Attractor solutions for general hessence dark energy

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As a candidate for the dark energy, the hessence model has been recently introduced. We discuss the critical points of this model in an almost general case, that is for arbitrary hessence potential and almost arbitrary hessence-background matter interaction. It is shown that, in all models, there always exist some stable late-time attractors. It is shown that our general results coincide with those solutions obtained earlier for special cases, but some of them are new. These new solutions have two unique characteristics. First, the hessence field has finite value in these solutions and, second, their stabilities depend on the second derivative of the hessence potential.

I. INTRODUCTION

In recent years, astronomical observations from type Ia supernova [1], WMAP data [2], and large scale structure surveys [3] have shown that the expansion of the universe is accelerated. Although there is no clear understanding of the mechanism leading to this acceleration, it is believed that about 70% of the total energy density of universe consists of this unknown energy, i.e. dark energy, which leads to this expansion. The simplest explanation of dark energy is a cosmological constant Λ of order $(10^{-3} \text{ eV})^4$. Unfortunately it is about 120 orders smaller than the naive expectations, giving rise to the idea of a dynamical nature of this energy. The possible dynamical explanations have been introduced in different frameworks, such as quintessence [4], phantom [5], k-essence [6], tachyons [7], etc.

In studying the dark energy, the equation of state parameter $w_{\text{de}} = p_{\text{de}}/\rho_{\text{de}}$ plays an important role, where p_{de} and ρ_{de} are the pressure and energy density of the dark energy, respectively. This parameter is always equal to constant -1 in cosmological constant model, but it can be a dynamical variable in the above mentioned dynamical models. This is an important point since the present data seem to slightly favor an evolving dark energy with w_{de} being below -1 around the present epoch [8], from $w_{\text{de}} > -1$ in the near past [9].

To be definite, we consider the following action:

$$S = \int d^4x \sqrt{-g} \left(-\frac{\mathcal{R}}{16\pi G} + \mathcal{L}_{DE} + \mathcal{L}_m \right), \quad (1)$$

where g is the determinant of the metric $g_{\mu\nu}$, $\mathcal{R}$ is the Ricci scalar, and $\mathcal{L}_{DE}$ and $\mathcal{L}_m$ are the Lagrangian densities of the dark energy and matter, respectively. In the case of quintessence, the Lagrangian density is

$$\mathcal{L}_{\text{quintessence}} = \tfrac{1}{2}(\partial_\mu \phi)^2 - V(\phi), \quad (2)$$

where ϕ is a real scalar field. In a spatially flat Friedmann-Robertson-Walker (FRW) universe with homogeneous ϕ,

w is

$$w_{\text{quintessence}} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}, \quad (3)$$

which results in $-1 \leq w_{\text{quintessence}}$. The Lagrangian density of the phantom scalar field is

$$\mathcal{L}_{\text{phantom}} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - V(\phi), \quad (4)$$

from which

$$w_{\text{phantom}} = \frac{-\dot{\phi}^2/2 - V(\phi)}{-\dot{\phi}^2/2 + V(\phi)}, \quad (5)$$

where for $\rho \geq 0$, which comes from $H^2 = (8\pi G/3)\rho$, it results $w_{\text{phantom}} \leq -1$. So we cannot cross the phantom divide line $w = -1$ in quintessence or the phantom model alone. A possible way to overcome this problem is considering two real fields, where one behaves as quintessence and other one as phantom field. The resulting model, called the quintom model, has the following Lagrangian [9,10]:

$$\mathcal{L}_{\text{quintom}} = \tfrac{1}{2}(\partial_\mu \phi_1)^2 - \tfrac{1}{2}(\partial_\mu \phi_2)^2 - V(\phi_1, \phi_2), \quad (6)$$

with

$$w_{\text{quintom}} = \frac{\dot{\phi}_1^2/2 - \dot{\phi}_2^2/2 - V(\phi_1, \phi_2)}{\dot{\phi}_1^2/2 - \dot{\phi}_2^2/2 + V(\phi_1, \phi_2)}. \quad (7)$$

Now it is obvious that $w_{\text{quintom}} \geq -1$ when $\dot{\phi}_1^2 \geq \dot{\phi}_2^2$ and $w_{\text{quintom}} < -1$ when $\dot{\phi}_1^2 < \dot{\phi}_2^2$. So crossing the phantom divide line is, in principle, possible in the quintom model. See, for example, [10,11].

Instead of introducing two independent real scalar fields to describe a quintom model, it is also natural to consider a single complex scalar field. The resulting spintessence model of dark energy [12–16] has the following Lagrangian density:

$$\mathcal{L}_{\text{spintessence}} = \tfrac{1}{2}(\partial^\mu \Phi^*)(\partial_\mu \Phi) - V(|\Phi|). \quad (8)$$

Using $\Phi = \phi_1 + i\phi_2$, the kinetic term of Eq. (8) reduces to the kinetic terms of Eq. (6). Also, the above Lagrangian

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is invariant under $\Phi \rightarrow e^{i\alpha}\Phi$ which leads to a conserved charge. Unfortunately, this model suffers from the problem of Q-ball formation [12,14]. Q-ball is a kind of nontopological soliton which, except in some special cases of spintessence with unnatural potentials, grows exponentially and depends on the potential, can be either stable at the late time to be a dark matter, or decay into other particles. Therefore the spintessence model cannot be a viable candidate for the dark energy.

To avoid the difficulty of Q-ball formation and also to introduce another possibility for mysterious dark energy problem, a noncanonical complex scalar field, called hessence, has been recently introduced in [17]. In the hessence model, the phantomlike role is played by the so-called internal motion $\dot{\theta}$, where θ is the internal degree of freedom of hessence. There is a conserved charge Q in this model which makes the physics of hessence more interesting, and the transition from $w_h > -1$ to $w_h < -1$ or vice versa is also possible. Another interesting feature of the hessence model is that it is free of big rip [18]. If $w < -1$ in an expanding FRW universe, then the positive energy density of a phantom matter generally becomes infinite in finite time, overcoming all other forms of matter and hence leads to the late-time singularity called the "big rip" [19].

By considering two specific hessence potentials, i.e. the exponential and the (inverse) power law, and four different interaction forms between hessence and background perfect fluid, the late-time attractors of the hessence model have been studied in [18]. In each case, different scaling and hessence-dominated solutions have been obtained and their stability properties have been studied.

In this paper we are going to study the late-time attractors of the almost general hessence model, with arbitrary hessence potential and almost arbitrary hessence-background matter interaction term. By almost arbitrary, we mean that the hessence potential and hessence-background matter interaction terms are arbitrary functions of dimensionless variables defined in (22). Specifically, we mean Eqs. (33) and (34). We show that there always exist some stable solutions, in scaling or hessence-dominated form, which some of them have not been appeared in special cases studied in [18]. These new solutions have two interesting unique characteristics which are absent in other solutions. First, their stability depends on the second derivative of the hessence potential and, second, the hessence field has finite value in these solutions. The significance of the second derivative of the potential in the late-time behaviors has been also revealed for the quintessence model in [20], in which some conditions have been imposed on the first and second derivatives of the potential.

The scheme of the paper is as follows. In Sec. II, we briefly introduce the main points of the hessence model and the system of equations which determine the critical points in terms of dimensionless variables. In Sec. III, we consider the general hessence potential, but assuming no

hessence-background matter interaction. It is shown that there exist five general solutions for critical points, three of which are stable under specific conditions. Finally, in Sec. IV, the hessence potential and hessence-background matter interaction are considered arbitrary and it is shown that there are, in general, six classes of solutions for critical points. The stability of solutions is discussed in special cases. It is shown that all the solutions of [18] can be obtained from our general results.

We use the units $\hbar = c = 1$, $\kappa^2 = 8\pi G$, and adopt the metric convention as $(+, -, -, -)$ throughout the paper.

II. HESSENCE MODEL

Following [17], the hessence field introduced by a non-canonical complex scalar field

$$\Phi = \phi_1 + i\phi_2, \tag{9}$$

with Lagrangian density

$$\begin{aligned}\mathcal{L}_h &= \tfrac{1}{4}[(\partial_\mu \Phi)^2 + (\partial_\mu \Phi^*)^2] - U(\Phi^2 + \Phi^{*2}) \\ &= \tfrac{1}{2}[(\partial_\mu \phi)^2 - \phi^2(\partial_\mu \theta)^2] - V(\phi),\end{aligned} \tag{10}$$

where the new fields (ϕ, θ) are defined through

$$\phi_1 = \phi \cosh\theta, \qquad \phi_2 = \phi \sinh\theta. \tag{11}$$

In a spatially flat FRW universe with scale factor $a(t)$, the equations of motion for ϕ and θ, when they are considered homogeneous, are

$$\ddot{\phi} + 3H\dot{\phi} + \phi\dot{\theta}^2 + V_{,\phi} = 0, \tag{12}$$

$$\phi^2\ddot{\theta} + (2\phi\dot{\phi} + 3H\phi^2)\dot{\theta} = 0, \tag{13}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and overdot and subscript "$,_\phi$" denote the derivatives with respect to cosmic time t and ϕ, respectively. Equation (13) implies

$$Q = a^3\phi^2\dot{\theta} = \text{const.,} \tag{14}$$

where Q is the total conserved charge due to the symmetry of Lagrangian (10) under the transformation $\phi \rightarrow \phi$ and $\theta \rightarrow \theta - i\alpha$. Substituting Eq. (14) into (12), one has

$$\ddot{\phi} + 3H\dot{\phi} + \frac{Q^2}{a^6\phi^3} + V_{,\phi} = 0. \tag{15}$$

The pressure and energy density of hessence are

$$\begin{aligned}p_h &= \frac{1}{2}\dot{\phi}^2 - \frac{Q^2}{2a^6\phi^2} - V(\phi), \\ \rho_h &= \frac{1}{2}\dot{\phi}^2 - \frac{Q^2}{2a^6\phi^2} + V(\phi).\end{aligned} \tag{16}$$

The Friedmann equation and the Raychaudhuri equation are given by, respectively,

$$H^2 = \frac{\kappa^2}{3}(\rho_h + \rho_m), \tag{17}$$

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_h + \rho_m + p_h + p_m), \qquad (18)$$

where p_m and ρ_m are the pressure and energy density of background matter, respectively. The background matter is described by a perfect fluid with barotropic equation of state

$$p_m = w_m \rho_m \equiv (\gamma - 1)\rho_m, \qquad (19)$$

where $0 < \gamma < 2$. In particular, $\gamma = 1$ and $4/3$ correspond to dust matter and radiation, respectively.

To introduce the interaction between hessence and background matter, it is assumed that it can be described by an interaction term C in the energy balance [18,21]

$$\dot{\rho}_h + 3H(\rho_h + p_h) = -C, \qquad (20)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = C, \qquad (21)$$

which preserves the total energy conservation equation $\dot{\rho}_{tot} + 3H(\rho_{tot} + p_{tot}) = 0$. $C = 0$ corresponds to no interaction between hessence and background matter and when $C \neq 0$, a new term due to C will appear in the right-hand side of Eq. (15).

Following [22] and many other papers, if we introduce the following dimensionless variables

$$x \equiv \frac{\kappa \dot{\phi}}{\sqrt{6}H}, \qquad y \equiv \frac{\kappa \sqrt{V}}{\sqrt{3}H}, \qquad z \equiv \frac{\kappa \sqrt{\rho_m}}{\sqrt{3}H},$$

$$u \equiv \frac{\sqrt{6}}{\kappa \phi}, \qquad v \equiv \frac{\kappa}{\sqrt{6}H}\frac{Q}{a^3 \phi}, \qquad (22)$$

then using Eqs. (16)–(18), (20), and (21), the evolution equations of these variables become

$$x' = 3x\left(x^2 - v^2 + \frac{\gamma}{2}z^2 - 1\right) - uv^2 - \sqrt{\frac{3}{2}}y^2 f - C_1, \qquad (23)$$

$$y' = 3y\left(x^2 - v^2 + \frac{\gamma}{2}z^2\right) + \sqrt{\frac{3}{2}}xyf, \qquad (24)$$

$$z' = 3z\left(x^2 - v^2 + \frac{\gamma}{2}z^2 - \frac{\gamma}{2}\right) + C_2, \qquad (25)$$

$$u' = -xu^2, \qquad (26)$$

$$v' = 3v\left(x^2 - v^2 + \frac{\gamma}{2}z^2 - 1\right) - xuv. \qquad (27)$$

Prime denotes the derivative with respect to the e-folding time $\mathcal{N} \equiv \ln a$, and

$$f \equiv \frac{V_{,\phi}}{\kappa V}, \qquad (28)$$

$$C_1 \equiv \frac{\kappa C}{\sqrt{6}H^2 \dot{\phi}}, \qquad C_2 \equiv \frac{\kappa C}{2\sqrt{3}H^2 \sqrt{\rho_m}} = \frac{x}{z}C_1. \qquad (29)$$

The Friedmann equation (17) becomes

$$x^2 + y^2 + z^2 - v^2 = 1, \qquad (30)$$

and the fractional energy densities are

$$\Omega_h = \frac{\rho_h}{\rho_c} = x^2 + y^2 - v^2, \qquad \Omega_m = \frac{\rho_m}{\rho_c} = z^2, \qquad (31)$$

where $\rho_c = \frac{3H^2}{\kappa}$ is the critical energy density. The equation of states of hessence and the whole system are

$$w_h = \frac{p_h}{\rho_h} = \frac{x^2 - v^2 - y^2}{x^2 - v^2 + y^2},$$

$$w_{eff} = \frac{p_h + p_m}{\rho_h + \rho_m} = x^2 - v^2 - y^2 + (\gamma - 1)z^2. \qquad (32)$$

The critical points $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v})$ are obtained by imposing the conditions $\bar{x}' = \bar{y}' = \bar{z}' = \bar{u}' = \bar{v}' = 0$.

Note that we take

$$f = f(u), \qquad (33)$$

and

$$C_1 = C_1(x, z, u), \qquad C_2 = C_2(x, z, u). \qquad (34)$$

It is because $V = V(\phi)$, so f is assumed to be a function of only one variable u. Also because of Eq. (30), only four of the variables (22) are independent, which we can take them x, y, z, and u. But the dependence of hessence-background matter interaction to the potential V (or variable y) is meaningless, so Cs are taken to be arbitrary functions of variables x, z, and u. Really, Eq. (33) does not constrain the potential V, but Eq. (34) restricts the possible interaction term C. In this way the Eqs. (23)–(27) become autonomous and we need not consider any further variables. For example, f is not an extra variable since $f' = 0$ leads to, for an arbitrary potential, $u' = 0$.

III. ATTRACTORS IN $C = 0$ CASE

To obtain the attractors for the arbitrary hessence potential and when there is no hessence-background matter interaction, we must solve Eq. (30) and the set of Eqs. (23)–(27), when setting zero, in $C_1 = C_2 = 0$. Equation (26) results in $\bar{u} = 0$ or $\bar{x} = 0$ and Eq. (25) results in $\bar{z} = 0$ or $\bar{x}^2 - \bar{v}^2 + (\gamma/2)\bar{z}^2 - \gamma/2 = 0$. So we have four possibilities: I = $(\bar{u} = 0, \bar{z} = 0)$, II = $(\bar{u} = 0, \bar{x}^2 - \bar{v}^2 + (\gamma/2)\bar{z}^2 - \gamma/2 = 0)$, III = $(\bar{x} = 0, \bar{z} = 0)$, and IV = $(\bar{x} = 0, \bar{x}^2 - \bar{v}^2 + (\gamma/2)\bar{z}^2 - \gamma/2 = 0)$. In the type I solution, Eq. (27), using (30), reduces to $vy^2 = 0$, so it divides to I.1 = $(\bar{u} = 0, \bar{z} = 0, \bar{v} = 0)$ and I.2 = $(\bar{u} = 0, \bar{z} = 0, \bar{y} = 0)$ solutions. The remaining variables can be easily found. The final results are represented in Table I.

The solutions with $\Omega_h = 1$, i.e. solutions I.1, I.2, and III, are hessence dominated, solution IV is background matter

TABLE I. Critical points for arbitrary hessence potential when there is no hessence-background matter interaction.

Label	Critical point $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v})$	Ω_h	Ω_m	w_h	w_{eff}
I.1	$-\frac{\bar{f}}{\sqrt{6}}, [1 - \frac{\bar{f}^2}{6}]^{1/2}, 0, 0, 0$	1	0	$-1 + \frac{\bar{f}^2}{3}$	$-1 + \frac{\bar{f}^2}{3}$
I.2	$\bar{x}^2 \geq 1, 0, 0, 0, \pm\sqrt{\bar{x}^2 - 1}$	1	0	1	1
II	$-[\frac{3}{2}]^{1/2}\frac{\gamma}{\bar{f}}, [\frac{3\gamma}{\bar{f}^2}(1 - \frac{\gamma}{2})]^{1/2}, [1 - \frac{3\gamma}{\bar{f}^2}]^{1/2}, 0, 0$	$\frac{3\gamma}{\bar{f}^2}$	$1 - \frac{3\gamma}{\bar{f}^2}$	$-1 + \gamma$	$-1 + \gamma$
III	$0, 1, 0, \bar{f} = 0, 0$	1	0	-1	-1
IV	$0, 0, 1, \text{any}, 0$	0	1	any	$-1 + \gamma$

dominated and solution II is the scaling solution. In solution III, $\bar{u}$ must be found by solving $\bar{f} \equiv f(\bar{u}) = 0$. Note that in all cases $w_{\text{eff}} > -1$. It is also interesting that for potentials where $f^2(\bar{u} = 0) = 3\gamma$, the solutions I.1 and II become degenerate.

In examples considered in [18], the potentials are

$$V_1 = V_0 e^{-\lambda\kappa\phi}, \tag{35}$$

$$V_2 = V_0(\kappa\phi)^n, \tag{36}$$

or, using (28),

$$f_1 = -\lambda, \tag{37}$$

$$f_2 = \frac{n}{\kappa\phi} = \frac{nu}{\sqrt{6}}. \tag{38}$$

It can be easily checked that our solutions (I.1, I.2, II, and IV) and (I.1, I.2, and IV) reduce to those obtained in [18] for V_1 and V_2, respectively. Solution II does not exist for potential V_2 since $f_2(\bar{u} = 0) = 0$. Solution III is a new solution which has not appeared in [18]. This is because the equation $\bar{f} = 0$ results in $\lambda = 0$ for V_1, which is not acceptable, and results in $\bar{u} = 0$ for V_2, which reduces solution III to I.1.

To study the stability of the critical points I.1–IV, we must consider a small perturbation about the critical point $(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v})$: $x \rightarrow \bar{x} + \delta x$, $y \rightarrow \bar{y} + \delta y$, $z \rightarrow \bar{z} + \delta z$, $u \rightarrow \bar{u} + \delta u$, and $v \rightarrow \bar{v} + \delta v$, in Eqs. (23)–(26) with $C_1 = C_2 = 0$, which due to Friedmann constraint (30), only four of them are independent. In this way, one can find a 4×4 matrix M defined through

$$\frac{d}{d\mathcal{N}}\begin{pmatrix} \delta q_1 \\ \delta q_2 \\ \delta q_3 \\ \delta q_4 \end{pmatrix} = M \begin{pmatrix} \delta q_1 \\ \delta q_2 \\ \delta q_3 \\ \delta q_4 \end{pmatrix}, \tag{39}$$

where $(q_1, \ldots, q_4)$ are four chosen independent variables. The critical solutions are stable if the real part of all the eigenvalues of matrix M are negative. The eigenvalues of matrix M for our solutions are as follows:

$$\text{I.1: } (-3\bar{y}^2, -6\bar{y}^2, \tfrac{1}{2}(\bar{f}^2 - 3\gamma), 0), \tag{40}$$

$$\text{I.2: } \left(0, 0, \frac{3}{2}(2 - \gamma), 3 + \sqrt{\frac{3}{2}}\bar{x}\bar{f}\right), \tag{41}$$

$$\text{III: } \left(-6, -\frac{3\gamma}{2}, -\frac{3}{2} + \frac{\sqrt{9 + 2\sqrt{6}\bar{u}^2\bar{f}'}}{2}, -\frac{3}{2} - \frac{\sqrt{9 + 2\sqrt{6}\bar{u}^2\bar{f}'}}{2}\right), \tag{42}$$

$$\text{IV: } \left(0, \frac{3\gamma}{2}, 3(\gamma - 2), \frac{3}{2}(\gamma - 2)\right), \tag{43}$$

in which $(q_1, \ldots, q_4) = (x, y, z, u)$ and $\bar{f}' = (df/du)_{\bar{u}}$. It is clear that the I.1 solution is *stable* if

$$\bar{f}^2 \leq 3\gamma, \tag{44}$$

the I.2 solution is *unstable* since $\gamma < 2$, the III solution is *stable* if

$$\left(\frac{df}{du}\right)_{\bar{u}} \leq 0, \tag{45}$$

and IV is an *unstable* solution since $\gamma > 0$.

For solution II, it is easier to use $(q_1, \ldots, q_4) = (x, z, u, v)$. Then it can be easily found that $\lambda_1 = 0$, $\lambda_2 = \frac{3}{2}(\gamma - 2)$, and λ_3 and λ_4 are roots of the equation $\lambda^2 + b\lambda + c = 0$ with

$$b = \frac{3}{2}(\gamma - 2), \qquad c = \frac{9\gamma}{2\bar{f}^2}(2 - \gamma)(\bar{f}^2 - 3\gamma). \tag{46}$$

λ_3 and λ_4 are nonpositive if $b \geq 0$ and $c \geq 0$. As $\gamma < 2$, solution II is *stable* if

$$\bar{f}^2 \geq 3\gamma. \tag{47}$$

So for any potential V, there *always* exists at least one stable attractor, a hessence-dominated attractor (I.1) if $\bar{f}^2 \leq 3\gamma$ or a scaling attractor (II) if $\bar{f}^2 \geq 3\gamma$.

It is interesting to note that solution III has two unique properties. First, it is the only stable attractor which has the nonvanishing $\bar{u}$ value, i.e. finite value of hessence field $\bar{\phi}$. Second, it is the only attractor whose stability depends on the derivative of f [Eq. (45)]. In other words, the stable attractors I.1 and II cannot distinguish between different potentials with the same $\bar{f}$ value, but the attractor III does.

As examples of potentials which have solution III as a stable attractor, we may consider $V_3 = V_0 \sin(\kappa\phi/\sqrt{6})$ and $V_4 = V_0 \cos(\kappa\phi/\sqrt{6})$. For V_3, we have $f_3 = -(1/\kappa u^2) \times \cot(1/u)$, which results in $\bar{u} = [(2n + 1)\pi/2]^{-1}$, with $n = 0, 1, 2, \ldots$, as the solution of the $\bar{f}_3 = 0$ equation. It really has infinite number of attractors. Then $(df/du)_{\bar{u}} = -(1/\kappa)[(2n + 1)\pi/2]^4 < 0$, which shows that the attractors are stable. The same is true for the V_4 potential.

IV. ATTRACTORS IN THE PRESENCE OF HESSENCE-BACKGROUND MATTER INTERACTION

In this case, we must solve Eq. (30) and the set of Eqs. (23)–(27), when setting zero, for the arbitrary C function. Equation (26) results in $\bar{u} = 0$ or $\bar{x} = 0$. In each case, we consider eight cases in which each of the variables $\bar{y}$, $\bar{z}$, and $\bar{v}$ has two possibilities, zero and not zero, and then check the consistency of the equations. The final results are as follows:

$$\text{solution 1:} \left\{ \bar{x} = -\frac{\bar{f}}{\sqrt{6}}, \bar{y} = \sqrt{1 - \frac{\bar{f}^2}{6}}, \bar{z} = \bar{u} = \bar{v} \right.$$
$$\left. = 0, \bar{C}_1 = \bar{C}_2 = 0 \right\}. \tag{48}$$

Note that the equations $\bar{C}_1 = \bar{C}_2 = 0$ imply that the functional form of C_1 and C_2 must be such that they are identically equal to zero at this critical point, otherwise this solution does not exist:

$$\text{solution 2:} \{ \bar{x}^2 \geq 1, \bar{y} = \bar{z} = \bar{u} = 0, \bar{v} = \pm\sqrt{\bar{x}^2 - 1}, \bar{C}_1$$
$$= \bar{C}_2 = 0 \}. \tag{49}$$

In this case, the equations $\bar{C}_1 = \bar{C}_2 = 0$ can generally determine the allowed value of $\bar{x}$. If $\bar{C}_1$ and $\bar{C}_2$ are identically equal to zero at $\bar{y} = \bar{z} = \bar{u} = 0$, as they are in the $C = 0$ case, then $\bar{x}$ can choose any arbitrary value:

$$\text{solution 3:} \{ \bar{x} = \bar{y} = 0, \bar{z} = 1, \bar{u}, \bar{v} = 0, \bar{C}_1 = \bar{C}_2 = 0 \}. \tag{50}$$

The value of $\bar{u}$ is generally determined by solving $\bar{C}_1 = \bar{C}_2 = 0$:

$$\text{solution 4:} \left\{ \bar{x} = 0, \bar{y} = 1, \bar{z} = 0, \bar{u}, \bar{v} = 0, \bar{C}_1 \right.$$
$$\left. = -\sqrt{\frac{3}{2}}\bar{f}, \bar{C}_2 = 0 \right\}, \tag{51}$$

where the last two equations can generally determine $\bar{u}$:

$$\text{solution 5:} \left\{ \bar{y} = \bar{u} = \bar{v} = 0, \bar{C}_1 \right.$$
$$= 3\bar{x}\left(\bar{x}^2 + \frac{\gamma}{2}\bar{z}^2 - 1 \right), \bar{C}_2$$
$$\left. = -3\bar{z}\left(\bar{x}^2 + \frac{\gamma}{2}\bar{z}^2 - \frac{\gamma}{2} \right), \bar{x}^2 + \bar{z}^2 = 1 \right\}. \tag{52}$$

$\bar{x}$ and $\bar{z}$ are found by solving the above equations. The last solution is

$$\text{solution 6:} \left\{ \bar{y} = \sqrt{1 - \bar{x}^2 - \bar{z}^2}, \bar{u} = \bar{v} = 0, \right.$$
$$\bar{C}_1 = 3\bar{x}\left(\bar{x}^2 + \frac{\gamma}{2}\bar{z}^2 - 1 \right) - \sqrt{\frac{3}{2}}\bar{y}^2\bar{f},$$
$$\bar{C}_2 = -3\bar{z}\left(\bar{x}^2 + \frac{\gamma}{2}\bar{z}^2 - \frac{\gamma}{2} \right),$$
$$\left. 3\left(\bar{x}^2 + \frac{\gamma}{2}\bar{z}^2 \right) + \sqrt{\frac{3}{2}}\bar{x}\bar{f} = 0 \right\}. \tag{53}$$

At $C = 0$, solutions 1, 2, 3, 4, and 6 reduce to I.1, I.2, IV, III, and II of Table I, respectively. Solutions 1, 2, and 4 are hessence dominated, solution 3 is background matter dominated and 5 and 6 are generally scaling solutions. Among these solutions, there are only two solutions 3 and 4 in which $\bar{u}$ can principally be different from zero, which have not been appeared in examples discussed in [18].

In [18], besides the noninteracting $C = 0$ case, three following interactions have been considered:

$$C_1^{(II)} = \sqrt{\frac{3}{2}}\alpha z^2,$$

$$C_1^{(III)} = \frac{3}{2}\frac{\beta}{x}, \qquad C_1^{(IV)} = \frac{3}{2}\eta\frac{z^2}{x}, \tag{54}$$

in which α, β, and η are some constants. In both of solutions 3 and 4, $\bar{x}$ is zero, so $C_1^{(III)}$ diverges and $C_1^{(IV)}$ is not generally well defined, and therefore these critical points do not exist in these cases. So we only consider the $C_1^{(II)}$ case. For the $V = V_1$ potential, with $f_1 = -\lambda$, solution 3 does not exist since $\bar{C}_1 = 0$ leads to $\alpha = 0$ which is not acceptable, and solution 4 does also not exist as $\bar{C}_1 = -[\frac{3}{2}]^{1/2}\bar{f}$ and results in $\lambda = 0$, which again is not acceptable. For the $V = V_2$ potential, with $f_2 = nu/\sqrt{6}$, solution 3 leads to $\alpha = 0$ which is not acceptable, and solution 4 results in $\bar{u} = 0$, which does not lead to a $\bar{u} \neq 0$ solution.

It may be useful to reproduce all the critical points of at least one of the cases studied in [18] in more detail. We consider, as an example, $V = V_1$ and $C = C_1^{(II)}$. So

$$f = -\lambda, \qquad C_1 = \sqrt{\frac{3}{2}}\alpha z^2, \qquad C_2 = \sqrt{\frac{3}{2}}\alpha xz. \quad (55)$$

Solutions 1–6 result in

solution 1: $\left\{\bar{x} = \frac{\lambda}{\sqrt{6}}, \bar{y} = \sqrt{1 - \frac{\lambda^2}{6}}, \bar{z} = \bar{u} = \bar{v} = 0\right\}$,

$$(56)$$

solution 2: $\{\bar{x}^2 \geq 1, \bar{y} = \bar{z} = \bar{u} = 0, \bar{v} = \pm\sqrt{\bar{x}^2 - 1}\}$,

$$(57)$$

solution 3: $\bar{C}_1 = 0 \rightarrow \alpha = 0$ (is not acceptable), (58)

solution 4: $\bar{C}_1 = -\sqrt{\frac{3}{2}}\bar{f} \rightarrow \lambda = 0$ (is not acceptable),

$$(59)$$

solution 5: $\left\{\bar{x} = \sqrt{\frac{2}{3}}\frac{\alpha}{\gamma - 2}, \bar{y} = 0, \bar{z}\right.$

$$\left. = \sqrt{1 - \frac{2\alpha^2}{3(\gamma - 2)^2}}, \bar{u} = \bar{v} = 0\right\}, \quad (60)$$

solution 6: $\left\{\bar{x} = \sqrt{\frac{3}{2}}\frac{\gamma}{\lambda + \alpha}, \bar{y}\right.$

$$= \sqrt{\frac{2\alpha^2 - 3(\gamma - 2)\gamma + 2\alpha\lambda}{2(\alpha + \lambda)^2}}, \bar{z}$$

$$\left. = \sqrt{\frac{\lambda(\lambda + \alpha) - 3\gamma}{(\lambda + \alpha)^2}}, \bar{u} = \bar{v} = 0\right\}, \quad (61)$$

which coincide with those in Table II of [18]. Note that in that table, four of the solutions (2p, 2m, 4, and 5) are not independent solutions and are special cases of the first solution.

The stability studies of these critical points depends on the precise value of the function C. But it may be interesting to study the conditions under which the derivative of f becomes important in the stability properties of the critical points. Consider the most general case $C_1 = C_1(x, z, u)$. It can be shown that the coefficient of $\bar{f}'$ term in the equation $\det(M - \hat{1}\lambda) = 0$, where $\hat{1}$ stands for the 4×4 unit matrix, is

$$\frac{\bar{y}^2\bar{u}^2}{\bar{z}^2}h(\bar{x}, \bar{y}, \bar{z}, \bar{u}), \quad (62)$$

which shows that the derivative of f survives only if

$$\bar{y} \neq 0 \quad \text{and} \quad \bar{u} \neq 0. \quad (63)$$

Therefore solution 4 is the only one in which we expect $\bar{f}'$ may be important. Focusing on this solution, the eigen-

value problem is still very complicated in the general case and it is better to consider the more restricted cases. As an example, we consider the potentials and C's in which $\bar{f} = \overline{(C_1/z)} = \overline{(\partial C_1/\partial x)} = 0$. Under these conditions, the equation $\det(M - \hat{1}\lambda) = 0$ leads to

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \quad (64)$$

where

$$a_1 = 9 + \frac{3}{2}\gamma, \qquad a_2 = 18 + \frac{27}{2}\gamma - \bar{u}^2\left(\bar{C}_{1,u} + \sqrt{\frac{3}{2}}\bar{f}'\right),$$

$$a_3 = -3\bar{u}^2\left(\bar{C}_{1,u} + \sqrt{\frac{3}{2}}\bar{f}'\right)\left(2 + \frac{\gamma}{2}\right) + 27\gamma, \quad (65)$$

$$a_4 = -9\gamma\bar{u}^2\left(\bar{C}_{1,u} + \sqrt{\frac{3}{2}}\bar{f}'\right),$$

where $\bar{C}_{1,u} = \overline{(\partial C_1/\partial u)}$. Now it is well known that $\sum_i \lambda_i = -a_1$, $\sum_{i \neq j} \lambda_i \lambda_j = a_2$, $\sum_{i \neq j \neq k} \lambda_i \lambda_j \lambda_k = -a_3$, and $\lambda_1\lambda_2\lambda_3\lambda_4 = a_4$, where $\lambda_1 \cdots \lambda_4$ are the roots of Eq. (64). So all the roots are nonpositive only if $a_i \geq 0$ ($i = 1, \ldots, 4$), provided if

$$\bar{f}' \leq -\sqrt{\frac{2}{3}}\bar{C}_{1,u}. \quad (66)$$

In $C = 0$, this condition of stability reduces to Eq. (45).

V. CONCLUSION

In this paper, we study the attractor solutions of the general hessence model by studying the 4-dimensional phase space of the theory. The hessence model is a non-canonical complex scalar theory which can be a candidate of dark energy with some interesting properties, among them is the possibility of crossing the $w = -1$ line. Comparing the Lagrangian of the hessence model with the quintom model shows that the θ (or equally charge Q) term plays the role of the phantom field. In $Q = 0$, the model reduced to the quintessence model with no $w = -1$ crossing.

We consider an arbitrary hessence potential $V(\phi)$ and almost arbitrary hessence-background matter interaction term C, and find several results. We show that, in $C = 0$, there is always at least one stable attractor which depends on the value $\bar{f} = f(\bar{u} = 0)$: For $\bar{f}^2 \leq 3\gamma$ the hessence-dominated attractor I.1 and for $\bar{f}^2 \geq 3\gamma$ the scaling attractor II of Table I. In all the attractor solutions we have $\bar{\phi} \rightarrow \infty$, except for attractor III where $\bar{\phi}$ is finite and also its stability depends on the derivative $(df/du)_{\bar{u}}$. This kind of attractor did not appear in the previously studied cases and can be seen in potentials like $\sin(\kappa\phi)$. $\bar{v}$ is zero in all the stable attractors of Table I, but for the arbitrary C-term, this is not the case.

For general C, we show that there generally exist six classes of attractor, which can all be stable in special cases (for example $V = V_1$ and $C_1 = C_1^{(\text{II})}$ of [18]). Among these

solutions, solution 2 is the only one with property $\bar{\nu} \neq 0$. Solutions 3 and 4 have a finite $\bar{\phi}$ value and solution 4 is the only one whose stability depends on $(df/du)_{\bar{u}}$, the features that cannot be seen in the previously studied potentials.

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