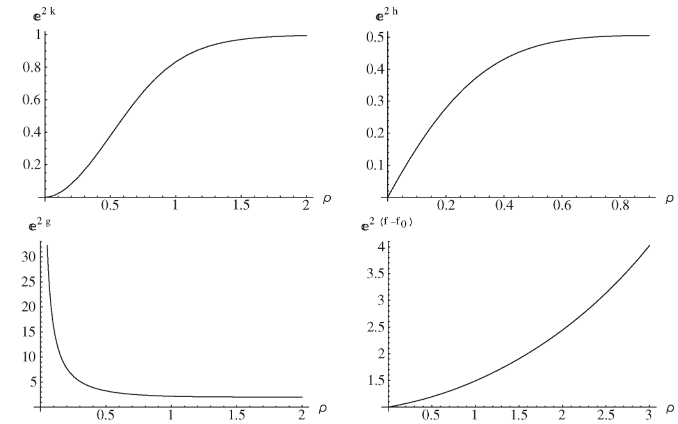

FIG. 5. Behavior of the background functions for the $x = 2$ solution with $a = \frac{1}{\cosh 2\rho}$. From the top left we have $e^{2k(\rho)}$, $e^{2h(\rho)}$, $e^{2g(\rho)}$, $e^{2(f(\rho)-f_0)}$. Here we have taken $N_c = 1$.

In the large ρ region, we can approximate $\coth 2\rho \simeq 1$ up to exponentially suppressed terms, which gives us that $e^{g+h} = N_c$ is an exact solution at large ρ. This also fixes the value of e^{2k} in the large ρ region, $e^{2k} = N_c$, and again the only corrections to this large ρ solution are exponentially suppressed. From (D2) and (D4) we find also

$$e^{2h} \simeq \frac{N_c}{2} + O(e^{-2\rho}), \qquad e^{2g} \simeq 2N_c + O(e^{-2\rho}), \qquad f \simeq \frac{\rho}{4} + \ldots = \frac{r}{4\sqrt{N_c}}. \tag{D6}$$

Notice that this behavior is very similar to the one of the other flavored $x \neq 2$ cases we described in Subsection IV C 1, even though the constants at infinity do not match.

We can now plug these asymptotic conditions in *Mathematica* and obtain the numerical solution to the BPS equations. The functions we obtain are shown in Fig. 5. Notice that this solution has a very nice property. In Section V G we had introduced the gravity dual description of Seiberg duality for the $\mathcal{N} = 1$ SQCD-like theories we are considering. The solution we have presented in this subsection is exactly Seiberg duality invariant. In fact, it is easy to check that this condition imposes a single equation

$$e^{2h} + \frac{e^{2g}}{4}(a^2 - 1) = 0 \tag{D7}$$

which is satisfied exactly by our $a = 1/\cosh 2\rho$ and $e^{h-g} = \frac{1}{2}\tanh 2\rho$.

APPENDIX E: SOME TECHNICAL DETAILS IN SECTION VIII

In this appendix, we will just cover the technical details that were left aside in the analysis of gauge theory aspects of the unflavored solutions.

1. Massless glueballs

Let us start with the fluctuated equations of motion, that lead us to the fact that a normalizable massless glueball does not exist.

The equations for the fluctuation were analyzed in [58] and read, for the fluctuated Ricci tensor equation

$$\frac{1}{2}[\nabla_\alpha\nabla_\mu h^\alpha_\nu + \nabla_\alpha\nabla_\nu h^\alpha_\mu - \nabla_\nu\nabla_\mu h^\alpha_\alpha - \nabla^2 h_{\mu\nu}] = \frac{1}{2}[\partial_\mu\phi\partial_\nu\delta\phi + \partial_\mu\delta\phi\partial_\nu\phi] + \frac{g_s e^\phi}{4}[\delta\phi F_{\mu**}F^{**}_\nu + \delta F_{\mu**}F^{**}_\nu + F_{\mu**}\delta F^{**}_\nu - 2h^{ab}F_{\mu a*}F^{*}_{\nu b}] - \frac{g_s e^\phi}{48}[g_{\mu\nu}(2F_3\delta F_3 - 3h^{ab}F_{a**}F^{**}_b + \delta\phi F_3^2) + h_{\mu\nu}F_3^2]. \quad \text{(E1)}$$

For the fluctuation of the Ricci scalar,

$$\nabla_\mu\nabla_\nu h^{\mu\nu} - \nabla^2 h^\mu_\mu - R_{\mu\nu}h^{\mu\nu} = g^{\mu\nu}\partial_\mu\phi\partial_\nu\delta\phi - \frac{h^{\mu\nu}}{2}\partial_\mu\phi\partial_\nu\phi + \frac{g_s e^\phi}{24}[2F_3\delta F_3 - 3h^{kl}F_{k**}F^{**}_l + \delta\phi F_3^2]. \quad \text{(E2)}$$

For the fluctuated dilaton equation,

$$\nabla^2\delta\phi - h^{\mu\nu}\nabla_\mu\partial_\nu\phi - \frac{1}{2}g^{rr}\partial_r\phi(2\nabla^\mu h_{r\mu} - \partial_r h^\mu_\mu) - \frac{g_s e^\phi}{12}(\delta\phi F^2 + 2F\delta F - 3h^{\mu\nu}F_{\mu**}F^{**}_\nu) = 0. \quad \text{(E3)}$$

For the fluctuated Maxwell and Bianchi equations

$$\partial_\mu[\sqrt{g}e^\phi((\tfrac{1}{2}h^\rho_\rho + \delta\phi)F^{\mu\nu\alpha} + \delta F^{\mu\nu\alpha} - h^{\alpha c}F^{\mu\nu}_c + h^{\nu c}F^{\mu\alpha}_c - h^{\mu c}F^{\nu\alpha}_c)] = 0, \qquad d\delta F = 0. \quad \text{(E4)}$$

One can easily see that a solution to these equations is the one found by the authors of [56],

$$\delta F_3 = *_4 da, \qquad h_{\mu\nu} = \delta\phi = 0; \qquad F_3.\delta F_3 = 0 \quad \text{(E5)}$$

the analysis of this solution shows that it is non-normalizable, hence it is not a mode in the dual gauge theory. The rest is covered in the main part of the paper.

2. Rotating strings

Now, let us turn to the analysis of new solutions for strings rotating in our backgrounds. So, the Polyakov action for the configuration we will propose reads (τ, σ are the world sheet coordinates with the usual metric diag$(-1, 1)$),

$$\mathcal{S} = \frac{1}{2\pi\alpha'}\int d\sigma d\tau G_{\mu\nu}\partial_\alpha X^\mu \partial_b X^\nu \eta^{\alpha\beta} \quad \text{(E6)}$$

and the Virasoro constraints:

$$\partial_\tau X^\mu\partial_\sigma X_\mu = 0, \qquad G_{\mu\nu}\partial_\tau X^\mu\partial_\tau X^\nu + G_{\mu\nu}\partial_\sigma X^\mu\partial_\sigma X^\nu = 0. \quad \text{(E7)}$$

Consider the ansatz,

$$t = \kappa\tau, \qquad x = x(\tau,\sigma), \qquad y = y(\tau,\sigma), \qquad \rho = \rho(\sigma), \qquad \psi = \psi(\tau). \quad \text{(E8)}$$

This is consistent if $\theta = \tilde{\theta} = \frac{\pi}{2}$. We consider the induced string frame metric

$$ds^2 = e^\phi\left(-dt^2 + dx^2 + dy^2 + e^{2k}d\rho^2 + \frac{e^{2k}}{4}d\psi^2\right), \quad \text{(E9)}$$

where $\phi(\rho)$ and $k(\rho)$. The equations of motion are consistent with:

$$x = R(\sigma)\cos(\omega_1\tau), \qquad y = R(\sigma)\sin(\omega_1\tau), \qquad \psi = 2\omega_2\tau. \quad \text{(E10)}$$

One obtains (prime is derivative with respect to σ):

$$R'' + (\partial_\rho\phi)\rho' R' + \omega_1^2 R = 0, \qquad \rho'' + (\partial_\rho k)\rho'^2 - (\partial_\rho\phi)e^{-2k}R'^2 + (\partial_\rho k)\omega_2^2 = 0, \qquad -\kappa^2 + \omega_1^2 R^2 + R'^2 + e^{2k}\omega_2^2 + e^{2k}\rho'^2 = 0. \quad \text{(E11)}$$

These equations are not independent since by deriving the last one and using the other two, one reaches an identity. The rest follows as in Section VIII.

3. Penrose limit and PP-waves

Now, let us present the technical details of the Penrose limit and plane wave geometry. The computation is very similar to the one in [66].

After doing a gauge transformation (that can be better understood as a coordinate transformation or reparametrization of the left invariant forms of $SU(2)$, see [19]), we leave the geometry in the form (4.1), but with the

$SU(2)$-valued oneform $A^a\sigma^a$ being

$$A = \mu\rho^2[(\cos\varphi d\theta - \cos\theta\sin\theta\sin\varphi d\varphi)\sigma^1 + (\sin\varphi d\theta + \cos\theta\sin\theta\cos\varphi d\varphi)\sigma^2 + \sin^2\theta d\varphi\sigma^3]. \quad \text{(E12)}$$

Now, we have to rescale and redefine the coordinates according to,

$$\rho = \frac{r}{L}, \qquad \tilde{\theta} = \frac{v}{L}, \qquad \vec{x} = \frac{\vec{x}}{L}, \qquad 2\phi_+ = \psi + \tilde{\varphi}. \quad \text{(E13)}$$

Then we take the limit of $L \to \infty$ and keep only terms suppressed in L^{-2}. We get a PP-wave that contains the so-called magnetic terms (that is crossed terms $d\phi_+ d\varphi$ and $d\phi_+ d\tilde{\varphi}$), that reads,

$$e^{-2k(0)-\phi(0)/2}L^2 ds^2 = L^2(-dt^2 e^{-2k(0)} + d\phi_+^2) + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{4}(dv^2 + v^2 d\tilde{\varphi}^2) - \left(\frac{v^2}{2}d\tilde{\varphi} + \mu r^2 \sin^2\theta d\varphi\right)d\phi_+. \quad \text{(E14)}$$

We remind the reader that μ is the parameter labelling the different solutions near $\rho = 0$. Even when this can be quantized, it is better in order to have a better intuition on the spectrum, to make a last change in variables

$$d\varphi \to d\varphi + \frac{\mu}{2}d\phi_+, \qquad d\tilde{\varphi} \to d\tilde{\varphi} + d\phi_+, \quad \text{(E15)}$$

then, defining as usual, $x^+ = te^{-k(0)}$, $x^- = L^2(t - \phi_+)$, we have that the geometry looks,

$$e^{-2k(0)-\phi(0)/2}L^2 ds^2 = -2dx^+ dx^- + d\vec{x}_3^2 + d\vec{y}_2^2 + dz^2 + \frac{1}{4}d\vec{v}_2^2 - \left(\frac{\vec{v}_2^2}{4} + \frac{\mu^2}{4}\vec{y}_2^2\right)d\phi_+^2. \quad \text{(E16)}$$

For the rest of the analysis, please go back to the relevant part in Section VIII.

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