

**Trace anomaly on a quantum spacetime manifold**Euro Spallucci,<sup>1,2,\*</sup> Anais Smailagic,<sup>2,†</sup> and Piero Nicolini<sup>2,3,‡</sup><sup>1</sup>*Dipartimento di Fisica Teorica, Università di Trieste, Strada Costiera 11, 34014, Trieste, Italy*<sup>2</sup>*INFN, Sezione di Trieste, Italy*<sup>3</sup>*Dipartimento di Matematica e Informatica, Università di Trieste, via A. Valerio 12, 34127, Trieste, Italy*

(Received 20 January 2006; published 5 April 2006)

In this paper we investigate the trace anomaly in a space-time where single events are delocalized as a consequence of short distance quantum coordinate fluctuations. We obtain a modified form of heat kernel asymptotic expansion which does not suffer from short distance divergences. Calculation of the trace anomaly is performed using an IR regulator in order to circumvent the absence of UV infinities. The explicit form of the trace anomaly is presented and the corresponding 2D Polyakov effective action and energy-momentum tensor are obtained. The vacuum expectation value of the energy-momentum tensor in the Boulware, Hartle-Hawking and Unruh vacua is explicitly calculated in a  $rt$  section of a recently found, noncommutative inspired, Schwarzschild-like solution of the Einstein equations. The standard short distance divergences in the vacuum expectation values are regularized in agreement with the absence of UV infinities removed by quantum coordinate fluctuations.

DOI: [10.1103/PhysRevD.73.084004](https://doi.org/10.1103/PhysRevD.73.084004)

PACS numbers: 04.62.+v, 04.70.Dy

**I. INTRODUCTION**

Quantum gravity has been considered for a long time the would-be quantized version of general relativity, or of any of its possible extensions. The success of string theory has shown that the former is actually untenable and that a fully consistent unified theory of all fundamental interactions, including gravity, requires a deeper level of quantization. Rather than quantization of fields propagating in a classical, nondynamical manifold, it is space-time itself that must be quantized. The boldest attempt in this direction is offered by  $M$ -theory [1], where classical space-time coordinates are expected to emerge as classical eigenvalues of noncommuting quantum string coordinates.

An alternative approach to space-time quantization is given by noncommutative geometry where the quantum features are encoded into a nonvanishing coordinate commutator. The technical difficulty of dealing with coordinates which are not  $c$ -numbers but “operators” is usually bypassed by introducing ordinary, commuting, coordinates and shifting noncommutativity (NCY) in a new multiplication rule, i.e.  $\ast$ -product, between functions. This approach has been widely exploited and it can be summarized in the following prescription: take commutative quantum field theory (QFT) results and substitute ordinary function multiplication by  $\ast$ -product multiplication. This prescription, however, leaves quadratic terms in the action unaffected, as the explicit form of the  $\ast$ -product leads, in this case, to surface terms only. Thus, free dynamics encoded into kinetic terms and Green functions remain the same as in the commutative case. The presence of a noncommutative product becomes relevant when more

than two field variables couple together. The  $\ast$ -product is a *nonlocal* operation giving rise to nonplanar contributions to Feynman diagrams at any perturbative order. These nonplanar graphs introduce an unexpected and unpleasant mixing of ultraviolet (UV) and infrared (IR) divergences [2]. So far there has been no solution to this problem and it has been accepted as an unavoidable by-product of NCY. This is only one of the technical problems in a series which includes breaking of Lorentz invariance and problems with unitarity [3]. Recently we have proposed an alternative approach to space-time quantization based on the use of coherent states of the quantum position operators  $\mathbf{x}^\mu$  [4] satisfying

$$[\mathbf{x}^\mu, \mathbf{x}^\nu] \neq 0 \quad (1)$$

The idea is that the nonvanishing commutator (1) excludes a common basis in coordinate representation. The best one can do is to define mean values between appropriately chosen states, i.e. coherent states. These mean values are the closest one can get to the classical commuting coordinates, since coherent states are minimum uncertainty states. In [4] we applied this procedure by constructing ladder operators built of quantum coordinates only, in analogy to the usual phase space ladder operators which are mixtures of coordinates and momenta. The use of mean values of quantum position operators as “classical coordinates” leads to the emergence of a *quasi-classical* space-time manifold where the position of any physical object is intrinsically uncertain. This uncertainty can be seen either as a Gaussian cutoff in momentum space Green functions, or as a substitution of position Dirac delta with minimal width Gaussian function.

It is important that the reader realizes that this cutoff is *not put by hand* as an auxiliary regularization, but is a direct consequence of the use of the position coherent states. In coherent states approach the problem of unitarity

\*Electronic address: [spallucci@trieste.infn.it](mailto:spallucci@trieste.infn.it)†Electronic address: [anais@ictp.trieste.it](mailto:anais@ictp.trieste.it)‡Electronic address: [nicolini@cmfd.univ.trieste.it](mailto:nicolini@cmfd.univ.trieste.it)

and Lorentz invariance can be handled properly and Gaussian cutoff renders the theory UV-finite. Furthermore, no UV/IR mixing can appear simply because there are no UV divergences.

It should be interesting to investigate the modification of some relevant results of standard QFT within position coherent states formulation.

One of the well-known effects, in both gauge theories and gravity, is the anomalous quantum breaking of classical symmetries. Anomalies, i.e. nonvanishing divergences of quantum symmetry currents, are usually extracted after suitable manipulations (regularization) of divergent quantities. These divergences are produced by coincidence limit of various operator products and are consequences of the pointlike structure of commutative space-time. It is interesting to look at the anomaly problem in NC space-time. Within the  $*$ -product approach the anomalies have been already investigated and, again, the only modification to standard results is the substitution of ordinary function multiplication with  $*$ -multiplication [5].

In the coherent states approach we do not expect the same result for two reasons: first, there are no UV divergences from which to extract anomalies; second, there is no  $*$ -product at all. Thus, either it could be that there are no anomalies at all, or there are modifications of the standard results different from the one obtained in the  $*$ -product.

The paper is organized as follows: in Sec. II we show that the standard trace anomaly can alternatively be extracted from IR divergences. This is a method followed in the case where there are no UV divergent Feynman diagrams, and thus suitable for our purposes. Second, we define heat kernel and effective action for a gauge D’Alambertian operator in a quasiclassical space-time. At the end, we perform an explicit calculation of the trace anomaly for massless fermions coupled to a background Abelian gauge field. Section III is devoted to the extension of the previous method to a curved space-time and to the consequent calculation of the gravitational trace anomaly. As an example, we integrate the  $2D$  trace anomaly and obtain the Polyakov effective action in a “quantum” string world manifold. In the appendices we introduce a useful relation between a Gaussian function and a Dirac delta, used throughout the paper.

## II. HEAT KERNEL IN FLAT, QUASI-CLASSICAL, SPACE-TIME

Quantum space-time manifold is defined by the coordinate commutator

$$[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\Theta^{\mu\nu}, \quad (2)$$

where  $\Theta^{\mu\nu}$  is a constant, antisymmetric, Lorentz tensor. In the case of coordinates satisfying (2), the usual notion of classical space-time is lost and is replaced by a quantum geometry. Thus, we face two levels of “quantization”: one due to space-time uncertainty and the other due to matter.

The interest for noncommutative quantum field theory (NCQFT) comes from the interplay between these two levels of quantization and the resulting physical effects.

In even dimensional space-time,  $\Theta^{\mu\nu}$  can be brought to a block-diagonal form by a suitable Lorentz rotation leading to

$$\Theta^{\mu\nu} = \text{diag}[\theta_1 \epsilon^{ab} \theta_2 \epsilon^{ab} \dots \theta_{d/2} \epsilon^{ab}] \quad (3)$$

with  $\epsilon^{ab}$  a  $2 \times 2$  antisymmetric Ricci Levi-Civita tensor. It has been shown [6] that the resulting field theory is Lorentz invariant if all the  $\theta$ -parameters are coincident:  $\theta_i \equiv \theta$ . Indeed, Lorentz invariance requires space-time homogeneity and isotropy. The  $\theta$  parameter is a UV cutoff in QFT on the quasiclassical space-time. At the same time,  $\theta$  has dimension  $(\text{mass})^{-2}$ , thus one would expect that its presence leads to an *explicit breaking* of scale invariance. It will be shown that there is no explicit  $\theta$ -breaking since the  $\theta$  parameter always comes in a particular scale invariant combination. Thus, scale anomaly is still a quantum mechanical effect in a quasiclassical space-time manifold, but now endowed with  $\theta$  corrections.

We briefly list methods for calculating anomalies. First, calculations dealt with axial current anomalous divergence which was obtained in perturbation theory by computing the appropriate Feynman diagram [7], and later repeated in the case of trace of the energy-momentum tensor [8].

Soon after Fujikawa noticed that anomalies in the path-integral formalism are due to the noninvariance of the functional measure under classical symmetries [9]. Heat kernel expansion [10] provides another very efficient method to extract divergent parts from quantum amplitudes [11]. Alternatively, one can resort to  $\zeta$ -function [12], or dimensional regularization [13]. As a result, all these different methods boil down to the introduction, at the quantum level, of some kind of regulating parameter, with dimension of mass, thus breaking classical scale invariance of the model.

In our case, the absence of UV divergences requires a suitable modification of the standard approach. Therefore, let us think of another reason to introduce some parameter with dimension of mass with the hope that it would still lead to the appearance of an anomalous trace. Fortunately, such a parameter is required to regulate IR divergences in massless QFTs. Starting the procedure is to define the path integral in an infrared regulated way and functionally integrate away matter fields. The resulting functional determinant will depend upon the infrared regulating parameter. Finally, the regulated trace of the vacuum averaged energy-momentum tensor is recovered after the infrared cutoff is removed. An eventual nonzero result will give us the desired trace anomaly.

Let us apply this procedure in the case of a massless fermionic field  $\psi(x)$  coupled to an Abelian background gauge field  $A_\mu(x)$ . The partition functional for the background field is given by

$$Z[A] = N \int D\bar{\psi} D\psi \exp\left[-\int d^d x \bar{\psi} i\gamma^\mu D_\mu \psi\right], \quad (4)$$

where  $D_\mu \equiv \partial_\mu - ieA_\mu$  is the  $U(1)$  gauge covariant derivative, and the normalization constant  $N$  is defined as

$$N^{-1} \equiv \int D\bar{\psi} D\psi \exp\left[-\int d^d x \bar{\psi} i\gamma^\mu \partial_\mu \psi\right]. \quad (5)$$

With this choice  $Z[0] = 1$ . The effective action for the background field is obtained by integrating out the matter field  $\psi$ , which gives

$$\begin{aligned} Z[A] &= \frac{\det[i\gamma^\mu D_\mu]}{\det[i\gamma^\mu \partial_\mu]} \\ &= \exp[\ln \det(i\gamma^\mu D_\mu) - \ln \det(i\gamma^\mu \partial_\mu)] \\ &= \exp\left[\frac{1}{2} \text{Tr}[\ln(i\gamma^\mu D_\mu)^2 - \ln(-\square)]\right]. \end{aligned} \quad (6)$$

As the starting theory is massless, let us take care of infrared divergences by introducing a fictitious mass parameter,  $\mu$ . This leads to the *IR regularized effective action*:

$$\begin{aligned} \Gamma[A]_{\text{IR-reg}} &\equiv \int d^d x L[A] \\ &= \frac{1}{2} \lim_{\mu^2 \rightarrow 0} \text{Tr}[\ln(\Delta + \mu^2) - \ln(-\square)], \end{aligned} \quad (7)$$

where  $\Delta \equiv (i\gamma^\mu D_\mu)^2$ . We are going to show that IR regularization procedure can be used to obtain the trace anomaly. Being an auxiliary parameter and not a physical mass,  $\mu$  will be set to zero at the end of calculations. It turns out that finite terms survive leading to the *trace anomaly*.

To prove this statement, let us consider an infinitesimal rescaling of the Euclidean metric:  $\delta_{\mu\nu} \rightarrow (1 - 2\epsilon)\delta_{\mu\nu}$ . Both  $D^\nu$  Alambertian operators in (7) scale with weight 2 and  $L$  varies as

$$dL = \frac{\partial L}{\partial \epsilon} d\epsilon = 2\mu^2 \text{Tr}\left[\frac{1}{1 + 2\epsilon} \frac{1}{\mu^2 - (1 + 2\epsilon)\Delta}\right] d\epsilon. \quad (8)$$

The trace anomaly represents the response of the effective Lagrangian under rescaling, namely

$$\langle 0|T^\nu{}_\nu|0\rangle = \lim_{\epsilon \rightarrow 0} \left(\frac{\partial L}{\partial \epsilon}\right). \quad (9)$$

By taking into account (8) we get

$$\lim_{\epsilon \rightarrow 0} \frac{\partial L}{\partial \epsilon} = 2\mu^2 \text{Tr}\left[\frac{1}{-\Delta^2 + \mu^2}\right]_{\mu^2=0} = 2 \lim_{\mu^2 \rightarrow 0} \mu^2 \left(\frac{\partial L}{\partial \mu^2}\right). \quad (10)$$

Thus, we can express the trace anomaly as

$$\langle 0|T^\nu{}_\nu|0\rangle = \left(\mu^2 \frac{\partial L}{\partial \mu^2}\right)_{\mu^2=0}. \quad (11)$$

Equation (11) is the definition of the trace anomaly in terms of the IR regulator. It shows that only the  $\mu$ -dependent part of  $L$  can contribute to the anomaly. Thus, in what follows we shall drop the field independent (infinite) term  $\ln(-\square)$ .

Our goal is to investigate the effects of quantum space-time manifold on the effective action and trace anomaly. Recently two of us have proposed a way of introducing space-time fluctuations in QFT formalism using coordinate coherent states [4]. The motivations are discussed in the introduction. The outcome of our approach is that the uncertainty in *coordinate* representation is seen through Gaussian function of width  $\sqrt{2\theta}$  in place of the Dirac delta function.

In view of the above prescription, the Green function equation for the square of the Dirac operator becomes

$$(\Delta + \mu^2)G_\theta(x, y) = \rho_\theta(x, y), \quad (12)$$

where  $\rho_\theta(x, y)$  is the above mentioned Gaussian source:

$$\rho_\theta(x, y) = \frac{1}{(4\pi\theta)^{d/2}} \exp\left(-\frac{(x-y)^2}{4\theta}\right). \quad (13)$$

Although there are no *physical* pointlike sources, one can use delta functions in the intermediate steps thanks to a useful representation of Gaussian of width  $\theta$  in terms of the Dirac delta as (see Appendix A):

$$e^{\theta\Box} \delta^{(d)}(x-y) = \rho_\theta(x, y). \quad (14)$$

The solution of Eq. (12) is given by

$$G_\theta(x-y) = \int d^d z G_0(x-z)\rho_\theta(z-y), \quad (15)$$

where  $G_0(x-z)$  is the commutative Green function satisfying

$$(\Delta + \mu^2)G_0(x-z) = \delta^{(d)}(x-z). \quad (16)$$

The commutative Green function  $G_0(x-z)$  can be expressed through the commutative heat kernel  $K_0(x-z; s)$  as

$$G_0(x-z) = \int_0^\infty ds K_0(x-z; s). \quad (17)$$

Thus, we exploit the relation between modified and commutative quantities to write

$$\begin{aligned}
G_\theta(x-y) &= \int_0^\infty ds \int d^d z K_0(x-z; s) \rho_\theta(z-y) \\
&\equiv \int_0^\infty ds K_\theta(x, y; s), \tag{18}
\end{aligned}$$

where we arrived to the definition of the *modified heat kernel* which can be further rewritten as

$$\begin{aligned}
K_\theta(x, y; s) &= \int d^d z K_0(x-z; s) e^{\theta \square_z} \delta^{(d)}(x-z) \\
&= e^{\theta \square_x} K_0(x, y; s) \tag{19}
\end{aligned}$$

in obtaining (19) we have exploited well-known integration properties of the Dirac delta. It can be shown that the noncommutative kernel (19) satisfies the standard heat equation but with modified initial condition which takes into account the fuzziness of the space-time

$$\Delta K_\theta(x, y; s) = \frac{\partial}{\partial s} K_\theta(x, y; s) \quad K_\theta(x, y; 0) = \rho_\theta(x, y). \tag{20}$$

### A. Asymptotic expansion

In order to calculate the trace anomaly in quantum space-time one starts from Eq. (19). First, we exploit the asymptotic expansion of the kernel in terms of the Seeley-DeWitt coefficients [14]

$$K_\theta(x, y; s) = e^{\theta \square_x} \left[ \frac{1}{(4\pi s)^{d/2}} e^{-\mu^2 s - (x-y)^2/4s} \sum_{n=0}^\infty s^n a_n(x, y) \right]. \tag{21}$$

Using the property that the first coefficient is the trace of the identity matrix we can write (21) as

$$\begin{aligned}
K_\theta(x, y; s) &= a_0 e^{\theta \square_x} \frac{1}{(4\pi s)^{d/2}} e^{-\mu^2 s - (x-y)^2/4s} + \frac{1}{(4\pi s)^{d/2}} \\
&\quad \times e^{\theta \square_x} \left[ e^{-\mu^2 s - (x-y)^2/4s} \sum_{n=1}^\infty s^n a_n(x, y) \right]. \tag{22}
\end{aligned}$$

The second term can be manipulated using the Fourier transform of the coefficient as follows

$$\begin{aligned}
e^{\theta \square_x} \int \frac{d^d p}{(2\pi)^d} e^{-s p^2 + i p(x-y)} \sum_{n=1}^\infty s^n a_n(x, y) &= \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} e^{-s p^2 - \theta(p+q)^2 + i(p+q)x - i p y} \sum_{n=1}^\infty s^n a_n(q, y) \\
&= \frac{e^{-(x-y)^2/4(s+\theta)}}{[(4\pi)(s+\theta)]^{d/2}} \int \frac{d^d q}{(2\pi)^d} e^{-q^2 \theta s / (\theta+s)} e^{i q [x + x\theta / (s+\theta) - y\theta / (s+\theta)]} \sum_{n=1}^\infty s^n a_n(q, y). \tag{23}
\end{aligned}$$

Thus, we arrive at the following asymptotic expansion for the heat kernel

$$K_\theta(x, y; s) = \rho_{\theta+s}(x-y) e^{-\mu^2 s} \left[ a_0 + \int \frac{d^d q}{(2\pi)^d} e^{-q^2 \theta s / (\theta+s)} e^{i q [x + (x-y)\theta s / (s+\theta)]} \sum_{n=1}^\infty s^n a_n(q, y) \right]. \tag{24}$$

In most of the actual applications of the heat kernel method one needs the trace of the kernel, i.e.  $K_\theta(x, x; s)$ :

$$\begin{aligned}
K_\theta(x, x; s) &= \rho_{\theta+s}(0) e^{-\mu^2 s} \left[ a_0 + e^{[\theta s / (\theta+s)] \square} \int \frac{d^d q}{(2\pi)^d} e^{i q x} \sum_{n=1}^\infty s^n a_n(q, x) \right] \\
&= \rho_{\theta+s}(0) e^{-\mu^2 s} \left[ a_0 + e^{[\theta s / (\theta+s)] \square} \sum_{n=1}^\infty s^n a_n(x, x) \right]. \tag{25}
\end{aligned}$$

We find the modified version of the asymptotic expansion for the heat kernel to be

$$\begin{aligned}
K_\theta(x, x; s) &= \frac{1}{[(4\pi)(s+\theta)]^{d/2}} \\
&\quad \times e^{-\mu^2 s} \left[ a_0 + e^{[\theta s / (\theta+s)] \square} \sum_{n=1}^\infty s^n a_n(x, x) \right]. \tag{26}
\end{aligned}$$

### B. Heat kernel representation of the effective action in quasiclassical space-time

To obtain the form of the effective action in terms of the heat kernel, one *formally* varies the differential operator  $\Delta$  and computes the corresponding variation of the effective action as

$$\delta\Gamma = \frac{1}{2} \delta \int d^d x \text{Tr} \ln(\Delta + \mu^2) = \frac{1}{2} \int d^d x \text{Tr} \left[ \frac{1}{\Delta + \mu^2} \rho_\theta(x, x) \right] \delta\Delta = \frac{1}{2} \int d^d x \text{Tr} \int_0^\infty ds e^{-s(\Delta + \mu^2)} e^{\theta \partial_x^2} \delta^{(d)}(x, x) \delta\Delta, \tag{27}$$

where we have taken into account (14) and momentarily dropped the limit  $\mu^2 \rightarrow 0$ . The operator  $\Delta$  can be always cast in the form

$$\Delta = -D^2 + X(x), \quad (28)$$

where  $D^2$  is the covariant D’Alambertian with respect to the local symmetry that one wants to maintain at the quantum level. The exact form of the kernel of  $\Delta$  is not

known, but it is sufficient to consider only the asymptotic form (26). Asymptotic expansion (26) is valid for the values of  $s$  for which the fields contained in  $X(x)$  are slowly varying, so that the condition

$$\square X(x) = 0 \rightarrow [\square, X(x)] = 0 \quad (29)$$

is satisfied. This assumption enables to “integrate” the functional variation of the effective action:

$$\begin{aligned} \delta\Gamma &= \frac{1}{2} \int d^d x \text{Tr} \int_0^\infty ds e^{-(s+\theta)\square - (s+\theta)(X(x)+\mu^2) + \theta(X(x)+\mu^2)} \delta^{(d)}(x, x) \delta\Delta \\ &= \frac{1}{2} \int d^d x \text{Tr} \int_0^\infty ds e^{-(s+\theta)(\Delta+\mu^2) + \theta(X(x)+\mu^2)} \delta^{(d)}(x, x) \delta\Delta \\ &= -\frac{1}{2} \int d^d x \delta \left[ \text{Tr} \int_0^\infty \frac{ds}{s+\theta} e^{\theta\square} e^{-s(\Delta+\mu^2)} \delta^{(d)}(x, x) \right]. \end{aligned} \quad (30)$$

The above manipulation enables us to write the final form of the modified effective action as

$$\Gamma = -\frac{1}{2} \lim_{\mu^2 \rightarrow 0} \int d^d x \int_0^\infty ds \frac{e^{-s\mu^2}}{s+\theta} e^{\theta\square} K_0(x, x; s). \quad (31)$$

The effective action (31) is ultraviolet *finite*. Short distance divergences have been removed in the quantum space-time. This is what Gaussian functions, representing position uncertainty, do to ordinary QFT [4,6].

### C. Trace anomaly calculation

Usually the anomaly is calculated from the divergent part of the asymptotic expansion of the effective action. However, in quasiclassical space-time these divergences have been cured and one cannot follow the same path. In Sec. II we have shown that, alternatively, the anomaly can be calculated through the IR regulator (11). Thus, let us calculate the derivative of the effective Lagrangian (31) with respect to  $\mu^2$ :

$$\begin{aligned} \mu^2 \frac{\partial L^{\text{eff}}}{\partial \mu^2} &= \frac{1}{2} \left( \frac{\mu^2}{4\pi} \right)^{d/2} \int_0^\infty d\tau \frac{\tau e^{-\tau}}{[(\tau + \mu^2\theta)]^{1+d/2}} \\ &\quad \times \text{Tr} \left[ a_0 + e^{[\theta\tau/(\mu^2\theta+\tau)]\square} \sum_{n=1}^{d/2} \left( \frac{\tau}{\mu^2} \right)^n a_n(x) \right]. \end{aligned} \quad (32)$$

In the limit  $\mu^2 \rightarrow 0$  the only nonvanishing and finite contribution comes from the  $n = d/2$  term in the sum. The IR divergent terms of (32) are subtracted in the usual manner and do not contribute to the anomaly. Therefore,

the trace anomaly turns out to be

$$\begin{aligned} \langle 0|T^\nu{}_\nu|0\rangle &\equiv 2 \lim_{\mu^2 \rightarrow 0} \left( \mu^2 \frac{\partial L^{\text{eff}}}{\partial \mu^2} \right) \\ &= \lim_{\mu^2 \rightarrow 0} \frac{(\mu^2)^{d/2}}{(4\pi)^{d/2}} \text{Tr} \int_0^\infty d\tau \frac{\tau e^{-\tau}}{\tau^{1+d/2}} \\ &\quad \times \left( \frac{\tau}{\mu^2} \right)^{d/2} e^{\theta\square} a_{d/2}(x) \\ &= \frac{1}{(4\pi)^{d/2}} e^{\theta\square_x} \text{Tra}_{d/2}(x). \end{aligned} \quad (33)$$

The result (33) holds in any dimension and is the modified version of the usual trace anomaly. As an explicit example, let us consider  $D$  trace anomaly of massless fermionic matter in an Abelian gauge field background. It is found to be

$$\langle 0|T^\lambda{}_\lambda|0\rangle = \frac{e^2 N_F}{12(4\pi)^2} e^{\theta\square} F_{\mu\nu} F^{\mu\nu}, \quad (34)$$

where  $N_F$  is the total number of fermionic degrees of freedom. Equation (34) displays in a clear way how space-time quantum fluctuations, described using coordinate coherent states, affects the standard result through the new term  $e^{\theta\square}$ .

### III. TRACE ANOMALY IN CURVED SPACE-TIME

So far, we have considered the trace anomaly in quasiclassical flat space-time. In view of the role of conformal invariance, and its breaking, in modern cosmological problems [16], we would like to extend the above consideration to the curved quasiclassical space-time. It is important to understand what modification will be sufficient in the passage from classical to quasiclassical geometry. We have already explained the effect of fluctuating quantum coordinates on the Dirac delta function and this applies equally in curved space-time. Furthermore, the question of

covariance has to be properly incorporated in operators connecting Gaussian and Dirac delta-functions. Thus, the curved modified space-time version of (12) is

$$(\Delta + \mu^2)G_\theta(x, y) = e^{\theta \nabla_\mu \nabla^\mu} \frac{1}{\sqrt{g}} \delta^{(d)}(x, y), \quad (35)$$

where  $\nabla_\mu$  is the generally covariant derivative in the metric  $g_{\mu\nu}$ . We would like to point out that the role of the IR regulator in (35) with respect to global scale transformations has been explained previously, and it remains such in the case of local Weyl symmetry. We are only interested in quantum (anomalous) Weyl symmetry breaking that will eventually remain after the limit  $\mu^2 \rightarrow 0$  is taken. On the other hand, the covariant D'Alambertian, is not Weyl covariant in an arbitrary dimension. We modify the D'Alambertian to achieve Weyl covariant form as

$$\nabla_\mu \nabla^\mu \rightarrow \nabla_\mu \nabla^\mu - \xi_d R \equiv \nabla_w^2, \quad \xi_d \equiv \frac{1}{4} \frac{d-2}{d-1} \quad (36)$$

which transforms under  $g_{\mu\nu} = \omega^2 \hat{g}_{\mu\nu}$  as  $\hat{\nabla}_w^2 = \omega^{-2}(x) \nabla_w^2$ .  $\theta$  as a component of the matrix  $\Theta^{\mu\nu}$  is a constant with dimension of length squared. For this reason the exponent in (35) is not Weyl invariant unless one expresses  $\theta$  in terms of  $\Theta^{\mu\nu}$  as

$$\theta \rightarrow \theta(x) \equiv \sqrt{\frac{1}{d} g_{\mu\alpha}(x) g_{\nu\beta}(x) \Theta^{\mu\nu} \Theta^{\alpha\beta}} \quad (37)$$

and such  $\theta(x)$  transforms under Weyl rescaling as  $\theta(x) = \omega^2(x) \hat{\theta}(x)$ . Now, we can safely claim the Weyl invariance of the exponent on the right-hand side of (12). To avoid any confusion, we remark that  $\Theta^{\mu\nu}$  remains a constant tensor while space-time dependence enters in the *invariant scalar*  $\theta(x)$  only through the metric tensor. As a consequence there will be additional contributions to the energy-momentum tensor due to the variation of  $\theta(x)$ . We calculate the variation of the curved space-time extension of the effective action (30) as

$$\begin{aligned} \delta\Gamma &= \frac{1}{2} \int d^d x \sqrt{g} \text{Tr} \int_0^\infty ds e^{-(s+\theta(x))\nabla_w^2 - s[(\beta - \xi_d)R + X(x) + \mu^2]} \frac{1}{\sqrt{g}} \delta^{(d)}(x, x) \delta\Delta \\ &= -\frac{1}{2} \delta \int d^d x \sqrt{g} \left[ \text{Tr} \int_0^\infty \frac{ds}{s + \theta(x)} e^{\theta(x)\nabla_w^2} e^{-s(\Delta + \mu^2)} \frac{1}{\sqrt{g}} \delta^{(d)}(x, x) \right]. \end{aligned} \quad (38)$$

In this manner, one obtains the generally covariant form of the heat kernel:

$$K_\theta(x, x; s) \equiv e^{\theta(x)\nabla_w^2} K(x, x; s) = \frac{e^{-\mu^2 s}}{[(4\pi)(s + \theta(x))]^{d/2}} \left[ a_0 e^{(s\theta(x)/s + s\theta(x))\xi_d R} + e^{(s\theta(x)/s + s\theta(x))\nabla_w^2} \sum_{n=1}^\infty s^n a_n(x, x) \right]. \quad (39)$$

Following the same steps as in (33) we find the trace anomaly to be

$$\begin{aligned} \mu^2 \frac{\partial L}{\partial \mu^2} &= \mu^2 \text{Tr} G(x, x) \\ &= \mu^2 \text{Tr} \left[ \frac{1}{\Delta + \mu^2} e^{(s\theta(x)/s + s\theta(x))\nabla_w^2} \frac{1}{\sqrt{g}} \delta^{(d)}(x, x) \right] \end{aligned} \quad (40)$$

$$\langle 0 | T^\nu{}_\nu | 0 \rangle = \frac{1}{(4\pi)^{d/2}} e^{\theta(x)\nabla_w^2} \text{Tr} [a_{d/2}(x)]. \quad (41)$$

### A. Anomalous effective action

Although the heat kernel method formally allows to calculate effective action it is possible in practice only if the complete Kernel is known. Since, this is never the case in QFT, one limits himself only to the calculation of the first few terms in the asymptotic expansion. The alternative is the calculation of few Feynman diagrams and covariantization of the result, but it turns out to be tedious and lengthy. The quickest way is to integrate the anomaly equation which in commutative two dimensions leads to

the Polyakov action. We find it the most convenient to follow the last approach since the modification due to noncommutativity can be integrated without big hassle to find

$$S = \frac{D-26}{48\pi} \int d^2 x \sqrt{g} R \frac{1}{\square} e^{\theta(x)\square} R. \quad (42)$$

It turns out that the effective action (42) generalizes the well-known  $2d$  anomalous effective action in a relatively simple way. The reason is that the exponent was introduced in a Weyl invariant way to avoid any explicit classical symmetry breaking. This condition has nothing to do with the effective action but turns out to be crucial in order to be able to integrate the anomaly Eq. (41). To be more precise, the trace of vacuum energy-momentum tensor couples to the Weyl degree of freedom. The anomaly comes from the term in the effective action which is linear in the Weyl degree of freedom which is already contained in the Ricci scalar curvature  $R$ . No other Weyl dependence should be present. In this case this is true due to the construction of the exponent. Thus, the anomaly (41) is a purely quantum mechanical effect.

Action (42) can be cast in the local form

$$S = \int d^2x \sqrt{g} \left[ \alpha \phi e^{(\theta/2)\square} R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right], \quad (43)$$

where the auxiliary field  $\phi$  must solve the field equation

$$\square \phi = -\alpha e^{(\theta/2)\square} R \quad (44)$$

and  $\alpha \equiv \sqrt{(26-D)/24\pi}$ .

In  $2D$  the above formula simplifies due to  $\theta^{\mu\nu} = \theta \varepsilon^{\mu\nu}$  where  $\varepsilon^{\mu\nu}$  is the totally antisymmetric symbol such that  $g_{\mu\alpha} g_{\nu\beta} \varepsilon^{\mu\nu} \varepsilon^{\alpha\beta} = \det(g_{\rho\tau})$ .

The energy-momentum tensor following from the Polyakov effective action is to be understood as the vacuum expectation value of a quantum energy-momentum tensor since it already incorporates anomaly induced quantum contributions. It is given by

$$\begin{aligned} T_{\mu\nu} = & \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla_\rho \phi)^2 + \\ & - \alpha \theta \phi (\nabla_\mu \nabla_\nu - \frac{1}{2} g_{\mu\nu} \square) e^{(\theta/2)\sqrt{-g}\square} R \\ & - 2\alpha (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) e^{(\theta/2)\sqrt{-g}\square} \phi. \end{aligned} \quad (45)$$

In order to perform an explicit calculation of the vacuum energy-momentum tensor in a  $2D$  black hole metric, the standard procedure amounts to consider as a background geometry the  $rt$  section of the corresponding  $4D$  black hole metric. This is known as the “ $s$ -wave approximation” [17]. The use of the Polyakov action derived from the anomaly already incorporates quantum effects. Thus, it provides a description of  $2D$  quantum black holes [18] and hopefully contains useful information about black hole evaporation in  $4D$ . Motivated by this argument we want to investigate coordinate noncommutativity in this framework. In order to do so, we adopt the  $rt$  section of a recently calculated, spherically symmetric solution of the noncommutative inspired Einstein equations in  $4D$  [19]. In this way we keep track of the effects of noncommutativity both on the metric itself and on the matter fields propagating in this background geometry.

The  $rt$  section of the metric obtained in [19], is

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2, \quad (46)$$

$$f(r) = 1 - \frac{4M}{r\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \equiv 1 - \frac{2M(r)}{r}, \quad (47)$$

$$\sqrt{-g} = 1, \quad (48)$$

$$\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \equiv \int_0^{r^2/4\theta} dt t^{1/2} e^{-t}, \quad (49)$$

where,  $\gamma(\frac{3}{2}, \frac{r^2}{4\theta})$  is an upper incomplete gamma function whose main properties are listed in Appendix B. The  $2D$  Ricci scalar turns out to be

$$R = -f'' = \frac{8M}{\sqrt{\pi} r^3} \left[ \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) - \frac{r^5}{16\theta^{5/2}} e^{-r^2/4\theta} \right], \quad (50)$$

where “prime” stands for  $d/dr$ .

The solution for the auxiliary field given by (44) is

$$\phi = e^{(\theta/2)\square} (At + \alpha \ln f + Br^* + C) \equiv e^{(\theta/2)\square} \phi^0, \quad (51)$$

where  $A$ ,  $B$ , and  $C$  are constants, while  $r^*$  is the tortoise coordinate defined as

$$r^* = \int \frac{dr}{f(r)}. \quad (52)$$

Fixing integration constants in (51) leads to the choice of different vacua [20]. The components of (45) are

$$\begin{aligned} T_{tt} = & (\partial_t \phi)^2 + \frac{f}{2} (\partial_r \phi)^2 - \frac{\alpha \theta}{2} \phi f (f' \partial_r + \square) e^{(\theta/2)\square} R \\ & + 2\alpha (f \square + \nabla_r \partial_r) e^{(\theta/2)\square} \phi, \end{aligned} \quad (53)$$

$$\begin{aligned} T_{rr} = & (\partial_r \phi)^2 - \frac{1}{2f} (\partial_t \phi)^2 + \frac{\alpha \theta}{2} \phi \left( -2\nabla_r \partial_r + \frac{1}{f} \square \right) \\ & \times e^{(\theta/2)\square} R - 2\alpha \left( \frac{1}{f} \square - \nabla_r \partial_r \right) e^{(\theta/2)\square} \phi, \end{aligned} \quad (54)$$

$$\begin{aligned} T_{tr} = & (\partial_t \phi)(\partial_r \phi) - \alpha \theta \phi \nabla_r \partial_t e^{(\theta/2)\square} R \\ & + 2\alpha \nabla_r \partial_t e^{(\theta/2)\square} \phi, \end{aligned} \quad (55)$$

where  $\square = f' \partial_r + f \partial_r^2 - f^{-1} \partial_t^2$ .

## B. Choice of vacua in quasiclassical metric

The components of the energy-momentum tensor (55) can be conveniently written as

$$\begin{aligned} T_{tt} = & \frac{\dot{\phi}^2}{2} + \frac{f^2}{2} \phi'^2 + 2\alpha f^2 \left( \tilde{\phi}'' + \frac{f'}{2f} \tilde{\phi}' \right) - \frac{\alpha \theta}{2} \phi f^2 \tilde{R}'', \\ T_{rr} = & \frac{\dot{\phi}^2}{2f^2} + \frac{\phi'^2}{2} - \alpha \frac{f'}{f} \tilde{\phi}' - \frac{\alpha \theta}{2} \phi \tilde{R}', \\ T_{tr} = & \dot{\phi} \left( \phi' - \alpha \frac{f'}{f} \right). \end{aligned} \quad (56)$$

We introduced field redefinitions  $\tilde{\phi} = e^{\theta \square} \phi^0$  and  $\tilde{R} = e^{\theta \square/2} R$  in order to have covariant  $\square$  act only on scalars. The light-cone components of momentum-energy tensor are useful for a choice of different vacua and are given by

TABLE I. We summarize the values of the integration constants leading to three different vacua.

| Vacuum         | Constant $A$ | Constant $B$                         | Physical meaning             |
|----------------|--------------|--------------------------------------|------------------------------|
| Boulware       | $A = 0$      | $B = 0$                              | Vacuum Polarization          |
| Hartle-Hawking | $A = 0$      | $B = \frac{\alpha}{2M_H}(1 - 2M_H')$ | Black Hole in a Thermal Bath |
| Unruh          | $A = -B$     | $B = \frac{\alpha}{4M_H}(1 - 2M_H')$ | Evaporating Black Hole       |

$$\begin{aligned}
4T_{uu} &\equiv T_{tt} + f^2 T_{rr} - 2fT_{tr} = \dot{\phi}^2 + f^2 \phi'^2 + 2\alpha f^2 \tilde{\phi}'' - \alpha\theta\phi f^2 \tilde{R}'' - 2\dot{\phi}(f\phi' - \alpha f') \\
&= (\dot{\phi} - f\phi')^2 + 2\alpha f^2 \tilde{\phi}'' - \alpha\theta\phi f^2 \tilde{R}'' + 2\alpha f' \dot{\phi} \\
4T_{vv} &\equiv T_{tt} + f^2 T_{rr} + 2fT_{tr} = \dot{\phi}^2 + f^2 \phi'^2 + 2\alpha f^2 \tilde{\phi}'' - \alpha\theta\phi f^2 \tilde{R}'' + 2\dot{\phi}(f\phi' - \alpha f') \\
&= (\dot{\phi} + f\phi')^2 + 2\alpha f^2 \tilde{\phi}'' - \alpha\theta\phi f^2 \tilde{R}'' - 2\alpha f' \dot{\phi}.
\end{aligned} \tag{57}$$

In order to study different vacuum states in the metric (47), we need to fix integration constants in (51). Let us first consider the case where there is no energy flux at any point. In our case this condition leads to

$$T_{tr}(r) = 0 \Rightarrow \dot{\phi} = 0 \rightarrow A = 0. \tag{58}$$

With this choice of constant  $A$ , light-cone components are equal and given by

$$4T_{uu} \equiv 4T_{vv} = (f\phi')^2 + 2\alpha f^2 \tilde{\phi}'' - \alpha\theta\phi f^2 \tilde{R}'' \tag{59}$$

In order to be able to calculate explicitly the above components, let us rewrite the scalar field  $\tilde{\phi}$  as

$$\tilde{\phi} = \phi^0 + (e^{\theta\Box} - 1)\phi^0 = \phi^0 + \sum_{n=1} \frac{(\theta\Box)^n}{n!} \phi^0 = \phi^0 - \alpha\theta \sum_{n=0} \frac{(\theta\Box)^n}{(n+1)!} R, \quad \tilde{\phi}' = \phi^{0'} - \alpha\theta \sum_{n=0} \frac{\theta^n}{(n+1)!} (\mathfrak{R}^n)', \tag{60}$$

$$\tilde{\phi}'' = \phi^{0''} - \alpha\theta \sum_{n=0} \frac{\theta^n}{(n+1)!} (\mathfrak{R}^n)'', \quad \mathfrak{R}^n = \Box^n R, \tag{61}$$

where we have used equation of motion  $\Box\phi^0 = -\alpha R$ . Now, we can calculate the field derivatives at the horizon. Keeping in mind that  $f(r_H) \equiv f_H = 0$ , the only components surviving in (57), when  $r = r_H$ , are

$$\begin{aligned}
(f^2 \tilde{\phi}'')_{r=r_H} &= (f^2 \phi^{0''})_{r=r_H} = -(\alpha f_H'^2 + B f_H') \\
(f^2 \tilde{\phi}'^2)_{r=r_H} &= (\alpha f_H' + B)^2.
\end{aligned} \tag{62}$$

The constant  $B$  is determined by the condition that there is no outgoing flux from the (future) horizon. This gives the Hartle-Hawking vacuum

$$\begin{aligned}
T_{uu}|_{r=r_H} &= B^2 - (\alpha f_H')^2 = 0 \Rightarrow B_{HH} \\
&= \frac{\alpha}{2M_H}(1 - 2M_H').
\end{aligned} \tag{63}$$

The Boulware vacuum is determined by the requirement that there is no outgoing flux at infinity which leads to

$$T_{tr} = 0 \Rightarrow A = 0 \tag{64}$$

$$\lim_{r \rightarrow \infty} T_{uu} = \lim_{r \rightarrow \infty} T_{vv} = 0 \Rightarrow B = 0 \tag{65}$$

Finally, the Unruh vacuum describing an evaporating black hole, originated by a collapsing body, requires no incoming flux from past infinity and no flux escaping out of the future horizon. These conditions lead to

$$\begin{aligned}
T_{vv}|_{r \rightarrow \infty} = 0 &\Rightarrow (\dot{\phi} + f\phi')|_{r \rightarrow \infty} = 0 \rightarrow A = -B \\
T_{uu}|_{r=r_H} = 0 &\Rightarrow B = \frac{\alpha f'}{2} = \frac{\alpha}{4M_H}(1 - 2M_H').
\end{aligned} \tag{66}$$

The values of the  $A$ ,  $B$  integration constants and the physical meaning of the corresponding vacuum state are summarized in Table I

### C. Short distance behavior

It is interesting to investigate the behavior of the energy-momentum tensor at short distances  $r^2/4\theta \ll 1$  where noncommutative effects should be important. In this regime the energy-momentum tensor components are given only by local part

$$T_{tt}^{\text{loc}} = \frac{A^2 + B^2}{2} - \frac{1}{2}\alpha^2 f'^2 + 2\alpha^2 f f'', \tag{67}$$



$$T_{rr}^{\text{loc}} = \frac{A^2 + B^2}{2f^2} - \frac{1}{2f^2} \alpha^2 f'^2, \quad (68)$$

$$T_{rt}^{\text{loc}} = \frac{AB}{f}. \quad (69)$$

Using the expansion of the incomplete  $\gamma$  function given in Appendix B, one finds  $f \simeq 1 - \Lambda r^2$ , where  $\Lambda = \frac{M}{3\sqrt{\pi}\theta^{3/2}}$ . Thus, at short distances the metrics is the de Sitter one and the components of energy-momentum tensor are

$$T_{tt}^{\text{short}} \simeq \frac{A^2 + B^2}{2} - 4\alpha^2 \Lambda + O(r^2), \quad (70)$$

$$T_{rr}^{\text{short}} \simeq \frac{A^2 + B^2}{2} + O(r^2), \quad (71)$$

$$T_{rt}^{\text{short}} \simeq AB + O(r^2). \quad (72)$$

The result shows that coordinate uncertainties remove the curvature singularity at the origin leading to a regular behavior in otherwise divergent quantities in all three vacua.

#### D. Long distance behavior

Long distance behavior shows no significant difference from ordinary results. This follows from the fact that noncommutative effects are exponentially small and the metric approaches the ordinary Schwarzschild solution at distances larger than  $\sqrt{\theta}$ . Indeed, one finds

$$\begin{aligned} f_{r^2/4\theta \gg 1}(r) &\approx 1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi}\theta} e^{-r^2/4\theta} \\ &\equiv f_S(r) + \frac{2M}{\sqrt{\pi}\theta} e^{-r^2/4\theta}. \end{aligned} \quad (73)$$

Energy-momentum tensor  $T_{\mu\nu}^{\text{loc}}$  is calculated with  $f$  given by (73). One obtains the usual Schwarzschild form of the vacuum expectation value of  $T_{\mu\nu}$  [20] plus exponentially damped corrections. This is in agreement with the general belief that the noncommutative effects modify, in a sensible way, only short distance physics. At large distances space-time looks as a smooth manifold and noncommutative effects are experimentally unobservable.

#### IV. CONCLUSIONS

We have shown how to modify the heat kernel asymptotic expansion to include the quantum coordinate fluctuations leading to an intrinsic delocalization of space-time events at short distances. As a result of the unavoidable position uncertainty, no UV divergencies are present. This result led us to propose an alternative way to compute the

trace anomaly by using an IR regulator. The resulting trace anomaly acquires a nonlocal character. Trace anomaly has been already used in  $2D$  as an essential ingredient to determine the vacuum expectation value of the energy-momentum tensor. Following these ideas, we have calculated corrections to the energy-momentum tensor mean value in the Boulware, Hartle-Hawking, and Unruh vacua. To carry out this calculation we have used the  $rt$  section of a recently found, noncommutative-inspired, Schwarzschild-like solution of the Einstein equations. The underlying motivation is that quantum coordinate fluctuations influence both matter *and* geometry itself. The metric found in [19] incorporates the latter effect. It turns out that the short distance behavior of the vacuum expectation values of  $T_{\mu\nu}$  is now regular. The metric itself is of de Sitter form near the origin, thus resolving the curvature singularity of the Schwarzschild solution.

#### APPENDIX A

In this appendix we prove the relation (14).

$$e^{\theta \square_x} \equiv \sum_0^\infty \frac{1}{n!} (\theta \square_x)^n, \quad (A1)$$

$$\begin{aligned} e^{\theta \square_x} e^{ikx} &= \sum_0^\infty \frac{1}{n!} (\theta \square_x)^n e^{ikx} = \sum_0^\infty \frac{1}{n!} (-\theta k^2)^n e^{ikx} \\ &= e^{-\theta k^2} e^{ikx}, \end{aligned} \quad (A2)$$

$$\begin{aligned} e^{\theta \square_x} \delta^{(d)}(x-y) &= e^{\theta \square_x} \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \\ &= \int \frac{d^d k}{(2\pi)^d} e^{-\theta k^2} e^{ik(x-y)} \\ &= \frac{1}{(\pi)^d} \left( \frac{2\pi}{\theta} \right)^{d/2} e^{-(x-y)^2/4\theta} \equiv \rho_\theta(x, y). \end{aligned} \quad (A3)$$

In the same way, one can show that the same widening effect occurs when the  $e^{\theta \square_x}$  operator is applied to a finite width Gaussian:

$$\begin{aligned} e^{\theta \square_x} \frac{1}{(2\pi s)^{d/2}} \exp\left(-\frac{x^2}{4s}\right) &= \frac{1}{[2\pi(s+\theta)]^{d/2}} \\ &\times \exp\left(-\frac{x^2}{4(s+\theta)}\right). \end{aligned} \quad (A4)$$

## APPENDIX B: PROPERTIES OF INCOMPLETE GAMMA FUNCTIONS

Definitions of incomplete lower  $\gamma$  and upper  $\Gamma$  functions

$$\begin{aligned}\gamma\left(\frac{n}{2} + 1, x^2\right) &\equiv \int_0^{x^2} dt t^{n/2} e^{-t} \\ \Gamma\left(\frac{n}{2} + 1, x^2\right) &\equiv \int_{x^2}^{\infty} dt t^{n/2} e^{-t} \\ &= \Gamma\left(\frac{n}{2} + 1\right) - \gamma\left(\frac{n}{2} + 1, x^2\right).\end{aligned}\quad (\text{B1})$$

Integral and differential properties of incomplete  $\gamma$  function

$$\begin{aligned}\int_0^r \frac{dx}{x^{n+1}} \gamma\left(\frac{n}{2} + 1, x^2\right) &= -\frac{1}{2} \frac{\gamma\left(\frac{n}{2}, r^2\right)}{r^n}, \\ \oint_{\text{sphere}} d^n x \rho_\theta(x^2) &= M \frac{\gamma\left(\frac{n}{2}, R^2\right)}{\Gamma\left(\frac{n}{2}\right)}, \\ \gamma'\left(\frac{n}{2}, x^2\right) &= 2x^{n-1} e^{-x^2}, \\ \gamma\left(\frac{n}{2} + 1, \frac{r^2}{4\theta}\right) &= \frac{n}{2} \gamma\left(\frac{n}{2}, \frac{r^2}{4\theta}\right) - \left(\frac{r}{2\sqrt{\theta}}\right)^n e^{-r^2/4\theta}.\end{aligned}\quad (\text{B2})$$

Long and short distance behavior of incomplete  $\gamma$  functions

$$\begin{aligned}\gamma\left(\frac{n}{2}, x^2\right) \Big|_{x \gg 1} &= \frac{2}{n} x^n e^{-x^2} \left[ 1 - \frac{2}{n+2} x^2 \right. \\ &\quad \left. + \frac{2}{n+2} \frac{2}{n+4} x^4 + \dots \right], \\ \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \Big|_{(r^2/4\theta) \ll 1} &\approx \frac{r^3}{12\sqrt{\theta^3}} \left( 1 - \frac{7}{20} \frac{r^2}{\theta} \right), \\ \Gamma\left(\frac{n}{2}, x^2\right) \Big|_{x \gg 1} &= x^{n-2} e^{-x^2} \left[ 1 + \left(\frac{n}{2} - 1\right) \frac{1}{x^2} \right. \\ &\quad \left. + \left(\frac{n}{2} - 1\right) \left(\frac{n}{2} - 2\right) \frac{1}{x^4} + \dots \right], \\ \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \Big|_{(r^2/4\theta) \gg 1} &= \frac{\sqrt{\pi}}{2} - \Gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \Big|_{(r^2/4\theta) \gg 1} \\ &\approx \frac{\sqrt{\pi}}{2} + \frac{1}{2} \frac{r}{\sqrt{\theta}} e^{-(r^2/4\theta)}.\end{aligned}\quad (\text{B3})$$

Long and short distance behavior of the metric

$$\begin{aligned}f &= \left( 1 - \frac{2M(r)}{r} \right) \\ &= 1 - \frac{2M}{r} - \frac{4m}{\sqrt{\pi}r} \Gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right), \\ f' &= \frac{2M(r)}{r^2} - \frac{2M'(r)}{r} \\ &= \frac{4M}{\sqrt{\pi}r^2} \left[ \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) - \frac{r^3}{4\theta^{3/2}} e^{-r^2/4\theta} \right], \\ f'' &= -\frac{4M(r)}{r^3} + \frac{4M'(r)}{r^2} - \frac{2M''(r)}{r} \\ &= -\frac{8M}{\sqrt{\pi}r^3} \left[ \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) - \frac{r^5}{16\theta^{5/2}} e^{-r^2/4\theta} \right], \\ M(r) &= \frac{2M}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right), \\ M'(r) &= 4\pi r^2 \rho_\theta(r) = \frac{Mr^2}{2\sqrt{\pi}\theta^{3/2}} e^{-r^2/4\theta}, \\ M''(r) &= \frac{Mr}{\sqrt{\pi}\theta^{3/2}} e^{-r^2/4\theta} \left( 1 - \frac{r^2}{4\theta} \right), \\ f_{r^2/4\theta \gg 1} &\approx 1 - \frac{2M}{r} + \frac{2M}{\sqrt{\pi}\theta} e^{-r^2/4\theta} = f_S + \frac{2M}{\sqrt{\pi}\theta} e^{-r^2/4\theta}, \\ f_{r^2/4\theta \ll 1} &\approx 1 - \frac{M}{3\sqrt{\pi}\theta^{3/2}} r^2.\end{aligned}\quad (\text{B4})$$

$$f_{r^2/4\theta \ll 1} \approx 1 - \frac{M}{3\sqrt{\pi}\theta^{3/2}} r^2. \quad (\text{B5})$$

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