

BPS black holes, quantum attractor flows, and automorphic formsMurat Gunaydin,^{1,*} Andrew Neitzke,^{2,†} Boris Pioline,^{3,4,‡} and Andrew Waldron^{5,§}¹*Department of Physics, Penn State University, University Park, Pennsylvania 16802, USA*²*School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540, USA*³*LPTHE, Universités Paris VI & VII, Jussieu, 75252 Paris, France*⁴*LPTENS, Département de Physique de l'ENS, 24, rue Lhomond, 75005 Paris, France*⁵*Department of Mathematics, University of California, Davis, California 95616, USA*

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We propose a program for counting microstates of four-dimensional BPS black holes in $\mathcal{N} \geq 2$ supergravities with symmetric-space valued scalars by exploiting the symmetries of timelike reduction to three dimensions. Inspired by the equivalence between the four-dimensional attractor flow and geodesic flow on the three-dimensional scalar manifold, we radially quantize stationary, spherically symmetric BPS geometries. Connections between the topological string amplitude, attractor wave function, the Ooguri-Strominger-Vafa conjecture and the theory of automorphic forms suggest that black hole degeneracies are counted by Fourier coefficients of modular forms for the three-dimensional U -duality group, associated to special unipotent representations which appear in the supersymmetric Hilbert space of the quantum attractor flow.

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I. INTRODUCTION

Understanding the microscopic origin of the thermodynamic entropy of black holes is a challenge for any quantum theory of gravity. In string theory a microscopic description of certain supersymmetric black holes is known and accounts for the expected number of microstates in the limit of large charges $Q \gg 1$ ([1,2] and much subsequent work). Recently there has been some progress in understanding the counting of microstates beyond the strict large Q limit (see e.g. [3] for a recent review).

Strikingly, Ooguri, Strominger and Vafa (OSV) [4] have argued that, in $\mathcal{N} = 2$ compactifications of Type II string theory on a Calabi-Yau threefold X , all subleading $1/Q$ corrections to the degeneracies should be computable by considering the squared topological string amplitude $|\Psi_{\text{top}}^2|$ as a generating function at fixed magnetic charge. A possible rationale for this phenomenon uses channel duality in the near horizon $\text{AdS}_2 \times S^2$ [5]: the partition function with compactified Euclidean time, which counts black hole microstates (possibly with signs), could equivalently be computed in radial quantization as an overlap of wave functions at each boundary of AdS_2 . Provided the topological string amplitude solves the radial Wheeler-DeWitt constraint, the general form of the OSV conjecture then follows in analogy with open/closed string duality on the cylinder. This equivalence, if correct, may be viewed as a mini-superspace AdS/CFT correspondence, where

the bulk evolution of the gravity fields is related to time evolution in a (as yet unknown) dual conformal quantum mechanics.

On the other hand, in theories with high $\mathcal{N} = 4$ or $\mathcal{N} = 8$ supersymmetry, the exact black hole degeneracies have long been suspected to be related to Fourier coefficients of some kind of automorphic form [6]. Indeed, the respective four-dimensional U -duality groups $G_4 = SL(2, \mathbb{Z}) \times SO(6, n, \mathbb{Z})$ or $E_{7(7)}(\mathbb{Z})$ provide strong constraints on microscopic degeneracies (see [7] for a U -duality review). Recently however, there have been indications that the *three*-dimensional U -duality groups $G_3 = SO(8, n + 2, \mathbb{Z})$ or $E_{8(8)}(\mathbb{Z})$ may be spectrum generating symmetries for *four*-dimensional BPS black holes [8–11]. This idea is partly motivated by the “quasiconformal” realization of G_3 [9] on the space of electric and magnetic charges (p^I, q_I) extended by one variable. This is parallel to the “conformal” realization of G_4 on the space of black hole charges in five dimensions [8–10]. Upon quantization, it leads to the minimal unipotent representation of G_3 [9,12], which is a crucial ingredient in the construction of a particular automorphic form for G_3 .

In this Letter, we outline a new approach to black hole microstate counting inspired by these ideas, postponing details to [13]. Taking the proposal in [5] literally, we study the radial quantization of stationary, spherically symmetric BPS black holes in four-dimensional $\mathcal{N} \geq 2$ supergravity theories. To avoid the complications of full-fledged $\mathcal{N} = 2$ supergravity, we restrict to special cases in which the scalars take values in a symmetric space G_4/K_4 . For simplicity, we further discard any higher-derivative corrections to the low energy effective action. This allows us to

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jettison Calabi-Yau geometry in favor of real Lie group representation theory, while retaining some essential features of a realistic $\mathcal{N} = 2$ supergravity. Higher $\mathcal{N} = 4$ or $\mathcal{N} = 8$ supergravities are also in this category, provided we again discard higher-derivative corrections.

Since our interest lies in stationary solutions, it is natural to perform a reduction along the time direction, leading to three-dimensional Euclidean gravity coupled to a nonlinear sigma model with an enlarged symmetry group G_3 [14]. An essential observation is that stationary, spherically symmetric solutions of the four-dimensional equations of motion are equivalent to geodesic trajectories on the three-dimensional scalar manifold $M_3^* = G_3/H_3$, where H_3 is a certain noncompact real form of the maximal compact subgroup K_3 . As such, they carry an action of G_3 relating black hole geometries with different charges.¹

This observation is key to the radial quantization of black holes in a mini-superspace approximation. Geodesic trajectories can be replaced by square-integrable wave functions on M_3^* (or sections of some bundle). The action of G_3 is then roughly the regular representation on the Hilbert space $L^2(G_3/H_3)$, which breaks up into a sum of unitary irreducible representations (irreps) of G_3 . As we shall see, the BPS component of the Hilbert space is a “unipotent” representation of unusually small functional dimension. Furthermore, in some cases there is a distinguished “spherical” vector in this space, which appears to play the role of a 1-parameter extension of the topological string amplitude Ψ_{top} , in broad accordance with the proposal of [5].

In the full quantum theory, pursuing the suggestion in [11], we conjecture that a discrete subgroup $G_3(\mathbb{Z})$ remains as a “spectrum generating” symmetry controlling the exact degeneracies of BPS black holes. This idea is supported by the fact that the above unipotent representations have been exploited by mathematicians to construct modular forms for G_3 . Thus it is tempting to identify the black hole degeneracies as particular generalized Fourier coefficients of these modular forms. As we shall see, the result (17) below bears a close resemblance to the OSV conjecture, albeit with striking differences.

The proposal that BPS microstates are counted by modular forms for G_3 should have interesting ramifications (if it can be verified). For example, it could provide an explanation of mysterious modular properties which have been observed in computations of the degeneracies of BPS states in four dimensions [6,11,16–20]. More generally, we are hopeful that understanding the spectrum generating symmetries of black hole degeneracies will improve our understanding of the conjecture of [4] and the nonperturbative meaning of the topological string.

¹We note that G_3 already made an appearance as a solution-generating symmetry in the $N = 4$ context [15].

Finally, it should be mentioned that the radial quantization of black holes has been a subject of much work by the canonical gravity community [21–26]. While this endeavor has so far yielded relatively little insight into black hole entropy, embedding this idea in supergravity may justify the mini-superspace truncation, at least for BPS black holes, and allow channel duality to teach us the nature of black hole microstates.

The organization of this Letter is as follows: in Sec. II, we describe a class of supergravities with symmetric scalar manifolds where our techniques apply most directly. In Sec. III, we interpret the radial evolution equations as geodesic motion on a scalar manifold M_3^* and determine the conditions on geodesics required for supersymmetric solutions. In Sec. IV, we explain our proposal for quantizing the attractor flow, and argue that the BPS Hilbert space furnishes a particular unipotent representation of G_3 . In Sec. V, we conjecture a relation between exact black hole degeneracies and Fourier coefficients of automorphic forms naturally associated to these representations. We close in Sec. VI with an outlook.

II. HOMOGENEOUS SUPERGRAVITIES

The analysis we describe applies to a class of $d = 4$, $\mathcal{N} = 2$ supergravity theories whose scalar fields are associated to vector multiplets and lie on a symmetric space $M_4 = G_4/K_4$. Such theories, known as “very special supergravities”, were first studied in [27–29]. The special geometry of M_4 turns out to be characterized by a cubic prepotential $F = N(X)/X^0$, with $N(X)$ the norm function of a degree 3 Jordan algebra J . The classification and study of such theories is therefore closely tied to the theory of Jordan algebras.

We also consider theories with higher supersymmetry, $\mathcal{N} = 4$ or even $\mathcal{N} = 8$. Here again one finds scalar fields living on some symmetric space $M_4 = G_4/K_4$. For example, in the $\mathcal{N} = 8$ case (the low energy limit of M -theory compactified on T^7) this space is $E_{7(7)}/(SU(8)/\mathbb{Z}_2)$. We list some examples in Table I, giving the number of supercharges, the moduli space in $d = 4$, and the moduli space in $d = 3$ after reduction along a timelike Killing vector (see the next Section). For brevity we omit discrete factors such as \mathbb{Z}_2 . We shall refer to these $\mathcal{N} \geq 2$ supergravities with symmetric-space-valued scalars as “homogeneous supergravities”.

Passing to the quantum theory, the continuous symmetry is generally broken to a discrete subgroup (the continuous symmetry would be inconsistent with charge quantization). In the $\mathcal{N} = 8$ case it is believed that the quantum theory is invariant under the largest subgroup consistent with this constraint, written $E_{7(7)}(\mathbb{Z})$. For the very special supergravity theories the quantum theory is not in general known to exist; we optimistically assume that the quantum theory exists and is invariant under a suitably large discrete sub-

TABLE I. Number of supercharges, vector fields in $d = 4$, moduli spaces before and after $d = 4 \rightarrow 3$ reduction along a timelike Killing vector, and associated Jordan algebra. Real forms of exceptional groups are indicated by the notation $E_{n(m)}$ where m is the difference between the number of noncompact and compact generators.

n_Q	n_v	M_4	M_3^*	J
8	1	\emptyset	$U(2, 1)/[U(1, 1) \times U(1)]$	\mathbb{R}
8	2	$SL(2, \mathbb{R})/U(1)$	$G_{2,2}/SO(2, 2)$	\mathbb{R}
8	7	$Sp(6, \mathbb{R})/(SU(3) \times U(1))$	$F_{4(4)}/[Sp(6, \mathbb{R}) \times SL(2, \mathbb{R})]$	$J_3^{\mathbb{R}}$
8	10	$SU(3, 3)/(SU(3) \times SU(3) \times U(1))$	$E_{6(2)}/[SU(3, 3) \times SL(2, \mathbb{R})]$	$J_3^{\mathbb{C}}$
8	16	$SO^*(12)/(SU(6) \times U(1))$	$E_{7(-5)}/[SO^*(12) \times SL(2, \mathbb{R})]$	$J_3^{\mathbb{H}}$
8	28	$E_{7(-25)}/(E_6 \times U(1))$	$E_{8(-24)}/[E_{7(-25)} \times SL(2, \mathbb{R})]$	$J_3^{\mathbb{O}}$
8	$n + 2$	$(SL(2, \mathbb{R})/U(1)) \times (SO(n, 2)/[SO(n) \times SO(2)])$	$SO(n + 2, 4)/[SO(n, 2) \times SO(2, 2)]$	$\mathbb{R} \oplus \Gamma_{n-1,1}$
16	$n + 2$	$(SL(2, \mathbb{R})/U(1)) \times (SO(n - 4, 6)/[SO(n - 4) \times SO(6)])$	$SO(n - 2, 8)/[SO(n - 4, 2) \times SO(2, 6)]$	$\mathbb{R} \oplus \Gamma_{n-5,5}$
32	28	$E_{7(7)}/SU(8)$	$E_{8(8)}/SO^*(16)$	$J_3^{\mathbb{O}_s}$

group $G_4(\mathbb{Z})$.² It is conceivable, but currently unproven, that our considerations extend to general nonhomogeneous $\mathcal{N} = 2$ theories, the monodromy group of the Calabi-Yau periods playing the role of $G_4(\mathbb{Z})$.

III. ATTRACTOR EQUATIONS AND GEODESIC FLOW

Since we are interested in stationary black hole solutions it is natural to dimensionally reduce the $d = 4$ theory along a timelike direction, using a Kaluza-Klein-type ansatz [14]

$$ds_4^2 = -e^{2U}(dt + \omega)^2 + e^{-2U}ds_3^2, \quad (1)$$

This leads to Euclidean gravity in three dimensions, coupled to scalars, vectors and fermions; upon dualizing the bosonic sector is described simply by the three-dimensional metric ds_3^2 and scalar fields ϕ^a . The ϕ^a include the scalars from the $d = 4$ theory, plus electric and magnetic potentials from the reduction of the gauge fields A_I^I and their duals, plus the scale factor U and the twist potential dual to the shift ω ; altogether they are organized into a manifold M_3^* of indefinite signature (the analytic continuation of the Riemannian signature manifold M_3 from a spacelike reduction). For generic $\mathcal{N} = 2$ theories, M_3 is a $4n_v$ dimensional quaternionic-Kähler manifold known as the c -map of M_4 [32]; M_3^* is an associated para-quaternionic space, studied independently in [33]. For homogeneous supergravities, $M_3 = G_3/K_3$ and

²Of the four $\mathcal{N} = 2$ supergravity theories defined by simple Jordan algebras of degree three, those with 7, 10 and 16 vector fields in four dimensions can be obtained by a consistent truncation of the maximal $\mathcal{N} = 8$ supergravity. The theory with 10 vector fields is known to describe the vector-multiplet sector of type IIA superstring compactified on the CY orbifold T^6/\mathbb{Z}_3 [30]. There are indications that the 6 extra scalars in the model based on $J_3^{\mathbb{H}}$ can be considered as generalized complex structure moduli [31]

$M_3^* = G_3/H_3$, where K_3 is the maximal compact subgroup of G_3 and H_3 is a noncompact real form of K_3 [14,34].

Stationary black hole configurations are identified with solutions of the nonlinear sigma model on M_3^* coupled to three-dimensional gravity. Supersymmetry requires a flat three-dimensional slicing, so that general BPS solutions are harmonic maps from \mathbb{R}^3 to M_3^* . Hence the problem of constructing black hole geometries, and possibly also counting their microstates, is related to the study of such harmonic maps.

As a first step, let us restrict to spherically symmetric configurations, whose metric on three-dimensional slices may be written

$$ds_3^2 = N^2(\rho)d\rho^2 + r^2(\rho)[d\theta^2 + \sin^2\theta d\phi^2]. \quad (2)$$

Considering ρ as a “radial time” the bosonic action reads

$$S = \int d\rho \left[\frac{N}{2} + \frac{1}{2N}(\dot{r}^2 - r^2 G_{ab} \dot{\phi}^a \dot{\phi}^b) \right], \quad (3)$$

where $\phi: \mathbb{R} \rightarrow M_3^*$ describes the radial scalar field evolution and G_{ab} is the metric on M_3^* . The 4-dimensional equations of motion are thus equivalent to geodesic motion of a fiducial particle on the real cone over M_3^* . The equation of motion for the lapse N imposes the Hamiltonian constraint

$$H = p_r^2 - \frac{1}{r^2} G^{ab} p_a p_b - 1 \equiv 0, \quad (4)$$

(where p_r and p_a are the canonical conjugates to r and ϕ^a) which fixes the mass of the fiducial particle on the cone to 1. For BPS black holes, one may set $N = 1$, $\rho = r$, $p_r = 1$ so that the problem reduces to *lightlike* geodesic motion on M_3^* , with affine parameter $\tau = 1/r$. It is convenient to retain the r variable to define observables such as the horizon area and the ADM mass.

The magnetic and electric charges of the black hole can be easily read off from this description: they are Noether charges p^I, q_I associated to the generators of gauge transformations in the isometry group G_3 acting on M_3^* . These charges do not commute; rather, they generate a Heisenberg subgroup $N \subset G_3$:

$$[p^I, q_I]_{\text{PB}} = 2\delta^I_J k, \quad (5)$$

where k is the NUT charge of the black hole [11]. Configurations with $k \neq 0$ have closed timelike curves when lifted back to four dimensions, due to the off-diagonal term $\omega = k \cos\theta d\phi$ in the metric (1); *bona fide* four-dimensional black holes are only obtained in the ‘‘classical’’ limit $k \rightarrow 0$. Nonetheless, it is advantageous to retain this variable to realize the full three-dimensional U -duality symmetry. In addition, the isometry associated to rescalings of g_{tt} leads to an additional conserved charge m identified as the ADM mass, obeying

$$[m, p^I]_{\text{PB}} = p^I, \quad [m, q_I]_{\text{PB}} = q_I, \quad [m, k]_{\text{PB}} = 2k. \quad (6)$$

Importantly, as already noted in [24], the ADM mass does not Poisson-commute with the charges.

For homogeneous supergravities where $M_3^* = G_3/H_3$, there exist additional conserved charges associated to the isometric action of G_3 on M_3^* . The full symmetry group G_3 includes the group G_4 already present in 4 dimensions, the unipotent subgroup N , and its opposite \bar{N} , corresponding to generalized Ehlers and Harrison transformations [14,35]. The corresponding set of conserved charges can be assembled into an element Q in the Lie algebra dual \mathfrak{g}_3^* of G_3 (which we canonically identify with \mathfrak{g}_3 via the Killing form)—the moment map of the symplectic action of G_3 on phase space. The space of charges Q naturally breaks up into orbits of various dimensions under the (co)adjoint action of G_3 , all equipped with a canonical Kirillov-Kostant symplectic structure. As we shall now see, the supersymmetry properties of the black hole solution are simply expressed in terms of the conjugacy class of Q .

While the flatness of the three-dimensional slices is necessary for supersymmetry, it is not a sufficient condition. For spherically symmetric stationary solutions, the supersymmetry variation of the fermionic partners λ^A of ϕ^a reads [36]

$$\delta\lambda^A = V_\alpha^A \epsilon^\alpha, \quad (7)$$

where ϵ^α is the supersymmetry parameter and V_α^A is a matrix linear in the velocities ϕ^a on M_3^* . For general $\mathcal{N} = 2$ supergravities, the indices $\alpha = 1, 2$ and $A = 1, \dots, 2n_v$ transform as fundamental representations of the restricted holonomy group $Sp(2, \mathbb{R}) \times Sp(2n_v, \mathbb{R})$ of paraquaternionic geometry. Supersymmetry is preserved when (7) vanishes for some nonzero ϵ^α , which implies that the quaternionic viel-bein V_α^A has a zero eigenvector. Expressing V_α^A in terms of the conserved charges, one may

show that this amounts to the system of equations

$$\frac{dz^i}{d\tau} = -e^{U+i\alpha} g^{i\bar{j}} \partial_{\bar{j}} |Z| \quad (8)$$

$$\frac{dU}{d\tau} + \frac{i}{2}k = -2e^{U+i\alpha}|Z|, \quad (9)$$

where

$$Z(p, q, k) = e^{K/2}[(q_I - 2k\tilde{\zeta}^I)X^I - (p^I + 2k\zeta^I)F_I] \quad (10)$$

is the central charge of the supersymmetry algebra, and the phase α is adjusted so that dU/dt is real. The scalars $\tilde{\zeta}^I, \zeta^I$, conjugate to the charges p^I, q_I , themselves evolve according to

$$\begin{aligned} \frac{d\zeta^I}{d\tau} &= -\frac{1}{2}e^{2U}[(\Im \mathcal{N})^{-1}]^{IJ}[q_I - 2k\tilde{\zeta}^I \\ &\quad - [\Re \mathcal{N}]_{JK}(p^K + 2k\zeta^K)] \\ \frac{d\tilde{\zeta}^I}{d\tau} &= -\frac{1}{2}e^{2U}[\Im \mathcal{N}]_{IJ}(p^J + 2k\zeta^I) - [\Re \mathcal{N}]_{IJ} \frac{d\zeta^J}{d\tau} \end{aligned} \quad (11)$$

where \mathcal{N}_{IJ} is the standard matrix of special geometry (we follow the conventions from [37]). For vanishing NUT charge³ $k = 0$, Eqs. (8) and (9) are recognized as the standard attractor flow equations describing the radial evolution of the scalars toward the black hole horizon [39–42]. The isomorphism between attractor flow on the special Kähler manifold M_4 and supersymmetric geodesic motion on the c -map of M_4 was in fact independently noticed in [43] in their study of spherically symmetric D -instantons in five-dimensional $\mathcal{N} = 2$ supergravity, and reflects mirror symmetry between D -instantons and black holes.

For homogeneous $\mathcal{N} = 2$ supergravity, the holonomy group is further restricted to $G_4 \times Sp(2, \mathbb{R})$, where $G_4 \subset Sp(2n_v, \mathbb{R})$. The matrix of velocities V_α^A can be viewed as an element of \mathfrak{p} in the decomposition $\mathfrak{g}_3 = \mathfrak{h} \oplus \mathfrak{p}$, and is conjugate to the matrix of conserved charges Q . The condition for supersymmetry is then⁴

$$[Ad(Q)]^5 = 0, \quad (12)$$

where $Ad(Q)$ denotes the adjoint representation of Q . In other words, Q should be an element of a *nilpotent orbit* \mathcal{O}_5 . The dimension of this orbit is $4n_v + 2$, much smaller than the dimension $8n_v$ of the unconstrained phase space. One may also consider yet smaller orbits $\mathcal{O}_4, \mathcal{O}_3, \mathcal{O}_2$, of dimensions⁵ $4n_v, (10n_v - 4)/3, 2n_v + 2$, corresponding to

³The attractor equations including NUT charge can also be obtained as a special case of the analysis in [38]; they agree with ours, although the proof is not immediate [13].

⁴One can see this by noting that \mathfrak{g}_3 has a 5-grading induced by the decomposition under a highest root of $Sp(2, \mathbb{R})$, and arguing that $V_1^A \propto V_2^A$ can be conjugated into the grade-1 subspace.

⁵The fact that the dimension of \mathcal{O}_3 is an even integer can be traced back to the Jordan algebra origin of homogeneous supergravities.

3-charge, 2-charge and 1-charge black holes, with zero entropy in tree-level supergravity. All of these preserve the same amount of supersymmetry, but belong to different duality orbits. The minimal orbit \mathcal{O}_2 will play an important role in the relation to the topological amplitude below.

This discussion may be repeated for theories with higher supersymmetry: for $\mathcal{N} = 8$, the fermionic variation is $\delta\lambda_A = \epsilon_\alpha \Gamma_{AA}^\alpha P^A$ where $\epsilon_\alpha, \lambda^A, P^A$ transform as a vector, spinor and conjugate spinor of the R -symmetry group $SO^*(16)$, respectively. A 1/2-BPS black hole in $\mathcal{N} = 8$ preserving 16 supercharges is obtained when the velocity P^A is a pure spinor of $SO^*(16)$, equivalently $Q \in \mathcal{O}_2$: this corresponds to the minimal orbit of E_8 , of dimension 58. Similarly, a 1/4-BPS black hole corresponds to $Q \in \mathcal{O}_3$, of dimension 92, while a generic 1/8-BPS black hole corresponds to $Q \in \mathcal{O}_5$, of dimension 114. The orbit \mathcal{O}_4 of dimension 112 corresponds to 1/8-BPS black holes with zero classical entropy, but does not enjoy any enhanced supersymmetry [8]. Note that the dimensions of these orbits agree with those given above for $n_v = 28$.

IV. THE QUANTUM ATTRACTOR FLOW

Having reduced the radial evolution of stationary, spherically symmetric solutions in 4 dimensions to geodesic flow on the three-dimensional scalar manifold M_3^* , quantization is straightforward: Classical geodesic motion of a particle on $\mathbb{R}^+ \times M_3^*$ is replaced by a wave function $\Psi \in L^2(\mathbb{R}^+ \times M_3^*)$, satisfying the quantum Hamiltonian constraint

$$\left[-\frac{\partial^2}{\partial r^2} + \frac{\Delta}{r^2} - 1 \right] \Psi(r, e) = 0, \quad (13)$$

where Δ is the Laplace-Beltrami operator on $M_3^* \ni e$. In our case we are really dealing with the motion of a superparticle, so Ψ should more properly be a section of a bundle over $\mathbb{R}^+ \times M_3^*$.

Furthermore, we are interested in the BPS Hilbert space, composed of states satisfying a quantum version of (7) and $p_r = 1$. The latter equation takes care of the radial dependence, $\Psi(r, e) = r\Psi(e)$. We shall discard the r variable from now on.

Let us first sketch the quantization of (7) for general $\mathcal{N} = 2$ supergravities. Note first that (after continuing from M_3^* to M_3) ϵ^α determines a complex structure at a point of M_3 . Classically, the relation (7) implies that the holomorphic part of the velocity V_α^A for this complex structure vanishes. We propose that quantum mechanically this should translate into the statement that Ψ belongs to a holomorphic sheaf cohomology group $H^1(T, \mathcal{O}(-h))$ on the twistor space⁶ T of the quaternionic-Kähler space M_3 .

⁶It may be fruitful to further extend T to the Hyperkähler cone over M_3 , along the lines of [44,45]. We thank M. Rocek and S. Vandoren for discussions on this approach.

(In particular, Ψ depends holomorphically on $2n_v + 1$ variables, giving the expected functional dimension for the BPS Hilbert space.) So we are proposing a relation between sections of bundles on M_3^* and sheaf cohomology on T ; this is likely related to the higher-dimensional quaternionic version of the Penrose transform, discussed in [46,47].

For homogeneous $\mathcal{N} \geq 2$ supergravities, this construction can be made more concrete as follows. We impose constraints on the phase space by fixing the values of all of the Casimir operators of G_3 acting on the Noether charge. After imposing these constraints, the phase space reduces to a union of coadjoint orbits of the Noether charges. In particular, if we impose the constraint (following from the BPS condition) that the Casimir operators all vanish, the reduced phase space is the union of nilpotent orbits. Correspondingly, the BPS Hilbert space \mathcal{H}_{BPS} decomposes into a direct sum of irreducible unitary representations of G_3 ,⁷ obtained by quantizing these nilpotent coadjoint orbits.⁸ The orbits relevant for us are the \mathcal{O}_i discussed in the previous Section, corresponding to duality orbits of the black hole charges.

Each representation (ρ, \mathcal{H}) obtained by orbit quantization can be thought of as embedded in the unconstrained Hilbert space of sections on G_3/H_3 ; namely, there is a spherical H_3 -covariant⁹ vector $f \in \mathcal{H}$, so given some $v \in \mathcal{H}$, we define a section $\Psi_v(g)$ by

$$\Psi_v(g) = \langle f, \rho(g)v \rangle. \quad (14)$$

Note that $\Psi_v(g)$ is well defined as a section on G_3/H_3 thanks to the H_3 -covariance of f .

So the radial quantization of supersymmetric stationary spherically symmetric geometries, or supersymmetric geodesic motion on M_3^* , is equivalent to the quantization of the small nilpotent orbits \mathcal{O}_i of G_3 , where $i = 2, 3, 4, 5$ depending on the fraction of supersymmetry preserved by the solution (or, in the $\mathcal{N} = 2$ case, on the number of independent charges carried by the black hole). This is a problem which has been considered at some length in the mathematics and physics literature. In particular, the minimal representation, based on the quantization of \mathcal{O}_2 , was

⁷Since G_3 is a noncompact group, its nontrivial unitary representations are infinite-dimensional; nevertheless they may be characterized by their *functional* (“Gelfand-Kirillov”) dimension, the number of variables needed to represent them.

⁸To be precise, the above construction provides a quantization of the observables associated to the action of G_3 . Fortunately, the most relevant observables p^I, q_I, k, m , are indeed generators of G_3 . Other observables need not act within \mathcal{H}_{BPS} , although they may be projected into it in the spirit of the lowest Landau level truncation in condensed matter physics.

⁹More precisely, the vector f transforms in a finite-dimensional representation of H_3 appropriate to the bundle over G_3/H_3 ; it is a spherical vector if this representation is trivial.

constructed for all simply laced Lie groups in their split real form in [48,49], while the small representations $\mathcal{H}_{2,3,4,5}$ associated to $\mathcal{O}_{2,3,4,5}$ were constructed in [50] for all simple Lie groups in their quaternionic real form. Their construction of \mathcal{O}_5 is in fact a particular case of the twistor construction for general $\mathcal{N} = 2$ supergravity sketched above. The quantization of \mathcal{O}_5 in the split case was also physically realized as a quasiconformal action leaving a quartic light-cone invariant in [9]; the quantization of \mathcal{O}_2 was obtained in [51] by “quantizing the quasiconformal action”; these constructions were extended to the nonsplit case in [52,53]. The same construction of the minimal representation in the split case was also independently arrived at in [12], albeit with very different physical motivations. In that work the distinguished spherical vector was also computed.

At this stage, we may try to make contact with the proposal of [5]: are any of the \mathcal{H}_i suitable Hilbert spaces to contain the topological string amplitude Ψ_{top} ? Taking [5] literally, the desired wave function must be a function of n_v variables. Alas, there is no unitary representation of G_3 on this number of variables, but the minimal representation comes very close: it has functional dimension $n_v + 1$. Furthermore, choosing an electric-magnetic splitting yields an explicit realization of the minimal representation as a function space [12]; n_v of the variables can be naturally identified with the magnetic charges p^I of the four-dimensional black hole, while the last one can be identified with the NUT charge k [11]. Modulo the extra variable, this is exactly the form expected for the topological string partition function, including the dependence on a choice of splitting.

Moreover, there is a natural way to eliminate the extra variable: letting H_ω denote the generator conjugate to the ADM mass, one may consider the limit $\lim_{\tau \rightarrow \infty} e^{\tau H_\omega} \Psi$ (which may be thought of as a “4-dimensional” or perhaps “near-horizon” limit). In this limit, G_3 is broken down to $G_4 \times \mathbb{R}$ commuting with H_ω , and the spherical vector computed in [12] reduces to the tree-level topological string amplitude $\Psi_{\text{top}} = e^{N(X)/X^0}$. It is thus tempting to interpret the topological string amplitude as a restriction of the spherical vector¹⁰ of the minimal representation for the three-dimensional duality group G_3 . It would obviously be very interesting to have a topological string interpretation of the extra variable k , and conversely to understand how perturbative and instanton corrections modify the notion of spherical vector.

¹⁰In particular, it is not accurate to say that Ψ_{top} is the wave function for the attractor flow: rather, (a one-parameter generalization of) it defines a map (14) from \mathcal{H}_{BPS} to the unconstrained Hilbert space; the wave function itself is determined only when a vector $v \in \mathcal{H}_{BPS}$ is specified. In the next section, we propose a principle that selects a unique v .

V. THE AUTOMORPHIC WAVE FUNCTION

In the last Section we discussed four unipotent representations \mathcal{H}_i of the three-dimensional continuous U -duality group G_3 , which arise upon quantization of spherically symmetric BPS black hole attractor flows. Mathematical interest in these representations lies in the fact that they allow for the construction of simple modular forms for G_3 ; so we begin by reviewing this notion and then explain its relevance for us.

Recall that an automorphic form for a group G is a function on the quotient $G(\mathbb{Z}) \backslash G$, where $G(\mathbb{Z})$ is a discrete subgroup of G (see e.g. [54] for a more precise definition). The space \mathcal{A} of automorphic forms has a natural G -action (by right multiplication). A modular form Ψ is abstractly defined as an equivariant map $\mathcal{H} \rightarrow \mathcal{A}$ where the space \mathcal{H} carries a representation ρ of G : \mathcal{H} characterizes the “modular weight” of Ψ , while the map encodes the precise modular form. In other words, Ψ denotes a particular realization of the representation ρ on a space of functions on $G(\mathbb{Z}) \backslash G$. One way to construct such a realization¹¹ is to find a $G(\mathbb{Z})$ -invariant distribution $f_{G(\mathbb{Z})}$ in \mathcal{H}^* : then the map

$$v \mapsto \Psi_v(g) = \langle f_{G(\mathbb{Z})}, \rho(g)v \rangle \quad (15)$$

defines a modular form. If the representation ρ admits a vector v_K invariant under $K \subset G$, the resulting function Ψ_{v_K} is K -invariant, hence a function on the double coset $G(\mathbb{Z}) \backslash G/K$. In some cases the invariant distribution $f_{G(\mathbb{Z})}$ can itself be obtained as an adelic spherical vector, expressible as the product over all primes p of spherical vectors over $G(\mathbb{Q}_p)$, where \mathbb{Q}_p is the field of p -adic rational numbers.

We briefly indicate how the usual modular forms for $SL(2, \mathbb{R})$ fit into this framework. Holomorphic modular forms $f(\tau)$ on the upper half-plane give functions on $SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R})$ by¹² $g \mapsto (f|_k g)(i)$. Under the action of $SL(2, \mathbb{R})$ this is a highest weight vector generating the k -th holomorphic discrete series representation. The Jacobi theta series also fits in this scheme via (15), upon choosing \mathcal{H} as the metaplectic representation, $v_K(x)$ as the ground state of the harmonic oscillator, and $f_{G(\mathbb{Z})}$ the “Dirac comb” distribution $\sum_{m \in \mathbb{Z}} \delta(x - m)$.

Now recall that given a modular form for $SL(2, \mathbb{R})$ its Fourier coefficients often have interesting number-theoretic properties—they are integers and answer counting problems. For example, for holomorphic modular forms of $SL(2, \mathbb{Z})$, one obtains these coefficients by computing

$$\hat{\Psi}(m) = \int_0^1 f(\tau_1 + i\tau_2) e^{-2\pi i m(\tau_1 + i\tau_2)} d\tau_1. \quad (16)$$

¹¹See e.g. [55] for a physicist’s introduction to this approach.

¹²As usual, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(f|_k g)(\tau) = (c\tau + d)^k f\left(\frac{a\tau + b}{c\tau + d}\right)$.

This can be equivalently viewed as integration over the parabolic subgroup

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\} \subset SL(2, \mathbb{R})$$

fixing the cusp $\tau = i\infty$, modulo the action of $SL(2, \mathbb{Z})$. In favorable cases, a similar prescription can be given for Fourier coefficients of modular forms of G , by performing integrals over the unipotent (“upper triangular”) parts of suitable parabolic subgroups; this procedure is not known to work in full generality, but does at least for the $\mathcal{N} = 2$ cases of interest here [56–58].

Namely, let us consider modular forms Ψ for $G = G_3$, the three-dimensional U -duality group appearing in the timelike reduction of stationary black holes, taking ρ to be one of the unipotent representations $\mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_5$ which are singled out by supersymmetry. This G_3 has a maximal parabolic subgroup $G_3 = LN$, where the Lévi component $L \simeq G_4 \times \mathbb{R}$ contains the four-dimensional U -duality along with H_ω , and N is the Heisenberg group of gauge transformations from (5). Given Ψ one can try to extract its Fourier coefficients. The analog of $e^{2\pi i n x}$ in (16) is now a character $\chi: N \rightarrow \mathbb{C}^\times$, with $\chi(n) = 1$ for $n \in G_3(\mathbb{Z})$. Such characters are naturally parameterized by vectors (p^I, q_I) of integrally quantized electric and magnetic black hole charges, the NUT charge k being necessarily¹³ set to 0. The Fourier coefficients $\hat{\Psi}(p, q)$ so obtained are by construction invariant under four-dimensional U -duality $G_4(\mathbb{Z})$; importantly they are constrained to fit together into a modular form for a larger group $G_3(\mathbb{Z})$. Furthermore, the “smallness” of the underlying representation implies that only a subset of the Fourier coefficients are nonzero, namely, those lying in the appropriate orbit of the U -duality group.

In particular, let us consider the minimal representation \mathcal{H}_2 . In this case it is believed that the modular form Ψ is unique. So our conjecture in this case would be that the Fourier coefficients of this Ψ count the microstates of 1-charge black holes. Although these microstate counts are not of much direct physical interest, the statement that they fit together into a modular form of G_3 is an illustration of our general philosophy.

When G_3 is a split group, the modular form attached to the minimal representation can be understood concretely by considering the spherical vector; its corresponding automorphic form was constructed in [59], essentially using the formula (15). Comparing (15) and (14), we see that this construction corresponds to a particular choice of wave function $v = f_{G(\mathbb{Z})}$ in the corresponding BPS Hilbert space, which we refer to as the “automorphic wave function”. Then the Fourier coefficient associated to the given

charges (p^I, q_I) can be formally expressed as

$$\hat{\Psi}(p, q) = \int d\zeta^I e^{iq_I \zeta^I} f_{G(\mathbb{Z})}^*(p^I - \zeta^I, 0) f_{K(\mathbb{R})}(p^I + \zeta^I, 0). \tag{17}$$

This expression strongly resembles the OSV conjecture but differs by involving the product of the real and adelic spherical vectors, rather than the squared modulus of the topological string amplitude.

For the other \mathcal{H}_i , coming from quantization of the larger \mathcal{O}_i , the situation becomes more complicated; the modular forms here are not expected to be unique, so some additional input will be required to pin down the desired ones. The OSV conjecture seems to suggest considering the tensor product of the representation \mathcal{H}_2 with itself and projecting onto \mathcal{H}_5 ; but this is unlikely to be the full story, since from the results of [60] it appears that considering nonspherically symmetric configurations requires the inclusion of higher powers of Ψ_{top} as well.

VI. SUMMARY AND OUTLOOK

In this work, motivated by recent conjectures about exact degeneracies of BPS black holes in four dimensions, we studied stationary, spherically symmetric solutions of $\mathcal{N} \geq 2$ supergravities, with emphasis on cases where the scalar manifold is a symmetric space. By utilizing the equivalence with geodesic motion on the three-dimensional scalar manifold, we quantized the radial attractor flow, and argued that the three-dimensional U -duality group G_3 acts as a spectrum generating symmetry for BPS black hole degeneracies in 4 dimensions. We suggested how these may be counted by Fourier coefficients of modular forms of G_3 . Clearly much work remains to be done in this direction. Other outstanding problems are to understand the rôle of rotating and multicentered black holes, as well as the effect of higher-derivative corrections, and to understand the appearance of the extra NUT-charge parameter k in the generalized topological string amplitude. It should also be pointed out that a similar line of reasoning may be developed for 5-dimensional black holes and black rings, leading us to expect that the 4-dimensional U -duality group plays a similar role in 5 dimensions. Finally, the extension of this approach to full-fledged $\mathcal{N} = 2$ string theory, if successful, is likely to uncover new relations between number theory, Calabi-Yau geometry and physics.

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¹³There exists a different unipotent subgroup of G_3 whose characters are parameterized by q_I and k , but then $p^I = 0$.

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