Local bulk operators in AdS/CFT correspondence: A boundary view of horizons and locality

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We develop the representation of local bulk fields in anti-de Sitter (AdS) space by nonlocal operators on the boundary, working in the semiclassical limit and using AdS_2 as our main example. In global coordinates we show that the boundary operator has support only at points which are spacelike separated from the bulk point. We construct boundary operators that represent local bulk operators inserted behind the horizon of the Poincaré patch and inside the Rindler horizon of a two-dimensional black hole. We show that these operators respect bulk locality and comment on the generalization of our construction to higher dimensional AdS black holes.

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I. INTRODUCTION

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence provides a nonperturbative definition of string theory in asymptotically AdS space [1,2]. In principle, all bulk observables are encoded in correlation functions of local operators in the CFT. In practice, however, many of the quantum gravity questions we would like to address are not simply related to local boundary correlators. These questions include the following: how does a quasilocal bulk spacetime emerge from the CFT? How does the region behind a horizon get encoded in the CFT? What is the CFT description (or perhaps resolution) of a black hole singularity?

In this paper we develop a set of tools for recovering local bulk physics from the CFT. We use the Lorentzian AdS/CFT correspondence developed in [3–5]. The basic idea is to express local operators in the bulk in terms of nonlocal operators on the boundary. We work in the leading semiclassical approximation—meaning both large N and large 't Hooft coupling—and consider free scalar fields in AdS space. The fields are taken to have normalizable falloff near the boundary of AdS space,

$$\phi(z, x) \sim z^{\Delta}\phi_0(x).$$

Here z is a radial coordinate which vanishes at the boundary. We will show that the bulk supergravity field can be expressed in terms of its behavior near the boundary via

$$\phi(z, x) = \int dx' K(x'|z, x)\phi_0(x').$$

We will refer to the kernel $K(x'|z, x)$ as a smearing function. General AdS/CFT considerations imply that the boundary behavior of the field corresponds to an operator of conformal dimension Δ in the CFT,

$$\phi_0(x) \leftrightarrow \mathcal{O}(x).$$

This implies a correspondence between local fields in the bulk and *nonlocal* operators in the CFT.

$$\phi(z, x) \leftrightarrow \int dx' K(x'|z, x)\mathcal{O}(x').$$

Bulk-to-bulk correlation functions, for example, are then equal to correlation functions of the corresponding nonlocal operators in the dual CFT,

$$\langle\phi(z_1, x_1)\phi(z_2, x_2)\rangle = \int dx_1' dx_2' K(x_1'|z_1, x_1)K(x_2'|z_2, x_2) \times\langle\mathcal{O}(x_1')\mathcal{O}(x_2')\rangle.$$

In this paper we construct smearing functions and show how certain aspects of bulk physics are encoded by these nonlocal operators. We will use AdS_2 as our main example, although many of the results presented here generalize to higher dimensions [6]. Smearing functions were discussed in [4,7], and smearing functions in AdS_5, as well as in some nonconformal variants, have been computed by Bena [8]. An algebraic formulation of the correspondence between local bulk fields and nonlocal boundary observables was developed in [9]. Other studies of bulk locality and causality include [10].

To avoid any possible confusion, we note that Witten [11] (see also [12]) introduced a bulk-to-boundary propagator whose Lorentzian continuation can be used to represent bulk fields having a prescribed *non-normalizable* behavior near the boundary [3]. Such bulk fields are dual to sources that deform the CFT action. Our approach is quite different: we work directly with the undeformed CFT, and introduce nonlocal operators that are dual to *normalizable* fluctuations in the bulk of AdS [13].

An outline of this paper is as follows. In Sec. II we construct smearing functions in AdS_2 in global coordinates. In Sec. III we construct smearing functions in

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Poincaré coordinates and show how the Poincaré horizon appears in the CFT. In Sec. IV we discuss the way in which bulk-to-bulk correlators are recovered from the CFT and show how coincident singularities arise. In Sec. V we work in Rindler coordinates and discuss the appearance of black hole horizons. In Sec. VI we extend the picture to general AdS black holes. We conclude in Sec. VII.

II. GLOBAL SMEARING FUNCTIONS

A. AdS_2 generalities

We begin by reviewing a few standard results; for more details see [3] or Appendix A. In global coordinates the AdS_2 metric is

$$ds^2 = \frac{R^2}{\cos^2\rho}(-d\tau^2 + d\rho^2) \tag{1}$$

where R is the radius of curvature and $-\infty < \tau < \infty$, $-\pi/2 < \rho < \pi/2$. It is convenient to introduce a distance function

$$\sigma(\tau, \rho|\tau', \rho') = \frac{\cos(\tau - \tau') - \sin\rho \sin\rho'}{\cos\rho \cos\rho'} \tag{2}$$

which is invariant under AdS isometries. Points in the unit cell $-\pi < \tau - \tau' < \pi$ that have $\sigma > -1$ can be connected by a geodesic; for such points

$$\sigma = \begin{cases} \cos(s/R) & \text{timelike } (s = \text{geodesic proper time}), \\ 1 & \text{null}, \\ \cosh(d/R) & \text{spacelike } (d = \text{geodesic proper distance}). \end{cases}$$

Points in the unit cell with $\sigma < -1$ are timelike separated but are not connected by a geodesic. A free scalar field of mass m can be expanded in a complete set of normalizable modes [16],

$$\phi(\tau, \rho) = \sum_{n=0}^{\infty} a_n e^{-i\omega_n\tau}\cos^\Delta\rho C_n^\Delta(\sin\rho) + \text{H.c.} \tag{3}$$

where $\omega_n = n + \Delta$, $\Delta = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2R^2}$ is the conformal dimension of the corresponding operator, and $C_n^\Delta(x)$ is a Gegenbauer polynomial. We have not bothered normalizing the modes.

The field vanishes at the boundary of AdS space. In global coordinates we define the right boundary value of the field by

$$\phi_0^{\text{global},R}(\tau) = \lim_{\rho\to\pi/2} \frac{\phi(\tau, \rho)}{\cos^\Delta\rho}. \tag{4}$$

Similarly the left boundary value is

$$\phi_0^{\text{global},L}(\tau) = \lim_{\rho\to-\pi/2} \frac{\phi(\tau, \rho)}{\cos^\Delta\rho}. \tag{5}$$

Some special simplifications occur when Δ is a positive integer. First of all, in this case the field is single valued on the AdS_2 hyperboloid (meaning that we can identify $\tau \approx \tau + 2\pi$). Also we define the antipodal map on AdS_2,

$$A\colon (\tau, \rho) \mapsto (\tau + \pi, -\rho). \tag{6}$$

Note that $\sigma(x|Ax') = -\sigma(x|x')$. When Δ is a positive integer we have

$$\phi(Ax) = (-1)^\Delta \phi(x)$$

in which case the boundary values are related by

$$\phi_0^{\text{global},L}(\tau) = (-1)^\Delta \phi_0^{\text{global},R}(\tau + \pi). \tag{7}$$

B. Green's function approach

In this subsection we construct smearing functions for AdS_2 in global coordinates starting from a suitable Green's function.

The Green's function should satisfy

$$(\Box - m^2)G(x|x') = \frac{1}{\sqrt{-g}}\delta^2(x - x'), \tag{8}$$

where $\delta^2(x - x')$ is defined on the universal cover of AdS space, $-\pi/2 \le \rho < \pi/2$, $-\infty < \tau < \infty$. We want a smearing function that is nonzero only at spacelike separation, so we make the ansatz

$$G(x|x') = f(\sigma(x|x'))\theta((\rho - \rho')^2 - (\tau - \tau')^2).$$

Here σ is the AdS-invariant distance defined in (2). Because of the step function, G is nonzero only at spacelike separation. By direct substitution one can check that (8) is satisfied provided that $f(\sigma)$ satisfies the homogeneous AdS-invariant wave equation

$$(\sigma^2 - 1)f''(\sigma) + 2\sigma f'(\sigma) - \Delta(\Delta - 1)f(\sigma) = 0 \tag{9}$$

with the boundary condition $f(1) = 1/4$ [18]. The solution

$$G(x|x') = \tfrac{1}{4}P_{\Delta-1}(\sigma)\theta((\rho - \rho')^2 - (\tau - \tau')^2)$$

is given by a Legendre function. It is worth emphasizing some curious properties of this Green's function. First of all, by construction, it is nonzero only at spacelike separation. It is finite (but discontinuous) on the light cone, when $G \to 1/4$ as the light cone is approached from a spacelike direction. However it is non-normalizable near the boundary of AdS space, with

$$G(x|x') \sim \frac{\Gamma(2\Delta - 1)}{2^{\Delta+1}\Gamma(\Delta)^2}\sigma^{\Delta-1}$$

at large spacelike separation. In AdS_2 we can simplify the discussion by working with a Green's function that is nonzero only in the right-hand part of the light cone [19]. With similar arguments it is easy to see that

$$G(x|x') = \tfrac{1}{2}P_{\Delta-1}(\sigma)\theta(\rho - \rho')\theta(\rho - \rho' - |\tau - \tau'|) \tag{10}$$

is a suitable Green's function.

Having constructed a Green's function that is nonzero only in the right light cone, we can make use of the Green's identity

$$\phi(x') = \int_{-\infty}^{\infty} d\tau \sqrt{-g}[\phi(\tau, \rho)\partial^\rho G(\tau, \rho|x') - G(\tau, \rho|x')\partial^\rho \phi(\tau, \rho)]|_{\rho=\rho_0}.$$

We are interested in sending the regulator $\rho_0 \to \pi/2$. In this limit only the leading behavior of both the field and the Green's function contributes, and we have

$$\phi(x') = \int_{-\infty}^{\infty} d\tau K(\tau|x')\phi_0^{\mathrm{global},R}(\tau) \tag{11}$$

where the smearing function can be variously expressed as

$$K(\tau|\tau', \rho') = (2\Delta - 1) \lim_{\rho\to\pi/2} \cos^{\Delta-1}\rho G(\tau, \rho|\tau', \rho') \tag{12}$$

$$= \frac{2^{\Delta-1}\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)} \lim_{\rho\to\pi/2} (\sigma\cos\rho)^{\Delta-1}\theta(\rho - \rho' - |\tau - \tau'|) \tag{13}$$

$$= \frac{2^{\Delta-1}\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)} \left(\frac{\cos(\tau - \tau') - \sin\rho'}{\cos\rho'}\right)^{\Delta-1} \times \theta\left(\frac{\pi}{2} - \rho' - |\tau - \tau'|\right). \tag{14}$$

These smearing functions have several important properties.

(i) The smearing function has compact support on the boundary of AdS space: K is nonzero on the boundary only within the right light cone of the point (τ', ρ'). Of course, we could have chosen to construct a smearing function that was nonzero only in the left light cone. Note that a local bulk operator near the left boundary could be described with a highly localized smearing function on the left boundary, or with a delocalized smearing function on the right boundary.

(ii) The whole setup is AdS covariant, since $\phi_0^{\mathrm{global},R}$ transforms as a primary field with dimension Δ under conformal transformations. This is clear when K is written in the form (13): the factor $\cos^{\Delta-1}\rho$ appearing in that expression cancels the conformal weight of the field together with the conformal weight of the measure $\int d\tau$.

(iii) As can be seen explicitly in (14), the smearing function has a finite limit as the regulator is removed, $\rho_0 \to \pi/2$.

Note that these properties all follow from the fact that we began with a Green's function that is non-normalizable near the boundary and nonzero only at spacelike separation. We have plotted the $\Delta = 3$ smearing function in Fig. 1.

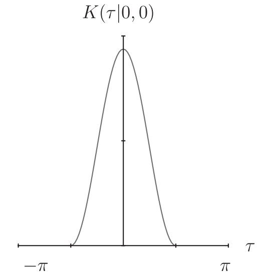

FIG. 1. Global smearing function for a $\Delta = 3$ bulk operator located at $\rho = 0$.

As a particularly simple example, consider a massless field in AdS_2. Such a field has $\Delta = 1$ and the general expression (14) reduces to

$$K = \frac{1}{2}\theta\left(\frac{\pi}{2} - \rho' - |\tau - \tau'|\right).$$

That is, a massless bulk field can be expressed in terms of its boundary value by

$$\phi(\tau', \rho') = \frac{1}{2}\int_{\tau'-(\pi/2-\rho')}^{\tau'+(\pi/2-\rho')} d\tau \phi_0^{\mathrm{global},R}(\tau).$$

As a check on our work, it is easy to see that this bulk field indeed satisfies the massless wave equation: it is annihilated by $(\partial_{\tau'}^2 - \partial_{\rho'}^2)$. One can also verify that it has the correct behavior near the boundary. As $\rho' \to \pi/2$ we can bring the boundary field out of the integral to obtain

$$\phi(\tau', \rho') \approx \tfrac{1}{2}\phi_0^{\mathrm{global},R}(\tau') \cdot (\pi - 2\rho').$$

This shows that the boundary conditions (4) are satisfied when $\Delta = 1$.

It is not much harder to check the boundary conditions for general Δ. As $\rho' \to \pi/2$ we have

$$\begin{aligned}\phi(\tau', \rho') &\approx \phi_0^{\mathrm{global},R}(\tau') \int d\tau K(\tau|\tau', \rho') \\ &\approx \phi_0^{\mathrm{global},R}(\tau') \frac{\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)\cos^{\Delta-1}\rho'} \\ &\quad\times \int_{\tau'-(\pi/2-\rho')}^{\tau'+(\pi/2-\rho')} d\tau((\pi/2 - \rho')^2 - (\tau - \tau')^2)^{\Delta-1} \\ &\approx \phi_0^{\mathrm{global},R}(\tau')(\pi/2 - \rho')^\Delta\end{aligned}$$

as required by (4).

C. Mode sum approach

In this subsection we take a different point of view and construct global smearing functions from a mode sum. We begin with integer conformal dimension then generalize.

Let us first suppose that Δ is a positive integer. Then, given an on-shell bulk field with mode expansion (3), we can reconstruct the bulk field from its right boundary value using

$$a_n = \frac{1}{C_n^\Delta(1)} \oint \frac{d\tau}{2\pi} e^{i\omega_n \tau} \phi_0^{\text{global},R}(\tau). \qquad (15)$$

The integral is over any 2π interval on the boundary. Plugging this back into the bulk mode expansion, we can write (as an operator identity)

$$\phi(\tau', \rho') = \oint d\tau K(\tau|\tau', \rho') \phi_0^{\text{global},R}(\tau) \qquad (16)$$

where the smearing function

$$K(\tau|\tau', \rho') = \frac{1}{2\pi} e^{i\Delta(\tau-\tau')} \cos^\Delta \rho' \sum_{n=0}^{\infty} e^{in(\tau-\tau'+i\epsilon)} \frac{C_n^\Delta(\sin\rho')}{C_n^\Delta(1)} + \text{c.c.} \qquad (17)$$

We have inserted an $i\epsilon$ to keep the mode sum convergent. Note that K is periodic in τ, with the same periodicity as the underlying modes, unlike the Green's function (10). To make contact with the results of the previous section we will eventually choose the range of integration in (16) to be $-\pi < \tau - \tau' < \pi$.

It is important to note that the smearing functions are not unique. For example, we could equally well have constructed a smearing function on the left boundary. More importantly, from (3) note that $\phi_0^{\text{global},R}$ does not have Fourier components with frequencies in the range $-\Delta + 1, \ldots, \Delta - 1$, so we are free to drop any Fourier components of K with frequencies in this range.

To evaluate K we use the integral representation [20]

$$\frac{C_n^\Delta(x)}{C_n^\Delta(1)} = \frac{\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)} \times \int_0^\pi d\theta \sin^{2\Delta-1}\theta (x + \sqrt{x^2 - 1} \cos\theta)^n.$$

Performing the sum on n gives

$$K(\tau|0, \rho') = \frac{1}{2\pi} e^{i\Delta\tau} \cos^\Delta \rho' \frac{\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)} \times \int_0^\pi d\theta \frac{\sin^{2\Delta-1}\theta}{1 - e^{i(\tau+i\epsilon)}(\sin\rho' + i\cos\rho' \cos\theta)} + \text{c.c.}$$

The integral is a polynomial in $e^{i\tau}$ plus a logarithm [20]. The polynomial only involves Fourier modes which do not appear in ϕ_0^{global}, so we may drop it leaving

$$K(\tau|0, \rho') \simeq \frac{2^{\Delta-1}\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)} \left(\frac{\cos\tau - \sin\rho'}{\cos\rho'}\right)^{\Delta-1} \frac{i}{2\pi} \times \log\left(\frac{1 - ie^{i(\tau-\rho'+i\epsilon)}}{1 + ie^{-i(\tau-\rho'-i\epsilon)}} \frac{1 - ie^{-i(\tau+\rho'-i\epsilon)}}{1 + ie^{i(\tau+\rho'+i\epsilon)}}\right).$$

Here $\simeq$ means up to terms whose time dependence is such that they vanish when integrated against ϕ_0^{global}. At this point it helps to note that $f(x) = -i \log[(1 + e^{i(x+i\epsilon)})/(1 + e^{-i(x-i\epsilon)})]$ is a sawtooth function, $f(x) = x$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$. Again, some Fourier modes do not contribute—only the discontinuities of the sawtooth function matter—and one is left with

$$K(\tau|\tau', \rho') \simeq \frac{2^{\Delta-1}\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)} \left(\frac{\cos(\tau - \tau') - \sin\rho'}{\cos\rho'}\right)^{\Delta-1} \times \theta\left(\cos(\tau - \tau') - \cos\left(\frac{\pi}{2} - \rho'\right)\right). \qquad (18)$$

Choosing the range of integration in (16) to be $-\pi < \tau - \tau' < \pi$, the step function in (18) reduces to the step function in (14), so this expression reproduces the result we obtained in the previous subsection starting from a Green's function [21].

What if Δ is not an integer? Recall that we are working on the universal cover of AdS space where $-\infty < \tau < \infty$. In this case the mode functions are no longer periodic. This means (15) is no longer valid, since for general Δ the positive- and negative-frequency modes are not orthogonal on the interval $-\pi < \tau < \pi$. The trick is to first decompose the field into positive- and negative-frequency pieces, $\phi(\tau, \rho) = \phi_+(\tau, \rho) + \phi_-(\tau, \rho)$ where

$$\phi_+(\tau, \rho) = \sum_{n=0}^{\infty} a_n e^{-i\omega_n \tau} \cos^\Delta \rho C_n^\Delta(\sin\rho) \qquad (19)$$

and $\phi_- = \phi_+^*$. We can recover $\phi_\pm$ from their boundary values, integrating over only the range $-\pi \le \tau - \tau' < \pi$, via $\phi_\pm(\tau', \rho') = \oint d\tau K_\pm(\tau|\tau', \rho') \phi_{0\pm}^{\text{global},R}(\tau)$ where

$$K_\pm(\tau|\tau', \rho') = \frac{1}{2\pi} \cos^\Delta \rho' \sum_{n=0}^{\infty} e^{\pm i\omega_n(\tau-\tau'\pm i\epsilon)} \frac{C_n^\Delta(\sin\rho')}{C_n^\Delta(1)}. \qquad (20)$$

These positive- and negative-frequency smearing functions are highly nonunique, since, for example, we can add Fourier modes to K_+ that $\sim e^{i(\Delta-1)(\tau-\tau')}, e^{i(\Delta-2)(\tau-\tau')}, \ldots$. By making use of this freedom we can put $K_\pm$ into the form of an image sum,

$$K_\pm \simeq \sum_{k=-\infty}^{\infty} e^{\pm i2\pi k\Delta} K(\tau|\tau' + 2\pi k, \rho') \qquad (21)$$

where for any Δ

$$K(\tau|\tau', \rho') \simeq \frac{2^{\Delta-1}\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)} \left(\frac{\cos(\tau - \tau') - \sin\rho'}{\cos\rho'}\right)^{\Delta-1} \times \theta\left(\frac{\pi}{2} - \rho' - |\tau - \tau'|\right) \qquad (22)$$

is defined over the range $-\infty < \tau - \tau' < \infty$. One can verify (21) by doing an inverse Fourier transform using a complete set of skew-periodic modes $e^{\pm i(n+\Delta)(\tau-\tau')}$, $n \in \mathbb{Z}$, and showing that the coefficients of the $n \ge 0$ modes appear in (20). On the restricted interval $-\pi < \tau - \tau' < \pi$ only the $k = 0$ term contributes, so the smearing functions all agree and

$$\begin{aligned}\phi &= \int_{\tau'-\pi}^{\tau'+\pi} d\tau (K_+ \phi_{0+}^{\text{global},R} + K_- \phi_{0-}^{\text{global},R}) \\ &= \int_{\tau'-\pi}^{\tau'+\pi} d\tau K \phi_0^{\text{global},R}\end{aligned}$$

reproduces our previous Green's function expression.

III. POINCARÉ COORDINATES

A. Smearing functions

It is worth asking how these nonlocal operators look in different coordinate systems. For example, one can introduce Poincaré coordinates

$$Z = \frac{R\cos\rho}{\cos\tau + \sin\rho}, \qquad T = \frac{R\sin\tau}{\cos\tau + \sin\rho}, \qquad 0 < Z < \infty, \qquad -\infty < T < \infty \tag{23}$$

in which

$$ds^2 = \frac{R^2}{Z^2}(-dT^2 + dZ^2).$$

These coordinates only cover an interval $-\pi < \tau < \pi$ of global time on the right boundary. The mode expansion reads

$$\phi(T, Z) = \int_0^\infty d\omega a_\omega e^{-i\omega T} \sqrt{Z} J_\nu(\omega Z) + \text{c.c.}$$

where $\nu = \Delta - \frac{1}{2}$. The field vanishes as $Z \to 0$. In Poincaré coordinates it is convenient to define the boundary value

$$\phi_0^{\text{Poincare}}(T) = \lim_{Z\to 0} \frac{\phi(T, Z)}{Z^\Delta}. \tag{24}$$

Note that this differs from the definition (4) we used in global coordinates.

It is straightforward to compute the smearing function in Poincaré coordinates directly from a mode sum. Given an on-shell bulk field we have

$$a_\omega = \int_{-\infty}^\infty dT e^{i\omega T} \frac{1}{2\pi\sqrt{Z} J_\nu(\omega Z)} \phi(T, Z),$$

or taking the limit $Z \to 0$,

$$a_\omega = \frac{\Gamma(\nu + 1)}{2\pi(\omega/2)^\nu} \int_{-\infty}^\infty dT e^{i\omega T} \phi_0^{\text{Poincare}}(T).$$

Plugging this back into the bulk mode expansion we can write

$$\phi(T', Z') = \int_{-\infty}^\infty dT K(T|T', Z') \phi_0^{\text{Poincare}}(T)$$

where the Poincaré smearing function is

$$K(T|T', Z') = \frac{1}{\pi} 2^\nu \Gamma(\nu + 1) \sqrt{Z'} \int_0^\infty d\omega \frac{1}{\omega^\nu} J_\nu(\omega Z') \times \cos\omega(T - T'). \tag{25}$$

The integral is well defined without an $i\epsilon$ prescription. It vanishes identically for $|T - T'| > Z'$, while for $|T - T'| < Z'$ we have [20]

$$K = \frac{\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)} Z'^{\Delta-1} F\left(\frac{1}{2}, 1 - \Delta, \frac{1}{2}, \frac{(T - T')^2}{Z'^2}\right).$$

Thus, in general,

$$K = \frac{\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)} \left(\frac{Z'^2 - (T - T')^2}{Z'}\right)^{\Delta-1} \theta(Z' - |T - T'|), \tag{26}$$

where we used $F(\alpha, \beta, \alpha, x) = (1 - x)^{-\beta}$. This expression is valid for any positive Δ. Note that, unlike the global mode sum (17), the Poincaré mode sum is nonzero only at spacelike separation [22]. It is also AdS covariant since

$$K(T|T', Z') = \frac{2^{\Delta-1}\Gamma(\Delta + 1/2)}{\sqrt{\pi}\Gamma(\Delta)} \lim_{Z\to 0} (Z\sigma(T, Z|T', Z'))^{\Delta-1} \times \theta(\sigma - 1) \tag{27}$$

where the AdS-invariant distance (2) expressed in Poincaré coordinates is

$$\sigma(Z, T|Z', T') = \frac{Z^2 + Z'^2 - (T - T')^2}{2ZZ'}. \tag{28}$$

Upon changing variables from Poincaré time to global time, this is equivalent to our previous result (13). To see this, one merely has to note that near the boundary

$$Z^\Delta \phi_0^{\text{Poincare}} \sim \cos^\Delta\rho \phi_0^{\text{global},R}$$

while the change of integration measure is

$$\frac{dT}{Z} = \frac{d\tau}{\cos\rho}.$$

B. Going behind the Poincaré horizon

In the previous subsection we showed how local bulk fields in the Poincaré patch are represented by smeared operators on the boundary. But what if the bulk point is outside the Poincaré patch? Can it still be represented as an operator on the boundary of the Poincaré patch? The answer turns out to be yes: we can still work in Poincaré coordinates on the boundary, but we have to use a different smearing function.

To obtain the correct smearing function, our strategy is to start with the global smearing function, manipulate it so that it is nonzero only on the boundary of the Poincaré patch, and then convert to Poincaré coordinates. We first assume that Δ is a positive integer. In this case $\phi_0^{\text{global},R}(\tau)$ is periodic in the global time coordinate τ with period 2π,

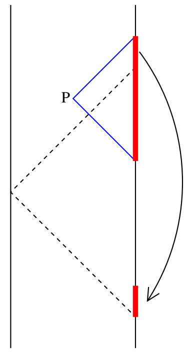

FIG. 2 (color online). Smearing function for a bulk operator with integer Δ located outside the Poincaré patch.

so one can take the global smearing function (18) and translate whatever part of it has left the Poincaré patch by a multiple of 2π in order to get it back inside the Poincaré patch. This is illustrated in Fig. 2.

This can be expressed quite simply in terms of the invariant distance (2). Noting that σ is 2π periodic in global time, for a general point P (not necessarily inside the Poincaré patch) we can express the smearing function in a form that, upon changing to Poincaré coordinates, looks identical to (27):

$$\phi(P) = \int_{-\infty}^{\infty} dT K(T|P)\phi_0^{\text{Poincare}}(T),$$
$$K(T|P) = \frac{2^{\Delta-1}\Gamma(\Delta+1/2)}{\sqrt{\pi}\Gamma(\Delta)} \lim_{Z\to 0}(Z\sigma(T,Z|P))^{\Delta-1} \times \theta(\sigma-1). \tag{29}$$

Note that the signal of a bulk operator approaching the future (past) Poincaré horizon is that the smearing function extends to Poincaré time $T \to +\infty$ ($T \to -\infty$).

What happens if Δ is not an integer? The trick is to note that, although the field itself is not periodic in τ, its positive- and negative-frequency components (19) are periodic up to a phase:

$$\phi_\pm(\tau + 2\pi, \rho) = e^{\mp i2\pi\Delta}\phi_\pm(\tau, \rho).$$

For a bulk point P with global coordinates (τ', ρ') we set $\tau' = \tau_0' + 2\pi n$ with $-\pi < \tau_0' < \pi$ and $n \in \mathbb{Z}$ and write

$$\phi(\tau', \rho') = \int d\tau K_{\text{global}}(\tau|\tau_0', \rho')(e^{-i2\pi n\Delta}\phi_{0+}^{\text{global},R}(\tau) + e^{i2\pi n\Delta}\phi_{0-}^{\text{global},R}(\tau)). \tag{30}$$

Converting to Poincaré coordinates this becomes

$$\phi(P) = \int_{-\infty}^{\infty} dT K_{\text{Poincare}}(T|P_0)(e^{-i2\pi n\Delta}\phi_{0+}^{\text{Poincare}}(T) + e^{i2\pi n\Delta}\phi_{0-}^{\text{Poincare}}(T)). \tag{31}$$

Here K_{Poincare} is given in (29) and P_0 is the bulk point with global coordinates (τ_0', ρ'). In deriving this we used the fact that the global and Poincaré vacua are identical [23,24]. This equivalence is discussed in more detail in Appendix B.

When expressed in terms of $\phi_{0+}^{\text{Poincare}}$ and $\phi_{0-}^{\text{Poincare}}$ the smearing function consists of one or two disconnected blobs on the boundary [25]. However, expressing $\phi_{0\pm}^{\text{Poincare}}$ in terms of ϕ_0^{Poincare} is completely nonlocal. For example, the positive-frequency part of the Poincaré boundary field is

$$\phi_{0+}^{\text{Poincare}}(T') = \int dT P_+(T|T')\phi_0^{\text{Poincare}}(T),$$
$$P_+(T|T') = \int_0^\infty \frac{d\omega}{2\pi} e^{i\omega(T-T'+i\epsilon)} = \frac{i}{2\pi(T-T'+i\epsilon)}. \tag{32}$$

For a point outside the Poincaré patch the smearing function in terms of ϕ_0^{Poincare} is nonzero everywhere. Just as for integer Δ, the signal of a bulk operator approaching the future (past) Poincaré horizon is that the smearing function extends to Poincaré time $T \to +\infty$ ($T \to -\infty$).

IV. RECOVERING BULK CORRELATORS

In the previous section we defined a set of smearing functions which enable us to reconstruct a normalizable bulk field from its behavior near the boundary of AdS space. We obtained these smearing functions by solving a wave equation in AdS space, so our expressions are valid for any state of the field provided we are in the limit of semiclassical supergravity where backreaction of the field on the geometry can be neglected.

Assuming the existence of a dual CFT, we can identify local operators in the bulk supergravity with nonlocal operators in the CFT,

$$\phi(\tau', \rho') \leftrightarrow \int d\tau K(\tau|\tau', \rho')\mathcal{O}(\tau).$$

This correspondence should hold for any state of the field (equivalently, any state of the CFT) provided we are in the limit of semiclassical supergravity where backreaction is negligible; note that in this limit the smearing functions are independent of the state. We can use the correspondence to recover bulk supergravity correlation functions from the CFT; for example,

$$_S\langle\psi|\phi(\tau,\rho)\phi(\tau',\rho')|\psi\rangle_S = \int ds \int ds' K(s|\tau,\rho)K(s'|\tau',\rho')_C \times \langle\psi|\mathcal{O}(s)\mathcal{O}(s')|\psi\rangle_C. \tag{33}$$

Here $|\psi\rangle_S$ is any supergravity state and $|\psi\rangle_C$ is the corresponding CFT state. To see this, note that our expression for the bulk field in terms of the boundary field holds as an operator identity at the level of supergravity. It enables us to express a bulk supergravity correlator in terms of a boundary supergravity correlator

$$S\langle\psi|\phi(\tau,\rho)\phi(\tau',\rho')|\psi\rangle_S = \int ds \int ds' K(s|\tau,\rho)K(s'|\tau',\rho')_S \times \langle\psi|\phi_0(s)\phi_0(s')|\psi\rangle_S. \quad (34)$$

The boundary supergravity correlator can be identified with a correlator in the CFT,

$$S\langle\psi|\phi_0(s)\phi_0(s')|\psi\rangle_S \underset{C}{=} \langle\psi|\mathcal{O}(s)\mathcal{O}(s')|\psi\rangle_C.$$

Combining these statements proves (33). One can also establish (33) somewhat more directly, by representing the CFT correlator as a sum over normalized positive-frequency boundary modes $f_n(s)$. For example, in the ground state of the CFT,

$$C\langle 0|\mathcal{O}(s)\mathcal{O}(s')|0\rangle_C = \sum_n f_n(s) f_n^*(s').$$

Plugging this into the right-hand side of (33) and performing the integrals over s and s' generates the appropriate bulk-to-bulk correlator, also in the form of a mode sum.

This gives a relationship between bulk and boundary expectation values (Wightman functions). All other bulk Green's functions can be expressed in terms of the bulk Wightman function; for example, the time-ordered Green's function is

$$iG_F(\tau,\rho|\tau',\rho') = \theta(\tau-\tau')\langle\phi(\tau,\rho)\phi(\tau',\rho')\rangle + \theta(\tau'-\tau)\langle\phi(\tau',\rho')\phi(\tau,\rho)\rangle.$$

Note however that (34) would not allow us to express a time-ordered Green's function in the bulk in terms of a time-ordered Green's function on the boundary.

Although true by construction, at first sight (33) is a rather surprising identity. For example, in the global AdS_2 vacuum the bulk and boundary Wightman functions are [24]

$$\langle\phi(\tau,\rho)\phi(\tau',\rho')\rangle = \frac{\Gamma(\Delta)}{2^{\Delta+1}\sqrt{\pi}\Gamma(\Delta+1/2)}\sigma^{-\Delta} \times F\left(\frac{\Delta}{2},\frac{\Delta+1}{2};\frac{2\Delta+1}{2};\frac{1}{\sigma^2}\right),$$
$$\langle\mathcal{O}(s)\mathcal{O}(s')\rangle = \frac{(-1)^\Delta\Gamma(\Delta)}{2^{2\Delta+1}\sqrt{\pi}\Gamma(\Delta+\frac{1}{2})\sin^{2\Delta}(\frac{s-s'}{2})}. \quad (35)$$

When $\sigma < -1$ (bulk points not connected by a geodesic) both the left- and right-hand sides of (33) are unambiguous. As a check on our work, we have verified numerically that the two sides are equal. To continue into the regime $\sigma > -1$ one has to check that the $i\epsilon$ prescriptions go through correctly. The bulk Wightman function is defined by a $\tau \to \tau - i\epsilon$ prescription, and, fortunately, by translation invariance of $K(s|\tau,\rho)$ this is equivalent to the correct $s \to s - i\epsilon$ prescription for the boundary Wightman function.

Given (33) we are guaranteed that—in the strict semiclassical limit—two nonlocal boundary operators will commute whenever the bulk points are spacelike separated. Bena [8] checked explicitly that the commutator vanishes from the boundary point of view.

Light-cone singularities in the bulk

Correlation functions of these nonlocal boundary operators diverge whenever the corresponding bulk points are coincident or lightlike separated. In this section we explain how these singularities arise from the boundary point of view.

It may seem a little surprising that the correlators diverge at all, since in field theory one usually introduces smeared operators to avoid singular correlators. To see what is going on let us look at how these operators are constructed. It is simplest to work in terms of a mode sum in frequency space. From (17) or (25) one sees that the smearing is done by multiplying each mode of the boundary operator by a frequency dependent phase. These phases act as a regulator which makes the correlator with most other operators nonsingular. However, for a given smeared operator one can find certain other smeared operators for which the phases cancel. For two such operators the correlator has a UV divergence.

To make this explicit, consider the correlator of two bulk operators in the Poincaré patch. Working in frequency space on the boundary the correlator is

$$\langle\phi(T,Z)\phi(T',Z')\rangle = \int_{-\infty}^{\infty} d\omega K(\omega|T,Z)K(-\omega|T',Z') \times \langle\mathcal{O}(\omega)\mathcal{O}(-\omega)\rangle. \quad (36)$$

The Poincaré smearing function in frequency space can be read off from (25),

$$K(\omega|T,Z) = 2^\nu\Gamma(\nu+1)\frac{\sqrt{Z}}{|\omega|^\nu}J_\nu(|\omega|Z)e^{-i\omega T} \sim \frac{1}{|\omega|^{\nu+1/2}}\cos(|\omega|Z + \text{const.})e^{-i\omega T}$$

at large $|\omega|$. For an operator of dimension Δ the boundary correlator in frequency space behaves as $\langle\mathcal{O}(\omega)\mathcal{O}(-\omega)\rangle \sim |\omega|^{2\nu}$. For generic bulk points the integrand in (36) has oscillating phases which regulate the behavior at large ω. But whenever (T,Z) and (T',Z') either coincide or are lightlike separated the phases cancel and the integral diverges logarithmically in the UV.

It is important to note that, even in the center of AdS space, the light-cone singularities of the bulk theory arise from the UV behavior of the boundary theory. This means, for example, that any attempt to put a UV cutoff on the boundary theory will modify locality everywhere in the bulk.

V. AdS_2 BLACK HOLES

A. Smearing functions

One can also introduce Rindler coordinates on AdS_2

$$\frac{r}{R\sqrt{M}} = \frac{\cos\tau}{\cos\rho}, \qquad \tanh\frac{t\sqrt{M}}{R} = \frac{\sin\tau}{\sin\rho},$$
$$R\sqrt{M} < r < \infty, \qquad -\infty < t < \infty, \tag{37}$$
$$ds^2 = -\left(\frac{r^2}{R^2} - M\right)dt^2 + \left(\frac{r^2}{R^2} - M\right)^{-1} dr^2.$$

The metric looks like a black hole [26] but the "mass" M is just an arbitrary dimensionless parameter which enters through the change of coordinates. These coordinates only cover an interval $-\pi/2 < \tau < \pi/2$ of global time on the right boundary, but they can be extended in the usual way to cover an identical interval on the left boundary.

What do smearing functions look like in Rindler coordinates? This depends on whether the bulk point is inside or outside the Rindler horizon. For a bulk point in the right Rindler wedge there is no difficulty: we merely have to transform the global smearing function (12) to Rindler coordinates. This gives a Rindler smearing function which is nonzero only at spacelike separation on the right boundary. Likewise, for a bulk point in the left Rindler wedge we can construct a smearing function that is nonzero only at spacelike separation on the left boundary. But what if the bulk point is inside the horizon? Then we need to modify the smearing function. Just as in Poincaré coordinates, our strategy will be to start with the global smearing function, manipulate it so that it is nonzero only on the boundary of the Rindler patch, and then convert to Rindler coordinates.

The analysis is simplest when Δ is an integer. Then we can use (7) to take those parts of the global smearing function that have left the boundary of the Rindler patch and bring them back inside. However, one obtains a smearing function with support on both the left and right boundaries of AdS space. This is illustrated in Fig. 3.

This can be expressed quite simply in terms of the invariant distance σ. Recall that σ is 2π periodic in global time, while under the antipodal map

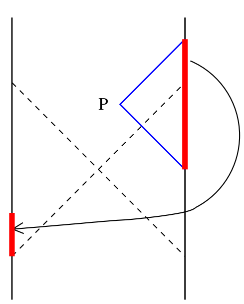

FIG. 3 (color online). Smearing function for an integer-dimension bulk operator located behind the Rindler horizon.

$$\sigma(x|Ax') = -\sigma(x|x'), \qquad \phi(Ax) = (-1)^\Delta \phi(x).$$

By starting with the global smearing function (12), decomposing it into pieces that are inside/outside the Rindler patch, and moving the outside part to the other boundary, we have

$$\phi(P) = \int_{-\pi/2}^{\pi/2} d\tau (K^R_{\text{global}}(\tau|P)\phi_0^{\text{global},R}(\tau) + (-1)^\Delta K^L_{\text{global}}(\tau|P)\phi_0^{\text{global},L}(\tau)),$$
$$K^R_{\text{global}}(\tau|P) = \frac{2^{\Delta-1}\Gamma(\Delta+1/2)}{\sqrt{\pi}\Gamma(\Delta)} \lim_{\rho\to\pi/2}(\sigma(\tau,\rho|P)\cos\rho)^{\Delta-1} \times\theta(\sigma-1),$$
$$K^L_{\text{global}}(\tau|P) = \frac{2^{\Delta-1}\Gamma(\Delta+1/2)}{\sqrt{\pi}\Gamma(\Delta)} \lim_{\rho\to-\pi/2}(-\sigma(\tau,\rho|P)\cos\rho)^{\Delta-1} \times\theta(-\sigma-1). \tag{38}$$

Here P is any point inside the Rindler horizon [28]. To express this in Rindler coordinates it is convenient to define the left and right Rindler boundary fields $\phi_0^{\text{Rindler},L/R}(t) = \lim_{r\to\infty} r^\Delta \phi(t, r)$ where L (R) refers to the left (right) Rindler wedge. Near the boundary

$$\frac{\phi_0^{\text{Rindler},L/R}}{r^\Delta} \sim \cos^\Delta\rho\,\phi_0^{\text{global},L/R}$$

while the change of integration measure is

$$\frac{d\tau}{\cos\rho} = \frac{rdt}{R^2}.$$

Putting this all together we have the final expression in Rindler coordinates,

$$\phi(P) = \int_{-\infty}^{\infty} dt (K^R_{\text{Rindler}}(t|P)\phi_0^{\text{Rindler},R}(t) + (-1)^\Delta K^L_{\text{Rindler}}(t|P)\phi_0^{\text{Rindler},L}(t)),$$
$$K^R_{\text{Rindler}}(t|P) = \frac{2^{\Delta-1}\Gamma(\Delta+1/2)}{\sqrt{\pi}\Gamma(\Delta)R^2} \lim_{r\to\infty}(\sigma/r)^{\Delta-1}\theta(\sigma-1),$$
$$K^L_{\text{Rindler}}(t|P) = \frac{2^{\Delta-1}\Gamma(\Delta+1/2)}{\sqrt{\pi}\Gamma(\Delta)R^2} \lim_{r\to\infty}(-\sigma/r)^{\Delta-1}\theta(-\sigma-1). \tag{39}$$

Here σ is the invariant distance from P to the point (t, r) on the appropriate boundary,

$$\sigma(t, r|t', r') = \frac{1}{M}\left[\frac{rr'}{R^2} \pm \left(\frac{r^2}{R^2} - M\right)^{1/2}\left(M - \frac{r'^2}{R^2}\right)^{1/2} \times \sinh\frac{\sqrt{M}(t-t')}{R}\right]$$

where P is inside the horizon with coordinates (t', r') and the upper (lower) sign applies for (t, r) near the right (left) boundary. Note that the smearing function on the left boundary has support only on points that are not connected to P by a geodesic (points with $\sigma < -1$).

What happens if Δ is not an integer? Although we no longer have (7), a similar property holds separately for the positive- and negative-frequency components of the boundary field (19):

$$\phi_{0\pm}^{\mathrm{global},R}(\tau) = e^{\pm i\pi\Delta}\phi_{0\pm}^{\mathrm{global},L}(\tau+\pi). \tag{40}$$

We can use this to rewrite the global smearing function in a form similar to (38), where it is nonzero only on the boundary of the Rindler patch. For a bulk point P inside the horizon with global coordinates (τ', ρ') we set $\tau' = \tau'_0 + 2\pi n$, $n \in \mathbb{Z}$ and

$$\begin{aligned}\phi(P) = \int_{-\pi/2}^{\pi/2} d\tau[&K_{\mathrm{global}}^{R}(\tau|P)(e^{-i2\pi n\Delta}\phi_{0+}^{\mathrm{global},R}(\tau) \\ &+ e^{i2\pi n\Delta}\phi_{0-}^{\mathrm{global},R}(\tau)) + K_{\mathrm{global}}^{L}(\tau|P) \\ &\times (e^{-i\pi(2n+1)\Delta}\phi_{0+}^{\mathrm{global},L}(\tau) \\ &+ e^{i\pi(2n+1)\Delta}\phi_{0-}^{\mathrm{global},L}(\tau))]\end{aligned} \tag{41}$$

where $K_{\mathrm{global}}^{R,L}(\tau|P)$ appear in (38). Expressed in terms of positive- and negative-frequency global fields, this smearing function consists of two disconnected blobs, one on the left boundary and one on the right [29]. One can switch to Rindler coordinates, however for points inside the horizon, in general, one obtains a smearing function that is completely nonlocal in terms of $\phi_0^{\mathrm{Rindler},L/R}$. This is worked out in Appendix B.

B. Thermofield interpretation and black holes

We can regard the AdS_2 metric in Rindler coordinates as a prototype for a black hole. A standard Euclidean calculation shows that the Hawking temperature is $1/\beta = \frac{\sqrt{M}}{2\pi R}$. By keeping modes in both the left and right Rindler wedges, the Hartle-Hawking state can be understood as a thermofield double [30,31].

How does this look on the boundary? As discussed in [33] an eternal AdS black hole is dual to two copies of the CFT in an entangled thermofield state. As we show in Appendix A the thermofield Hamiltonian is

$$H_{TF} = \frac{\sqrt{M}}{R}(\hat{S}_1 \otimes \mathbb{1}_2 - \mathbb{1}_1 \otimes \hat{S}_2).$$

Here $\hat{S}$ is a noncompact conformal generator on the boundary. The thermofield state is annihilated by H_{TF}, and can be formally expressed in terms of $\hat{S}$ eigenstates as

$$|\psi\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\beta E_n/2}|n\rangle_1 \otimes |n\rangle_2.$$

Since the global vacuum is $SL(2,\mathbb{R})$ invariant, this state should also be annihilated by the global Hamiltonian $\hat{R} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{R}$ and the Poincaré Hamiltonian $\mathbb{1} \otimes \hat{H}$.

What does this mean as far as constructing local bulk operators? The expressions we derived in Sec. VA are perfectly applicable. They imply that to put a bulk operator behind the horizon of the black hole we need boundary operators that act nontrivially on both copies of the Hilbert space. Related observations were made in [33].

Let us summarize the picture we have developed. A bulk operator outside the horizon corresponds to a nonlocal operator that acts on a single copy of the Hilbert space. As the bulk point approaches the future (past) horizon the smearing function extends to cover an infinite range of coordinate time on the boundary: it has support as $t \to +\infty$ ($t \to -\infty$) [34]. To insert a bulk operator behind the horizon we need a nonlocal operator that acts on both copies of the Hilbert space.

What does this mean from the point of view of an observer outside the black hole, who can only interact with a single copy of the CFT? Such an observer must trace over the second copy of the CFT. If no operators are inserted behind the horizon then the trace leads to a precisely thermal density matrix that describes the black hole. But operator insertions behind the horizon act on the other copy of the CFT, and modify the resulting density matrix. In general, these modifications will not have a thermal character. Thus, from the point of view of an outside observer, *operator insertions behind the horizon are seen as nonthermal modifications to the black hole density matrix*.

VI. HIGHER DIMENSIONAL BLACK HOLES

In the previous section we saw that in order to describe an object inside the Rindler horizon of a two-dimensional black hole one needs a nonlocal operator that acts on both copies of the Hilbert space [35]. In this section we show that a similar property holds for a general AdS black hole. Analogous arguments can be made for bulk points outside of the Poincaré patch.

A local field $\phi(U, V, \Omega)$ anywhere in the extended Kruskal diagram can be expanded in terms of Kruskal creation and annihilation operators,

$$\phi = \sum_i f_i(U, V, \Omega)a_K^i + f_i^*(U, V, \Omega)a_K^{i\dagger}. \tag{42}$$

The Kruskal creation and annihilation operators can be expressed in terms of left and right creation and annihilation operators using Bogolubov coefficients

$$\begin{aligned} a_K^i &= \alpha_{ij}^L a_L^j + \beta_{ij}^L a_L^{j\dagger} + \alpha_{ij}^R a_R^j + \beta_{ij}^R a_R^{j\dagger}, \\ a_K^{i\dagger} &= \alpha_{ij}^{L*} a_L^{j\dagger} + \beta_{ij}^{L*} a_L^j + \alpha_{ij}^{R*} a_R^{j\dagger} + \beta_{ij}^{R*} a_R^j. \end{aligned} \tag{43}$$

Since the left and right creation and annihilation operators can be written as a Fourier transform of operators in one of the copies of the CFT,

$$a_L^j \sim \int dt d^d x e^{-i\omega_j t + i k_j x} \mathcal{O}_L(t, x),$$
$$a_R^j \sim \int dt d^d x e^{-i\omega_j t + i k_j x} \mathcal{O}_R(t, x), \quad (44)$$

we see that a local bulk field anywhere in the Kruskal diagram can be represented as a linear combination of two operators, one acting on the left and one acting on the right. If the bulk point is in either the left or the right region, then its representation reduces to a single operator on the appropriate copy.

VII. CONCLUSIONS

In this paper we related local bulk fields in AdS space to nonlocal operators on the boundary. In global coordinates we found AdS-covariant smearing functions with support purely at spacelike separation. We then showed how to represent bulk operators that are inserted behind the Poincaré horizon, or inside the horizon of a black hole. By construction these boundary operators reproduce all bulk correlation functions; they therefore respect bulk locality. Light-cone singularities in the bulk arise from the UV behavior of the boundary theory. Although we concentrated on AdS_2, similar results hold in higher dimensions [6].

It is curious that local operators in the interior of a black hole correspond to boundary operators that act on both copies of the thermofield double. Note that in the thermal AdS phase, such operators do not exist: any operator that acts on both copies of the CFT will at best be bilocal in the (disconnected) bulk spacetime. It is only in the black hole phase that operators which act on both copies of the CFT can be local in the bulk.

One can even give a boundary description of the fact that the black hole time coordinate switches from timelike to spacelike at the horizon. In the right Rindler wedge, or more generally whenever the Killing vector $\frac{\partial}{\partial t}$ is timelike, chronology in the bulk corresponds to chronology on the boundary: bulk operators which are inserted at the same spatial position but different values of t correspond to smeared operators on a single boundary that are related by time translation. This holds in the left and right Rindler wedges, inside the Poincaré horizon, or even globally if one uses global time. But for bulk points inside the horizon, a Rindler time translation will move the left and right boundary operators in opposite directions. This corresponds to the fact that $\frac{\partial}{\partial t}$ is spacelike inside the horizon.

There are many open questions and directions for future work. Our construction is applicable in the limit of semiclassical supergravity; it would be extremely interesting to understand $1/N$ and α' corrections. Nonlocal operators in the CFT should provide a new tool to study bulk phenomena such as black hole singularities [36,37], the interplay of holography and locality [38], or even nonlocal deformations of AdS space [39]. But, at a more fundamental level, one might ask: purely from the boundary CFT point of view, what if anything would allow one to identify a particular set of nonlocal operators as appropriate for describing bulk spacetime?

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APPENDIX A: ADS ISOMETRIES

We review the relationship between isometries of AdS_2 and conformal transformations on the boundary. For a more extensive discussion see [40,41].

AdS_2 can be embedded as a hypersurface

$$-(X^0)^2 - (X^1)^2 + (X^2)^2 = -R^2$$

in $\mathbb{R}^{2,1}$ with a $(- - +)$ metric. Isometries of AdS_2 arise from Lorentz transformations of the embedding space, with generators

$$J_{\mu\nu} = \eta_{\mu\sigma} X^\sigma \partial_\nu - \eta_{\nu\sigma} X^\sigma \partial_\mu.$$

For example, the $SL(2, \mathbb{R})$-invariant distance function is

$$\sigma(x|x') = \frac{1}{2R^2}(X - X')_\mu (X - X')^\mu + 1$$

and the antipodal map is simply A: $X^\mu \to -X^\mu$.

We would like to understand the action of these isometry generators on the boundary in different coordinate systems (global, Poincaré and Rindler). These coordinate systems are defined as follows [3]:

(i)
$$\text{Global}: \begin{cases} X^0 = R\sec\rho\cos\tau, \\ X^1 = R\sec\rho\sin\tau, \\ X^2 = R\tan\rho, \end{cases}$$

with metric $ds^2 = \frac{R^2}{\cos^2\rho}(-d\tau^2 + d\rho^2)$,

(ii)
$$\text{Poincaré}: \begin{cases} X^0 = (R^2 + Z^2 - T^2)/2Z, \\ X^1 = RT/Z, \\ X^2 = (R^2 - Z^2 + T^2)/2Z, \end{cases}$$

with metric $ds^2 = \frac{R^2}{Z^2}(-dT^2 + dZ^2)$,

(iii) Rindler (with arbitrary dimensionless M):

$$\begin{cases} X^0 = r/\sqrt{M}, \\ X^1 = \sqrt{\frac{r^2}{M} - R^2}\sinh(\sqrt{M}t/R), \\ X^2 = \sqrt{\frac{r^2}{M} - R^2}\cosh(\sqrt{M}t/R), \end{cases}$$

with metric $ds^2 = -(\frac{r^2}{R^2} - M)dt^2 + (\frac{r^2}{R^2} - M)^{-1}dr^2$.

Near the boundary ($\rho \to \pi/2, Z \to 0, r \to \infty$) the isometry generators approach

(i)
$$\text{Global}: \begin{cases} J_{01} = -\partial_\tau, \\ J_{02} = \sin\tau\partial_\tau, \\ J_{12} = -\cos\tau\partial_\tau, \end{cases}$$

(ii)
$$\text{Poincaré}: \begin{cases} J_{01} = -\frac{1}{2R}(T^2 + R^2)\partial_T, \\ J_{02} = T\partial_T, \\ J_{12} = \frac{1}{2R}(T^2 - R^2)\partial_T, \end{cases}$$

(iii)
$$\text{Rindler}: \begin{cases} J_{01} = -\frac{R}{\sqrt{M}}\cosh(\sqrt{M}t/R)\partial_t, \\ J_{02} = \frac{R}{\sqrt{M}}\sinh(\sqrt{M}t/R)\partial_t, \\ J_{12} = -\frac{R}{\sqrt{M}}\partial_t. \end{cases}$$

Here we are keeping only the leading (divergent) behavior of the vector field near the boundary.

In Poincaré coordinates the isometries give rise to the usual representation for the conformal generators on a line,

$$\hat{H} = i\partial_T = -(i/R)(J_{01} + J_{12}), \qquad \hat{D} = iT\partial_T = iJ_{02},$$
$$\hat{K} = iT^2\partial_T = -iR(J_{01} - J_{12}). \tag{A1}$$

Rather than adopting $\hat{H}$ as the Hamiltonian, one can use a different linear combination of the conformal generators to evolve in time [42]. This is exactly what is done when using the other two coordinate systems:

(i) Global: $i\partial_\tau = \frac{1}{2}(R\hat{H} + \frac{1}{R}\hat{K}) = \hat{R}$.

(ii) Rindler: $i\partial_t = \frac{\sqrt{M}}{2R}(R\hat{H} - \frac{1}{R}\hat{K}) = -\frac{\sqrt{M}}{R}\hat{S}$.

Here $\hat{R}$ (not to be confused with the AdS radius) and $\hat{S}$ are compact rotation and noncompact boost generators, respectively. One can show that when evolving in time using noncompact generators (such as $\hat{S}$), one cannot cover the entire range $-\infty < T < \infty$. This is exactly what happens in Rindler coordinates—Rindler time t only covers half of the boundary of the Poincaré patch. When evolving with compact generators such as $\hat{R}$, one does cover the entire range of T (as global coordinates do) [42].

APPENDIX B: ADS VACUA

At various points in the paper we made use of the equivalence between the global, Poincaré and Hartle-Hawking vacua [23,24,32]. In this appendix we review these results and show how they relate different boundary fields.

We begin with the equivalence between the global and Poincaré vacua. Bulk modes that are positive frequency with respect to global time are also positive frequency with respect to Poincaré time [23]. This implies that the two bulk vacua are equivalent. To understand the corresponding relationship between the global and Poincaré boundary fields, let us start with a global boundary field that is purely positive frequency,

$$\phi_{0+}^{\text{global},R}(\tau) = \sum_{n=0}^{\infty} c_n e^{-i\omega_n\tau}. \tag{B1}$$

Near the boundary,

$$Z^\Delta\phi_0^{\text{Poincare}} \sim \cos^\Delta\rho\,\phi_0^{\text{global},R}$$

so the corresponding Poincaré boundary field is equal to the global field times a Jacobian

$$\phi_0^{\text{Poincare}}(T) = \lim_{\rho\to\pi/2}\left(\frac{\cos\rho}{Z}\right)^\Delta \phi_{0+}^{\text{global},R}(\tau)$$
$$= \left(\frac{2R}{T^2 + R^2}\right)^\Delta \phi_{0+}^{\text{global},R}(\tau). \tag{B2}$$

Here $\tau = 2\tan^{-1}(T/R) = -i\log\frac{1+iT/R}{1-iT/R}$ so

$$\phi_0^{\text{Poincare}}(T) = \left(\frac{2}{R}\right)^\Delta \sum_{n=0}^{\infty} c_n \frac{(1 - iT/R)^n}{(1 + iT/R)^{n+2\Delta}}. \tag{B3}$$

The Poincaré boundary field is analytic in the lower half of the complex T plane, so its Fourier transform

$$\tilde{\phi}_0^{\text{Poincare}}(\omega) = \int dT e^{i\omega T}\phi_0^{\text{Poincare}}(T)$$

vanishes if $\omega < 0$. Thus it follows from the equivalence of the bulk Poincaré and global vacua that a positive-frequency global boundary field corresponds to a positive-frequency Poincaré boundary field. We used this in Sec. III B to go from (30) and (31). Note that the Jacobian appearing in (B2) is crucial: the global boundary field (B1) by itself, when expressed in terms of Poincaré time, is *not* positive frequency.

Now let us turn to the equivalence between the global and Hartle-Hawking vacua. For these purposes it is convenient to introduce null Kruskal coordinates

$$u = \tan\frac{\tau + \rho}{2}, \qquad v = \tan\frac{\tau - \rho}{2} \tag{B4}$$

in which

$$ds^2 = -\frac{4R^2 du dv}{(1 + uv)^2}. \tag{B5}$$

These coordinates cover the region shown in Fig. 4; the restriction $-\pi/2 < \rho < \pi/2$ corresponds to $uv > -1$. For $u < 0$ note that points with $uv = -1$ make up the boundary of the left Rindler wedge, while for $u > 0$ points with $uv = -1$ make up the boundary of the right Rindler wedge.

Working in the right Rindler wedge ($u > 0$ and $v < 0$) mode solutions with normalizable falloff near the right boundary of AdS space are

$$\phi(u, v) = u^{-i\omega}(1 + uv)^\Delta F(\Delta, \Delta - i\omega, 2\Delta, 1 + uv). \tag{B6}$$

Here $-\infty < \omega < \infty$ is a parameter which can be under-

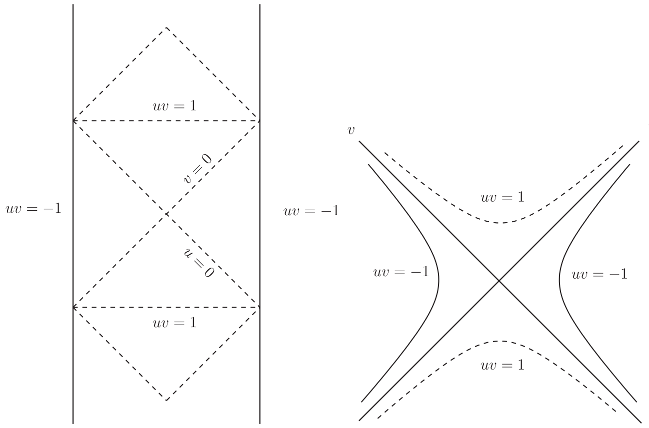

FIG. 4. The Kruskal patch, indicated on the AdS Penrose diagram (left) and in the uv plane (right).

stood as the frequency measured with respect to Rindler time, since under the isometry corresponding to a Rindler time translation,

$$u \to e^{\lambda} u, \qquad v \to e^{-\lambda} v,$$

these modes transform by a definite phase $\phi \to e^{-i\omega\lambda}\phi$. Note that ω does not correspond to the frequency measured with respect to a Kruskal time coordinate.

We want to find a set of positive-frequency Kruskal modes [43] for which (by construction) the Kruskal vacuum will be equivalent to the global vacuum. This depends on making the correct analytic continuation from right to left. To do this we first write the modes (B6) in a more symmetric form, using a transformation formula for the hypergeometric function

$$\phi(u, v) = (1 + uv)^{\Delta}\bigg[\frac{\Gamma(2\Delta)\Gamma(i\omega)}{\Gamma(\Delta)\Gamma(\Delta + i\omega)} u^{-i\omega} F(\Delta, \Delta - i\omega, 1 - i\omega, -uv) + \frac{\Gamma(2\Delta)\Gamma(-i\omega)}{\Gamma(\Delta)\Gamma(\Delta - i\omega)} \times (-v)^{i\omega} F(\Delta, \Delta + i\omega, 1 + i\omega, -uv)\bigg]. \quad \text{(B7)}$$

This makes it clear that, aside from the prescribed behavior near the boundary of AdS space, the modes have branch points on the horizon (at $u = 0$ or $v = 0$). A positive-frequency Kruskal mode is defined by analytically continuing from right to left going through the lower half of the complex u and v planes, while a negative-frequency Kruskal mode is defined by analytically continuing through the upper half u and v planes [44]. With this prescription the global and Kruskal bulk vacua are equivalent.

It is straightforward to obtain the corresponding analyticity properties on the boundary. In Kruskal coordinates we define the boundary field by

$$\phi_0^{\mathrm{Kruskal}}(u) = \lim_{uv \to -1} \frac{\phi(u, v)}{(1 + uv)^{\Delta}}. \quad \text{(B8)}$$

The relation between Kruskal and global boundary fields is then

$$\phi_0^{\mathrm{Kruskal}}(u) = \begin{cases} \lim_{uv\to -1} \frac{\cos^{\Delta}\rho}{(1+uv)^{\Delta}} \phi_0^{\mathrm{global},R}(\tau) & \text{for } u > 0, \\ \lim_{uv\to -1} \frac{\cos^{\Delta}\rho}{(1+uv)^{\Delta}} \phi_0^{\mathrm{global},L}(\tau) & \text{for } u < 0. \end{cases} \quad \text{(B9)}$$

A positive-frequency global boundary field takes the form

$$\begin{aligned} \phi_{0+}^{\mathrm{global},R}(\tau) &= \sum_{n=0}^{\infty} c_n e^{-i\omega_n \tau}, \\ \phi_{0+}^{\mathrm{global},L}(\tau) &= \sum_{n=0}^{\infty} c_n (-1)^n e^{-i\omega_n \tau}. \end{aligned} \quad \text{(B10)}$$

Note that

$$\lim_{\tau\to +\pi/2} \phi_{0+}^{\mathrm{global},L}(\tau) = e^{-i\pi\Delta} \lim_{\tau\to -\pi/2} \phi_{0+}^{\mathrm{global},R}(\tau). \quad \text{(B11)}$$

Inserting (B10) into (B9) we find that a positive-frequency global boundary field maps to a Kruskal boundary field given by

$$\phi_0^{\mathrm{Kruskal}}(u) = \sum_{n=0}^{\infty} c_n \frac{(-1)^{n/2}(-iu)^{\Delta}(u + i)^n}{(u - i)^{2\Delta+n}}. \quad \text{(B12)}$$

The Kruskal field has a branch point at $u = 0$. In order to reproduce (B11) we must analytically continue from $u > 0$ to $u < 0$ by going through the lower half of the complex u plane. We take this analyticity condition to define a positive-frequency Kruskal boundary mode; this makes the global and Kruskal vacua equivalent on the boundary, just as they were equivalent in the bulk.

With this definition, from (B7) a positive-frequency Kruskal mode can be characterized by the boundary behavior

$$\phi_{0+}^{\mathrm{Kruskal}}(u) = \begin{cases} u^{-i\omega} & \text{for } u > 0, \\ e^{-\pi\omega}(-u)^{-i\omega} & \text{for } u < 0, \end{cases}$$

while a negative-frequency Kruskal mode is characterized by

$$\phi_{0-}^{\mathrm{Kruskal}}(u) = \begin{cases} u^{-i\omega} & \text{for } u > 0, \\ e^{+\pi\omega}(-u)^{-i\omega} & \text{for } u < 0. \end{cases}$$

At this point it is convenient to switch to Rindler coordinates. The relation is

$$\begin{aligned} u &= \pm\left(\frac{r - \sqrt{M}R}{r + \sqrt{M}R}\right)^{1/2} e^{\sqrt{M}t/R}, \\ v &= \mp\left(\frac{r - \sqrt{M}R}{r + \sqrt{M}R}\right)^{1/2} e^{-\sqrt{M}t/R} \end{aligned}$$

where the upper (lower) sign holds in the right (left)

Rindler wedge. The Jacobian for switching from Kruskal to Rindler is a constant,

$$\phi_0^{\text{Rindler}} = \lim_{r\to\infty}(r(1+uv))^\Delta \phi_0^{\text{Kruskal}} = (2\sqrt{M}R)^\Delta \phi_0^{\text{Kruskal}},$$

so we can characterize Rindler modes which are positive or negative frequency with respect to Kruskal time as having the behavior

$$\phi_{0\pm}^{\text{Rindler},R} \sim e^{-i\omega t}, \qquad \phi_{0\pm}^{\text{Rindler},L} \sim e^{\mp\beta\omega/2}e^{-i\omega t}$$

where $\beta = 2\pi R/\sqrt{M}$ is the inverse temperature of the black hole. Finally, this lets us define a projection operator which picks out the part of the Rindler boundary field which is positive or negative frequency with respect to Kruskal time. Let

$$\tilde{\phi}_0^{\text{Rindler},L/R}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}\phi_0^{\text{Rindler},L/R}(t)$$

be the Fourier transform of the Rindler boundary field. Then Rindler boundary fields which are positive or negative frequency with respect to Kruskal time are given by

$$\begin{aligned}\phi_{0\pm}^{\text{Rindler},R}(t) &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}f_\pm(\omega)e^{-i\omega t},\\ \phi_{0\pm}^{\text{Rindler},L}(t) &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}f_\pm(\omega)e^{\mp\beta\omega/2}e^{-i\omega t}\end{aligned} \tag{B13}$$

where the conditions

$$\begin{aligned}f_+ + f_- &= \tilde{\phi}_0^{\text{Rindler},R},\\ f_+e^{-\beta\omega/2} + f_-e^{\beta\omega/2} &= \tilde{\phi}_0^{\text{Rindler},L}\end{aligned} \tag{B14}$$

fix

$$\begin{aligned}f_+(\omega) &= \frac{\tilde{\phi}_0^{\text{Rindler},R}(\omega)e^{\beta\omega/2} - \tilde{\phi}_0^{\text{Rindler},L}(\omega)}{2\sinh\beta\omega/2},\\ f_-(\omega) &= \frac{\tilde{\phi}_0^{\text{Rindler},L}(\omega) - \tilde{\phi}_0^{\text{Rindler},R}(\omega)e^{-\beta\omega/2}}{2\sinh\beta\omega/2}.\end{aligned} \tag{B!5}$$

Multiplying by the Rindler-to-global Jacobian

$$\lim_{r\to\infty}\frac{1}{(r\cos\rho)^\Delta} = \left(\frac{\cosh(\sqrt{M}t/R)}{\sqrt{M}R}\right)^\Delta \tag{B16}$$

the boundary fields (B13) become positive or negative frequency with respect to global rather than Kruskal time. They can then be substituted in (41) to obtain the Rindler smearing function for a point inside the horizon.

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