Equivalent Hermitian Hamiltonian for the non-Hermitian $-x^4$ potential

H. F. Jones

Physics Department, Imperial College, London SW7, United Kingdom

J. Mateo

Departamento de Física Teórica, Facultad de Ciencias, E-47011, Valladolid, Spain

The potential $V(x) = -x^4$, which is unbounded below on the real line, can give rise to a well-posed bound state problem when x is taken on a contour in the lower-half complex plane. It is then PT-symmetric rather than Hermitian. Nonetheless it has been shown numerically to have a real spectrum, and a proof of reality, involving the correspondence between ordinary differential equations and integrable systems, was subsequently constructed for the general class of potentials $-(ix)^N$. For such Hamiltonians the natural PT metric is not positive definite, but a dynamically-defined positive-definite metric can be defined, depending on an operator Q. Further, with the help of this operator an equivalent Hermitian Hamiltonian h can be constructed. This programme has been carried out exactly for a few soluble models, and the first few terms of a perturbative expansion have been found for the potential $m^2 x^2 + i g x^3$. However, until now, the $-x^4$ potential has proved intractable. In the present paper we give explicit, closed form expressions for Q and h, which are made possible by a particular parametrization of the contour in the complex plane on which the problem is defined. This constitutes an explicit proof of the reality of the spectrum. The resulting equivalent Hamiltonian has a potential with a positive quartic term together with a linear term.

I. INTRODUCTION

There has been a great deal of interest in non-Hermitian Hamiltonians since the numerical observation by Bender and Boettcher [1] that Hamiltonians of the form

$$H = p^2 - g(ix)^N \qquad (1)$$

have a real positive spectrum for $N \geq 2$. As illustrated in Fig. 1 (from Ref. [1], where $g = 1$), their spectra constitute a smooth extrapolation from the simple harmonic oscillator, for which $N = 2$. The reality of their spectra is understood as being due to their unbroken PT symmetry, but there is no simple way of telling in advance whether or not this symmetry is broken, as indeed it is for $N < 2$, where the spectra are partly complex. Eventually a rather intricate proof of the reality of the spectrum, involving the correspondence between the differential equations for such potentials and integrable models, was constructed by Dorey *et al.* [2].

A potential problem with such Hamiltonians is their physical interpretation, since the natural PT norm on the Hilbert space, $\int dx \psi^*(x)\psi(-x)$, is not positive definite, in contrast to the usual norm $\int dx \psi^*(x)\psi(x)$. However, it turns out to be possible to construct an alternative norm, the CPT norm [3], which is indeed positive definite. This norm is different from the usual norm, in that it is dynamically determined by the Hamiltonian itself, and needs to be calculated in each individual case.

Such calculations were encompassed by Mostafazadeh [4] in the more general framework of pseudo-Hermiticity, whereby

$$H^\dagger = \eta H \eta^{-1} \qquad (2)$$

Here the operator η is Hermitian and positive definite, and may usefully be written as $\eta = e^{-Q}$, in order to connect with the notation of Ref. [5], where, for PT-symmetric Hamiltonians, $\eta = PC$ and Q was defined by $C = e^Q P$. For calculational purposes it is much easier to deal with Q rather than η directly. Mostafazadeh showed further that
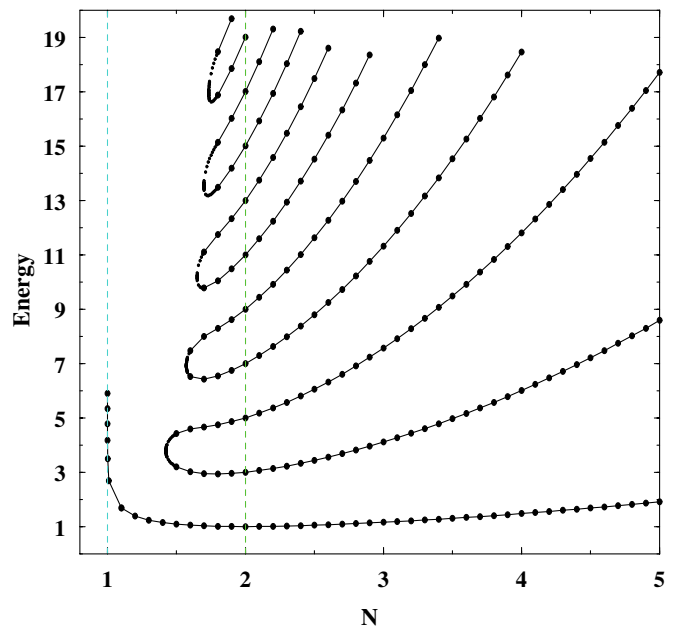

FIG. 1 (color online). Energy levels of Eq. (1) for $N = 4$ with $g = 1$, from Ref. [1]

$$h \equiv e^{(-1/2)Q} H e^{1/2Q} \tag{3}$$

is an equivalent Hermitian Hamiltonian, obtained from H by a similarity (Darboux) transformation.

In general it is difficult to solve Eqs. (2) and (3) exactly; instead one uses perturbation theory in a small parameter ε. If H is of the form $H = H_0 + \varepsilon H_1$, where H_0 is Hermitian and H_1 anti-Hermitian, then Q can be taken as $Q = \sum_{r \, \text{odd}} Q_r \varepsilon^r$, which then gives $h = \sum_{r \, \text{even}} h_r \varepsilon^r$. In this case the first few equations for the Q_r, arising from the expansion of Eq. (2), read [1]

$$[Q_1, H_0] = 2H_1$$

$$[Q_3, H_0] = \frac{1}{6}[Q_1, [Q_1, H_1]]$$

$$[Q_5, H_0] = \frac{1}{6}([Q_3, [Q_1, H_1]] + [Q_1, [Q_3, H_1]]) \tag{4}$$

$$\qquad\qquad - \frac{1}{360}[Q_1, [Q_1, [Q_1, [Q_1, H_1]]]]$$

and so on. Using these, the first few equations for the h_r, arising from the expansion of Eq. (3), can be cast in the form

$$h_0 = H_0$$

$$h_2 = -\frac{1}{4}[Q_1, H_1] \tag{5}$$

$$h_4 = \frac{1}{192}[Q_1, [Q_1, [Q_1, H_1]]] - \frac{1}{4}[Q_3, H_1].$$

The smooth continuation from the harmonic oscillator, and the ODE-IM correspondence of Ref. [2], rest on the fact that the Schrödinger differential equation has various different sectors, defined by wedges in the complex x-plane. Along the center of the wedges the wave-function decays exponentially at infinity, while along the edges the wave-function is purely oscillatory. Fig. 2, taken from Ref. [1], shows the particular wedge that connects smoothly with that for the harmonic oscillator.

The critical case, where the upper edge of the wedge coincides with the real axis is the case $N = 4$, i.e. the potential $-x^4$. For $N < 4$, it is possible to stay on the real axis, where the wave-function decays exponentially, albeit with an oscillatory modulation, but for $N \geq 4$ we have no option but to formulate the problem on a contour in the lower-half x plane.

This is the fundamental reason why the $-gx^4$ potential has proved so intractable. At first sight it appears Hermitian: it is only because of the contour on which it must be defined that it is non-Hermitian. The problem is inherently nonperturbative, so any expansion to be attempted can not be in the coupling constant g. A previous

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[1]The equations of even order are satisfied identically by $Q_{2n} = 0$.
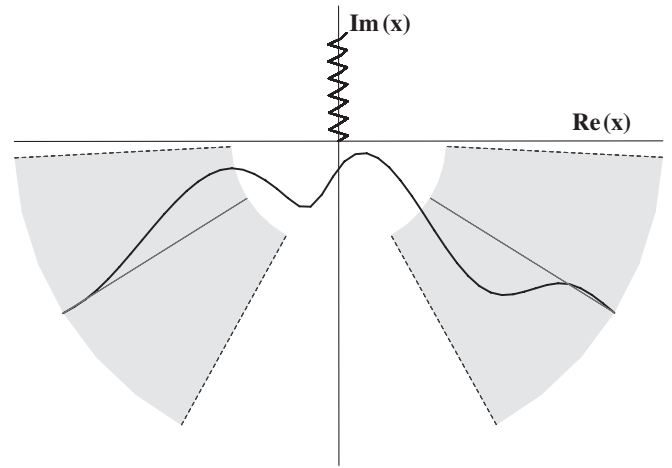

FIG. 2. Wedges in the complex plane in which the Schrödinger equation for Eq. (1) is posed.

attempt [6] used WKB methods, and was able to calculate Q to leading nonperturbative order.

II. CHOICE OF CONTOUR

Our present approach starts with the idea of Mostafazadeh [7], to map the problem back onto the real axis using a real parametrization of a suitable contour. We recall that a wide variety of contours are possible, as long as they go off to infinity at an angle within the wedges. Taking $N = 4$ and writing the original x variable of Eq. (1) as z to reflect its complex character, so that

$$H = -\frac{d^2}{dz^2} - gz^4, \tag{6}$$

the parametrization used in Ref. [7] was

$$z = x \cos\theta - i|x| \sin\theta, \tag{7}$$

corresponding to straightline contours, with an infinitesimal rounding off near the origin. Here θ was taken as $\pi/6$, the optimal angle for $N = 4$. The resulting Hamiltonian was

$$H = e^{i \, \text{sgn}(x)\pi/3}\left(-\frac{d^2}{dx^2} + g|x|^4\right). \tag{8}$$

Because of rounding, there are nontrivial boundary conditions at $x = 0$, namely, (i) ψ is real and continuous, (ii) $\psi'(0^-) = e^{4i\theta}\psi'(0^+)$. Consequently, in calculating $H^\dagger$ there is an additional term δH beyond the obvious one.

A calculation of Q with this Hamiltonian is very difficult because of the boundary conditions at $x = 0$ and the lack of an obvious expansion parameter. In Ref. [8] we attempted to make an expansion in θ, freeing it from its optimal value for $N = 4$, noting that any positive value for θ would suffice to make the wave function vanish with an exponential component. In addition we smoothed out the curve chosen in Ref. [7], taking the hyperbola

$$z = x\cos\theta - i\sin\theta\sqrt{(1 + x^2)}, \tag{9}$$

in order to remove the boundary conditions at $x = 0$. Unfortunately this calculation did not produce a very useful h, but rather one which still had a $-x^4$ term, so that the asymptotic behavior of the wave function was oscillatory, with only a power suppression.

In the present paper we adopt a different approach. First we choose a new parametrization, whose asymptotes are not in fact in the center of the wedges, but rather are inclined at $\pi/4$ to the real axis, and then we introduce an artificial parameter ε multiplying H_1, the non-Hermitian part of $H(x)$.

The contour that turns out to give particularly simple results is of the form

$$z = -2i\sqrt{(1 + ix)}. \tag{10}$$

Notice that with this choice, the *PT*-symmetry of the original Hamiltonian, which is a real function of iz, will be respected by the new Hamiltonian, written in terms of x. This new Hamiltonian is in fact

$$H = \frac{1}{2}\{(1 + ix), p^2\} - \frac{1}{2}p - \alpha(1 + ix)^2, \tag{11}$$

where $\{.., ..\}$ denotes the anticommutator, $p \equiv d/dx$, and for convenience we have introduced $\alpha \equiv 16g$. Separating H into its Hermitian and anti-Hermitian parts, and multiplying the latter by the artificial parameter ε, which at the end will be set equal to one, we write

$$H = H_0 + \varepsilon H_1, \tag{12}$$

where

$$H_0 = p^2 - \frac{1}{2}p + \alpha(x^2 - 1) \qquad H_1 = \frac{1}{2}i\{x, p^2\} - 2i\alpha x \tag{13}$$

III. CALCULATION OF Q AND h

First we calculate Q_1 from the first of Eqs. (4), namely $[Q_1, H_0] = 2H_1$. As a general, systematic procedure for such problems we would write the Hermitian operator Q_1 as a sum of anticommutators of the form $Q_1 = \sum_{n \text{ odd}}\{f_n(x), p^n\}$, where $f_n(x)$ is a real function of x, and gradually increase the order n. However, in this case H_0 and H_1 are so simple that the solution can essentially be found by inspection. Thus a p^3 term in Q_1 will produce the desired structure $i\{x, p^2\}$ when commuted with the x^2 term of H_0, while a term in p will produce the x term of H_1. By equating coefficients we find that

$$Q_1 = -\frac{p^3}{3\alpha} + 2p. \tag{14}$$

In order to calculate Q_3 from the second of Eqs. (4) we need the double commutator $[Q_1, [Q_1, H_1]]$. First let us calculate the inner commutator $[Q_1, H_1]$, which will also

be needed for the computation of h:

$$[Q_1, H_1] = -\frac{p^4}{\alpha} + 4p^2 - 4\alpha. \tag{15}$$

The crucial point is that this is a function of p only, and therefore commutes with Q_1. Thus $[Q_1, [Q_1, H_1]] = 0$, which means that $Q_3 = 0$. Then the third of Eqs. (4) shows that $Q_5 = 0$ and so on. Thus we have an *exact* solution for Q, after setting $\varepsilon = 1$, namely

$$Q = -\frac{p^3}{3\alpha} + 2p. \tag{16}$$

Having obtained the metric operator Q we are in a position to calculate the equivalent Hermitian Hamiltonian h of Eq. (3). Because the expansion for Q has truncated, so does that for h, namely $h = H_0 + h_2$. The commutator required for the evaluation of h_2 has already been calculated in Eq. (15), so it is straightforward to evaluate h, with the remarkably simple result that

$$h = \frac{p^4}{4\alpha} - \frac{1}{2}p + \alpha x^2. \tag{17}$$

We emphasize that this Hermitian Hamiltonian, defined on the real line, has the same energy spectrum as that of the original H of Eq. (6) defined on a complex contour. The only unusual feature of h is that it does not have the standard form of a quadratic kinetic term plus a potential. However, just such a Hamiltonian results if we take the Fourier transform. In terms of the transformed variable y, and after a rescaling $y \rightarrow y\sqrt{\alpha}$, we have

$$\tilde{h} = p_y^2 + \frac{1}{4}\alpha y^4 - \frac{1}{2}\sqrt{\alpha}y \tag{18}$$

IV. DISCUSSION

Eqs. (16) and (18) constitute our main results. The latter exhibits a standard Hermitian Hamiltonian, with a positive quartic potential plus a linear term, shown in Fig. 3, whose spectrum is the same as that of the original problem, with a $-z^4$ potential posed on a contour in the complex plane. It constitutes the first direct, constructive proof of the reality of the spectrum of Eq. (6). In accordance with our introductory remarks, we note that $\tilde{h}$ is completely nonperturbative, since, without a harmonic term m^2x^2 term in the potential, g can rescaled to 1.

We have performed a numerical calculation of the energy eigenvalues of Eq. (18), using both Runge-Kutta integration and the variational truncated matrix method of Ref. [9]. Both methods give eigenvalues that are indistinguishable from those cited by Bender and Boettcher (calculated by Runge-Kutta integration along a complex contour) in their original paper [1].

A simple extension of the above result can be obtained when an additional harmonic term m^2z^2 is introduced into Eq. (6). The only change in Eq. (17) is that h becomes
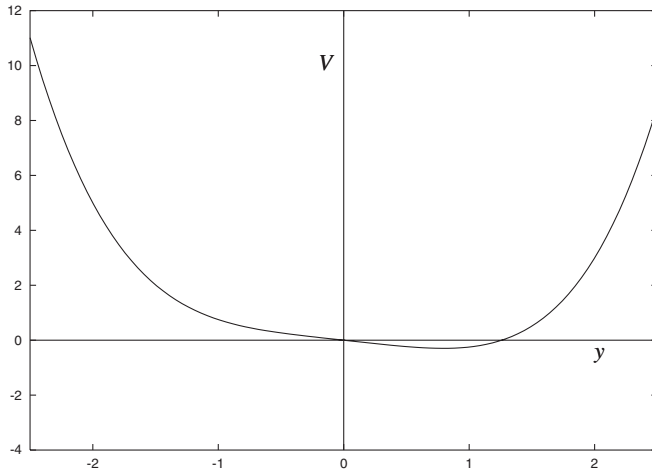
FIG. 3. The potential of Eq. (18), with $\alpha = 16$ ($g = 1$).

$$h = \frac{(p^2 - 4m^2)^2}{4\alpha} - \frac{1}{2}p + \alpha x^2 \qquad (19)$$

with corresponding scaled Fourier transform

$$\tilde{h} = p_y^2 + \frac{1}{4\alpha}(\alpha y^2 - 4m^2)^2 - \frac{1}{2}\sqrt{\alpha}y \qquad (20)$$

After completion of this work we were made aware of an earlier paper by Buslaev and Grecchi [10], which showed the spectral equivalence of the massive version of the $-x^4$ theory (their $H_\varepsilon(ig, j)$, with $j = 1$), formulated on the line $z = x - i\eta$, with a Hermitian Hamiltonian that can be identified with Eq. (20) on setting $\alpha = 4g^2$, $m = 1/2$. Their method made use of the perturbation series for the energy eigenvalues of the two Hamiltonians, which only exists for $m \neq 0$. However, they were subsequently able to go the massless limit by rescaling and taking g to ∞. In this way they obtained the spectral equivalence between Eq. (6) and (17) (see their Theorem 6, with $j = 1$, $\alpha = 0$ after a simple rescaling).

The present paper approaches the problem from a completely different perspective and offers a simple, explicit and transparent derivation of these spectral equivalences, together with the operator Q required to define the positive-definite metric, and the observables [4], of the non-Hermitian Hamiltonians.

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