Dark energy in modified Gauss-Bonnet gravity: Late-time acceleration and the hierarchy problem

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(Received 31 December 2005; published 6 April 2006)

Dark energy cosmology is considered in a modified Gauss-Bonnet (GB) model of gravity where an arbitrary function of the GB invariant, $f(G)$, is added to the general relativity action. We show that a theory of this kind is endowed with a quite rich cosmological structure: it may naturally lead to an effective cosmological constant, quintessence, or phantom cosmic acceleration, with a possibility for the transition from deceleration to acceleration. It is demonstrated in the paper that this theory is perfectly viable, since it is compliant with the solar system constraints. Specific properties of $f(G)$ gravity in a de Sitter (dS) universe, such as dS and SdS solutions, their entropy, and its explicit one-loop quantization are studied. The issue of a possible solution of the hierarchy problem in modified gravities is also addressed.

DOI: 10.1103/PhysRevD.73.084007 PACS numbers: 04.50.+h, 95.36.+x

I. INTRODUCTION

Recent observational data indicate that our universe is accelerating. This acceleration is explained in terms of the so-called dark energy (DE) which could result from a cosmological constant, an ideal fluid with a (complicated) equation of state and negative pressure, the manifestation of vacuum effects, a scalar (or more sophisticated) field, with quintessencelike or phantomlike behavior, etc. (For a very complete review of a dynamical DE see [1] and references therein; for an earlier review, see [2].) The choice of possibilities reflects the undisputable fact that the true nature and origin of the dark energy has not been convincingly explained yet. It is not even clear what type of DE (cosmological constant, quintessence, or phantom) occurs in the present, late universe.

A quite appealing possibility for the gravitational origin of the DE is the modification of general relativity. Actually, there is no compelling reason why standard general relativity should be trusted at large cosmological scales. For a rather minimal modification, one assumes that the gravitational action may contain some additional terms which start to grow slowly with decreasing curvature (of type $1/R$ [3,4], $\ln R$ [5], $\mathrm{Tr}1/R$ [6], string-inspired dilaton gravities [7], etc.), and which could be responsible for the current accelerated expansion. In fact, there are stringent constraints on these apparently harmless modifications of general relativity coming from precise solar system tests, and thus not many of these modified gravities may be viable in the end. In such a situation, a quite natural explanation for both the cosmic speed-up issue and also of the first and second coincidence problems (for a recent discussion of the same, see [8]) could be to say that all of them are caused, in fact, by the universe expansion itself. Never-
theless, one should not forget that some duality exists between the ideal fluid equation of state (EoS) description, the scalar-tensor theories, and modified gravity [9]. Such duality leads to *the same* Friedmann-Robertson-Walker (FRW) dynamics, starting from three physically different—but mathematically equivalent—theories. Moreover, even for modified gravity, different actions may lead to the same FRW dynamics [10]. Hence, additional evidence in favor of one or another DE model (with the same FRW scale factor) should be clearly exhibited [9].

As a simple example, let us now see how different types of DE may actually show up in different ways at large distances. It is well known that cold dark matter is localized near galaxy clusters but, quite on the contrary, dark energy distributes uniformly in the universe. The reason for that could be explained by a difference in the EoS parameter $w = p/\rho$. As we will see in the following, the effect of gravity on the cosmological fluid depends on w and even when $-1 < w < 0$ gravity can act sometimes as a repulsive force.

To see the w-dependence on the fluid distribution in a quite simple example, we consider cosmology in anti-de Sitter (AdS) space, whose metric is given by

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$$ds^2 = dy^2 + e^{2y/l}\left(-dt^2 + \sum_{i=1,2}(dx^i)^2\right). \quad (1)$$

Then the conservation law of the energy momentum tensor, $\nabla_\mu T^{\mu\nu}$ gives, by putting $\nu = y$,

$$\frac{dp}{dy} + \frac{1}{l}(p + \rho) = 0, \quad (2)$$

if we assume matter to depend only on the coordinate y. When w is a constant, we can solve Eq. (2) explicitly:

$$\rho = \rho_0 \exp\left[-\frac{1}{l}\left(1 + \frac{1}{w}\right)y\right]. \quad (3)$$

Here ρ_0 is a constant. We should note that $1 + 1/w > 0$ when $w > 0$ or $w < -1$, and $1 + 1/w < 0$ when $-1 < w < 0$. Then, for usual matter with $w > 0$, the density ρ becomes large when y is negative and large. In particular, for dust $w \sim 0$ but $w > 0$, and a collapse would occur. On the other hand, when $-1 < w < 0$, like for quintessence, the density ρ becomes large when y is positive. In the phantom case, with $w < -1$, the density ρ becomes large when y is negative, although a collapse does not occur. When $w = -1$, ρ becomes constant and uniform.

We may also consider a Schwarzschild-like metric:

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{-2\nu(r)}dr^2 + r^2 d\Omega_{(2)}^2. \quad (4)$$

Here $d\Omega_{(2)}^2$ expresses the metric of a two-dimensional sphere of unit radius. Then, the conservation law $\nabla_\mu T^{\mu\nu}$ with $\nu = r$ gives

$$\frac{dp}{dr} + \frac{d\nu}{dr}(p + \rho) = 0. \quad (5)$$

When w is constant, we can solve (5) and obtain

$$\rho = \rho_0 \exp\left[-\left(1 + \frac{1}{w}\right)\nu\right]. \quad (6)$$

Here ρ_0 is a constant again. In particular, in the case of the Schwarzschild metric,

$$e^{2\nu(r)} = 1 - \frac{r_0}{r}, \quad (7)$$

with horizon radius r_0, we find

$$\rho = \rho_0\left(1 - \frac{r_0}{r}\right)^{-(1/2)(1+1/w)}. \quad (8)$$

Then, when $w > 0$ or $w < -1$, ρ is a decreasing function of r; that is, the fluid is localized near the horizon. Specifically, in the case of dust with $w = 0$, the fluid collapses. On the other hand, when $-1 < w < 0$, ρ is an increasing function of r, which means that the fluid delocalizes. When $w = 0$, the distribution of the fluid is uniform.

The above results tell us that the effect of gravity on matter with $-1 < w < 0$ is opposite to that on usual matter. Usual matter becomes dense near a star but matter with $-1 < w < 0$ becomes less dense when approaching a star. As is known, cold dark matter localizes near galaxy clusters but dark energy distributes uniformly within the universe, which would be indeed consistent, since the EoS parameter of dark energy is almost -1. If dark energy is of phantom nature ($w < -1$), its density becomes large near the cluster, but if dark energy is of quintessence type ($-1 < w < -1/3$), its density becomes smaller.

In the present paper the (mainly late-time) cosmology coming from modified Gauss-Bonnet (GB) gravity, introduced in Ref. [11], is investigated in detail. In the next section, general FRW equations of motion in modified GB gravity with matter are derived. Late-time solutions thereof, for various choices of the function $f(G)$, are found. It is shown that modified GB gravity may indeed play the role of a gravitational alternative for DE. In particular, we demonstrate that this model may naturally lead to a plausible, effective cosmological constant, quintessence or a phantom era. In addition, $f(G)$ gravity has the possibility to describe the inflationary era (unifying then inflation with late-time acceleration), and to yield a transition from deceleration to acceleration, as well as a natural crossing of the phantom divide. It also passes the stringent solar system tests, as it shows no correction to Newton's law in flat space for an arbitrary choice of $f(G)$, as well as no instabilities. Section III is devoted to the study of the de Sitter universe solution in such a model. The entropies of a Schwarzschilde-de Sitter (SdS) black hole (BH) and of a de Sitter (dS) universe are derived, and possible applications to the calculation of the nucleation rate are discussed. In Sec. IV, the quantization program at one-loop order for modified GB gravity is presented. This issue is of the essence for the phantom era, where quantum gravity effects eventually become important near the big rip singularity. Section V is devoted to the generalization of modified gravity where $F = F(G, R)$. This family of models looks less attractive, given that only some of its specific realizations may pass the solar system tests. Nevertheless, it can serve to discuss the origin of the cosmic speed-up as well as a possible transition from deceleration to acceleration. In Sec. VI, the important hierarchy problem of particle physics is addressed in the framework of those modified gravity theories. It is demonstrated there that this issue may have a natural solution in the frame of $\mathcal{F}(R)$ or $F(G, R)$ gravity. The last section is devoted to a summary and an outlook. In the Appendix, an attempt is made to construct zero-curvature black hole solutions in the theory under discussion.

II. LATE-TIME COSMOLOGY IN MODIFIED GAUSS-BONNET GRAVITY

Let us start from the following, quite general action for modified gravity [12]:

$$S = \int d^4x\sqrt{-g}(\tilde{f}(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \mathcal{L}_m). \quad (9)$$

Here $\mathcal{L}_m$ is the matter Lagrangian density. It is not easy to construct a viable theory directly from this general class, which allows for nonlinear forms for the action. One must soon make use of symmetry considerations, which lead to theories which are more friendly, e.g., to the common solar system tests. Specifically, we shall restrict the action to the following form:

$$S = \int d^4x\sqrt{-g}(F(G, R) + \mathcal{L}_m). \tag{10}$$

Here G is the GB invariant:

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}. \tag{11}$$

Varying over $g_{\mu\nu}$,

$$\begin{aligned} 0 = {} & T^{\mu\nu} + \tfrac{1}{2}g^{\mu\nu}F(G,R) - 2F_G(G,R)RR^{\mu\nu} + 4F_G(G,R)R^{\mu}{}_{\rho}R^{\nu\rho} - 2F_G(G,R)R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} - 4F_G(G,R)R^{\mu\rho\sigma\nu}R_{\rho\sigma} \\ & + 2(\nabla^\mu\nabla^\nu F_G(G,R))R - 2g^{\mu\nu}(\nabla^2 F_G(G,R))R - 4(\nabla_\rho\nabla^\mu F_G(G,R))R^{\nu\rho} - 4(\nabla_\rho\nabla^\nu F_G(G,R))R^{\mu\rho} \\ & + 4(\nabla^2 F_G(G,R))R^{\mu\nu} + 4g^{\mu\nu}(\nabla_\rho\nabla_\sigma F_G(G,R))R^{\rho\sigma} - 4(\nabla_\rho\nabla_\sigma F_G(G,R))R^{\mu\rho\nu\sigma} - F_R(G,R)R^{\mu\nu} \\ & + \nabla^\mu\nabla^\nu F_R(G,R) - g^{\mu\nu}\nabla^2 F_R(G,R), \end{aligned} \tag{12}$$

$T^{\mu\nu}$ being the matter-energy momentum tensor, and where the following expressions are used:

$$F_G(G,R) = \frac{\partial F(G,R)}{\partial G}, \qquad F_R(G,R) = \frac{\partial F(G,R)}{\partial R}. \tag{13}$$

The spatially flat FRW universe metric is chosen as

$$ds^2 = -dt^2 + a(t)^2\sum_{i=1}^{3}(dx^i)^2. \tag{14}$$

Then the (t, t)-component of (12) has the following form:

$$\begin{aligned} 0 = {} & GF_G(G,R) - F(G,R) - 24H^3\frac{dF_G(G,R)}{dt} \\ & + 6\left(\frac{dH}{dt} + H^2\right)F_R(G,R) - 6H\frac{dF_R(G,R)}{dt} + \rho_m, \end{aligned} \tag{15}$$

where ρ_m is the energy density corresponding to matter. Here, G and R have the following form:

$$G = 24(H^2\dot{H} + H^4), \qquad R = 6(\dot{H} + 2H^2). \tag{16}$$

In absence of matter ($\rho_m = 0$), there can be a de Sitter solution ($H = H_0 =$ constant) for (15), in general (see [12]). One finds H_0 by solving the algebraic equation

$$0 = 24H_0^4F_G(G,R) - F(G,R) + 6H_0^2F_R(G,R). \tag{17}$$

For a large number of choices of the function $F(G, R)$, Eq. (20) has a nontrivial ($H_0 \neq 0$) real solution for H_0 (the de Sitter universe). The late-time cosmology for the above theory without matter has been discussed for a number of examples in Ref. [11].

In this section, we restrict the form of $F(G, R)$ to be

$$F(G,R) = \frac{1}{2\kappa^2}R + f(G), \tag{18}$$

where $\kappa^2 = 8\pi G_N$, G_N being the Newton constant. As will be shown, such an action may pass the solar system tests quite easily. Let us consider now several different forms of such action. By introducing two auxiliary fields, A and B, one can rewrite action (10) with (18) as

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + B(G - A) + f(A) + \mathcal{L}_m\right). \tag{19}$$

Varying over B, it follows that $A = G$. Using this in (19), the action (10) with (18) is recovered. On the other hand, varying over A in (19), one gets $B = f'(A)$, and hence

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + f'(A)G - Af'(A) + f(A)\right). \tag{20}$$

By varying over A, the relation $A = G$ is obtained again. The scalar is not dynamical and it has no kinetic term. We may add, however, a kinetic term to the action by hand:

$$\begin{aligned} S = \int d^4x\sqrt{-g}\bigg(& \frac{1}{2\kappa^2}R - \frac{\epsilon}{2}\partial_\mu A\partial^\mu A + f'(A)G \\ & - Af'(A) + f(A)\bigg). \end{aligned} \tag{21}$$

Here ϵ is a positive constant parameter. Then, one obtains a dynamical scalar theory coupled with the Gauss-Bonnet invariant and with a potential. It is known that a theory of this kind has no ghosts and it is stable, in general. Actually, it is related with string-inspired dilaton gravity, proposed as an alternative for dark energy [7]. Thus, in the case that the limit $\epsilon \to 0$ can be obtained smoothly, the corresponding $f(G)$ theory would not have a ghost and could actually be stable. This question deserves further investigation.

We now consider the case $\rho_m \neq 0$. Assuming that the EoS parameter $w \equiv p_m/\rho_m$ for matter (p_m is the pressure of matter) is a constant, then, by using the conservation of energy: $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, we find $\rho = \rho_0 a^{-3(1+w)}$. We also assume that $f(G)$ is given by

$$f(G) = f_0|G|^\beta, \tag{22}$$

with constants f_0 and β. If $\beta < 1/2$, the $f(G)$ term becomes dominant, as compared with the Einstein term, when the curvature is small. If we neglect the contribution from the Einstein term in (15) with (18), assuming that

$$a = \begin{cases} a_0 t^{h_0}, & \text{when } h_0 > 0 \\ a_0(t_s - t)^{h_0}, & \text{when } h_0 < 0, \end{cases} \tag{23}$$

the following solution is found:

$$h_0 = \frac{4\beta}{3(1+w)},$$
$$a_0 = \left[-\frac{f_0(\beta - 1)}{(h_0 - 1)\rho_0} \{24|h_0^3(-1 + h_0)|\}^\beta \times (h_0 - 1 + 4\beta) \right]^{-\{1/[3(1+w)]\}}. \tag{24}$$

One can define the effective EoS parameter $w_{\rm eff}$ as

$$w_{\rm eff} = \frac{p}{\rho} = -1 - \frac{2\dot{H}}{3H^2}, \tag{25}$$

which is less than -1 if $\beta < 0$, and for $w > -1$ as

$$w_{\rm eff} = -1 + \frac{2}{3h_0} = -1 + \frac{1+w}{2\beta}, \tag{26}$$

which is again less than -1 for $\beta < 0$. Thus, if $\beta < 0$, we obtain an effective phantom with negative h_0 even in the case when $w > -1$. In the phantom phase [13], a singularity of big rip type at $t = t_s$ [14] seems to appear (for the classification of these singularities, see [15]). Near this sort of big rip singularity, however, the curvature becomes dominant and then the Einstein term dominates, so that the $f(G)$-term can be neglected. Therefore, the universe behaves as $a = a_0 t^{2/3(w+1)}$ and, as a consequence, the big rip singularity will not eventually appear. The phantom era is transient.

A similar model has been found in [16] by using a consistent version [4] of $1/R$-gravity [3]. In general, in the case of $\mathcal{F}(R)$-gravity instabilities appear [17]. These instabilities do not show up for the case of $f(G)$-gravity.

Note that under the assumption (23), the GB invariant G and the scalar curvature R behave as

$$G = \frac{24h_0^3(h_0 - 1)}{t^4}, \quad \text{or} \quad \frac{24h_0^3(h_0 - 1)}{(t_s - t)^4},$$
$$R = \frac{6h_0(2h_0 - 1)}{t^2}, \quad \text{or} \quad \frac{6h_0(2h_0 - 1)}{(t_s - t)^2}. \tag{27}$$

As a consequence, when the scalar curvature R becomes small, that is, when t or $t_s - t$ becomes large, the GB invariant G becomes small more rapidly than R. When R becomes large, that is, if t or $t_s - t$ becomes small, then G becomes large more rapidly than R. Thus, if $f(G)$ is given by (22) with $\beta < 1/2$, the $f(G)$-term in the action (10) with (18) becomes more dominant for small curvature than the Einstein term, but becomes less dominant in the case of large curvature. Therefore, Eq. (24) follows when the curvature is small. There are, however, some exceptions to this. As is clear from the expressions in (27), when $h_0 = -1/2$, which corresponds to $w_{\rm eff} = -7/3$, R vanishes, and when $h_0 = -1$, corresponding to $w_{\rm eff} = -5/3$, G vanishes. In both of these cases, only one of the Einstein and $f(G)$ terms survives.

In the case when $\beta < 0$, if the curvature is large, the Einstein term in the action (10) with (18) dominates, and we have a nonphantom universe, but when the curvature is small, the $f(G)$-term dominates and we obtain an effective phantom one. Since the universe starts with large curvature, and the curvature becomes gradually smaller, the transition between the nonphantom and phantom cases can naturally occur in the present model.

The case when $0 < \beta < 1/2$ may be also considered. As β is positive, the universe does not reach here the phantom phase. When the curvature is strong, the $f(G)$-term in the action (10) with (18) can be neglected and we can work with Einstein's gravity. Then, if w is positive, the matter energy density ρ_m should behave as $\rho_m \sim t^{-2}$, but $f(G)$ goes as $f(G) \sim t^{-4\beta}$. Then, for late times (large t), the $f(G)$-term may become dominant as compared with the matter one. If we neglect the contribution from matter, Eq. (15) with (18) has a de Sitter universe solution where H, and therefore G, are constant. If $H = H_0$ with constant H_0, Eq. (15) with (18) looks as (17) with (18). As a consequence, even if we start from the deceleration phase with $w > -1/3$, we may also reach an asymptotically de Sitter universe, which is an accelerated universe. Correspondingly, also here there could be a transition from acceleration to deceleration of the universe.

Now, we consider the case when the contributions coming from the Einstein and matter terms can be neglected. Then, Eq. (15) with (18) reduces to

$$0 = Gf'(G) - f(G) - 24\dot{G}f''(G)H^3. \tag{28}$$

If $f(G)$ behaves as (22), from assumption (23), we obtain

$$0 = (\beta - 1)h_0^6(h_0 - 1)(h_0 - 1 + 4\beta). \tag{29}$$

As $h_0 = 1$ implies $G = 0$, one may choose

$$h_0 = 1 - 4\beta, \tag{30}$$

and Eq. (25) gives

$$w_{\rm eff} = -1 + \frac{2}{3(1 - 4\beta)}. \tag{31}$$

Therefore, if $\beta > 0$, the universe is accelerating ($w_{\rm eff} < -1/3$) and if $\beta > 1/4$, the universe is in a phantom phase ($w_{\rm eff} < -1$). Thus, we are led to consider the following model:

$$f(G) = f_i|G|^{\beta_i} + f_l|G|^{\beta_l}, \tag{32}$$

where we assume that

$$\beta_i > \tfrac{1}{2}, \qquad \tfrac{1}{2} > \beta_l > \tfrac{1}{4}. \tag{33}$$

Here, when the curvature is large, as in the primordial universe, the first term dominates, compared with the second one and the Einstein term, and gives

$$-1 > w_{\rm eff} = -1 + \frac{2}{3(1-4\beta_i)} > -5/3. \tag{34}$$

On the other hand, when the curvature is small, as is the case in the present universe, the second term in (32) dominates, compared with the first one and the Einstein term, and yields

$$w_{\rm eff} = -1 + \frac{2}{3(1-4\beta_l)} < -5/3. \tag{35}$$

Therefore, the theory (32) can in fact produce a model which is able to describe both inflation and the late-time acceleration of our universe in a unified way.

Instead of (33), one may also choose β_l as

$$\tfrac{1}{4} > \beta_l > 0, \tag{36}$$

which gives

$$-\tfrac{1}{3} > w_{\rm eff} > -1. \tag{37}$$

Then, what we obtain is effective quintessence. Moreover, by properly adjusting the couplings f_i and f_l in (32), we can obtain a period where the Einstein term dominates and the universe is in a deceleration phase. After that, there would come a transition from deceleration to acceleration, where the GB term becomes the dominant one.

One can consider the system (32) coupled with matter as in (15) with (18). To this end we just choose

$$\beta_i > \tfrac{1}{2} > \beta_l. \tag{38}$$

Then, when the curvature is large, as in the primordial universe, the first term dominates, as compared with the second one and the Einstein term. When the curvature is small, as in the present universe, the second term in (32) is dominant as compared with the first and Einstein's. Then, an effective $w_{\rm eff}$ can be obtained from (26). In the primordial universe, matter could be radiation with $w = 1/3$, and hence the effective w is given by

$$w_{i,\rm eff} = -1 + \frac{2}{3\beta_i}, \tag{39}$$

which can be less than $-1/3$; that is, the universe is accelerating, when $\beta_i > 1$. On the other hand, in the late-time universe matter could be dust with $w = 0$, and then we would obtain

$$w_{l,\rm eff} = -1 + \frac{1}{2\beta_l}, \tag{40}$$

which is larger than 0, if $0 < \beta_l < 1/2$, or less than -1, if β_l is negative. Thus, acceleration could occur in both the primordial and late-time universes, if

$$\beta_i > 1, \qquad \beta_l < 0. \tag{41}$$

Similarly, one can consider DE cosmology for other choices of $f(G)$, for instance, $\ln G$ or other function f increasing with the decrease of G (late universe).

Let us address the issue of the correction to Newton's law. Let $g_{(0)}$ be a solution of (12) with (18) and represent the perturbation of the metric as $g_{\mu\nu} = g_{(0)\mu\nu} + h_{\mu\nu}$. First, we consider the perturbation around the de Sitter background which is a solution of (17) with (18). We write the de Sitter space metric as $g_{(0)\mu\nu}$, which gives the following Riemann tensor:

$$R_{(0)\mu\nu\rho\sigma} = H_0^2(g_{(0)\mu\rho}g_{(0)\nu\sigma} - g_{(0)\mu\sigma}g_{(0)\nu\rho}). \tag{42}$$

The flat background corresponds to the limit of $H_0 \to 0$. For simplicity, the following gauge condition is chosen: $g_{(0)}^{\mu\nu}h_{\mu\nu} = \nabla_{(0)}^{\mu}h_{\mu\nu} = 0$. Then Eq. (12) with (18) gives

$$0 = \frac{1}{4\kappa^2}(\nabla^2 h_{\mu\nu} - 2H_0^2 h_{\mu\nu}) + T_{\mu\nu}. \tag{43}$$

The GB term contribution does not appear except in the length parameter $1/H_0$ of the de Sitter space, which is determined with account to the GB term. This may occur due to the special structure of the GB invariant. Equation (43) tells us that there is no correction to Newton's law in de Sitter and even in the flat background corresponding to $H_0 \to 0$, whatever the form of f (at least, with the above gauge condition). [Note that a study of the Newtonian limit in $1/R$ gravity (where significant corrections to Newton's law may appear), and its extension has been done in [4,18].] For most $1/R$ models the corrections to Newton's law do not comply with solar system tests.

Expression (43) can be actually valid in the de Sitter background only. In a more general FRW universe, there can appear corrections coming from the $f(G)$ term. We should also note that, in deriving (43), a gauge condition $g_{(0)}^{\mu\nu}h_{\mu\nu} = 0$ was used, but if the mode corresponding to $g_{(0)}^{\mu\nu}h_{\mu\nu}$ is included, there might appear corrections coming from the $f(G)$ term. The mode corresponding to $g_{(0)}^{\mu\nu}h_{\mu\nu}$ gives an infinitesimal scale transformation of the metric. Then, it is convenient to write the metric as

$$g_{\mu\nu} = e^{\phi}g_{(0)\mu\nu}. \tag{44}$$

Here $g_{(0)\mu\nu}$ expresses the metric of de Sitter space in (42). The Gauss-Bonnet invariant G is correspondingly given by

$$G = e^{-2\phi}\{24H_0^4 - 12H_0^2\nabla_{(0)}^2\phi - 6H_0^2\partial_\mu\phi\partial^\mu\phi + 2(\nabla_{(0)}^2\phi)^2 - 2\nabla_{(0)\mu}\nabla_{(0)\nu}\phi\nabla_{(0)}^\mu\nabla_{(0)}^\nu\phi + \nabla_{(0)}^2\partial_\mu\phi\partial^\mu\phi + 2\nabla_{(0)\mu}\nabla_{(0)\nu}\phi\partial_{(0)}^\mu\phi\partial_{(0)}^\nu\phi\} = e^{-2\phi}\{24H_0^4 + \nabla_{(0)}^\mu(-12H_0^2\nabla_{(0)\mu}\phi + 2\partial_\mu\phi\nabla_{(0)}^2\phi - 2\nabla_{(0)\mu}\nabla_{(0)\nu}\phi\nabla_{(0)}^\nu\phi + \partial_\mu\phi\partial_{(0)}^\nu\phi\partial_{(0)\nu}\phi)\}. \quad (45)$$

The covariant derivative associated with $g_{(0)\mu\nu}$ is written here as $\nabla_{(0)\mu}$. The expansion of $f(G)$, with respect to ϕ, is

$$\sqrt{-g}f(G) = \sqrt{-g_{(0)}}\{f(24H_0^4) + 2f(24H_0^4)\phi^2 - 48f'(24H_0^4)H_0^4\phi^2 + 72f''(24H_0^4)H_0^4(\nabla_{(0)}^2\phi + 4H_0^2\phi)^2 + \mathcal{O}(\phi^3) + \text{total derivative terms}\}. \quad (46)$$

Since the last term contains $(\nabla_{(0)}^2\phi)^2$, in general, there could be an instability. A way to avoid the problem is to fine-tune $f(G)$ so that $f''(24H_0^4)$ vanishes for the solution H_0 in (17) with (18).

In order to consider a more general case, one expands $f(G)$ in the action (10) with (18) as

$$f(G) = f(G_{(0)}) + \tfrac{1}{2}f''(G_{(0)})\delta G^2 + f'(G_{(0)})\delta^2 G + \mathcal{O}(h^3), \quad (47)$$

where $G_{(0)}$ is the Gauss-Bonnet invariant given by $g_{(0)\mu\nu}$ and

$$\delta G = 4(R\nabla^\mu\nabla^\nu h_{\nu\mu} - 4R^\rho{}_\nu R^{\nu\sigma}h_{\rho\sigma} + 4R^{\nu\sigma}R^\rho{}_\sigma{}^\mu{}_\nu h_{\mu\rho} - 4\nabla_\nu\nabla^\mu h_{\sigma\mu} + R^{\mu\nu\rho\sigma}\nabla_\rho\nabla_\nu h_{\sigma\mu}), \quad (48)$$

$$\begin{aligned}\delta^2 G = [h_{\zeta\eta}\{&-R(\nabla^\eta\nabla^\mu h_\mu{}^\zeta + \nabla^\mu\nabla^\eta h_\mu{}^\zeta) - R^{\zeta\eta}\nabla^\rho\nabla^\nu h_{\nu\rho} + 4R^{\nu\sigma}\nabla^\eta\nabla_\nu h_\sigma{}^\zeta + 2R^{\zeta\sigma}\nabla^\mu\nabla^\eta h_{\sigma\mu} + 4R^{\nu\zeta}\nabla^\mu\nabla_\nu h_\mu{}^\eta \\ &+ 2R^{\zeta\nu\eta\sigma}\nabla^\mu\nabla_\nu h_{\sigma\mu} - R^{\mu\nu\zeta\sigma}\nabla^\eta\nabla_\nu h_{\sigma\mu} - \tfrac{1}{2}R^{\zeta\nu\rho\sigma}(\nabla_\rho\nabla_\nu h_\sigma{}^\eta - \nabla_\rho\nabla^\eta h_{\sigma\nu}) - R^{\mu\nu\rho\zeta}\nabla_\rho\nabla_\nu h^\eta{}_\mu\} \\ &+ \tfrac{1}{2}\{2\nabla^\mu\nabla^\nu h_{\nu\mu}(\nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla^2 h_\alpha^\alpha) - 4\nabla^\mu\nabla_\nu h_{\sigma\mu}(\nabla^\alpha\nabla^\nu h_\alpha{}^\sigma - \nabla^2 h^{\sigma\nu} - \nabla^\sigma\nabla^\nu h_\alpha{}^\alpha + \nabla^\sigma\nabla^\alpha h_\alpha{}^\nu) \\ &+ \nabla_\rho\nabla_\nu h_{\sigma\mu}(\nabla^\rho\nabla^\nu h^{\sigma\mu} - \nabla^\rho\nabla^\mu h^{\sigma\nu} - \nabla^\sigma\nabla^\nu h^{\rho\mu} + \nabla^\sigma\nabla^\mu h^{\sigma\nu})\} - \tfrac{1}{2}\{R(2\nabla^\mu h_{\nu\lambda}\nabla^\lambda h^\nu{}_\mu + 2\nabla^\mu h_{\mu\nu}\nabla_\nu h^{\nu\lambda} \\ &- \nabla_\lambda h_\alpha{}^\alpha\nabla_\nu h^{\nu\lambda} + \nabla_\mu(2h^{\lambda\mu}\nabla^\nu h_{\nu\lambda} - h^{\lambda\mu}\nabla_\lambda h_\alpha{}^\alpha + h^{\nu\lambda}\nabla^\mu h_{\nu\lambda})) - 4R^{\nu\sigma}\{(\nabla^\mu h_{\nu\lambda} + \nabla_\nu h^\mu{}_\lambda - \nabla_\lambda h_\nu{}^\mu)\nabla^\lambda h_{\sigma\mu} \\ &+ (\nabla_\sigma h^\mu{}_\lambda + \nabla^\mu h_{\sigma\lambda} - \nabla_\lambda h^\mu{}_\sigma)\nabla_\nu h^\lambda{}_\mu + (2\nabla^\mu h_{\mu\lambda} - \nabla_\lambda h_\alpha{}^\alpha)\nabla_\nu h_\sigma{}^\lambda + \nabla_\mu(2h^{\lambda\mu}\nabla_\nu h_{\sigma\lambda} \\ &+ (\nabla_\nu h_{\mu\lambda} + \nabla_\mu h_{\nu\lambda} - \nabla_\lambda h_{\nu\mu})h_\sigma{}^\lambda))\} + R^{\mu\nu\rho\sigma}((\nabla_\rho h_{\nu\lambda} + \nabla_\nu h_{\rho\lambda} - \nabla_\lambda h_{\nu\rho})\nabla^\lambda h_{\sigma\mu} + (\nabla_\mu h_{\rho\lambda} + \nabla_\rho h_{\mu\lambda} \\ &- \nabla_\lambda h_{\rho\mu})\nabla_\nu h_{\sigma\lambda} + \nabla_\rho((\nabla_\nu h_{\sigma\lambda} + \nabla_\sigma h_{\nu\lambda} - \nabla_\lambda h_{\nu\sigma})h^\lambda{}_\mu)\}]. \end{aligned} \quad (49)$$

Note that the term proportional to δG does not appear in (47) since the background metric is a solution of (42). Here and in the following, the index (0) is always suppressed, when there cannot be confusion. If we choose the gauge condition $\nabla^\mu h_{\mu\nu} = 0$, Eqs. (48) and (49) have the following form:

$$\delta G = 4(-4R^\rho{}_\nu R^{\nu\sigma}h_{\rho\sigma} + 4R^{\nu\sigma}R^\rho{}_\sigma{}^\mu{}_\nu h_{\mu\rho} + R^{\mu\nu\rho\sigma}\nabla_\rho\nabla_\nu h_{\sigma\mu}), \quad (50)$$

$$\begin{aligned}\delta^2 G = [h_{\zeta\eta}\{&-R(h^{\rho\zeta}R_\rho{}^\eta - h_{\mu\rho}R^{\rho\zeta\mu\eta}) + 4R^{\nu\sigma}\nabla^\eta\nabla_\nu h_\sigma{}^\zeta + 2R^{\zeta\sigma}(h_{\rho\sigma}R^{\rho\eta} - h_{\mu\rho}R^\rho{}_\sigma{}^{\mu\eta}) + 4R^{\nu\zeta}(h^{\rho\eta}R_{\rho\nu} - h_{\mu\rho}R^{\rho\eta\mu}{}_\nu) \\ &+ 2R^{\zeta\nu\eta\sigma}(h_{\rho\sigma}R^\rho{}_\nu - h_{\mu\rho}R^\rho{}_\sigma{}^\mu{}_\nu) - R^{\mu\nu\zeta\sigma}\nabla^\eta\nabla_\nu h_{\sigma\mu} - \tfrac{1}{2}R^{\zeta\nu\rho\sigma}(\nabla_\rho\nabla_\nu h_\sigma{}^\eta - \nabla_\rho\nabla^\eta h_{\sigma\nu}) - R^{\mu\nu\rho\zeta}\nabla_\rho\nabla_\nu h^\eta{}_\mu\} \\ &+ \tfrac{1}{2}\{-4(h_{\rho\sigma}R^\rho{}_\nu - h_{\mu\rho}R^\rho{}_\sigma{}^\mu{}_\nu)(h_\rho{}^\sigma R^{\rho\nu} - h_{\mu\rho}R^{\rho\sigma\mu\nu} - \nabla^2 h^{\sigma\nu} - \nabla^\sigma\nabla^\nu h_\alpha{}^\alpha) + \nabla_\rho\nabla_\nu h_{\sigma\mu}(\nabla^\rho\nabla^\nu h^{\sigma\mu} \\ &- \nabla^\rho\nabla^\mu h^{\sigma\nu} - \nabla^\sigma\nabla^\nu h^{\rho\mu} + \nabla^\sigma\nabla^\mu h^{\sigma\nu})\} - \tfrac{1}{2}\{R(2\nabla^\mu h_{\nu\lambda}\nabla^\lambda h^\nu{}_\mu - \nabla_\lambda h_\alpha{}^\alpha + \nabla_\mu(-h^{\lambda\mu}\nabla_\lambda h_\alpha{}^\alpha + h^{\nu\lambda}\nabla^\mu h_{\nu\lambda})) \\ &- 4R^{\nu\sigma}\{(\nabla^\mu h_{\nu\lambda} + \nabla_\nu h^\mu{}_\lambda - \nabla_\lambda h_\nu{}^\mu)\nabla^\lambda h_{\sigma\mu} + (\nabla_\sigma h^\mu{}_\lambda + \nabla^\mu h_{\sigma\lambda} - \nabla_\lambda h^\mu{}_\sigma)\nabla_\nu h^\lambda{}_\mu - \nabla_\lambda h_\alpha{}^\alpha\nabla_\nu h_\sigma{}^\lambda \\ &+ \nabla_\mu(2h^{\lambda\mu}\nabla_\nu h_{\sigma\lambda} + (\nabla_\nu h_{\mu\lambda} + \nabla_\mu h_{\nu\lambda} - \nabla_\lambda h_{\nu\mu})h_\sigma{}^\lambda))\} + R^{\mu\nu\rho\sigma}((\nabla_\rho h_{\nu\lambda} + \nabla_\nu h_{\rho\lambda} - \nabla_\lambda h_{\nu\rho})\nabla^\lambda h_{\sigma\mu} \\ &+ (\nabla_\mu h_{\rho\lambda} + \nabla_\rho h_{\mu\lambda} - \nabla_\lambda h_{\rho\mu})\nabla_\nu h_{\sigma\lambda} + \nabla_\rho((\nabla_\nu h_{\sigma\lambda} + \nabla_\sigma h_{\nu\lambda} - \nabla_\lambda h_{\nu\sigma})h^\lambda{}_\mu)\}]. \end{aligned} \quad (51)$$

Now, we consider the case that $\dot{H} \sim H^2$ in the FRW universe (14). Then, by specifying the dimension, the following structure is found:

$$\delta G \sim H^4 h + H^2\nabla^2 h, \qquad \delta^2 G \sim H^4 h^2 + H^2 h\nabla^2 h + H^2(\nabla h)^2 + H^3 h\nabla h + (\nabla^2 h)^2. \quad (52)$$

For the qualitative arguments that follow, we have abbreviated the vector indices and coefficients. Since Eq. (47) contains δG^2 and $\delta^2 G$ terms, the $H^2\nabla^2 h$-term in δG and the $(\nabla^2 h)^2$-term in $\delta^2 G$ have a possibility to generate the instability. Explicit calculations in the FRW universe tell us that the $(\nabla^2 h)^2$ term in $\delta^2 G$ vanishes identically, while the $H^2\nabla^2 h$ term in δG has the following form:

$$H^2\nabla^2 h \text{ term in } \delta G = -4\dot{H}\nabla^2 h_{tt} \tag{53}$$

For simplicity, we have chosen again the gauge condition $g_{(0)}^{\mu\nu}h_{\mu\nu} = \nabla^\mu h_{\mu\nu} = 0$. Then, except for the $\dot{H} = 0$ case, which describes the de Sitter universe, there might be an instability.

Since the δG^2-term has a factor $f''(G_{(0)})$, if one properly chooses the form of $f(G)$ and fine-tunes the coefficients, it could occur that $f''(G_{(0)}) = 0$ in the present universe. Correspondingly, the term $(\nabla^2 h)^2$ does not appear in the action and no instability appears.

To summarize, in both cases (46) and (52), if we choose $f'' = 0$ in the present universe, the instability does *not* appear. As an example, one can consider the model (32). As

$$f''(G) = f_i\beta_i(\beta_i - 1)|G|^{\beta_i-2} + f_l\beta_l(\beta_l - 1)|G|^{\beta_l-2}, \tag{54}$$

if we choose the parameters f_i, β_i, f_l, and β_l to satisfy

$$0 = f_i\beta_i(\beta_i - 1)|G_{(0)}|^{\beta_i-\beta_l} + f_l\beta_l(\beta_l - 1), \tag{55}$$

we find $f'(G_{(0)}) = 0$ and thus there will be no instability. In (55), $G_{(0)}$ is the value of the Gauss-Bonnet invariant given by the curvature in the present universe.

One now rewrites (12) with (18) as an FRW equation:

$$0 = -\frac{3}{\kappa^2}H^2 + \rho_G + \rho_M, \tag{56}$$

$$0 = \frac{1}{\kappa^2}(2\dot{H} + 3H^2) + p_G + p_m. \tag{57}$$

Here ρ_G and p_G express the contribution from the $f(G)$ term in the action (10) with (18):

$$\begin{aligned}\rho_G &= Gf'(G) - f(G) - 24\dot{G}f''(G)H^3,\\ p_G &= -Gf'(G) + f(G) + 24\dot{G}f''(G)H^3 + 8\dot{G}^2 f'''(G)H^2\\ &\quad - 192f''(G)(-8H^3\dot{H}\ddot{H} - 6H^2\dot{H}^3 - H^4\dddot{H} - 3H^5\ddot{H}\\ &\quad - 18H^4\dot{H}^2 + 4H^6\dot{H}).\end{aligned} \tag{58}$$

One can view the contribution from the $f(G)$ term as a sort of matter satisfying a special (inhomogeneous) EoS [19] (or usual EoS with time-dependent bulk viscosity [20]) of the form

$$\begin{aligned}0 &= \rho_G + p_G - 8\dot{G}^2 f'''(G)H^2 + 192f''(G)(-8H^3\dot{H}\ddot{H}\\ &\quad - 6H^2\dot{H}^3 - H^4\dddot{H} - 3H^5\ddot{H} - 18H^4\dot{H}^2 + 4H^6\dot{H}).\end{aligned} \tag{59}$$

In particular, in the case of de Sitter space, owing to the fact that the Hubble rate H, and therefore G, are constant, we find $0 = \rho_G + p_G$. In the case that $f(G)$ is given by (22) and one further assumes (23), one gets

$$\begin{aligned}\rho_G &= f_0|G|^\beta(\beta - 1)\frac{h_0 - 1 + 4\beta}{h_0 - 1},\\ p_G &= f_0|G|^\beta(\beta - 1)\frac{3h_0^2 - (3 + 8\beta)h_0 + 16\beta^2 - 4\beta}{3h_0(h_0 - 1)}.\end{aligned} \tag{60}$$

It follows that the effective EoS $w_G \equiv p_G/\rho_G$ for the $f(G)$ part is given by

$$w_G = \frac{3h_0^2 - (3 + 8\beta)h_0 + 16\beta^2 - 4\beta}{3h_0(h_0 - 1 + 4\beta)}. \tag{61}$$

In absence of matter ($\rho_m = p_m = 0$), Eq. (56) may be rewritten as

$$H^2 = \frac{\kappa^2}{6}\rho_G, \qquad \dot{H} = -\frac{\kappa^2}{2}(\rho_G + p_G). \tag{62}$$

Then, by using the expression (16), we find

$$\begin{aligned}G &= -\frac{2\kappa^4}{3}\rho_G(2\rho_G + 3p_G),\\ \dot{G} &= 4\left\{\kappa^2\ddot{H}\rho + \kappa^4(\rho_G + p_G)(\rho_G + 3p_G)\sqrt{\frac{\kappa^2\rho}{6}}\right\},\end{aligned} \tag{63}$$

and by using the first equation in (58), the effective equation of state is

$$\begin{aligned}0 &= -\rho_G - \frac{2\kappa^4}{3}\rho_G(2\rho_G + 3p_G)f'\left(-\frac{2\kappa^4}{3}\rho_G(2\rho_G + 3p_G)\right) - f\left(-\frac{2\kappa^4}{3}\rho_G(2\rho_G + 3p_G)\right)\\ &\quad - (24)^2\left\{\ddot{H}\left(\frac{\kappa^2}{6}\rho_G\right)^{5/2} + \frac{\kappa^2}{6}(\rho_G + p_G)(\rho_G + 3p_G)\left(\frac{\kappa^2}{6}\rho_G\right)^2\right\}f''\left(-\frac{2\kappa^4}{3}\rho_G(2\rho_G + 3p_G)\right),\end{aligned} \tag{64}$$

which has the form of $\tilde{F}(\rho_G, p_G, \ddot{H}) = 0$ [19]. At the dynamical level, this demonstrates the equivalency between modified GB gravity and the effective inhomogeneous EoS description. It may be of interest to study the choice of $f(G)$ which leads to the inhomogeneous generalization of EoS $p = -\rho - A\rho^\alpha$ suggested and studied in [21] as this may be easily compared with standard ΛCDM cosmology.

Summing up, late-time cosmology in modified GB gravity with matter was studied. It has been shown that effective DE of quintessence, phantom, or cosmological constant type can be actually produced within such theory, with the possibility to unify it also with primordial inflation. Moreover, the transition from the deceleration to the acceleration era easily occurs in some versions of the $f(G)$ theory which comply with the solar system tests (no instabilities appear; no corrections to Newton's law follow).

III. DE SITTER SOLUTION IN MODIFIED GAUSS-BONNET GRAVITY MODELS

In this section, we will investigate the properties of some black hole solutions within the modified GB gravity scheme. The role played by static, spherically symmetric solutions, as the Schwarzschild one, with regard to solar system tests, is well known. Because of the absence of this kind of zero-curvature BH in the $f(G)$ theory (see the Appendix) de Sitter space as well as SdS BH need her special consideration, which is done below.

For the general model, it can be shown that the equations of motion have the following structure (no-matter case)

$$\begin{aligned} 0 = {}& \tfrac{1}{2}F(G,R)g_{\mu\nu} - F_R(G,R)R_{\mu\nu} \\ & - 2F_G(G,R)(RR_{\mu\nu} - 2R_{\mu\rho}R^\rho_\nu + R_{\mu\rho\alpha\beta}R_\nu{}^{\rho\alpha\beta} \\ & - 2R_{\mu\alpha\nu\beta}R^{\alpha\beta}) + \Xi_{\mu\nu}(\nabla_\alpha F_G(G,R), \nabla_\alpha F_R(G,R)), \end{aligned}$$

where $\Xi_{\mu\nu}$ is vanishing when G and R are constant.

In this section we are interested in a finding condition assuring the existence of solutions of the de Sitter type (including SdS BH). In such a case, the Gauss-Bonnet invariant and the Ricci scalar are constant and the Gauss-Bonnet invariant reads

$$R = R_0, \qquad G = G_0 = \tfrac{1}{6}R_0^2. \tag{65}$$

Assuming the maximally symmetric metric solution, one gets

$$2F_R(G,R)R^0_{\mu\nu} = (F(G_0,R_0) - G_0F_{G_0}(G_0,R_0))g^0_{\mu\nu}. \tag{66}$$

Note that if $F(G,R) = R + \alpha G$, namely, there is only a linear term in G, one gets the ordinary Einstein equation in vacuum, as it should be, because in this case, $F(G,R)$ contains the Hilbert-Einstein term plus a topological invariant.

Taking the trace of (66), the condition follows

$$\frac{R_0F_{R_0}(G_0,R_0)}{2} = [F(G_0,R_0) - G_0F_{G_0}(G_0,R_0)]. \tag{67}$$

This condition will play an important role in the following and it is equivalent to the condition (17). As an example, when $F(G,R) = R + f(G)$, (here $2\kappa^2 = 1$), one has

$$G_0f'(G_0) - f(G_0) = \frac{R_0}{2}. \tag{68}$$

In general, solving Eq. (67) in terms of R_0, one can rewrite the maximally symmetric solution as

$$R^0_{\mu\nu} = \frac{R_0}{4}g^0_{\mu\nu} = \Lambda_{\rm eff}g^0_{\mu\nu}, \tag{69}$$

which defines an effective cosmological constant. For example, when $F(G,R) = R + f(G)$, one has

$$\Lambda_{\rm eff} = \tfrac{1}{2}(G_0f'(G_0) - f(G_0)). \tag{70}$$

Thus, if $f(G) = -\alpha G^\beta$, one has,

$$2\alpha(1-\beta)(\tfrac{1}{6})^\beta = R_0^{1-2\beta}. \tag{71}$$

When β is small, $\alpha > 0$, one obtains $R_0 \sim \alpha$, while with the choice $\beta = -1/2$, $\alpha > 0$, one has

$$R_0 = 6^{1/4}\sqrt{3\alpha}, \tag{72}$$

and the corresponding effective cosmological constant reads

$$\Lambda_{\rm eff} = \tfrac{1}{4}6^{1/4}\sqrt{3\alpha}. \tag{73}$$

As in the pure Einstein case, one is confronted with the black hole nucleation problem [22]. We review here the discussion reported in Refs. [22,23].

To begin with, we recall that we shall deal with a tunneling process in quantum gravity. On general backgrounds, this process is mediated by the associated gravitational instantons, namely, stationary solutions of Euclidean gravitational action, which dominate the path integral of Euclidean quantum gravity. It is a well-known fact that, as soon as an imaginary part appears in the one-loop partition function, one has a metastable thermal state and thus a nonvanishing decay rate. Typically, this imaginary part comes from the existence of a negative mode in the one-loop functional determinant. Here, the semiclassical and one-loop approximations are the only techniques at disposal, even though one should bear in mind their limitations as well as their merits.

Let us consider a general model described by $F(G,R)$, satisfying the condition (67) and with $\Lambda_{\rm eff} > 0$. Thus, we may have a de Sitter Euclidean instanton. In the Euclidean version, the associated manifold is S_4.

Making use of the instanton approach, for the Euclidean partition function we have

$$Z \simeq Z(S_4) = Z^{(1)}(S_4)e^{-I(S_4)}, \tag{74}$$

where I is the classical action and $Z^{(1)}$ the quantum correction, typically a ratio of functional determinants. The classical action can be easily evaluated and reads

$$I(S_4) = -\frac{192\pi^2}{\kappa^2 R_0^2} F(G_0, R_0). \tag{75}$$

At this point we make a brief digression regarding the entropy of the above black hole solution. To this aim, we follow the arguments reported in Ref. [24]. If one make use of the Noether charge method [25] for evaluating the entropy associated with black hole solutions with constant Gauss-Bonnet and Ricci invariants in modified gravity models, a direct computation gives

$$S = \frac{2\pi A_H}{\kappa^2}\left(F_{R_0}(G_0, R_0) + \frac{R_0}{3} F_{G_0}(G_0, R_0)\right). \tag{76}$$

In the above equations, $A_H = 4\pi r_H^2$, r_H being the radius of the event horizon or cosmological horizon related to a black hole solution. This turns out to be model dependent.

Another consequence, as is well known, is the modification of the "Area Law," which reads instead

$$S = \frac{2\pi A_H}{\kappa^2} = \frac{A_H}{4G_N}. \tag{77}$$

One should also stress that, since the above entropy formula depends on $F_G(G, R)$, there is always an indetermination: any linear term in G appearing in the classical action is irrelevant as far as the equations of motion are concerned, while in the entropy formula it gives a constant nonvanishing contribution. This is a kind of indetermination associated with the Noether method [25].

Furthermore, as in principle the quantity which modifies in a nontrivial way the usual Area Law

$$F_{R_0}(G_0, R_0) + \frac{R_0}{3} F_{G_0}(G_0, R_0) \tag{78}$$

might be negative, there exists the possibility of having negative BH entropies, according to specific choices of $F(G, R)$.

Let us consider an example. For the model defined by $F(G, R) = R + f(G)$,

$$1 + \frac{R_0}{3} f'(G_0) = 2\left(1 + \frac{f(G_0)}{R_0}\right). \tag{79}$$

Thus, with the choice $f(G) = -\alpha G^\beta$, $\alpha > 0$, one has

$$S = \frac{4\pi A_H}{\kappa^2}\frac{1}{\beta - 1}. \tag{80}$$

As a result, modulo the Noether charge method indetermination, the entropy may be negative.

Coming back to the general case, we recall that we are interested in the de Sitter metric, which reads

$$ds^2 = -\left(1 - \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{1 - \frac{r^2}{l^2}} + r^2 dS_2^2, \tag{81}$$

with $\Lambda_{\mathrm{eff}} = \frac{3}{l^2}$. The Ricci scalar is $R_0 = 4\Lambda$.

Since $r_H = l$, one has

$$A_H = \frac{12\pi}{\Lambda_{\mathrm{eff}}} = \frac{48\pi}{R_0}. \tag{82}$$

As a consequence,

$$S(S_4) = \frac{96\pi^2}{\kappa^2 R_0}\left(F_{R_0}(G_0, R_0) + \frac{R_0}{3} F_{G_0}(G_0, R_0)\right). \tag{83}$$

Taking Eqs. (67) and (75) into account, from the last equation, one obtains

$$S(S_4) = -I(S_4), \tag{84}$$

which is a good check of our entropy formula (76).

IV. ONE-LOOP QUANTIZATION OF MODIFIED GAUSS-BONNET GRAVITY ON DE SITTER SPACE

Here we discuss the one-loop quantization of the class of models we are dealing with, on a maximally symmetric space. One-loop contributions are certainly important during the inflationary phase, but as it has been shown in [26], they also provide a powerful method in order to study the stability of the solutions.

We start by recalling some properties of the classical model defined by the choice $F(G, R) = \frac{1}{2\kappa^2}(R + \hat{f}(G))$, where now the generic function $\hat{f}(G)$ is supposed to satisfy the "on shell" condition (68). Such a condition ensures the existence of a constant GB invariant, maximally symmetric solution of the field equations (12) with (18).

In accordance with the background field method, we now consider the small fluctuations of the fields around the de Sitter manifold. Then, for the arbitrary solutions of the field equations we set

$$g_{\mu\nu} = g_{0\mu\nu} + h_{\mu\nu} \tag{85}$$

and perform a Taylor expansion of the action around the de Sitter manifold. Up to second order in $h_{\mu\nu}$, we get

$$S[h] \sim \frac{1}{2\kappa^2}\int d^4x\sqrt{-g_0}\left[R_0 + \hat{f}_0 + \left(\frac{R_0}{4} + \frac{\hat{f}_0}{2} - \frac{R_0^2 \hat{f}_1}{2}\right)h + \mathcal{L}_2\right], \tag{86}$$

where $\mathcal{L}_2$ represents the quadratic contribution in the fluctuation field $h_{\mu\nu}$ and, in contrast with previous sections, here ∇_μ represents the covariant derivative in the unperturbed metric $g_{0\mu\nu}$. For the sake of simplicity, we have also used the notation $\hat{f}_0 = \hat{f}(G_0)$, $\hat{f}_1 = \hat{f}'(G_0)$, and $\hat{f}_2 = \hat{f}''(G_0)$.

For technical reasons it is convenient to carry out the standard expansion of the tensor field $h_{\mu\nu}$ in irreducible components [27], that is

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \nabla_\mu \nabla_\nu \sigma + \tfrac{1}{4} g_{\mu\nu}(h - \Delta\sigma), \tag{87}$$

where σ is the scalar component, while ξ_μ and $\hat{h}_{\mu\nu}$ are the vector and tensor components, with the properties

$$\nabla_\mu \xi^\mu = 0, \qquad \nabla_\mu \hat{h}^\mu_\nu = 0, \qquad \hat{h}^\mu_\mu = 0. \tag{88}$$

In terms of the irreducible components of the $h_{\mu\nu}$ field, the quadratic part of the Lagrangian density, disregarding total derivatives, reads

$$\mathcal{L}_2 = \mathcal{L}_{\rm tensor} + \mathcal{L}_{\rm vector} + \mathcal{L}_{\rm scalar}, \tag{89}$$

where $\mathcal{L}_{\rm tensor}$, $\mathcal{L}_{\rm vector}$, and $\mathcal{L}_{\rm scalar}$ represent the tensor, vector, and scalar contributions, respectively. They have the form

$$\mathcal{L}_{\rm tensor} = \hat{h}_{\mu\nu}\left[-\frac{\hat{f}_0}{4} - \frac{R_0}{6} + \frac{\hat{f}_1 R_0^2}{6} + \left(\frac{1}{4} + \frac{\hat{f}_1 R_0}{6}\right)\Delta_2\right]\hat{h}^{\mu\nu}, \tag{90}$$

$$\mathcal{L}_{\rm vector} = \xi_\mu\left[\frac{\hat{f}_0 R_0}{8} + \frac{R_0^2}{16} - \frac{31\hat{f}_1 R_0^3}{288} - \frac{\hat{f}_2 R_0^5}{432} + \left(\frac{\hat{f}_0}{2} + \frac{R_0}{4} - \frac{17\hat{f}_1 R_0^2}{36} + \frac{\hat{f}_2 R_0^4}{108}\right)\Delta_1 - \frac{\hat{f}_1 R_0}{32}\Delta_1^2\right]\xi^\mu, \tag{91}$$

$$\begin{aligned}\mathcal{L}_{\rm scalar} = {}& h\left[\frac{\hat{f}_0}{16} + \frac{\hat{f}_1 R_0^2}{48} + \frac{\hat{f}_2 R_0^4}{72} + \left(-\frac{3}{32} + \frac{5\hat{f}_1 R_0}{32} + \frac{\hat{f}_2 R_0^3}{24}\right)\Delta_0 + \frac{\hat{f}_2 R_0^2}{32}\Delta_0^2\right]h \\ & + \sigma\left[-\frac{\hat{f}_0 R_0}{16} - \frac{R_0^2}{32} + \frac{17\hat{f}_1 R_0^3}{288} + \left(-\frac{3\hat{f}_0}{16} - \frac{R_0}{8} + \frac{3\hat{f}_1 R_0^2}{16} + \frac{\hat{f}_2 R_0^4}{288}\right)\Delta_0 + \left(-\frac{3}{32} + \frac{\hat{f}_1 R_0}{32} + \frac{\hat{f}_2 R_0^3}{48}\right)\Delta_0^2 \right.\\ & \left. + \frac{\hat{f}_2 R_0^2}{32}\Delta_0^3\right]\Delta_0\sigma + h\left[\frac{R_0}{16} - \frac{\hat{f}_1 R_0^2}{16} - \frac{\hat{f}_2 R_0^4}{72} + \left(\frac{3}{16} - \frac{3\hat{f}_1 R_0}{16} - \frac{\hat{f}_2 R_0^3}{16}\right)\Delta_0 - \frac{\hat{f}_2 R_0^2}{16}\Delta_0^2\right]\Delta_0\sigma,\end{aligned} \tag{92}$$

where Δ_0, Δ_1, and Δ_2 are the Laplace-Beltrami operators acting on scalars, transverse vector, and traceless-transverse tensor fields, respectively. The latter expression is valid off-shell, that is, for an arbitrary choice of the function $f(G_0)$.

As is well known, invariance under diffeomorphisms renders the operator related to the latter quadratic form not invertible in the (h, σ) sector. One needs a gauge-fixing term and a corresponding ghost compensating term. We can use the same class of gauge conditions chosen in Ref. [28] and for this reason we refer the reader to that paper, the gauge-fixing $\mathcal{L}_{gf}$ and ghost $\mathcal{L}_{gh}$ contributions to the quadratic Lagrangian $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{gf} + \mathcal{L}_{gh}$ being the very same, that is

$$\begin{aligned}\mathcal{L}_{gf} = {}& \frac{\alpha}{2}\left[\xi^\mu\left(\Delta_1 + \frac{R_0}{4}\right)^2\xi_\mu + \frac{3\rho}{8}h\left(\Delta_0 + \frac{R_0}{3}\right)\Delta_0\sigma - \frac{\rho^2}{16}h\Delta_0 h - \frac{9}{16}\sigma\left(\Delta_0 + \frac{R_0}{3}\right)^2\Delta_0\sigma\right] \\ & + \frac{\beta}{2}\left[\xi^k\left(\Delta_1 + \frac{R_0}{4}\right)^2\Delta_1\xi_k + \frac{3\rho}{8}h\left(\Delta_0 + \frac{R_0}{4}\right)\left(\Delta_0 + \frac{R_0}{3}\right)\Delta_0\sigma - \frac{\rho^2}{16}h\left(\Delta_0 + \frac{R_0}{4}\right)\Delta_0 h \right.\\ & \left. - \frac{9}{16}\sigma\left(\Delta_0 + \frac{R_0}{4}\right)\left(\Delta_0 + \frac{R_0}{3}\right)^2\Delta_0\sigma\right],\end{aligned} \tag{93}$$

$$\begin{aligned}\mathcal{L}_{gh} = {}& \alpha\left[\hat{B}^\mu\left(\Delta_1 + \frac{R_0}{4}\right)\hat{C}_\mu + \frac{\rho - 3}{2}b\left(\Delta_0 - \frac{R_0}{\rho - 3}\right)\Delta_0 c\right] + \beta\left[\hat{B}^\mu\left(\Delta_1 + \frac{R_0}{4}\right)\Delta_1\hat{C}_\mu \right.\\ & \left. + \frac{\rho - 3}{2}b\left(\Delta_0 + \frac{R_0}{4}\right)\left(\Delta_0 - \frac{R_0}{\rho - 3}\right)\Delta_0 c\right],\end{aligned} \tag{94}$$

where α, β, ρ are arbitrary parameters and $\hat{C}^\mu$, c, $\hat{B}^\mu$, b are the irreducible components of ghost and anti-ghost fields.

Now, via standard path integral quantization and zeta-function regularization [27,29,30], one can compute the one-loop effective action as a determinant of a differential operator, exactly in the same way as it has been done in Ref. [28]. Here the technical difficulty comes from the fact that the determinant which gives the one-loop effective action, also after simplifications, is a polynomial of fifth order in the Laplace operator and thus, in general, it is not possible to write it as a product of determinants of Laplace-like operators. The general structure of the one-loop effective action $\Gamma^{(1)}$ is of the form

$$e^{-\Gamma^{(1)}} = \det(\Delta_0 - R_0)\det\left(\Delta_1 + \frac{R_0}{4}\right)\det\left[\sum_{k=0}^{5} W_k \Delta_0^k\right]^{-1/2} \times \det[(\Delta_1 + Y)(\Delta_1 + Z)]^{-1/2}\det(\Delta_2 + X)^{-1/2}, \tag{95}$$

where X, Y, Z, W_k are complicated functions of $\hat{f}_0$, $\hat{f}_1$, $\hat{f}_2$. In principle, they can be exactly computed, but we will not write them explicitly here since, as they stand, they are not really convenient for direct applications to the dark energy problem. However, Eq. (95) is certainly useful for a numerical analysis of the models, since the eigenvalues of Laplace-like operators on de Sitter manifolds are exactly known and, as a consequence, the one-loop effective action can be obtained in closed form. They could be also useful for the study of the stability of the de Sitter solutions of GB modified gravity, as it happens for the simpler case of $\mathcal{F}(R)$ models [26].

One can see that the structure of Eq. (95), here written for the gauge parameter $\beta = 0$, is similar to the one for the analog equation (3.24) in Ref. [26], the only difference being due to the fact that (95) contains some additional contributions, since the starting classical action here depends not only on curvature, but also on other invariants.

V. THE TRANSITION FROM THE DECELERATION TO THE ACCELERATION ERA IN MODIFIED CURVATURE-GAUSS-BONNET GRAVITY

It is interesting to study late-time cosmology in generalized theories, which include both the functional dependence from curvature as well as from the Gauss-Bonnet term (for some investigation of the related Ricci-squared gravity cosmology, see [31]). Our starting action is (10). In the case $\rho_m = 0$, we can consider the situation where $F(G, R)$ has the following form, as an explicitly solvable example:

$$F(G, R) = R\tilde{f}\left(\frac{G}{R^2}\right). \tag{96}$$

If we assume that $H = h_0/t$, with a constant h_0, Eq. (15) reduces to an algebraic equation:

$$0 = 2(3h_0 - 1)\tilde{f}'(C) + 3(2h_0 - 1)^2\tilde{f}(C), \qquad C \equiv \frac{2h_0(h_0 - 1)}{3(2h_0 - 1)^2}. \tag{97}$$

As a further example, we consider the model where

$$\tilde{f}\left(\frac{G}{R^2}\right) = \frac{1}{2\kappa^2} + f_0\left(\frac{G}{R^2}\right). \tag{98}$$

Here, it may be shown that

$$0 = \left(\frac{6}{\kappa^2} + 2f_0\right)h_0^2 - 2\left(\frac{3}{\kappa^2} - 2f_0\right)h_0 + \frac{3}{2\kappa^2} + 2f_0. \tag{99}$$

When $f_0 < 0$ and $f_0 > 3/8\kappa^2$, Eq. (99) has the following solutions:

$$h_0 = \frac{\frac{3}{\kappa^2} - 2f_0 \pm \sqrt{8f_0(f_0 - \frac{3}{8\kappa^2})}}{\frac{6}{\kappa^2} + 2f_0}. \tag{100}$$

One can solve (99) with respect to $\kappa^2 f_0$:

$$\kappa^2 f_0 = \tilde{G}(h_0) \equiv -\frac{3(2h_0 - 1)^2}{4(h_0^2 + 2h_0 - 1)} \tag{101}$$

and, therefore, by properly choosing f_0, we obtain a theory for any specified h_0. It is easy to check that $\tilde{G}(\infty) = 3$, $\tilde{G}(1/2) = 0$, and $\tilde{G}(-1 \pm \sqrt{2}) = \infty$. Since

$$\tilde{G}'(h_0) = -\frac{3(2h_0 - 1)(3h_0 - 1)}{2(h_0^2 + 2h_0 - 1)^2}, \tag{102}$$

we also get that $\tilde{G}(h_0)$ has extrema for $h_0 = 1/2$, $1/3$. Moreover, $\tilde{G}(1/3) = 3/8$. The qualitative behavior of $\kappa^2 f_0 = \tilde{G}(h_0)$ is given in Fig. 1. Then, for $\kappa^2 f_0 < -3$, there is a solution describing a phantom with $h_0 < -1 - \sqrt{2}$ and a solution describing effective matter with $h_0 > -1 + \sqrt{2}$. When $-3 < \kappa^2 f_0 < 0$, there are two solutions, describing effective matter with $h_0 > -1 + \sqrt{2}$. When $0 < \kappa^2 f_0 < 3/8$, there is no solution. When $3/8 < \kappa^2 f_0 < 3/2$, there are two solutions describing matter with $0 < h_0 < -1 + \sqrt{2} < 1$. When $\kappa^2 f_0 > 3/2$, there is a solution describing a phantom era with $-1 - \sqrt{2} < h_0 < 0$ and a solution describing an effective matter era with $<1/3 < h_0 < -1 + \sqrt{2}$. As one sees, there can be indeed solutions describing an effective phantom era, in general.

Observational data hint towards the fact that the deceleration of the universe turned into acceleration about 5×10^9 years ago. We now investigate if we can construct a model describing the transition from the deceleration phase

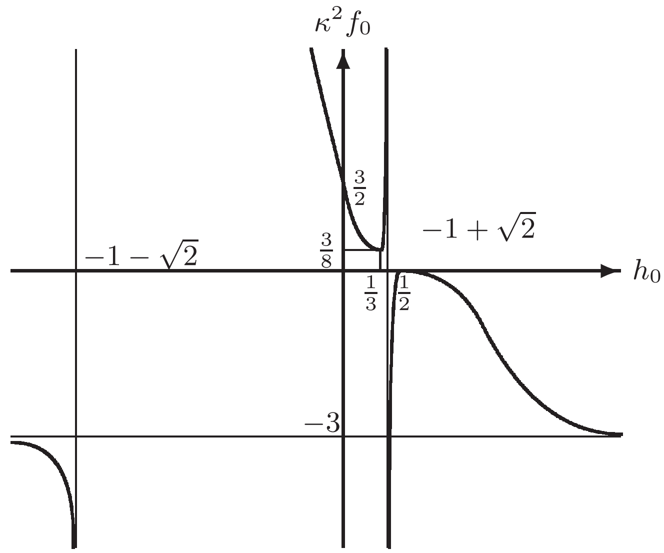

FIG. 1. The qualitative behavior of $\tilde{G}(h_0)$.

to the acceleration one, in the present formulation. To this end, we consider the following model:

$$F(G,R) = R\left(\frac{1}{2\kappa^2} + f_0\frac{G}{R^2}\right) + \frac{g_0}{G}. \tag{103}$$

When the curvature is large, as in the case of the primordial universe, the last term can be neglected and the model (103) reduces to (98). Then, with the choice $3/8 < \kappa^2 f_0 < 3/2$, it follows $0 < h_0 < -1 + \sqrt{2} < 1$ and therefore the universe is decelerating. If the curvature becomes small, as in the present universe, the last term becomes large. But, by including the last term, we have a solution describing a de Sitter universe in (17), which has the form

$$0 = -\frac{H_0^2}{\kappa^2}(\kappa^2 f_0 + 3) - \frac{g_0}{12H_0^4}. \tag{104}$$

As $\kappa^2 f_0 + 3$ is positive when $3/8 < \kappa^2 f_0 < 3/2$, Eq. (105) has a real solution for $g_0 < 0$:

$$H_0 = \left\{-\frac{\kappa^2 g_0}{12(\kappa^2 f_0 + 3)}\right\}^{1/6}. \tag{105}$$

Then, the decelerating universe can indeed turn to a de Sitter universe, which is accelerating. Therefore, the model (103) could perfectly describe the transition from deceleration to acceleration. More complicated, dark energy cosmologies may be constructed in frames of such theory.

VI. HIERARCHY PROBLEM IN MODIFIED GRAVITY

Recently the hierarchy problem has been investigated, in [32], by using a scalar-tensor theory. Here, we will give somewhat similar but seemingly more natural models in the scalar-tensor family with modified gravity. In [32], in order to generate the hierarchy, a small scale, which is the vacuum decay rate $\Gamma_{\rm vac}$, was considered. Instead of $\Gamma_{\rm vac}$, we here use the age of the universe, $\sim 10^{-33}$ eV, as the small mass scale.

The following scalar-tensor theory [33] can be considered, as an example:

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\,e^{\alpha\phi}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_0 e^{-2\phi/\phi_0}\right) + \int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - U(\chi)\right), \tag{106}$$

where α, V_0, and ϕ_0 are constant parameters and $U(\chi)$ is the potential for χ. As the matter scalar, χ does not couple with ϕ directly, the equivalence principle is not violated, although the effective gravitational coupling depends on ϕ as

$$\tilde{\kappa} = \kappa e^{-[(\alpha\phi)/2]}. \tag{107}$$

When $\chi = 0$, the following FRW solution exists:

$$\begin{aligned} a(t) &= a_0\left(\frac{t}{t_0}\right)^{h_0}, \qquad \phi = \phi_0\ln\frac{t}{t_0}, \\ h_0 &\equiv \frac{2\alpha^2\phi_0^2 + \phi_0^2 - 2\alpha\phi_0}{2(\alpha\phi_0 + 2)}, \\ t_0 &\equiv \frac{\phi_0^2}{(\alpha\phi_0 + 2)^2}\times\sqrt{\frac{(\alpha^2 + \frac{1}{3})(24\alpha^2\phi_0^2 - 4\alpha\phi_0 + 9\phi_0^2 - 4)}{2V_0}}. \end{aligned} \tag{108}$$

The effective EoS parameter is

$$w_{\rm eff} = -1 + \frac{4(\alpha\phi_0 + 2)}{3(2\alpha^2\phi_0^2 + \phi_0^2 - 2\alpha\phi_0)}, \tag{109}$$

which can be less than -1, in general.

We may perfectly assume the dimensionful parameters κ and V_0, and therefore t_0 in (106), could be the scale of the weak interaction $\sim 10^2$ GeV $= 10^{11}$ eV. If t is of the order of the age of the universe, $\sim 10^{-33}$ eV, Eq. (108) gives

$$e^{-\alpha\phi/2} \sim 10^{-22\alpha\phi_0}. \tag{110}$$

Then, if

$$\alpha\phi_0 = \frac{17}{22}, \tag{111}$$

$\tilde{\kappa}$ (107) is of the order of the Planck length $(10^{19}$ GeV$)^{-1}$. Therefore, using a model whose action is given by (106), the important hierarchy problem might be solved. By substituting (111) into (109), one obtains

$$w_{\rm eff} = -1 + \frac{61}{33(\phi_0^2 - \frac{85}{242})}. \tag{112}$$

It is seen that $w_{\rm eff}$ can be less than -1 if $\phi_0^2 < 85/242$, and t_0 is given by

$$t_0 = \frac{22^2\phi_0^2}{61^2}\sqrt{\frac{(\frac{17^2}{22^2\phi_0^2} + \frac{1}{3})(\frac{876}{121} + 9\phi_0^2)}{2V_0}}, \tag{113}$$

which is real and positive, as far as $V_0 > 0$. A similar method can be applied in the solution of the hierarchy problem in a generalized scalar-tensor theory including a nonminimal coupling with the curvature [34].

We now start from the following action for modified gravity coupled with matter:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{\kappa^2}\mathcal{F}(R) + \mathcal{L}_{\rm matter}\right), \tag{114}$$

$\mathcal{F}(R)$ being some arbitrary function. Introducing the auxiliary fields, A and B, one can rewrite the action (114) as follows:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{\kappa^2}\{B(R - A) + \mathcal{F}(A)\} + \mathcal{L}_{\rm matter}\right]. \tag{115}$$

One is able to eliminate B, and obtain

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{\kappa^2}\{\mathcal{F}'(A)(R-A) + \mathcal{F}(A)\} + \mathcal{L}_{\rm matter}\right], \tag{116}$$

and by using the conformal transformation

$$g_{\mu\nu} \to {\rm e}^{\sigma} g_{\mu\nu}, \tag{117}$$

with

$$\sigma = -\ln\mathcal{F}'(A), \tag{118}$$

the action (116) is rewritten as the Einstein-frame action:

$$S_E = \int d^4x\sqrt{-g}\left[\frac{1}{\kappa^2}\left(R - \frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma)\right) + \mathcal{L}^{\sigma}_{\rm matter}\right]. \tag{119}$$

Here,

$$V(\sigma) = {\rm e}^{\sigma}\mathcal{G}({\rm e}^{-\sigma}) - {\rm e}^{2\sigma}\mathcal{F}(\mathcal{G}({\rm e}^{-\sigma})) = \frac{A}{\mathcal{F}'(A)} - \frac{\mathcal{F}(A)}{\mathcal{F}'(A)^2}, \tag{120}$$

and we solve (118) with respect to A as $A = \mathcal{G}(\sigma)$. Let us assume that the matter Lagrangian density $\mathcal{L}_{\rm matter}$ contains a Higgs-like scalar field φ

$$\mathcal{L}_{\rm matter} = -\frac{1}{2}\nabla^\mu\varphi\nabla_\mu\varphi + \frac{\mu^2}{2}\varphi^2 - \lambda\varphi^4 + \cdots. \tag{121}$$

Under the conformal transformation (117), the matter Lagrangian density $\mathcal{L}_{\rm matter}$ is transformed as

$$\mathcal{L}_{\rm matter} \to \mathcal{L}^{\sigma}_{\rm matter} = -\frac{{\rm e}^{\sigma}}{2}\nabla^\mu\varphi\nabla_\mu\varphi + \frac{\mu^2{\rm e}^{2\sigma}}{2}\varphi^2 - \lambda{\rm e}^{2\sigma}\varphi^4 + \cdots. \tag{122}$$

By redefining φ as

$$\varphi \to {\rm e}^{-\sigma/2}\varphi, \tag{123}$$

$\mathcal{L}^{\sigma}_{\rm matter}$ acquires the following form:

$$\mathcal{L}^{\sigma}_{\rm matter} \sim -\frac{1}{2}\nabla^\mu\varphi\nabla_\mu\varphi + \frac{\mu^2{\rm e}^{\sigma}}{2}\varphi^2 - \lambda\varphi^4 + \cdots, \tag{124}$$

where the time derivative of σ is neglected. Then, the massive parameter μ, which determines the weak scale, is effectively transformed as

$$\mu \to \tilde{\mu} \equiv {\rm e}^{\sigma/2}\mu. \tag{125}$$

In principle, μ can be of the order of the Planck scale 10^{19} GeV, but if ${\rm e}^{\sigma/2} \sim 10^{-17}$ in the present universe, $\tilde{\mu}$ could be 10^2 GeV, which is the scale of the weak interaction. Therefore, there is a quite natural possibility that the hierarchy problem can be solved by using the above version of modified gravity.

We may consider the model

$$\mathcal{F}(R) = R + f_0 R^{\alpha}, \tag{126}$$

with constant f_0 and α. If $\alpha < 1$, the second term dominates, when the curvature is small. Assuming that the EoS parameter w of matter is constant, one gets [16]

$$a = a_0 t^{h_0}, \qquad h_0 \equiv \frac{2\alpha}{3(1+w)},$$
$$a_0 \equiv \left[-\frac{6f_0h_0}{\rho_0}(-6h_0 + 12h_0^2)^{\alpha-1}\{(1-2\alpha)(1-\alpha) - (2-\alpha)h_0\}\right]^{-\{1/[3(1+w)]\}}, \tag{127}$$

and, by using (127), we find the effective $w_{\rm eff}$ to be given by

$$w_{\rm eff} = -1 + \frac{1+w}{\alpha}. \tag{128}$$

Hence, if w is larger than -1 (as for effective quintessence or even for a usual ideal fluid with positive w), when α is negative, an effective phantom phase occurs where $w_{\rm eff}$ is less than -1. Note that this is different from the case of pure modified gravity.

By using (118) and neglecting the first term in (126), it follows that

$${\rm e}^{\sigma/2} \sim \frac{1}{\sqrt{f_0\alpha R^{\alpha-1}}}. \tag{129}$$

In the present universe, $R \sim (10^{-33}\ {\rm eV})^2$. Assume now that f_0 could be given by the Planck scale $\sim 10^{19}$ GeV $= 10^{28}$ eV as $f_0 \sim (10^{28}\ {\rm eV})^{1/2(\alpha-1)}$. Then, Eq. (129) would yield

$${\rm e}^{\sigma/2} \sim 10^{61(\alpha-1)}. \tag{130}$$

If we furthermore assume that $-17 = 61(\alpha - 1)$, we find $\alpha = 44/61$. In that case, if $w > -1$, $w_{\rm eff} > -1$ and the universe is not phantomlike, but (130) hints towards the possibility that modified gravity can solve in fact the hierarchy problem.

Let us write the action of the scalar-tensor theory as

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}{\rm e}^{\alpha\phi}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_0{\rm e}^{-2\phi/\phi_0}\right) + \int d^4x\sqrt{-g}\mathcal{L}_{\rm matter},$$
$$\mathcal{L}_{\rm matter} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + \frac{\mu^2}{2}\varphi^2 - \lambda\varphi^4 + \cdots. \tag{131}$$

This action could be regarded as the Jordan frame action. Scalar field φ could be identified with the Higgs field in the weak (electromagnetic) interaction. Now the ratio of the inverse of the effective gravitational coupling $\tilde{\kappa} = \kappa e^{-(\alpha\phi/2)}$ (107) and the Higgs mass μ is given by

$$\frac{\frac{1}{\tilde{\kappa}}}{\mu} = \frac{e^{(\alpha\phi)/2}}{\kappa\mu}. \tag{132}$$

Hence, even if both of $1/\kappa$ and μ are of the order of the weak interaction scale, if $e^{\alpha\phi/2} \sim 10^{17}$, $1/\tilde{\kappa}$ could be of the order of the Planck scale.

By rescaling the metric and the Higgs scalar φ as

$$g_{\mu\nu} \to e^{-\alpha\phi} g_{\mu\nu}, \qquad \varphi \to e^{(\alpha\phi)/2}\varphi, \tag{133}$$

the Einstein-frame action is obtained:

$$\begin{aligned} S &= \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\Big(R - \frac{1}{2}(1+3\alpha^2)\partial_\mu\phi\partial^\mu\phi \\ &\quad - V_0 e^{-2\phi(1/\phi_0+\alpha)}\Big) + \int d^4x\sqrt{-g}\mathcal{L}_{\rm matter}, \\ \mathcal{L}_{\rm matter} &= -\frac{1}{2}\Big(\partial^\mu\varphi + \frac{\alpha}{2}\partial^\mu\phi\varphi\Big)\Big(\partial_\mu\varphi + \frac{\alpha}{2}\partial_\mu\phi\varphi\Big) \\ &\quad + \frac{\mu^2 e^{-\alpha\phi}}{2}\varphi^2 - \lambda\varphi^4 + \cdots. \end{aligned} \tag{134}$$

In the Einstein frame, the gravitational coupling κ is constant but the effective Higgs mass, $\tilde{\mu}$, defined by

$$\tilde{\mu} \equiv e^{-[(\alpha\phi)/2]}\mu, \tag{135}$$

can be time dependent. Hence, the ratio of $1/\kappa$ and $\tilde{\mu}$ is given by

$$\frac{\frac{1}{\kappa}}{\tilde{\mu}} = \frac{e^{(\alpha\phi)/2}}{\kappa\mu}, \tag{136}$$

which is identical with (132). Then, even if both of $1/\kappa$ and μ are of the order of the Planck scale, if $e^{\alpha\phi/2} \sim 10^{17}$, $\tilde{\mu}$ could be an order of the weak interaction scale. Therefore the solution of the hierarchy problem does not essentially depend on the choice of frame.

Nevertheless, note that the cosmological time variables in the two frames could be different due to the scale transformation (133), as

$$dt \to d\tilde{t} = e^{-[(\alpha\phi)/2]}dt. \tag{137}$$

Therefore, the time intervals are different in the two frames. The units of time and length are now defined by electromagnetism. Then, the frame where the electromagnetic fields do not couple with the scalar field ϕ could be physically more preferable. Since the electromagnetic interaction is a part of the electroweak interaction, the Jordan frame in (131) should be more preferable from the point of view of the solution of the hierarchy problem.

In the case of $f(G)$-gravity, whose action is given by (10) with (18), it is rather difficult to solve the hierarchy problem in the same way, since the factor in front of the scalar curvature, which should be the inverse of the Newton constant, although it is indeed a constant, in the above cases this factor depends on time. However, when including a term like $g(G)R$, where $g(G)$ is a proper function of the Gauss-Bonnet invariant, the effective Newton constant could become time dependent and might indeed help solve the hierarchy problem.

A similar mechanism can work also in $F(G, R)$-gravity (10). Introducing the auxiliary fields, A, B, C, and D, one can rewrite the action (114) as follows:

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[\frac{1}{\kappa^2}\{B(R-A) + D(G-C) \\ + F(A,C)\} + \mathcal{L}_{\rm matter}\Big]. \end{aligned} \tag{138}$$

One is able to eliminate B and D, and obtain

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[\frac{1}{\kappa^2}\{\partial_A F(A,C)(R-A) \\ + \partial_C F(A,C)(G-C) + F(A,C)\} + \mathcal{L}_{\rm matter}\Big]. \end{aligned} \tag{139}$$

If a scalar field is deleted by

$$e^{-\sigma} \equiv \partial_A F(A,C), \tag{140}$$

one can solve (140) with respect to A as $A = A(\sigma, C)$. Then, we obtain

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[\frac{1}{\kappa^2}\{e^{-\sigma}(R - A(\sigma,C)) \\ + \partial_C F(A(\sigma,C),C)(G-C) + F(A(\sigma,C),C)\} \\ + \mathcal{L}_{\rm matter}\Big]. \end{aligned} \tag{141}$$

Varying over C, it follows that $C = G$, which allows one to eliminate C:

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[\frac{1}{\kappa^2}\{e^{-\sigma}(R - A(\sigma,G)) \\ + F(A(\sigma,G),G)\} + \mathcal{L}_{\rm matter}\Big]. \end{aligned} \tag{142}$$

Performing the scale transformation (117), we obtain the Einstein-frame action. If we consider matter (Higgs) scalar as in (121), we can redefine φ as in (123). The same scenario as in the case of $\mathcal{F}(R)$-gravity is applied, if $e^{\sigma/2} \sim 10^{-17}$ in the present universe, there is a possibility that the hierarchy problem can be solved by working in the above version of $F(G, R)$ gravity. Hence, we have proven that the hierarchy problem can indeed be solved in modified gravity which contains a $F(G, R)$ term.

VII. DISCUSSION

To summarize, various types of dark energy cosmologies in modified Gauss-Bonnet gravity—which can be viewed as inspired by string considerations—have been investigated in this paper. We have shown, in particular, that effective quintessence, phantom, and cosmological constant eras can naturally emerge in this framework, without the need to introduce scalar fields of any sort explicitly. Actually, the cosmic acceleration we observe may result from the expansion of the universe due to the growing of the extra terms in the gravitational action when the curvature decreases. In addition, with the help of several examples, corresponding to explicit choices of the function $f(G)$, we have shown that the unification of early-time inflation with late-time acceleration in those theories occurs also quite naturally. Moreover, the framework is attractive in the sense that it leads to a reasonable behavior in the solar system limit (no corrections to Newton's law, no instabilities, no Brans-Dicke problems appear), whatever the particular choice of $f(G)$. Finally, the transition from the deceleration to the acceleration epoch, or from the nonphantom to the phantom regime—provided the current universe is in its phantom phase—may both be natural ingredients of our theory, without the necessity to invoke any sort of exotic matter (quintessence or phantom) with an explicit negative EoS parameter.

We have also shown in the paper that modified GB gravity has de Sitter or SdS BH solutions, for which the corresponding entropy has been calculated. It has been explicitly demonstrated that our theory can be consistently quantized to one-loop order in de Sitter space, in the same way as modified $\mathcal{F}(R)$ gravity.

Dark energy cosmologies in a more complicated $F(G, R)$ framework can be constructed in a similar fashion, too. An attempt to address fundamental particle physics issues (as the hierarchy problem), as resulting from a modification of gravity, has shown that some natural solution may possibly be achieved in $\mathcal{F}(R)$-gravity, but probably not in $f(G)$-gravity [albeit the case $F(G, R)$ opens again a new possibility]. In this respect, it may also be of interest to study other modified gravities, where a nonminimal coupling of the sort L_d-$f(G)$, with L_d being some matter Lagrangian which includes also a kinetic term is introduced. In the specific case of a L_d-$\mathcal{F}(R)$ nonminimal coupling, such terms may be able to explain the current dark energy dominance [35] as a gravitationally assisted one.

The next step should be to fit the specific astrophysical predictions of the above theory with current observational data (for a recent summary and comparison of such data from various sources, see [36]), which ought to be modified accordingly, as most of them are derived under the (often implicit) assumption that standard general relativity is correct. One immediate possibility is to study the perturbation structure in close analogy with what has been done for $\mathcal{F}(R)$-gravity [19,37]. This will be reported elsewhere.

ACKNOWLEDGMENTS

We thank L. Vanzo for useful discussions. Support from the AI program INFN(Italy)-CICYT(Spain), from MEC (Spain), Project No. FIS2005-01181 (S. D. O.), and SEEU Grant No. PR2004-0126 (E. E.), is gratefully acknowledged.

APPENDIX

Having in mind the importance of spherically symmetric BH solutions in gravity theories, let us consider the possibility that $F(G, R)$-gravity has the Schwarzschild black hole solution:

$$ds^2 = -e^{2\nu}dt^2 + e^{-2\nu}dr^2 + r^2 d\Omega_2^2, \qquad e^{2\nu} = 1 - \frac{r_0}{r}, \tag{A1}$$

For simplicity, we consider the vacuum case ($T_{\mu\nu} = 0$) and concentrate on the model (18). Then, by multiplying $g_{\mu\nu}$ with (12) for the action (18), we find

$$\begin{aligned} 0 = {}& \frac{1}{2\kappa^2}R + 2f(G) - 2f'(G)R^2 + 4f'(G)R_{\mu\rho}R^{\mu\rho} \\ & - 2f'(G)R^{\mu\rho\sigma\tau}R_{\mu\rho\sigma\tau} + 4f'(G)R^{\mu\nu}R_{\mu\nu} \\ & - 2(\nabla^2 f'(G))R + 4(\nabla_\nu\nabla^\mu f'(G))R^{\nu\mu}. \end{aligned} \tag{A2}$$

In the case of the Schwarzschild black hole (A1), one has

$$R = R_{\mu\nu} = 0, \qquad G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12r_0^2}{r^6}. \tag{A3}$$

Then, Eq. (A2) is reduced to be

$$0 = f(G) - Gf'(G), \tag{A4}$$

which gives

$$f(G) = f_0 G, \tag{A5}$$

with a constant f_0. Since G is a total derivative, one can drop $f(G)$ in (10) with (18) for (A5). Hence, the Schwarzschild black hole geometry is not a solution for a nontrivial $f(G)$-gravity which justifies our interest for SdS BH in Sec. III as for the spherically symmetric solution of the above $f(G)$ theory. Note that a theory of this sort may contain a Schwarzschild solution in higher dimensions, where G is not a topological invariant (for a recent example, see [38]). It should be also stressed that modified gravity of the $F(G, R)$ form containing more complicated R-dependent terms might admit the standard Schwarzschild black hole solution.

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