Large non-Gaussianity in multiple-field inflation

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(Received 22 December 2005; published 18 April 2006)

We investigate non-Gaussianity in general multiple-field inflation using the formalism we developed in earlier papers. We use a perturbative expansion of the nonlinear equations to calculate the three-point correlator of the curvature perturbation analytically. We derive a general expression that involves only a time integral over background and linear perturbation quantities. We work out this expression explicitly for the two-field slow-roll case, and find that non-Gaussianity can be orders of magnitude larger than in the single-field case. In particular, the bispectrum divided by the square of the power spectrum can easily be of $\mathcal{O}(1\text{--}10)$, depending on the model. Our result also shows the explicit momentum dependence of the bispectrum. This conclusion of large non-Gaussianity is confirmed in a semianalytic investigation of a simple quadratic two-field model.

DOI: 10.1103/PhysRevD.73.083522 PACS numbers: 98.80.Cq

I. INTRODUCTION

The assumption that inflationary fluctuations are Gaussian is a good starting point for the study of cosmological perturbations, but it is only true to linear order in perturbation theory. Since gravity is inherently nonlinear, and most inflation models have (self-)interacting potentials, nonlinearity must be present at some level in all inflation models. Hence the issue is not whether inflation is non-Gaussian, but how large the non-Gaussianity is. With increasingly precise CMB data becoming available in the near future from the WMAP and Planck satellites and other experiments, we might well hope to detect this non-Gaussianity. This would offer us another key observable to help constrain or confirm specific inflation models and the underlying high-energy theories from which they are derived. As a rough order of magnitude estimate, we note that non-Gaussianity will be detectable by Planck if the bispectrum (the Fourier transform of the three-point correlator) is of the order of the square of the power spectrum [1].

It follows that to compute the predicted amount of non-Gaussianity in specific inflation models we need to go beyond linear-order perturbation theory. In [2,3] we introduced a new formalism to deal with the nonlinearity during inflation. We will not again summarize the other work dealing with this subject, references for which can be found in [3] or a recent review [4]. Our formalism is distinguished by being based on a system of fully nonlinear equations for long wavelengths, while stochastic sources take into account the continuous influx of short-wavelength fluctuations into the long-wavelength system as the inflationary comoving horizon shrinks. The variables used incorporate both scalar metric and matter perturbations self-consistently and they are invariant under changes of time slicing in the long-wavelength limit.

The advantages of our method are threefold: (i) it is physically intuitive and relatively simple to use for quantitative analytic and semianalytic calculations; (ii) it is valid in a very general multiple-field inflation setting, which includes the possibility of a nontrivial field metric; and (iii) it is well-suited for direct numerical implementation. The first point was already demonstrated in [5], where we computed the non-Gaussianity in single-field slow-roll inflation, while the third point is the subject of a forthcoming paper [6]. The present paper is dedicated to the exploration of the second point, as well as the first.

In [5] we found, confirming what was known in the literature beforehand (see e.g. [7]), that non-Gaussianity in the single-field case is too small to be realistically observable, because it is suppressed by slow-roll factors (actually the scalar spectral index $n - 1$). However, there have been long-standing claims in the literature (see e.g. [8,9]) that specific multiple-field models can, in principle, create significant non-Gaussianity. Indeed, there has been growing recent interest in models which can produce large primordial non-Gaussianity [10]. A feature shared by most of these models though is that this non-Gaussianity involves some mechanism operating *after* inflation, usually (p)reheating or later domination of a curvaton field. In this paper we investigate for the first time general multiple-field inflation, not just specific models, presenting a method by which to accurately calculate the resulting three-point correlator *during* inflation. We find that it is possible to get significant primordial non-Gaussianity without invoking some post-inflationary mechanism even for the simplest two-field quadratic potential.

The key mechanism for the production of this large non-Gaussianity is the superhorizon influence of isocurvature perturbations on the adiabatic mode. The former feed into the latter when the background follows a curved trajectory in field space. Note that the example studied in Sec. V D illustrates that there is no need for the potential to be interacting. Our aim is to push forward towards a tractable non-Gaussian methodology for the new era of precision cosmology which confronts us.

This work is organized as follows. In Sec. II we give the equations from [3] that are used as the starting point for the present investigations. In Sec. III we then derive the general solution for the relevant quantities in multiple-field inflation, culminating in a general expression for the three-point correlator of the adiabatic component of the curvature perturbation, without any slow-roll approximations. This integral solution—Eq. (29)—is a very useful calculational tool because it gives the three-point correlator entirely in terms of background quantities and linear perturbation quantities at horizon crossing. In Sec. IV we make a leading-order slow-roll approximation to work out the various contributions in the general solution more explicitly. Finally in Sec. V we calculate the bispectrum in an analytic treatment of the two-field case with constant slow-roll parameters. We find that the result can be orders of magnitude larger than for single-field inflation. This result is confirmed with a semianalytic calculation of an explicit model with a quadratic potential in Sec. V D. Our method yields the full momentum dependence, not just an overall magnitude, and we find that there can be a difference of the order of a few between opposite extreme momentum limits. We conclude in Sec. VI. Parts of this paper, in particular Sec. V, are rather technical, so some readers might be interested in referring to [11] first, which contains a simplified derivation of only the dominant non-Gaussian contributions in multiple-field inflation, along with a summarized discussion.

II. MULTIPLE-FIELD SETUP

Since in this paper we are explicitly working out the general nonlinear multiple-field formalism of [3], we refer the reader to that paper for derivations and more details of the initial equations. Here we just briefly describe the context and give the relevant equations and definitions to be used as starting point for further calculations.

We start from a completely general inflation model, with an arbitrary number of scalar fields ϕ^A (where A labels the different fields) and a potential $V(\phi^A)$ with arbitrary interactions. We also allow for the possibility of a nontrivial field manifold with field metric G_{AB}. We will consider only scalar modes and make the long-wavelength approximation (i.e. consider only wavelengths larger than the Hubble radius $1/(aH)$, where second-order spatial gradients can be neglected compared to time derivatives).[1] The spacetime metric $g_{\mu\nu}$ and matter Lagrangean are given by

$$\begin{aligned} \mathrm{d}s^2 &= -N^2(t,\boldsymbol{x})\mathrm{d}t^2 + a^2(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}^2, \\ \mathcal{L}_m &= -\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi^A G_{AB}\partial_\nu\phi^B - V(\phi^A), \end{aligned} \tag{1}$$

with a the local scale factor and N the lapse function. The local expansion rate is defined as $H \equiv \dot{a}/(Na)$, where the dot denotes a derivative with respect to t. The proper field velocity is $\Pi^A \equiv \dot{\phi}^A/N$, with length Π. We also define local slow-roll parameters as

$$\tilde{\epsilon}(t,\boldsymbol{x}) \equiv \frac{\kappa^2\Pi^2}{2H^2}, \qquad \tilde{\eta}^A(t,\boldsymbol{x}) \equiv -\frac{3H\Pi^A + G^{AB}\partial_B V}{H\Pi},$$
$$\tilde{\xi}^A(t,\boldsymbol{x}) \equiv -\frac{\mathcal{D}_B\partial^A V}{H^2}\frac{\Pi^B}{\Pi} + 3\tilde{\epsilon}\frac{\Pi^A}{\Pi} - 3\tilde{\eta}^A, \tag{2}$$

where $\kappa^2 \equiv 8\pi G = 8\pi/m_{\mathrm{Pl}}^2$ and $\mathcal{D}_B$ is a covariant derivative with respect to the field ϕ^B. For the first part of the paper we will not make a slow-roll approximation, and consider these definitions as just a short-hand notation. When we do make this approximation, from Sec. IV $\tilde{\epsilon}$ and $\tilde{\eta}^A$ are first order in slow roll, while $\tilde{\xi}^A$ is second order. Finally, we choose the gauge where

$$t = \ln(aH) \quad \Leftrightarrow \quad NH = (1-\tilde{\epsilon})^{-1}. \tag{3}$$

In this gauge horizon exit of a mode, $k = aH$, occurs simultaneously for all spatial points and calculations are simpler.

We will make use of a preferred basis in field space, defined as follows. The first basis vector e_1^A is the direction of the field velocity. Next, e_2^A is defined as the direction of that part of the field acceleration that is perpendicular to e_1^A. One continues this orthogonalisation process with higher derivatives until a complete basis is found. Explicit expressions can be found in [3], here we only give $e_1^A \equiv \Pi^A/\Pi$. Now one can take components of vectors in this basis and we define, for example, for ζ_i^A [defined below in (6)] and $\tilde{\eta}^A$:

$$\zeta_i^m \equiv e_{mA}\zeta_i^A, \qquad \tilde{\eta}^\parallel \equiv e_1^A\tilde{\eta}_A, \qquad \tilde{\eta}^\perp \equiv e_2^A\tilde{\eta}_A. \tag{4}$$

Note that, unlike for the index A, there is no difference between upper and lower indices for the m. By construction there are no other components of $\tilde{\eta}^A$, so that one can write $\tilde{\eta}^\perp = |\tilde{\eta}^A - \tilde{\eta}^\parallel e_1^A|$. We also define

$$Z_{mn} \equiv \frac{1}{NH} e_{mA}\mathcal{D}_t e_n^A, \tag{5}$$

where $\mathcal{D}_t$ is the covariant time derivative containing the connection of the field manifold. Z_{mn} is antisymmetric and only nonzero just above and below the diagonal, and first order in slow roll. Its explicit form in terms of slow-roll parameters can be found in [13]; here we only need that $Z_{12} = -Z_{21} = -\tilde{\eta}^\perp$.

As discussed in [2,3] it is useful to work with the following combination of spatial gradients to describe the fully nonlinear inhomogeneities:

[1]Formally this corresponds with taking only the leading-order terms in the gradient expansion. We expect higher-order terms to be subdominant on long wavelengths during inflation, but this statement has only been rigorously verified at the linear level. A calculation to higher order in spatial gradients, or, even better, a full proof of convergence of the expansion, would be desirable. See [12] for more details on the validity of the gradient expansion beyond linear theory.

$$\zeta_i^A(t,\boldsymbol{x}) \equiv e_1^A(t,\boldsymbol{x})\partial_i \ln a(t,\boldsymbol{x}) - \frac{\kappa}{\sqrt{2\tilde{\epsilon}(t,\boldsymbol{x})}}\partial_i\phi^A(t,\boldsymbol{x}) \quad \Leftrightarrow$$

$$\zeta_i^m \equiv \delta_{m1}\partial_i \ln a - \frac{\kappa}{\sqrt{2\tilde{\epsilon}}} e_{mA}\partial_i\phi^A, \tag{6}$$

which is invariant under changes of time slicing, up to second-order spatial gradients [3,14]. Note that, when linearized, ζ_i^1 (the $m = 1$ component of ζ_i^m) is the spatial gradient of the well-known ζ from the literature, the curvature perturbation. In [3] we derived a fully nonlinear equation of motion for ζ_i^m without any slow-roll approximations:

$$\dot{\zeta}_i^m - \theta_i^m = \mathcal{S}_i^m,$$
$$\dot{\theta}_i^m + \left(\frac{(3-2\tilde{\epsilon}+2\tilde{\eta}^\parallel-3\tilde{\epsilon}^2-4\tilde{\epsilon}\tilde{\eta}^\parallel)\delta_{mn}}{(1-\tilde{\epsilon})^2} + \frac{2Z_{mn}}{1-\tilde{\epsilon}}\right)\theta_i^n + \frac{\Xi_{mn}}{(1-\tilde{\epsilon})^2}\zeta_i^n = \mathcal{J}_i^m \tag{7}$$

where θ_i^m is the velocity corresponding with ζ_i^m and

$$\begin{aligned}\Xi_{mn} \equiv{}& \frac{V_{mn}}{H^2} - \frac{2\tilde{\epsilon}}{\kappa^2}R_{m11n} + (1-\tilde{\epsilon})\dot{Z}_{mn} + Z_{mp}Z_{pn} \\ &+ (3-2\tilde{\epsilon}+2\tilde{\eta}^\parallel-\tilde{\epsilon}^2-2\tilde{\epsilon}\tilde{\eta}^\parallel)\frac{Z_{mn}}{1-\tilde{\epsilon}} \\ &+ (3\tilde{\epsilon}+3\tilde{\eta}^\parallel+2\tilde{\epsilon}^2+4\tilde{\epsilon}\tilde{\eta}^\parallel+(\tilde{\eta}^\perp)^2+\tilde{\xi}^\parallel)\delta_{mn} \\ &- 2\tilde{\epsilon}((3+\tilde{\epsilon}+2\tilde{\eta}^\parallel)\delta_{m1}\delta_{n1} + \tilde{\eta}^\perp(\delta_{m1}\delta_{n2}+\delta_{m2}\delta_{n1})),\end{aligned} \tag{8}$$

where $V_{mn} \equiv e_m^A(\mathcal{D}_B\partial_A V)e_n^B$ and $R_{m11n} \equiv e_m^A R_{ABCD} e_1^B e_1^C e_n^D$ with R^A_{BCD} the curvature tensor of the field manifold. Although for the first part of the paper we will not make a slow-roll approximation, we give here immediately the leading-order slow-roll approximation of (7), which we will be using in the second part, to show that things simplify considerably in that case:

$$\dot{\zeta}_i^m - \theta_i^m = \mathcal{S}_i^m, \qquad \dot{\theta}_i^m + 3\theta_i^m + \left(\frac{V_{mn}}{H^2} + 3Z_{mn} + 3(\tilde{\epsilon}+\tilde{\eta}^\parallel)\delta_{mn} - 6\tilde{\epsilon}\delta_{m1}\delta_{n1}\right)\zeta_i^n = \mathcal{J}_i^m. \tag{9}$$

The stochastic source terms $\mathcal{S}_i^m$ and $\mathcal{J}_i^m$ are given by

$$\mathcal{S}_i^m = -\frac{\kappa}{a\sqrt{2\tilde{\epsilon}}}\int\frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}}\dot{\mathcal{W}}(k)Q_{mn}^{\rm lin}(k)\alpha_n(\boldsymbol{k})ik_ie^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \text{c.c.},$$

$$\mathcal{J}_i^m = -\frac{\kappa}{a\sqrt{2\tilde{\epsilon}}}\int\frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}}\dot{\mathcal{W}}(k) \times\left[\dot{Q}_{mn}^{\rm lin}(k) - \frac{1+\tilde{\epsilon}+\tilde{\eta}^\parallel}{1-\tilde{\epsilon}}Q_{mn}^{\rm lin}(k)\right]\alpha_n(\boldsymbol{k})ik_ie^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \text{c.c.}, \tag{10}$$

where c.c. denotes the complex conjugate. The perturbation quantity $Q_{mn}^{\rm lin}$ is the solution from linear theory for the multiple-field generalization of the Sasaki-Mukhanov variable $Q \equiv -a\sqrt{2\tilde{\epsilon}}\zeta/\kappa$. It can be computed exactly numerically, or analytically within the slow-roll approximation [13]. The $\alpha_m(\boldsymbol{k})$ are a set of Gaussian complex random numbers satisfying

$$\langle\alpha_m(\boldsymbol{k})\alpha_n^*(\boldsymbol{k}')\rangle = \delta_{mn}\delta^3(\boldsymbol{k}-\boldsymbol{k}'), \qquad \langle\alpha_m(\boldsymbol{k})\alpha_n(\boldsymbol{k}')\rangle = 0. \tag{11}$$

The quantity $\mathcal{W}(k)$ is the Fourier transform of an appropriate smoothing window function which cuts off modes with wavelengths smaller than the Hubble radius; we choose it to be a Gaussian with smoothing length $R \equiv c/(aH) = ce^{-t}$, where $c \approx 3$–5:

$$\mathcal{W}(k) = e^{-k^2R^2/2} \Rightarrow \dot{\mathcal{W}}(k) = k^2R^2e^{-k^2R^2/2}. \tag{12}$$

Since ζ_i^m and θ_i^m are smoothed long-wavelength variables, the appropriate initial conditions are that they should be zero at early times when all the modes are subhorizon. Hence,

$$\lim_{t\to-\infty}\zeta_i^m = 0, \qquad \lim_{t\to-\infty}\theta_i^m = 0. \tag{13}$$

A key aspect of the system (7) or (9) is that it is fully nonlinear. All functions in the coefficients on the left-hand side of the equation, like $\tilde{\epsilon}(t,\boldsymbol{x})$, and in the sources on the right-hand side are inhomogeneous and depend on ζ_i^m and θ_i^m via a basic set of three constraint equations:

$$\partial_i \ln a = -\partial_i \ln H = -\frac{\tilde{\epsilon}}{1-\tilde{\epsilon}}e_{1A}\zeta_i^A, \tag{14}$$

$$\partial_i\phi^A = -\frac{\sqrt{2\tilde{\epsilon}}}{\kappa}\left(\delta_B^A + \frac{\tilde{\epsilon}}{1-\tilde{\epsilon}}e_1^Ae_{1B}\right)\zeta_i^B, \tag{15}$$

$$\mathcal{D}_i\Pi^A = -\frac{\sqrt{2\tilde{\epsilon}}}{\kappa}H\left[(1-\tilde{\epsilon})\theta_i^A + \left((\tilde{\epsilon}+\tilde{\eta}^\parallel)\delta_B^A - \tilde{\epsilon}e_1^Ae_{1B} + \frac{\tilde{\epsilon}}{1-\tilde{\epsilon}}\tilde{\eta}^Ae_{1B}\right)\zeta_i^B\right]. \tag{16}$$

Using only these three constraints one can compute the spatial derivative of all relevant quantities, keeping in mind that $\theta_i^m = e_{mA}\theta_i^A - (1-\tilde{\epsilon})^{-1}Z_{mn}\zeta_i^n$. Note that in our gauge $\mathcal{W}$ depends on t only and does not get any nonlinear contributions.

III. GENERAL ANALYTIC SOLUTION

In this section we investigate how to solve the system (7) analytically and give formal expressions for the solution. In the next sections we will investigate cases where we can determine the solution more explicitly. We start by rewriting the system (7) into a single vector equation:

$$\dot{v}_{ia}(t,\boldsymbol{x}) + A_{ab}(t,\boldsymbol{x})v_{ib}(t,\boldsymbol{x}) = b_{ia}(t,\boldsymbol{x}),$$
$$\lim_{t\to-\infty} v_{ia} = 0, \qquad v_i \equiv \begin{pmatrix} \zeta_i^1 \\ \theta_i^1 \\ \zeta_i^2 \\ \theta_i^2 \\ \vdots \end{pmatrix}, \qquad b_i \equiv \begin{pmatrix} \mathcal{S}_i^1 \\ \mathcal{J}_i^1 \\ \mathcal{S}_i^2 \\ \mathcal{J}_i^2 \\ \vdots \end{pmatrix}. \tag{17}$$

Here the indices $a, b, \ldots$ label the components within this $2N$-dimensional space (with N the number of fields). The matrix A can be read off from (7) and has the following form: $A_{2m-1,2m} = -1$, $A_{2m,2n} = \Theta_{mn}$ and $A_{2m,2n-1} = \Xi_{mn}/(1-\tilde{\epsilon})^2$, where Θ_{mn} is the matrix between parentheses in the second equation of (7) and $m, n = 1, 2, \ldots, N$. All other entries of A are zero.

Equation (17) is nonlinear since the matrix $A(t,\boldsymbol{x})$ and the the source term $b_i(t,\boldsymbol{x})$ are inhomogeneous functions in space and depend on the v_i through (14)–(16). It can be solved perturbatively as an infinite hierarchy of linear equations with known source terms at each order (see also [3]). We expand the relevant quantities as

$$v_i = v_i^{(1)} + v_i^{(2)} + \ldots, \qquad b_i = b_i^{(1)} + b_i^{(2)} + \ldots,$$
$$A(t,\boldsymbol{x}) = A^{(0)}(t) + A^{(1)}(t,\boldsymbol{x}) + A^{(2)}(t,\boldsymbol{x}) + \ldots. \tag{18}$$

Then the equation of motion for $v_i^{(m)}$ is

$$\dot{v}_{ia}^{(m)}(t,\boldsymbol{x}) + A_{ab}^{(0)}(t)v_{ib}^{(m)}(t,\boldsymbol{x}) = \tilde{b}_{ia}^{(m)}(t,\boldsymbol{x}),$$
$$\lim_{t\to-\infty} v_{ia}^{(m)} = 0, \qquad \tilde{b}_{ia}^{(m)} \equiv b_{ia}^{(m)} - \sum_{j=1}^{m-1} A_{ab}^{(m-j)} v_{ib}^{(j)}. \tag{19}$$

Let us recapitulate the meaning of the various indices, to avoid confusion. The index $i = 1, 2, 3$ labels the components of spatial vectors. The indices $A, B, \ldots = 1, \ldots, N$ label components in field space. These indices will not occur in the rest of the paper, since they have been replaced by the indices $m, n, \ldots = 1, \ldots, N$ that label components in field space within the special basis as defined in (4). Next, the indices $a, b, \ldots = 1, \ldots, 2N$ label components within the $2N$-dimensional space consisting of both ζ and θ as defined in (17). Finally there are the labels within parentheses that denote the order in the perturbative expansion defined above. Only with the i and $A, B, \ldots$ is there a difference between upper and lower indices.

We now show that the source term $\tilde{b}_i^{(m)}$ is known from the solutions for v_i up to order $(m-1)$. The equation for $v_i^{(1)}$ is linear by construction: $A^{(0)}$ depends only on the homogeneous background quantities, and the only $\boldsymbol{x}$ dependence in $b_i^{(1)}$ is in the $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, for the rest it depends on homogeneous background quantities. All of these are in the end functions of just H, ϕ^A, and Π^A via their definitions. To go beyond linear order all these background quantities are perturbed as follows (C stands for any of the quantities to be perturbed, for example $\tilde{\epsilon}$, $\tilde{\eta}^{\perp}$, etc.):

$$\begin{aligned} C(t,\boldsymbol{x}) &= C^{(0)}(t) + C^{(1)}(t,\boldsymbol{x}) + \ldots \\ &= C^{(0)} + \partial^{-2}\partial^i(\partial_i C)^{(1)} + \ldots \\ &= C^{(0)} + \bar{C}_a^{(0)}\partial^{-2}\partial^i v_{ia}^{(1)} + \ldots \end{aligned} \tag{20}$$

where we use (14)–(16) to compute $\partial_i C$ and $\bar{C}^{(0)}$ is some homogeneous (space-independent) vector that is the result of that calculation. Next, to compute $C^{(2)}$ one simply repeats this process with the vector $\bar{C}$, and continues in this way order by order (of course there is also a $\bar{C}_a^{(0)}\partial^{-2}\partial^i v_{ia}^{(2)}$ term at second order, etc.). Then it is easy to see that $\tilde{b}_i^{(m)}$ depends only on $v_i^{(1)}, \ldots, v_i^{(m-1)}$, and hence is a known quantity at each order.

The solution of Eq. (19) for $v_i^{(m)}$ can be written as

$$v_{ia}^{(m)}(t,\boldsymbol{x}) = \int_{-\infty}^{t} \mathrm{d}t' G_{ab}(t,t')\tilde{b}_{ib}^{(m)}(t',\boldsymbol{x}), \tag{21}$$

with the Green's function $G_{ab}(t,t')$ satisfying[2]

$$\frac{\mathrm{d}}{\mathrm{d}t}G_{ab}(t,t') + A_{ac}^{(0)}(t)G_{cb}(t,t') = 0, \qquad \lim_{t\to t'} G_{ab}(t,t') = \delta_{ab}. \tag{22}$$

It is important to note that this Green's function is homogeneous, a solution of the background equation. It has to be computed only once, and can then be used to calculate the solution at each order using the different source terms as in (21). We write explicitly for the first two orders:

$$\begin{aligned} b_{ia}^{(1)}(t,\boldsymbol{x}) &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \dot{\mathcal{W}}(k,t)X_{am}^{(1)}(k,t)\alpha_m(\boldsymbol{k})ik_i e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \text{c.c.}, \\ b_{ia}^{(2)}(t,\boldsymbol{x}) &= (\partial^{-2}\partial^i v_{ic}^{(1)}(t,\boldsymbol{x}))\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \dot{\mathcal{W}}(k,t)\bar{X}_{amc}^{(1)}(k,t) \\ &\quad \times \alpha_m(\boldsymbol{k})ik_i e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \text{c.c.} \end{aligned} \tag{23}$$

Comparison with (10) shows that X_{am} is given by the following equations:

$$X_{2n-1,m} = -\frac{\kappa}{a\sqrt{2\tilde{\epsilon}}}Q_{nm}^{\text{lin}}, \qquad X_{2n,m} = -\frac{\kappa}{a\sqrt{2\tilde{\epsilon}}}\left[\dot{Q}_{nm}^{\text{lin}} - \frac{1+\tilde{\epsilon}+\tilde{\eta}^{\parallel}}{1-\tilde{\epsilon}}Q_{nm}^{\text{lin}}\right]. \tag{24}$$

[2] To be precise, the Green's function is actually defined as the solution of (22) with $\delta(t-t')$ on the right-hand side instead of zero. The solution is then a step function times what we call the Green's function. This step function has been taken into account by changing the upper limit of the integral in (21) from ∞ to t.

The quantity $\bar{X}_{amc}$ is derived from X_{am} as in (20). In the same way we define $A^{(1)}_{ab}(t,\boldsymbol{x}) = \bar{A}^{(0)}_{abc}(t)\partial^{-2}\partial^i v^{(1)}_{ic}(t,\boldsymbol{x})$.

Using the solution (21), valid at each order, and the relations (11) to compute the averages, it is now straightforward to write down the general expressions for the two-point and three-point correlators of the adiabatic ($m = 1$) component of $\zeta^m \equiv \partial^{-2}\partial^i \zeta^m_i$, which is the $a = 1$ component of v_{ia}, or rather their Fourier transforms, the power spectrum and the bispectrum. Making use of the short-hand notation

$$v^{(1)}_{am}(k,t) \equiv \int_{-\infty}^{t} \mathrm{d}t' G_{ab}(t,t')\dot{\mathcal{W}}(k,t')X^{(1)}_{bm}(k,t'), \tag{25}$$

we find for the power spectrum:

$$\langle|\zeta^{(1)1}(k,t)|^2\rangle = v^{(1)}_{1m}(k,t)v^{(1)*}_{1m}(k,t) + \text{c.c.} \tag{26}$$

We emphasize here that the quantity $v^{(1)}_{am}$ defined by (25) is simply made up of the linear $\zeta^{\rm lin}_{mn}$, $\theta^{\rm lin}_{mn}$ mode functions. One needs to evaluate the Green's function $G_{ab}(t,t')$ and to perform the integral (25) when the linear source terms $Q^{\rm lin}_{mn}$, $\dot{Q}^{\rm lin}_{mn}$ in $X^{(1)}_{bm}$ are known only up to horizon crossing, $k \approx aH$. However, where analytic solutions are available for $k \ll aH$, or in a fully numerical approach, we can dispense with the integral (25) by using the linear solution on superhorizon scales.

To find the bispectrum the calculation is slightly longer. One first has to compute $\zeta^{(2)1}(t,\boldsymbol{x})$. As we noted in the single-field case in [5], $\langle\zeta^{(2)1}\rangle$ is indeterminate. To remove this ambiguity and also require that perturbations have a zero average, we define $\tilde{\zeta}^m \equiv \zeta^m - \langle\zeta^m\rangle$. Expanding $\tilde{\zeta}^m = \tilde{\zeta}^{(1)m} + \tilde{\zeta}^{(2)m}$, the three-point correlator will be a combination of the different permutations of $\langle\tilde{\zeta}^{(2)1}(t,\boldsymbol{x}_1)\tilde{\zeta}^{(1)1}(t,\boldsymbol{x}_2)\tilde{\zeta}^{(1)1}(t,\boldsymbol{x}_3)\rangle$, and the bispectrum is its Fourier transform. The intermediate steps are given in more detail in the explicit calculation in Sec. V; here we go directly to the end result for the bispectrum of the adiabatic component:

$$\begin{aligned}&\langle\tilde{\zeta}^1(t,\boldsymbol{x}_1)\tilde{\zeta}^1(t,\boldsymbol{x}_2)\tilde{\zeta}^1(t,\boldsymbol{x}_3)\rangle^{(2)}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\\ &\quad = (2\pi)^3\delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)[f(\boldsymbol{k}_1,\boldsymbol{k}_2)+f(\boldsymbol{k}_1,\boldsymbol{k}_3)+f(\boldsymbol{k}_2,\boldsymbol{k}_3)]\end{aligned} \tag{27}$$

with

$$\begin{aligned} f(\boldsymbol{k},\boldsymbol{k}') \equiv{}& \frac{k^2+\boldsymbol{k}\cdot\boldsymbol{k}'}{|\boldsymbol{k}+\boldsymbol{k}'|^2} v^{(1)*}_{1m}(k,t)\int_{-\infty}^{t}\mathrm{d}t' G_{1a}(t,t')\\ &\times[\bar{X}^{(1)}_{amc}(k,t')\dot{\mathcal{W}}(k,t') - \bar{A}^{(0)}_{abc}(t')v^{(1)}_{bm}(k,t')]\\ &\times[v^{(1)*}_{1n}(k',t)v^{(1)}_{cn}(k',t') + v^{(1)}_{1n}(k',t)v^{(1)*}_{cn}(k',t')]\\ &+\text{c.c.} + \boldsymbol{k}\leftrightarrow\boldsymbol{k}'.\end{aligned} \tag{28}$$

If $Q^{\rm lin}_{mn}$ is real, this simplifies to

$$\begin{aligned} f(\boldsymbol{k},\boldsymbol{k}') \equiv{}& 4\frac{k^2+\boldsymbol{k}\cdot\boldsymbol{k}'}{|\boldsymbol{k}+\boldsymbol{k}'|^2} v^{(1)}_{1m}(k,t)v^{(1)}_{1n}(k',t)\int_{-\infty}^{t}\mathrm{d}t' G_{1a}(t,t')\\ &\times[\bar{X}^{(1)}_{amc}(k,t')\dot{\mathcal{W}}(k,t') - \bar{A}^{(0)}_{abc}(t')v^{(1)}_{bm}(k,t')]\\ &\times v^{(1)}_{cn}(k',t') + \boldsymbol{k}\leftrightarrow\boldsymbol{k}'.\end{aligned} \tag{29}$$

This integral expression is a key result of this paper. Using our methodology, the three-point correlator with full momentum dependence has been expressed as a single time integral over quantities determined by the background model and the linear perturbations, that is, respectively, the matrix $\bar{A}^{(0)}_{abc}$ and the solution $Q^{\rm lin}_{mn}$ embedded in $\bar{X}^{(1)}_{amc}$ (24) and $v^{(1)}_{am}$ (25) (in both of which the background is also implicit). Of course, one also has to find the Green's function G_{ab} from (22), but, like the equation for $Q^{A}_{{\rm lin}B}$ in [3], this is a linear ordinary differential equation for which there is no serious impediment to finding a numerical solution, in cases where an analytic or semianalytic solution is unknown. The integral solution (29), then, demonstrates that the calculation of the three-point correlator is straightforward and tractable. It is, in principle, similar to calculations of the power spectrum, where accurate estimates can be found from background quantities, for example, in the slow-roll approximation. Here, we only have to supplement this with the amplitudes of the linear perturbation mode functions $Q^{\rm lin}_{mn}(k,t)$ and the closely related Green's function $G_{ab}(t,t')$. In Sec. V D, using this methodology, we provide some quantitative semianalytic results for the bispectrum of a two-field inflation model with a quadratic potential.

Before closing this section a final comment is in order. A feature of (29) which may at first sight cast doubt on its utility for quantitative calculations is its apparent dependence on the ad hoc choice for the functional form of the window function $\mathcal{W}(k)$. Closer inspection reveals that the second term of (29) (the $\bar{A}$ term) does not depend on $\mathcal{W}(k)$ for scales sufficiently larger than the horizon. This is evident from the fact that (25) is simply the solution to linear theory smoothed on scales larger than the horizon. Any properly normalised window function with $\mathcal{W}(k) \to 1$ for scales sufficiently larger than the horizon will produce the same final answer. The $\bar{A}$ term represents the nonlinear evolution on superhorizon scales and, as we show below, it describes an integrated effect which can lead to large non-Gaussianity. In contrast, the $\bar{X}$ term arises from perturbations around horizon crossing and may depend on $\mathcal{W}$. We find below that this term is localized around horizon crossing and that it does not give rise to observationally interesting effects. Section V E provides more discussion on these points. In [11] we explicitly show that taking a step function instead of a Gaussian as window function does not change the leading-order integrated effects.

IV. SLOW-ROLL APPROXIMATION

The perturbation quantity $Q^{\rm lin}_{mn}$ can be computed exactly numerically, or analytically within the slow-roll approximation where all slow-roll parameters are assumed to be smaller than unity. The latter was done in [13] to next-to-leading order in slow roll[3]:

$$Q^{\rm lin}_{mn} = \frac{1}{2\sqrt{k}}\left[E\left(\frac{c}{kR}\right)^{1+D}\right]_{mn}, \tag{30}$$

where the matrices D and E are defined by

$$\begin{aligned} D_{mn} &\equiv \tilde{\epsilon}\delta_{mn} + 2\tilde{\epsilon}\delta_{m1}\delta_{n1} - \frac{V_{mn}}{3H^2},\\ E_{mn} &\equiv (1-\tilde{\epsilon})\delta_{mn} + (2-\gamma_E - \ln 2)D_{mn}, \end{aligned} \tag{31}$$

with γ_E Euler's constant. Overall unitary factors that are physically irrelevant have been omitted. Using this expression the source terms are given by

$$\begin{aligned} \mathcal{S}^m_i &= -\frac{\kappa}{a\sqrt{2\tilde{\epsilon}}}\int\frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\dot{\mathcal{W}}(k)Q^{\rm lin}_{mn}(k)\alpha_n(\boldsymbol{k})ik_i e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\\ &\quad + {\rm c.c.},\\ \mathcal{J}^m_i &= -\frac{\kappa}{a\sqrt{2\tilde{\epsilon}}}[D_{mn} - (2\tilde{\epsilon}+\tilde{\eta}^{\parallel})\delta_{mn}]\\ &\quad\times\int\frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\dot{\mathcal{W}}(k)Q^{\rm lin}_{np}(k)\alpha_p(\boldsymbol{k})ik_i e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + {\rm c.c.} \end{aligned} \tag{32}$$

Now when computing $\bar{X}^{(1)}_{amc}$ as defined in (23), or any higher-order terms in the perturbative expansion, we would in principle have to make the background quantities in $Q^{\rm lin}_{mn}$ dependent on $\boldsymbol{x}$ and perturb them according to (20). However, from (30) we see that $Q^{\rm lin}_{mn}$ depends on $\boldsymbol{x}$ only beyond leading order in slow roll (to leading order it is just given by $Q^{\rm lin}_{mn} = c/(2k^{3/2}R)\delta_{mn}$). Hence in a leading-order slow-roll approximation the only nonlinear parts in the source terms are the factors in front of the integrals in (32).

Within the leading-order slow-roll approximation, we now look at the two-field case, to make things a bit more explicit. In that case the matrix A in (17) is given by

$$A = \begin{pmatrix} 0 & -1 & 0 & 0\\ 0 & 3 & -6\tilde{\eta}^{\perp} & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 3\chi & 3 \end{pmatrix}, \qquad \chi \equiv \frac{V_{22}}{3H^2} + \tilde{\epsilon} + \tilde{\eta}^{\parallel}. \tag{33}$$

The quantity χ is first order in slow roll. Here we used (9) and the relation, valid to leading order in slow roll (see e.g. [13]),

$$\frac{V_{1m}}{3H^2} = \frac{V_{m1}}{3H^2} = (\tilde{\epsilon}-\tilde{\eta}^{\parallel})\delta_{m1} - \tilde{\eta}^{\perp}\delta_{m2}. \tag{34}$$

Using the constraints (14)–(16) we can compute the spatial derivatives that are needed to calculate $\bar{A}$. Some of these were given in a general form in [3]; to first order in slow roll for the θ coefficients and to second order for the ζ coefficients we find in the two-field case,

$$\begin{aligned} &(\partial_i \ln a)^{(1)} = -(\partial_i \ln H)^{(1)} = -\tilde{\epsilon}\zeta^{(1)1}_i, \qquad (\partial_i \ln\tilde{\epsilon})^{(1)} = -2\theta^{(1)1}_i - 2(\tilde{\epsilon}+\tilde{\eta}^{\parallel})\zeta^{(1)1}_i + 2\tilde{\eta}^{\perp}\zeta^{(1)2}_i,\\ &(\partial_i\tilde{\eta}^{\perp})^{(1)} = \tilde{\eta}^{\perp}\theta^{(1)1}_i + (3-3\tilde{\epsilon}+\tilde{\eta}^{\parallel})\theta^{(1)2}_i - ((\tilde{\epsilon}-2\tilde{\eta}^{\parallel})\tilde{\eta}^{\perp} + \tilde{\xi}^{\perp})\zeta^{(1)1}_i + (3\chi + (\tilde{\epsilon}+\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel} - (\tilde{\eta}^{\perp})^2)\zeta^{(1)2}_i,\\ &(\partial_i\chi)^{(1)} = (3-5\tilde{\epsilon}+\tilde{\eta}^{\parallel})\theta^{(1)1}_i - 3\tilde{\eta}^{\perp}\theta^{(1)2}_i - \psi_1\zeta^{(1)1}_i - (\psi_2 + (6-10\tilde{\epsilon}+2\tilde{\eta}^{\parallel}+3\chi)\tilde{\eta}^{\perp})\zeta^{(1)2}_i, \end{aligned} \tag{35}$$

introducing the two second-order slow-roll quantities ψ_1 and ψ_2 as short-hand notation:

$$\begin{aligned} \psi_1 &\equiv 2\tilde{\epsilon}\chi + (\tilde{\epsilon}-\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel} + 3(\tilde{\eta}^{\perp})^2 + \tilde{\xi}^{\parallel} + \frac{\sqrt{2\tilde{\epsilon}}}{\kappa}\frac{V_{221}}{3H^2} = 2\tilde{\epsilon}(\chi + 2\tilde{\eta}^{\parallel}) + 4(\tilde{\eta}^{\perp})^2 - \frac{\sqrt{2\tilde{\epsilon}}}{\kappa}\frac{1}{3H^2}(V_{111} - V_{221}),\\ \psi_2 &\equiv (11\tilde{\epsilon} + 2\tilde{\eta}^{\parallel} - 3\chi)\tilde{\eta}^{\perp} + \tilde{\xi}^{\perp} + \frac{\sqrt{2\tilde{\epsilon}}}{\kappa}\frac{V_{222}}{3H^2}, \end{aligned} \tag{36}$$

with $V_{mnp} \equiv e^A_m e^B_n e^C_p \mathcal{D}_C\mathcal{D}_B\partial_A V$. The reason for this specific definition of ψ_2 will become clear later on. Since we have only two fields, the notation $\tilde{\xi}^{\perp}$ is unambiguous. To compute $\partial_i\chi$ we used that in the two-field case, because of the orthonormality of the basis vectors, $\mathcal{D}_i e^A_2 = -e^A_1(e_{2B}\mathcal{D}_i e^B_1)$. All slow-roll parameters in these expressions take their homogeneous background values. From this we find that the rank-3 matrix $\bar{A}_{abc}$ [defined below (24)] is

$$\bar{A} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -6(-(\tilde{\epsilon}-2\tilde{\eta}^{\parallel})\tilde{\eta}^{\perp} - \tilde{\xi}^{\perp}, \tilde{\eta}^{\perp}, 3\chi + (\tilde{\epsilon}+\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel} - (\tilde{\eta}^{\perp})^2, 3-3\tilde{\epsilon}+\tilde{\eta}^{\parallel}) & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & 3(-\psi_1, 3-5\tilde{\epsilon}+\tilde{\eta}^{\parallel}, -\psi_2 - (6-10\tilde{\epsilon}+2\tilde{\eta}^{\parallel}+3\chi)\tilde{\eta}^{\perp}, -3\tilde{\eta}^{\perp}) & \mathbf{0} \end{pmatrix}. \tag{37}$$

In the same way we find that the matrices X_{am} and $\bar{X}_{amc}$, defined in (24), are given by

[3] Compared with the solution in [13] there is an extra factor of $1/\sqrt{2}$. It has to be introduced to take into account the difference between the classical Gaussian random numbers α, which have $\langle\alpha\alpha^*\rangle = \langle\alpha^*\alpha\rangle$, and the quantum creation/annihilation operators $\hat{a}^\dagger$, $\hat{a}$, which have $\langle\hat{a}^\dagger\hat{a}\rangle = 0$. In [5] we introduced this factor of $1/\sqrt{2}$ in the analogue of Eq. (10), which leads of course to identical results.

$$X = -\frac{\kappa}{a\sqrt{2\tilde{\epsilon}}}\begin{pmatrix} 1 & 0 \\ 0 & \tilde{\eta}^{\perp} \\ 0 & 1 \\ \tilde{\eta}^{\perp} & -\chi \end{pmatrix}\frac{e^t}{2k^{3/2}}, \qquad \bar{X} = -\frac{\kappa}{a\sqrt{2\tilde{\epsilon}}}\begin{pmatrix} (2\tilde{\epsilon} + \tilde{\eta}^{\parallel}, 1, -\tilde{\eta}^{\perp}, 0) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{X}_{22} \\ \mathbf{0} & (2\tilde{\epsilon} + \tilde{\eta}^{\parallel}, 1, -\tilde{\eta}^{\perp}, 0) \\ \boldsymbol{X}_{22} & \boldsymbol{X}_{42} \end{pmatrix}\frac{e^t}{2k^{3/2}}, \tag{38}$$

$$\boldsymbol{X}_{22} \equiv ((\tilde{\epsilon} + 3\tilde{\eta}^{\parallel})\tilde{\eta}^{\perp} - \tilde{\xi}^{\perp}, 2\tilde{\eta}^{\perp}, 3\chi + (\tilde{\epsilon} + \tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel} - 2(\tilde{\eta}^{\perp})^2, 3 - 3\tilde{\epsilon} + \tilde{\eta}^{\parallel}),$$

$$\boldsymbol{X}_{42} \equiv (\psi_1 - (2\tilde{\epsilon} + \tilde{\eta}^{\parallel})\chi, -3 + 5\tilde{\epsilon} - \tilde{\eta}^{\parallel} - \chi, \psi_2 + (6 - 10\tilde{\epsilon} + 2\tilde{\eta}^{\parallel} + 4\chi)\tilde{\eta}^{\perp}, 3\tilde{\eta}^{\perp}),$$

where we used (30) and (32).

In general the Green's function $G_{ab}(t, t')$ cannot be expressed in closed form, since the time-dependent matrix A does not commute at different times. It can be formally represented as

$$G(t, t') = \mathcal{T}\exp\left[-\int_{t'}^{t} A(s)\mathrm{d}s\right], \tag{39}$$

where $\mathcal{T}$ denotes a time-ordered exponential:

$$\mathcal{T}\exp\left[-\int_{t'}^{t} A(s)\mathrm{d}s\right] \equiv 1 - \int_{t'}^{t} A(s)\mathrm{d}s + \int_{t'}^{t}\mathrm{d}s\int_{t'}^{s}\mathrm{d}s' A(s)A(s') - \int_{t'}^{t}\mathrm{d}s\int_{t'}^{s}\mathrm{d}s'\int_{t'}^{s'}\mathrm{d}s'' A(s)A(s')A(s'') + \ldots. \tag{40}$$

This formal expression is standard in quantum mechanics and quantum field theory (see e.g. [15]) where, viewed as a perturbative expansion, the first few terms in the series are kept when the operator A contains a small parameter. In our case, however, not all elements of A are first order in slow roll, so that a truncation at any finite order is a bad approximation. Moreover, even if A were first order in slow roll, one should still be careful, because the time interval in the integrations can easily be of the order of an inverse slow-roll parameter. The time-ordered exponential can be written as an ordinary exponential plus terms which contain (nested) commutators. For example, the second and third order terms in the series (40) can be written as

$$\int_{t'}^{t}\mathrm{d}s\int_{t'}^{s}\mathrm{d}s' A(s)A(s') = \frac{1}{2}\left(\int_{t'}^{t} A(s)\mathrm{d}s\right)^2 + \frac{1}{2}\int_{t'}^{t}\mathrm{d}s\int_{t'}^{t}\mathrm{d}s'[A(s), A(s')]\Theta(s - s') \tag{41}$$

and

$$\int_{t'}^{t}\mathrm{d}s\int_{t'}^{s}\mathrm{d}s'\int_{t'}^{s'}\mathrm{d}s'' A(s)A(s')A(s'') = \frac{1}{3!}\left(\int_{t'}^{t} A(s)ds\right)^3 + \frac{1}{2}\int_{t'}^{t}\mathrm{d}s\int_{t'}^{t}\mathrm{d}s'\int_{t'}^{t}\mathrm{d}s'' A(s)[A(s'), A(s'')]\Theta(s' - s'')$$
$$+ \frac{1}{3}\int_{t'}^{t}\mathrm{d}s\int_{t'}^{t}\mathrm{d}s'\int_{t'}^{t}\mathrm{d}s''([A(s), A(s')]A(s'') - A(s'')[A(s), A(s')])\Theta(s - s')\Theta(s - s'') \tag{42}$$

respectively, where Θ is the step function, and similarly for higher orders (see [16] for general expressions).

There are basically three ways to proceed with this expression. In the first place we can, if we are interested only in relatively short time intervals, neglect the commutator terms in the expansion of the time-ordered exponential and write it as an ordinary exponential. The commutator terms all contain a difference of slow-roll parameters at different times, as opposed to the terms of the ordinary exponential that have just a slow-roll parameter at one time. Hence, for small time intervals, the commutator terms are a slow-roll order of magnitude smaller. Then we have an exact analytic solution in closed form for the Green's function. Secondly, as will be the case in the explicit example in the next section, we can consider examples where A does commute with itself at different times, in which case the time-ordered exponential simplifies to an ordinary exponential exactly. Finally, we can compute the Green's function numerically and use it in a semianalytic calculation (remember that the Green's function has to be computed only once). That will be worked out in a future publication, though we give some results in Sec. V D.

V. EXPLICIT SOLUTION FOR TWO-FIELD SLOW-ROLL CASE

In this section we provide an analytic solution for the bispectrum in two-field slow-roll inflation. We assume slow roll as in Sec. IV but in order to obtain explicit solutions in Secs. V A, V B, and V C, and we make the *further* assumption that all slow-roll parameters are constant. The semianalytic results of Sec. V D are *not* bound by this further assumption and all slow-roll parameters are calculated numerically for a quadratic model. No slow-roll parameter takes values greater than unity. However, a semianalytic approach is feasible for the most general non-slow-roll case and will be the study of a future publication [6].

A. Power spectrum

We now restrict ourselves to just two fields. Moreover, we assume the background values of H and all slow-roll parameters, including perpendicular ones, to be completely constant in time whenever they are the leading-order (in slow roll) nonzero terms in our expressions. Then we can actually solve the system explicitly, i.e. do the time integrals. We start from the equation of motion (17) for $\boldsymbol{v}_i$ together with the definitions (33), or rather from (19) for the mth order $\boldsymbol{v}_i^{(m)}$ in an expansion in perturbation orders. We assume everywhere that $\chi > 0$. At first order in the perturbations this reads as

$$\dot{\boldsymbol{v}}_i^{(1)} + A^{(0)}\boldsymbol{v}_i^{(1)} = \boldsymbol{b}_i^{(1)}, \qquad \lim_{t\to-\infty}\boldsymbol{v}_i^{(1)} = 0. \tag{43}$$

The matrix $A^{(0)}$ contains just background quantities, which by assumption are constant. Hence we circumvent the issues of noncommutativity and time-ordered exponentials, and we can write down the solution immediately as

$$\boldsymbol{v}_i^{(1)}(t,\boldsymbol{x}) = e^{-A^{(0)}t}\int_{-\infty}^{t} \mathrm{d}t' e^{A^{(0)}t'}\boldsymbol{b}_i^{(1)}(t',\boldsymbol{x}). \tag{44}$$

(In the terminology of Sec. III, the Green's function is $G(t,t') = \exp[-A^{(0)}(t-t')]$.) The exponential can be worked out using its eigenvalues and eigenvectors. When multiplied with its inverse (at a different time) and expanded to first order in slow roll, we obtain

$$e^{-A^{(0)}t}e^{A^{(0)}t'} = \begin{pmatrix} 1 & \frac{1}{3}(1-(y/y')^3) & \frac{2\tilde{\eta}^\perp}{\chi}(1-(1+\frac{\chi}{3})(y/y')^\chi + \frac{\chi}{3}(y/y')^{3-\chi}) & \frac{2\tilde{\eta}^\perp}{3\chi}(1-(y/y')^3-(1+\frac{2\chi}{3})((y/y')^\chi-(y/y')^{3-\chi})) \\ 0 & (y/y')^3 & 2\tilde{\eta}^\perp((y/y')^\chi-(y/y')^{3-\chi}) & \frac{2\tilde{\eta}^\perp}{\chi}((y/y')^3+\frac{\chi}{3}(y/y')^\chi-(1+\frac{\chi}{3})(y/y')^{3-\chi}) \\ 0 & 0 & (1+\frac{\chi}{3})(y/y')^\chi - \frac{\chi}{3}(y/y')^{3-\chi} & \frac{1}{3}(1+\frac{2\chi}{3})((y/y')^\chi-(y/y')^{3-\chi}) \\ 0 & 0 & -\chi((y/y')^\chi-(y/y')^{3-\chi}) & -\frac{\chi}{3}(y/y')^\chi+(1+\frac{\chi}{3})(y/y')^{3-\chi} \end{pmatrix}. \tag{45}$$

Obviously, it is the identity matrix if $y' = y$. For calculational simplicity here we have defined the new time variable y, as well as the relative momentum p,

$$y \equiv \frac{k_* c}{\sqrt{2}}e^{-t} = \frac{k_* R}{\sqrt{2}} = \frac{c}{\sqrt{2}}e^{-\Delta t_*}, \qquad p \equiv \frac{k}{k_*} \quad \Rightarrow \quad py = \frac{kR}{\sqrt{2}} = \frac{c}{\sqrt{2}}e^{-\Delta t_k}, \tag{46}$$

with $\Delta t_* \equiv t - t_*$, the time since horizon crossing of a reference mode k_*, and we have used the fact that $k_*\exp(-t_*) = 1$ by definition (Δt_k is defined similarly for the mode k). The fixed reference mode k_* is most conveniently chosen to be one of the observable modes, say the one that crossed the horizon 50 e folds before the end of inflation. From (23) and (38) and the relation $a = kc/(pyH\sqrt{2})$ we find that $\boldsymbol{b}_i^{(1)}$ can be written to leading order in slow roll as

$$\boldsymbol{b}_i^{(1)} = -\frac{\kappa}{2\sqrt{2}}\frac{H}{\sqrt{\tilde{\epsilon}}}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\frac{1}{k^{3/2}}2p^2y^2e^{-p^2y^2}ik_ie^{i\boldsymbol{k}\cdot\boldsymbol{x}} \times \begin{pmatrix} \alpha_1(\boldsymbol{k}) \\ \tilde{\eta}^\perp\alpha_2(\boldsymbol{k}) \\ \alpha_2(\boldsymbol{k}) \\ \tilde{\eta}^\perp\alpha_1(\boldsymbol{k}) - \chi\alpha_2(\boldsymbol{k}) \end{pmatrix} + \text{c.c.} \tag{47}$$

Changing to y as integration variable, we can then do the integral in (44) explicitly to find the solution

$$\begin{aligned}\boldsymbol{v}_i^{(1)}(y,\boldsymbol{x}) &= -\frac{\kappa}{2\sqrt{2}}\frac{H}{\sqrt{\tilde{\epsilon}}}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\frac{1}{k^{3/2}}ik_ie^{i\boldsymbol{k}\cdot\boldsymbol{x}}\begin{pmatrix} \alpha_1 + 2\frac{\tilde{\eta}^\perp}{\chi}\alpha_2 & -\frac{2}{3}\frac{(\tilde{\eta}^\perp)^2}{\chi}\alpha_1 + \frac{1}{3}\tilde{\eta}^\perp\alpha_2 & -2\frac{\tilde{\eta}^\perp}{\chi}\alpha_2 & \frac{2}{3}\frac{(\tilde{\eta}^\perp)^2}{\chi}\alpha_1 \\ 0 & 2\frac{(\tilde{\eta}^\perp)^2}{\chi}\alpha_1 - \tilde{\eta}^\perp\alpha_2 & 2\tilde{\eta}^\perp\alpha_2 & -2\frac{(\tilde{\eta}^\perp)^2}{\chi}\alpha_1 \\ 0 & 0 & \alpha_2 & -\frac{1}{3}\tilde{\eta}^\perp\alpha_1 \\ 0 & 0 & -\chi\alpha_2 & \tilde{\eta}^\perp\alpha_1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} e^{-p^2y^2} \\ p^3y^3\Gamma(-\frac{1}{2}, p^2y^2) \\ p^\chi y^\chi\Gamma(1-\frac{\chi}{2}, p^2y^2) \\ p^{3-\chi}y^{3-\chi}\Gamma(-\frac{1}{2}+\frac{\chi}{2}, p^2y^2) \end{pmatrix} + \text{c.c.} \\ &\equiv -\frac{\kappa}{2\sqrt{2}}\frac{H}{\sqrt{\tilde{\epsilon}}}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\frac{1}{k^{3/2}}ik_ie^{i\boldsymbol{k}\cdot\boldsymbol{x}}B(\boldsymbol{k})u(py) + \text{c.c.}\end{aligned} \tag{48}$$

where we have omitted the explicit $\boldsymbol{k}$ dependence of the α's and the final expression defines the matrix $B(\boldsymbol{k})$ and vector $u(py)$.

It is interesting to look at the time behavior of (48) in more detail. Note that in our leading-order slow-roll approximation, differences Δt in the time variable defined in (3) are equal to differences in the number of e folds. A few e folds[4] after horizon crossing the vector $u(py)$ in (48) can be approximated by $(1, 0, p^\chi y^\chi \Gamma(1-\chi/2), 0)^T$. The third entry can be approximated even further as just $1-\chi\Delta t_k$, where Δt_k is the number of e folds after horizon crossing of the mode k, and the expression is valid for $\chi\Delta t_k$ sufficiently smaller than unity, but $\Delta t_k \gtrsim 3$.[5] With this approximation the solution (48) can be written as

$$v_i^{(1)}(t,\boldsymbol{x}) \approx -\frac{\kappa}{2\sqrt{2}}\frac{H}{\sqrt{\tilde{\epsilon}}}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \times \frac{1}{k^{3/2}} ik_i e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \begin{pmatrix} \alpha_1 + 2\tilde{\eta}^\perp \Delta t_k \alpha_2 \\ 2\tilde{\eta}^\perp(1-\chi\Delta t_k)\alpha_2 \\ (1-\chi\Delta t_k)\alpha_2 \\ -\chi(1-\chi\Delta t_k)\alpha_2 \end{pmatrix} + \text{c.c.} \tag{49}$$

As expected (see e.g. [13]) we find that the effectively single-field (α_1) component of ζ_i^1 reaches its constant final value right after horizon crossing, while the influence of the perpendicular field direction (α_2, "isocurvature mode") on ζ_i^1 continues to grow with time on superhorizon scales. The velocities θ_i^1 and θ_i^2 are both suppressed by an additional slow-roll factor compared to the ζ_i's. In the limit of $\Delta t_k \to \infty$ (or $py \to 0$), where the above approximation is no longer valid, the exact result (48) leads to the limit

$$v_i^{(1)}(\boldsymbol{x}) \approx -\frac{\kappa}{2\sqrt{2}}\frac{H}{\sqrt{\tilde{\epsilon}}}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \times \frac{1}{k^{3/2}} ik_i e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \begin{pmatrix} \alpha_1 + 2\frac{\tilde{\eta}^\perp}{\chi}\alpha_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \text{c.c.} \tag{50}$$

Hence the expression does not diverge as t grows, but reaches a well-defined value, which is independent of the smoothing parameter c.

Concentrating now on the adiabatic (e_1) component of $\zeta \equiv \partial^{-2}\partial^i\zeta_i$ we find from (48) to leading order in slow roll:

$$\zeta^{(1)1}(t,\boldsymbol{x}) = -\frac{\kappa}{2\sqrt{2}}\frac{H}{\sqrt{\tilde{\epsilon}}}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\frac{1}{k^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left[e^{-p^2y^2(\Delta t_*)}\alpha_1(\boldsymbol{k}) + 2\frac{\tilde{\eta}^\perp}{\chi} g(p,\chi,\Delta t_*)\alpha_2(\boldsymbol{k})\right] + \text{c.c.}, \tag{51}$$

where y as a function of Δt_* is given in (46), $p = k/k_*$, and we have defined

$$g(p,\chi,\Delta t_*) \equiv e^{-p^2y^2} - p^\chi y^\chi \Gamma\left(1-\frac{\chi}{2}, p^2y^2\right) \approx 1 - p^\chi e^{-\chi\Delta t_*} = 1 - e^{-\chi\Delta t_k}, \tag{52}$$

where the approximation is good from a few e folds after horizon crossing. Hence the two-point correlator is given by

$$\langle\zeta^{(1)1}(t,\boldsymbol{x})\zeta^{(1)1}(t,\boldsymbol{x}')\rangle = \frac{\kappa^2}{8}\frac{H^2}{\tilde{\epsilon}}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{k^3}\left[e^{-2p^2y^2(\Delta t_*)} + 4\left(\frac{\tilde{\eta}^\perp}{\chi}\right)^2 g^2(p,\chi,\Delta t_*)\right]e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} + \text{c.c.}, \tag{53}$$

or, equivalently, for the power spectrum:

$$\langle\zeta^{(1)1}(k,t)\zeta^{(1)1}(k,t)\rangle = \frac{\kappa^2}{4}\frac{H^2}{\tilde{\epsilon}}\frac{1}{k^3}\left[e^{-2p^2y^2(\Delta t_*)} + 4\left(\frac{\tilde{\eta}^\perp}{\chi}\right)^2 g^2(p,\chi,\Delta t_*)\right]. \tag{54}$$

Here we used (11) to take the average. Alternatively, we could have used (26) directly. From a few e folds after horizon crossing, $\exp(-2p^2y^2) \approx 1$ and $g(p,\chi,\Delta t_*)$ is given by the final expression in (52), so that the power spectrum is independent of the smoothing parameter c. Finally, we can compute the adiabatic spectral index using the expressions in [17], where the U_{Pe} in that paper can be read off from (54), once the transient horizon-crossing effects have disappeared, to be $2(\tilde{\eta}^\perp/\chi)g(p,\chi,\Delta t_*)e_2$,

$$n_{\text{ad}} - 1 = -4\tilde{\epsilon} - 2\tilde{\eta}^\parallel - 8\tilde{\eta}^\perp\frac{\tilde{\eta}^\perp}{\chi}g(p,\chi,\Delta t_*) \times \frac{1-g(p,\chi,\Delta t_*)}{1+4(\frac{\tilde{\eta}^\perp}{\chi})^2 g^2(p,\chi,\Delta t_*)}. \tag{55}$$

B. Second-order solution

At second order in the perturbations we expand all quantities in A and b_i as explained in (20), using (35), resulting in the expressions in (37) and (38). Remember that superscripts within parentheses denote the order in perturbation theory, while the superscripts without paren

[4]For example, 3 e folds is good enough if $c = 3$ and $\chi = 0.05$, and this result depends only weakly on the values of c and χ.

[5]See the previous footnote. A logarithmic dependence on c has been ignored here. For $c = 3$ this term is 4 times smaller than $\chi\Delta t_k$ when $\Delta t_k = 3$, and becomes even less important as Δt_k grows.

theses indicate the component of the vector within the field basis as defined in (4). The resulting equation for $v_i^{(2)}$ has the same structure as (43), but with a different source term:

$$\dot{v}_i^{(2)} + A^{(0)}v_i^{(2)} = b_i^{(2)} - \begin{pmatrix} 0 \\ -6\tilde{\eta}^{\perp(1)} \\ 0 \\ 3\chi^{(1)} \end{pmatrix} \zeta_i^{(1)2}, \qquad \lim_{t\to-\infty} v_i^{(2)} = 0, \tag{56}$$

where $b_i^{(2)}$ is the vector obtained by perturbing H, $\tilde{\epsilon}$, $\tilde{\eta}^{\perp}$, and χ in $b_i^{(1)}$ given in (47). Explicitly, the right-hand side of Eq. (56) is to leading order in slow roll given by

$$\frac{\kappa^2}{8}\frac{H^2}{\tilde{\epsilon}}\iint \frac{d^3k d^3k'}{(2\pi)^3}\frac{ik_i e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{k^{3/2}k'^{3/2}}\left[2p^2y^2e^{-p^2y^2}\begin{pmatrix} (2\tilde{\epsilon}+\tilde{\eta}^{\parallel})\alpha_1(\boldsymbol{k}) & \alpha_1(\boldsymbol{k}) & -\tilde{\eta}^{\perp}\alpha_1(\boldsymbol{k}) & 0 \\ * & * & * & * \\ (2\tilde{\epsilon}+\tilde{\eta}^{\parallel})\alpha_2(\boldsymbol{k}) & \alpha_2(\boldsymbol{k}) & -\tilde{\eta}^{\perp}\alpha_2(\boldsymbol{k}) & 0 \\ * & * & * & * \end{pmatrix}\right.$$

$$\left. - 3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}^T B(\boldsymbol{k})u(py)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2(\tilde{\epsilon}-2\tilde{\eta}^{\parallel})\tilde{\eta}^{\perp}+2\tilde{\xi}^{\perp} & -2\tilde{\eta}^{\perp} & -6\chi-2(\tilde{\epsilon}+\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}+2(\tilde{\eta}^{\perp})^2 & -6+6\tilde{\epsilon}-2\tilde{\eta}^{\parallel} \\ 0 & 0 & 0 & 0 \\ -\psi_1 & 3-5\tilde{\epsilon}+\tilde{\eta}^{\parallel} & -\psi_2-(6-10\tilde{\epsilon}+2\tilde{\eta}^{\parallel}+3\chi)\tilde{\eta}^{\perp} & -3\tilde{\eta}^{\perp} \end{pmatrix}\right]$$

$$\times (B(\boldsymbol{k}')u(qy)e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} + \text{c.c.}) + \text{c.c.}$$

$$\equiv \frac{\kappa^2}{8}\frac{H^2}{\tilde{\epsilon}}\iint \frac{d^3k d^3k'}{(2\pi)^3}\frac{ik_i e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{k^{3/2}k'^{3/2}}[2p^2y^2e^{-p^2y^2}\tilde{B}(\boldsymbol{k}) - 3(0,0,1,0)B(\boldsymbol{k})u(py)F](B(\boldsymbol{k}')u(qy)e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} + \text{c.c.}) + \text{c.c.} \tag{57}$$

where we have defined $q \equiv k'/k_*$, as well as the matrices $\tilde{B}(\boldsymbol{k})$ and F in the last expression (the matrix B and vector u were defined in (48)). The entries indicated by an asterisk in the matrix $\tilde{B}$ are not given explicitly here, but can be read off from (38); they do not contribute to $\zeta_i^{(1)1}$ and $\zeta_i^{(1)2}$ to leading order in slow roll, because they are one order higher than the corresponding entries in the first and third row, after cancellations in the final result have been taken into account. The solution for $v_i^{(2)}(t,\boldsymbol{x})$ is now given by the same expression (44) as $v_i^{(1)}(t,\boldsymbol{x})$, if one replaces $b_i^{(1)}$ in that expression by (57), though actually calculating the time integral to obtain a completely explicit expression is clearly more difficult.

To get all the time-dependent terms together, it is useful to change from the matrix notation used above to a component notation, as defined in (17). We define the indices a, b, c, d, e, f running from 1 to 4 to label the components in the 4-dimensional $\{\zeta_i^1, \theta_i^1, \zeta_i^2, \theta_i^2\}$ space. Moreover, we rewrite the matrix in (45) as

$$e^{-A^{(0)}t}e^{A^{(0)}t'} = K_{abc}w_c(y,y'),$$

$$K \equiv \begin{pmatrix} (1,0,0,0) & \frac{1}{3}(1,-1,0,0) & \frac{2\tilde{\eta}^{\perp}}{\chi}(1,0,-1-\frac{\chi}{3},\frac{\chi}{3}) & \frac{2\tilde{\eta}^{\perp}}{3\chi}(1,-1,-1-\frac{2\chi}{3},1+\frac{2\chi}{3}) \\ \mathbf{0} & (0,1,0,0) & 2\tilde{\eta}^{\perp}(0,0,1,-1) & \frac{2\tilde{\eta}^{\perp}}{\chi}(0,1,\frac{\chi}{3},-1-\frac{\chi}{3}) \\ \mathbf{0} & \mathbf{0} & (0,0,1+\frac{\chi}{3},-\frac{\chi}{3}) & \frac{1}{3}(0,0,1+\frac{2\chi}{3},-1-\frac{2\chi}{3}) \\ \mathbf{0} & \mathbf{0} & \chi(0,0,-1,1) & (0,0,-\frac{\chi}{3},1+\frac{\chi}{3}) \end{pmatrix}, \tag{58}$$

$$w(y,y') \equiv \begin{pmatrix} 1 \\ (y/y')^3 \\ (y/y')^{\chi} \\ (y/y')^{3-\chi} \end{pmatrix},$$

which defines the rank-3 matrix K and the vector $w(y,y')$. Then the solution for $v_i^{(2)}$ can be written as

$$v_{ia}^{(2)}(y,\boldsymbol{x}) = \frac{\kappa^2}{8}\frac{H^2}{\tilde{\epsilon}}\iint \frac{\mathrm{d}^3\boldsymbol{k}\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3}\frac{ik_i e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{k^{3/2}k'^{3/2}} K_{abc}(B_{de}(\boldsymbol{k}')e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} + \text{c.c.}) \times\left[\tilde{B}_{bd}(\boldsymbol{k})\int_y^\infty \mathrm{d}y' 2p^2 y' e^{-p^2y'^2} w_c(y,y')u_e(qy') - 3B_{3f}(\boldsymbol{k})F_{bd}\int_y^\infty \frac{\mathrm{d}y'}{y'} w_c(y,y')u_e(qy')u_f(py')\right] + \text{c.c.} \tag{59}$$

(To be precise, c, e, and f are actually indices in a completely different space than the other indices, but it is also 4-dimensional.) It is useful to consider the contributions of the two integral terms within the square brackets separately, since they have a different origin [cf. (29)]. The first term is the variation of the stochastic source, represented in (29) by the $\bar{X}$ term, and because of the window function the integral only picks up a contribution around horizon crossing (although this contribution is time-dependent even later on, because of the dependence on y, not just y', of the Green's function). The second term is the variation of the coefficients in the equation of motion, represented in (29) by the $\bar{A}$ term, which is an integrated effect up to the end of inflation, and is not present in single-field inflation. The leading-order coefficients in front of the first term are first order in slow roll, while the ones in front of the second term are second order[6], however, this can be more than compensated by the larger integration interval.

In principle there are 80 different integrals here: 16 from the first term and 64 from the second one. Some of the integrals can be done analytically, but most have to be studied numerically. However, of those 64 from the second term the only integrals that matter are those that are secular, i.e. continue to grow with time (up to a time of order $\Delta t \sim \chi^{-1}$), since these will be, roughly speaking, a slow-roll order of magnitude larger at the end of inflation than the other integrals. Although we studied all integrals more carefully, one can easily get an idea of which integrals in the second term will be secular by looking at the behavior of the integrand for $y' \to 0$ (i.e. $t \to \infty$): only the components with e and f either 1 or 3 are secular, since these are close to y'^{-1} in that limit. A slightly more careful analysis shows that, roughly speaking, the $c = 2$ and $c = 4$ components of those terms will be a factor χ smaller (one gets a 3^{-1} instead of a χ^{-1} when integrating). Given that $B_{31} = 0$, f cannot be equal to 1, so in the end one expects the 4 integrals in the second term with c and e both either 1 or 3 and $f = 3$ to be dominant, and that is confirmed by a careful numerical study. We denote the $(c, e, f) = (1, 1, 3)$, $(1, 3, 3)$, $(3, 3, 3)$, and $(3, 1, 3)$ integrals in the second term within the square brackets of (59) by $I_1(p, q, \chi, \Delta t_*)$, $I_2(p, q, \chi, \Delta t_*)$, $I_3(p, q, \chi, \Delta t_*)$, and $I_4(p, q, \chi, \Delta t_*)$, respectively.

Regarding the 16 integrals of the first term the following can be said. Because of the y^3 factor in front of the $c = 2, 4$ integrals, which cannot be completely canceled by factors coming from the integral, these terms will become negligible after just a few e folds after horizon crossing. Of the remaining integrals those with $e = 2$ and $e = 4$ will be practically equal, because $\chi \ll 3$. Hence there are only 6 distinct integrals here that have to be considered: those with $c = 1, 3$ and $e = 1, 2, 3$; the 2 integrals with $c = 1, 3$ and $e = 4$ are taken to be equal to those with $e = 2$. Again, these simple estimates are confirmed by careful numerical study of the integrals. We denote the $(c, e) = (1, 1)$, $(1, 2)$, $(1, 3)$, $(3, 1)$, $(3, 2)$, and $(3, 3)$ integrals in the first term within the square brackets of (59) by $J_1(p, q, \Delta t_*)$, $J_2(p, q, \Delta t_*)$, $J_3(p, q, \chi, \Delta t_*)$, $J_4(p, q, \chi, \Delta t_*)$, $J_5(p, q, \chi, \Delta t_*)$, and $J_6(p, q, \chi, \Delta t_*)$, respectively.

Let us now investigate these 10 integrals. Half of them, viz. I_3, I_4, J_4, J_5, and J_6, are zero in the limit of $t \to \infty$, but decrease slowly enough with time that they should not be neglected at the end of inflation. Three of the integrals can be done analytically:

$$\begin{aligned}
J_1(p, q, \Delta t_*) &= \int_{y(\Delta t_*)}^\infty \mathrm{d}y' 2p^2 y' e^{-(p^2+q^2)y'^2} = \frac{p^2}{p^2+q^2} e^{-(p^2+q^2)y^2},\\
J_4(p, q, \chi, \Delta t_*) &= y^\chi(\Delta t_*)\int_{y(\Delta t_*)}^\infty \mathrm{d}y' 2p^2 y'^{1-\chi} e^{-(p^2+q^2)y'^2} = \frac{p^2}{p^2+q^2}(p^2+q^2)^{\chi/2} y^\chi \Gamma\left(1-\frac{\chi}{2}, (p^2+q^2)y^2\right),\\
J_6(p, q, \chi, \Delta t_*) &= q^\chi y^\chi(\Delta t_*)\int_{y(\Delta t_*)}^\infty \mathrm{d}y' 2p^2 y' e^{-p^2y'^2}\Gamma\left(1-\frac{\chi}{2}, q^2y'^2\right)\\
&= -\frac{q^2}{p^2+q^2}(p^2+q^2)^{\chi/2} y^\chi \Gamma\left(1-\frac{\chi}{2}, (p^2+q^2)y^2\right) + q^\chi y^\chi e^{-p^2y^2}\Gamma\left(1-\frac{\chi}{2}, q^2y^2\right),
\end{aligned} \tag{60}$$

where $y(\Delta t_*)$ is given in (46). It is also interesting to look at the behavior of the integrals in the limits of $p \to 0$ ($k \to 0$) and $q \to 0$ ($k' \to 0$). For $p \to 0$ all integrals are zero. For $q \to 0$ only I_1, I_4, J_1, and J_4 are nonzero. The integrals J_1 and J_4 are given above, but in this limit also I_1 and I_4 can be computed analytically (the expression for I_4 is only valid from a few e folds after horizon crossing, i.e. for $py \ll 1$):

[6] There are some entries in the product $F_{bd}B_{de}$ that are first order in slow roll, but these exactly cancel when (59) is worked out explicitly, so that the nonvanishing leading-order coefficients are second order in slow roll.

$$
\begin{aligned}
I_1(p,0,\chi,\Delta t_*) &= \int_{y(\Delta t_*)}^{\infty} \mathrm{d}y' p^\chi y'^{-1+\chi}\Gamma\Big(1-\frac{\chi}{2}, p^2 y'^2\Big) = \frac{1}{\chi}\Big(e^{-p^2y^2} - p^\chi y^\chi \Gamma\Big(1-\frac{\chi}{2}, p^2y^2\Big)\Big) = \frac{1}{\chi} g(p,\chi,\Delta t_*),\\
I_4(p,0,\chi,\Delta t_*) &= p^\chi y^\chi(\Delta t_*)\int_{y(\Delta t_*)}^{\infty} \frac{\mathrm{d}y'}{y'}\Gamma\Big(1-\frac{\chi}{2}, p^2y'^2\Big) \simeq -p^\chi y^\chi \ln(py)\Gamma\Big(1-\frac{\chi}{2}\Big) \approx (\Delta t_* - \ln p)p^\chi e^{-\chi\Delta t_*},
\end{aligned} \tag{61}
$$

where $g(p,\chi,\Delta t_*)$ is defined in (52). Using the results discussed in the text above Eq. (49), one sees that from a few e folds after horizon crossing both start growing linearly with Δt_k ($=\Delta t_* - \ln p$), although finally the limit $1/\chi$ is reached for I_1, while I_4 goes to zero.

For $q > 0$ the four I-integrals have to be evaluated numerically; the resuls are plotted in Fig. 1 as a function of q/p for $\Delta t_* = 50$, for various values of the parameters χ and p. Although we will be using the exact numerical results for all integrals when plotting the three-point correlator, one can get an approximation by neglecting the $\chi/2$ inside the gamma function in $u_3(py')$ [see (59) and (48)]. Then the I-integrals can be done analytically, with the results

$$
\begin{aligned}
I_1(p,q,\chi,\Delta t_*) &\approx \frac{1}{2}p^\chi(p^2+q^2)^{-\chi/2}\Gamma\Big(\frac{\chi}{2},(p^2+q^2)y^2\Big),\\
I_2(p,q,\chi,\Delta t_*) &\approx \frac{1}{2}p^\chi q^\chi(p^2+q^2)^{-\chi}\Gamma(\chi,(p^2+q^2)y^2),\\
I_3(p,q,\chi,\Delta t_*) &\approx \frac{1}{2}p^\chi q^\chi(p^2+q^2)^{-\chi/2}y^\chi\Gamma\Big(\frac{\chi}{2},(p^2+q^2)y^2\Big),\\
I_4(p,q,\chi,\Delta t_*) &\approx \frac{1}{2}p^\chi y^\chi\Gamma(0,(p^2+q^2)y^2),
\end{aligned} \tag{62}
$$

so that we can make the following estimates:

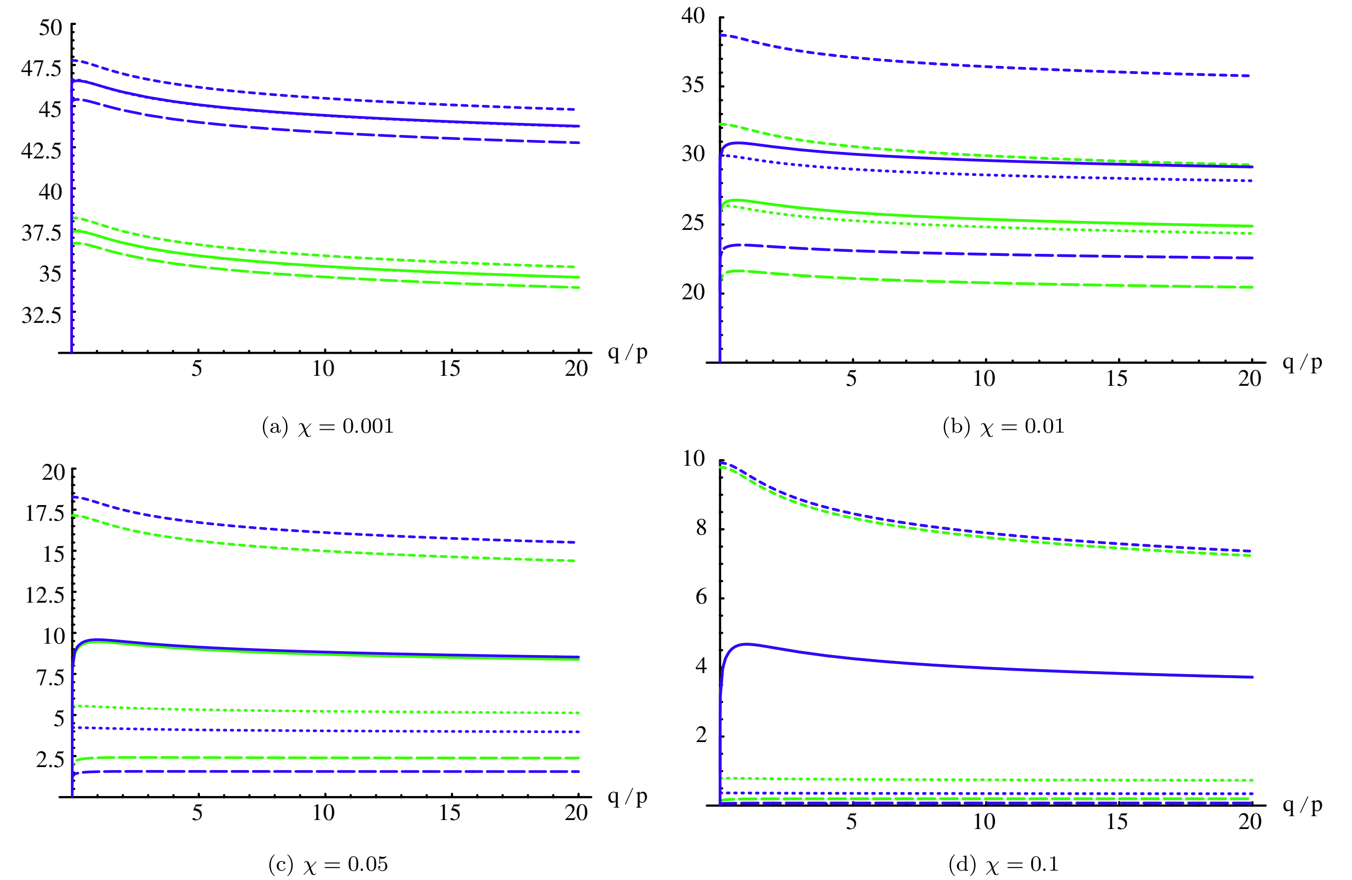

FIG. 1 (color online). The integrals $I_1(p,q,\chi,\Delta t_*)$ (small dashes), $I_2(p,q,\chi,\Delta t_*)$ (solid), $I_3(p,q,\chi,\Delta t_*)$ (large dashes), and $I_4(p,q,\chi,\Delta t_*)$ (dots) plotted as a function of q/p for $\Delta t_* = 50$, both for $p = 1$ (blue, darker) and for $p = \exp(10)$ (green, lighter) (i.e. 50 and 40 e folds after horizon crossing of the mode k). The smoothing parameter $c = 3$, although the dependence on c is negligible. The different figures correspond with different values of χ, as indicated. Note that I_2 and I_4 almost coincide in the first plot.

$$\begin{aligned} I_1 &\approx \Delta t_k(1 - \tfrac{1}{2}\chi\Delta t_k + \tfrac{1}{6}(\chi\Delta t_k)^2),\\ I_2 &\approx \Delta t_k(1 - \chi\Delta t_k + \tfrac{2}{3}(\chi\Delta t_k)^2),\\ I_3 &\approx \Delta t_k(1 - \tfrac{3}{2}\chi\Delta t_k + \tfrac{7}{6}(\chi\Delta t_k)^2),\\ I_4 &\approx \Delta t_k(1 - \chi\Delta t_k + \tfrac{1}{2}(\chi\Delta t_k)^2), \end{aligned} \qquad (63)$$

for $\chi^{-1} \gg \Delta t_k$, and

$$I_1 \approx \chi^{-1}, \qquad I_2 \approx \tfrac{1}{2}\chi^{-1}, \qquad I_3 \approx 0, \qquad I_4 \approx 0 \quad (64)$$

for $\chi^{-1} \ll \Delta t_k$. As a rough approximation, they can be taken independent of q for reasonable ranges, say up to $q/p \sim 100$. For I_2 and I_3 this range has a lower limit as well: $100^{-1} \lesssim q/p \lesssim 100$; they are zero for $q = 0$. Note that these secular I-integrals typically give a result which is of the order of an inverse slow-roll parameter.

For the J-integrals the results are much smaller, since the integration interval is restricted because of the window function. The integrals J_1, J_2, and J_3 become completely independent of Δt_k from a few e folds after horizon crossing of the mode k. Moreover, J_1 and J_2 are independent of χ, while J_3 has a relatively weak dependence on χ. On the other hand, all three depend strongly on q/p. They are plotted in Fig. 2(a). The integrals J_4, J_5, and J_6 depend strongly on both q/p and $\chi\Delta t_k$. In the limit $\chi\Delta t_k \ll 1$ they become equal to J_1, J_2, and J_3, respectively, while in the opposite limit they all go to zero. They are plotted in Fig. 2(b). For $q = 0$ we have an exact analytic result; for $q = p$ (i.e. $k = k'$) it is sometimes useful to have an analytic approximation:

$$\begin{aligned} &J_1 \approx \tfrac{1}{2}, \quad J_2 \approx 0.14, \quad J_3 \approx \tfrac{1}{2}, \quad J_4 \approx \tfrac{1}{2}(1-\chi\Delta t_k),\\ &J_5 \approx 0.14(1-\chi\Delta t_k), \quad J_6 \approx \tfrac{1}{2}(1-\chi\Delta t_k), \end{aligned} \qquad (65)$$

for $\chi^{-1} \gg \Delta t_k$, and

$$\begin{aligned} &J_1 \approx \tfrac{1}{2}, \qquad J_2 \approx 0.14, \qquad J_3 \approx \tfrac{1}{2},\\ &J_4 \approx 0, \qquad J_5 \approx 0, \qquad J_6 \approx 0, \end{aligned} \qquad (66)$$

for $\chi^{-1} \ll \Delta t_k$.

Having studied all the integrals, we can now work out (59) explicitly. We focus on the $a = 1$ component of v_{ia}, that is ζ_i^1 (the adiabatic component of ζ_i), since that is the quantity we want to compute the three-point correlator of in the end. The final result for $\zeta_i^{(2)1}$ in the two-field case, in a leading-order slow-roll approximation (constant slow-roll parameters) and valid well after horizon crossing, is

$$\begin{aligned} \zeta_i^{(2)1}(t,\boldsymbol{k}) = \frac{\kappa^2}{8}\frac{H^2}{\tilde\epsilon}\iint\frac{\mathrm{d}^3\boldsymbol{k}\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3}\frac{1}{k^{3/2}k'^{3/2}}e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} \\ \times\Bigg[(ik_ie^{i\boldsymbol{k}\cdot\boldsymbol{x}}\alpha_1(\boldsymbol{k})+\text{c.c.})\Big\{[(2\tilde\epsilon+\tilde\eta^{\parallel})J_1]\alpha_1(\boldsymbol{k}')+2\frac{\tilde\eta^{\perp}}{\chi}\Big[(2\tilde\epsilon+\tilde\eta^{\parallel})(J_1-J_3)+\frac{\chi}{2}(J_3-J_2)\Big]\alpha_2(\boldsymbol{k}')\Big\} \\ +2\frac{\tilde\eta^{\perp}}{\chi}(ik_ie^{i\boldsymbol{k}\cdot\boldsymbol{x}}\alpha_2(\boldsymbol{k})+\text{c.c.})\Big\{\Big[(2\tilde\epsilon+\tilde\eta^{\parallel})(J_1-J_4)+\psi_1(I_1-I_4)+\chi\Big(-\tilde\epsilon+2\tilde\eta^{\parallel}-\frac{\tilde\xi^{\perp}}{\tilde\eta^{\perp}}\Big)I_1\Big]\alpha_1(\boldsymbol{k}') \\ +2\frac{\tilde\eta^{\perp}}{\chi}\Big[(2\tilde\epsilon+\tilde\eta^{\parallel})(J_1-J_3-J_4+J_6)+\frac{\chi}{2}(J_3-J_2-J_6+J_5)+\psi_1(I_1-I_4-I_2+I_3) \\ +\chi\Big(-\tilde\epsilon+2\tilde\eta^{\parallel}-\frac{\tilde\xi^{\perp}}{\tilde\eta^{\perp}}\Big)(I_1-I_2)+\frac{\chi}{2\tilde\eta^{\perp}}\psi_2(I_2-I_3)+\frac{\chi^2}{2(\tilde\eta^{\perp})^2}\omega I_2\Big]\alpha_2(\boldsymbol{k}')\Big\}\Bigg]+\text{c.c.}, \end{aligned} \qquad (67)$$

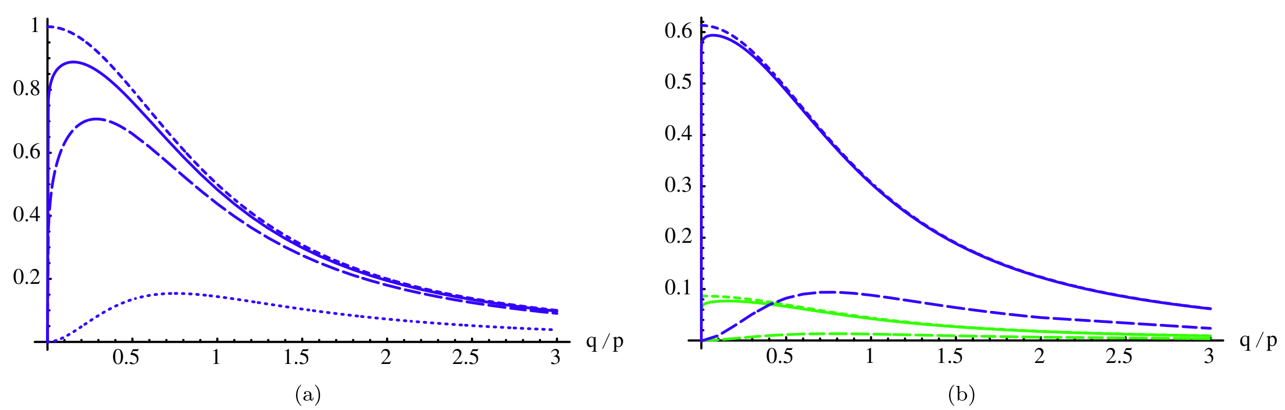

FIG. 2 (color online). (a) The integrals $J_1(p,q,\Delta t_*)$ (small dashes), $J_2(p,q,\Delta t_*)$ (dots), and $J_3(p,q,\chi,\Delta t_*)$ for $\chi = 0.05$ (solid) and $\chi = 0.2$ (large dashes) plotted as a function of q/p, a few e folds after horizon crossing of the mode k when the dependence on $\Delta t_k = \Delta t_* - \ln p$ has become negligible. (b) The integrals $J_4(p,q,\chi,\Delta t_*)$ (small dashes), $J_5(p,q,\chi,\Delta t_*)$ (large dashes), and $J_6(p,q,\chi,\Delta t_*)$ (solid) plotted as a function of q/p for $p = 1$ and $\Delta t_* = 50$, for both $\chi = 0.01$ (blue, darker) and $\chi = 0.05$ (green, lighter).

with $\omega \equiv (\tilde{\epsilon} + \tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel} + (3\tilde{\epsilon} - \tilde{\eta}^{\parallel})\chi + (\tilde{\eta}^{\perp})^2$ defined as short-hand notation. The arguments $(p, q, \chi, \Delta t_*)$ of the integrals have been suppressed, but it should of course be kept in mind that that is where the time dependence resides in this expression. Note that in the single-field limit, where all terms with α_2 disappear, we recover the result of [5]. For the three-point correlator we need to know $\zeta^{(2)1} \equiv \partial^{-2}\partial^i \zeta_i^{(2)1}$, which is given by the same expression (67), but with $e^{i\boldsymbol{k}'\cdot\boldsymbol{x}}(ik_i e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\alpha_1(\boldsymbol{k}) + \text{c.c.})$ replaced by

$$\left(\frac{k^2 + \boldsymbol{k}\cdot\boldsymbol{k}'}{|\boldsymbol{k}+\boldsymbol{k}'|^2} e^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}}\alpha_1(\boldsymbol{k}) + \frac{k^2 - \boldsymbol{k}\cdot\boldsymbol{k}'}{|\boldsymbol{k}-\boldsymbol{k}'|^2} e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}\alpha_1^*(\boldsymbol{k})\right) \tag{68}$$

and the same for $\alpha_2(\boldsymbol{k})$.

C. Bispectrum

As in the single-field case, $\langle\zeta^{(2)1}\rangle$ is indeterminate. To remove this ambiguity and also require that perturbations have a zero average, we define $\tilde{\zeta}^m \equiv \zeta^m - \langle\zeta^m\rangle$. Expanding $\tilde{\zeta}^m = \tilde{\zeta}^{(1)m} + \tilde{\zeta}^{(2)m}$ and switching over to Fourier space, we finally arrive at our end result for the three-point correlator (or rather the bispectrum) of the adiabatic component:

$$\langle\tilde{\zeta}^1(t,\boldsymbol{x}_1)\tilde{\zeta}^1(t,\boldsymbol{x}_2)\tilde{\zeta}^1(t,\boldsymbol{x}_3)\rangle^{(2)}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = (2\pi)^3\delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)[f(\boldsymbol{k}_1,\boldsymbol{k}_2)+f(\boldsymbol{k}_1,\boldsymbol{k}_3)+f(\boldsymbol{k}_2,\boldsymbol{k}_3)] \tag{69}$$

with

$$\begin{aligned} f(\boldsymbol{k},\boldsymbol{k}') \equiv{}& \frac{\kappa^4}{16}\frac{1}{k^3k'^3}\frac{H^4}{\tilde{\epsilon}^2}\frac{k^2+\boldsymbol{k}\cdot\boldsymbol{k}'}{|\boldsymbol{k}+\boldsymbol{k}'|^2}\Bigg\{(2\tilde{\epsilon}+\tilde{\eta}^{\parallel})J_1 \\ &+ 4\left(\frac{\tilde{\eta}^{\perp}}{\chi}\right)^2\Bigg[g(p,\chi,\Delta t_*)\bigg[(2\tilde{\epsilon}+\tilde{\eta}^{\parallel})(J_1-J_4) + \psi_1(I_1-I_4) + \chi\left(-\tilde{\epsilon}+2\tilde{\eta}^{\parallel}-\frac{\tilde{\xi}^{\perp}}{\tilde{\eta}^{\perp}}\right)I_1\bigg] \\ &+ g(q,\chi,\Delta t_*)\bigg[(2\tilde{\epsilon}+\tilde{\eta}^{\parallel})(J_1-J_3)+\frac{\chi}{2}(J_3-J_2)\bigg]\Bigg] \\ &+ 16\left(\frac{\tilde{\eta}^{\perp}}{\chi}\right)^4 g(p,\chi,\Delta t_*)g(q,\chi,\Delta t_*)\bigg[(2\tilde{\epsilon}+\tilde{\eta}^{\parallel})(J_1-J_3-J_4+J_6)+\frac{\chi}{2}(J_3-J_2-J_6+J_5) \\ &+ \psi_1(I_1-I_4-I_2+I_3) + \chi\left(-\tilde{\epsilon}+2\tilde{\eta}^{\parallel}-\frac{\tilde{\xi}^{\perp}}{\tilde{\eta}^{\perp}}\right)(I_1-I_2) + \frac{\chi}{2\tilde{\eta}^{\perp}}\psi_2(I_2-I_3)+\frac{\chi^2}{2(\tilde{\eta}^{\perp})^2}\omega I_2\bigg]\Bigg\} + \boldsymbol{k}\leftrightarrow\boldsymbol{k}'. \end{aligned} \tag{70}$$

Again, this result is valid in the two-field case, in a leading-order slow-roll approximation (constant slow-roll parameters) and valid from a sufficient number of e folds after horizon crossing that transient effects have disappeared. The function $g(p, \chi, \Delta t_*)$ is given in (52), χ, ψ_1, and ψ_2 are defined in Sec. IV, and ω is defined below (67). Remember that all the integrals and the function $g(p, \chi, \Delta t_*)$ depend on the momenta via $p = k/k_*$ and $q = k'/k_*$ and hence are affected by the interchange of $\boldsymbol{k}$ and $\boldsymbol{k}'$. In the single-field limit only the first line of (70) remains, which agrees exactly with [5].

In the limit $k_3 \ll k_1, k_2$ (and hence $\boldsymbol{k}_1 = -\boldsymbol{k}_2 \equiv \boldsymbol{k}$, while we also fix $\boldsymbol{k}_* = \boldsymbol{k}$ so that we do not need to write a subscript on Δt), all the integrals can be performed analytically and the result is (leaving aside the overall factor of $(2\pi)^3\delta^3(\sum_s \boldsymbol{k}_s)$):

$$\begin{aligned} \langle\tilde{\zeta}^1\tilde{\zeta}^1\tilde{\zeta}^1\rangle^{(2)} ={}& \frac{\kappa^4}{8}\frac{1}{k^3k_3^3}\frac{H^4}{\tilde{\epsilon}^2}\left(1+4\left(\frac{\tilde{\eta}^{\perp}}{\chi}\right)^2\right) \\ &\times\Bigg\{(2\tilde{\epsilon}+\tilde{\eta}^{\parallel})\left(1+4\left(\frac{\tilde{\eta}^{\perp}}{\chi}\right)^2(1-e^{-\chi\Delta t})^2\right) \\ &+4\left(\frac{\tilde{\eta}^{\perp}}{\chi}\right)^2(1-e^{-\chi\Delta t})\bigg[\frac{\psi_1}{\chi}(1-(1+\chi\Delta t)e^{-\chi\Delta t}) \\ &+\left(-\tilde{\epsilon}+2\tilde{\eta}^{\parallel}-\frac{\tilde{\xi}^{\perp}}{\tilde{\eta}^{\perp}}\right)(1-e^{-\chi\Delta t})\bigg]\Bigg\}, \end{aligned} \tag{71}$$

where the term on the second line comes from the J-integrals, and the terms on the third and fourth lines from the I-integrals. Again, this agrees with the single-field result in the limit $\tilde{\eta}^{\perp} \to 0$. Unlike the single-field case, the multiple-field result cannot be expressed in terms of the scalar spectral index and the power spectrum only (see (54) and (55) and [17] for expressions for the isocurvature and mixing components). Instead of the three-point correlator itself, it is actually more useful to look at the ratio of the bispectrum to the square of the power spectrum, since that ratio is related to observables like the $f_{\rm NL}$ parameter (more about that later). Dividing (71) by the square of (54) (one with momentum k and the other with k_3) and taking the limit of $\tilde{\eta}^{\perp}/\chi \gg 1$, we get for the two opposite limits of $\chi\Delta t$ that

$$\frac{\langle\tilde{\zeta}^1\tilde{\zeta}^1\tilde{\zeta}^1\rangle}{(\langle\tilde{\zeta}^1\tilde{\zeta}^1\rangle)^2} = \begin{cases} 2\left(\tilde{\epsilon}+3\tilde{\eta}^{\parallel}-\frac{\tilde{\xi}^{\perp}}{\tilde{\eta}^{\perp}}\right)+\psi_1\Delta t & \text{for } \chi^{-1}\gg\Delta t, \\ 2\left(\tilde{\epsilon}+3\tilde{\eta}^{\parallel}-\frac{\tilde{\xi}^{\perp}}{\tilde{\eta}^{\perp}}+\frac{\psi_1}{\chi}\right) & \text{for } \chi^{-1}\ll\Delta t. \end{cases} \tag{72}$$

Now if we assume that $\tilde{\eta}^{\perp}$ is larger than the other slow-roll parameters, the dominating term in both expressions will be the $4(\tilde{\eta}^{\perp})^2$ in ψ_1 (36), so that the two expressions in (72) will go to $4(\tilde{\eta}^{\perp})^2\Delta t$ and $8(\tilde{\eta}^{\perp})^2/\chi$, respectively. Hence,

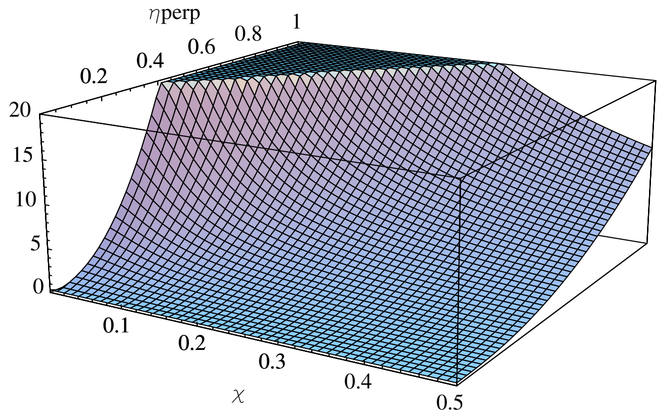

FIG. 3 (color online). The bispectrum (71) divided by the square of the power spectrum (54) (with two different momenta) in the limit where one of the momenta is much smaller than the other two, plotted as a function of $\tilde{\eta}^\perp$ and χ for $\Delta t = 50$, $\tilde{\epsilon} = \tilde{\eta}^\parallel = 0.05$, and $\tilde{\xi}^\parallel = \tilde{\xi}^\perp = (\sqrt{2\tilde{\epsilon}}/\kappa)V_{221}/(3H^2) = 0.003$.

while this ratio of the bispectrum to the square of the power spectrum is first order in slow roll by naive power counting (counting $1/\Delta t$ as a slow-roll parameter), as in the single-field case, it can be much larger for models with a relatively small χ and relatively large $\tilde{\eta}^\perp$. For example, $\tilde{\eta}^\perp = 0.07$ and $\Delta t = 1/\chi = 50$ would already give a ratio of more than unity, so that a value about 100 times larger than in the single-field case seems well within range for multiple-field models. This is confirmed by the full plot of (71) divided by the square of (54) as a function of $\tilde{\eta}^\perp$ and χ given in Fig. 3. It is also interesting to see that in the cases where non-Gaussianity is large, this is caused by the I-integrals (i.e. the superhorizon integrated background effects that are absent in single-field inflation): roughly speaking it boils down to $\tilde{\epsilon}J_1$ versus $(\tilde{\eta}^\perp)^2 I_1$, which gives $\tilde{\epsilon}$ versus the smaller of $(\tilde{\eta}^\perp)^2\Delta t$ and $(\tilde{\eta}^\perp)^2/\chi$, either of which can easily be 2 orders of magnitude larger.

In the opposite limit of $k_1 = k_2 = k_3 \equiv k$ (where we again set $k_* = k$ so that Δt is unambiguous) we do not have an exact analytic result for all integrals, but we can use the approximations (63)–(66). We find the following results for the bispectrum divided by the square of the power spectrum, in the limit of $\tilde{\eta}^\perp/\chi \gg 1$:

$$\frac{\langle \tilde{\zeta}^1 \tilde{\zeta}^1 \tilde{\zeta}^1 \rangle}{(\langle \tilde{\zeta}^1 \tilde{\zeta}^1 \rangle)^2} = \begin{cases} \frac{3}{2}\left(\frac{0.36+\omega/(\tilde{\eta}^\perp)^2}{\Delta t} - \tilde{\epsilon} + 2\tilde{\eta}^\parallel - \frac{\tilde{\xi}^\perp}{\tilde{\eta}^\perp} + \frac{\psi_2}{2\tilde{\eta}^\perp} + \frac{1}{3}\psi_1 \Delta t\right) & \text{for } \chi^{-1} \gg \Delta t, \\ \frac{3}{2}\left(0.36\chi + \frac{\chi\omega}{2(\tilde{\eta}^\perp)^2} - \tilde{\epsilon} + 2\tilde{\eta}^\parallel - \frac{\tilde{\xi}^\perp}{\tilde{\eta}^\perp} + \frac{\psi_2}{2\tilde{\eta}^\perp} + \frac{\psi_1}{\chi}\right) & \text{for } \chi^{-1} \ll \Delta t. \end{cases} \tag{73}$$

If we assume once again that $\tilde{\eta}^\perp$ is larger than the other slow-roll parameters, the dominating term in both expressions will again be the $4(\tilde{\eta}^\perp)^2$ in ψ_1, so that the two expressions will go to $2(\tilde{\eta}^\perp)^2\Delta t$ and $6(\tilde{\eta}^\perp)^2/\chi$, respectively. Finally we should check that all the limits that produce large non-Gaussianity do not produce an unacceptably large spectral index at the same time. Fortunately that is not the case: from (55) we derive, under the same limits as in (72) and (73),

$$n_{\rm ad} - 1 = \begin{cases} -4\tilde{\epsilon} - 2\tilde{\eta}^\parallel - \frac{8(\tilde{\eta}^\perp)^2\Delta t}{1+4(\tilde{\eta}^\perp)^2(\Delta t)^2} & \text{for} \quad \chi^{-1} \gg \Delta t, \\ -4\tilde{\epsilon} - 2\tilde{\eta}^\parallel & \text{for} \quad \chi^{-1} \ll \Delta t. \end{cases} \tag{74}$$

After having discussed the various momentum limits, we finally show the full dependence on the relative magnitude of the momenta of the bispectrum divided by the square of the power spectrum in Fig. 4(a), where we did not use any analytic approximations for the integrals. To be precise, it is actually the bispectrum given in (69) and (70), without the overall $(2\pi)^3\delta^3(\sum_s \boldsymbol{k}_s)$ factor (but taking into account the relation between the momenta that the δ-function implies), divided by the sum of products of power spectra (54) with different momenta, as follows:

$$\tilde{f}_{\rm NL} \equiv \frac{\langle \tilde{\zeta}^1 \tilde{\zeta}^1 \tilde{\zeta}^1 \rangle (k_1, k_2, k_3)}{[\langle \tilde{\zeta}^1 \tilde{\zeta}^1 \rangle (k_1) \langle \tilde{\zeta}^1 \tilde{\zeta}^1 \rangle (k_2) + \langle \tilde{\zeta}^1 \tilde{\zeta}^1 \rangle (k_1) \langle \tilde{\zeta}^1 \tilde{\zeta}^1 \rangle (k_3) + \langle \tilde{\zeta}^1 \tilde{\zeta}^1 \rangle (k_2) \langle \tilde{\zeta}^1 \tilde{\zeta}^1 \rangle (k_3)]/3}. \tag{75}$$

This quantity can be seen as a momentum-dependent version of the $f_{\rm NL}$ parameter often used in the literature (see e.g. [4]).[7] We now choose k_* to be the mode that crossed the horizon 50 e folds before the end of inflation (i.e. we set $\Delta t_* = 50$). The function $\tilde{f}_{\rm NL}$ depends on the three scalars k_1, k_2, k_3, but we can plot it in a two-dimensional triangular domain if we fix their sum, which we do by setting $(k_1 + k_2 + k_3)/k_* = 3$. This convenient way of plotting the three-point correlator in a triangle, clearly demonstrating its symmetries, was introduced in [5], and is illustrated in Fig. 4(b).[8] The quantities on the axes are

[7]There is a difference of a factor of order unity between $\tilde{f}_{\rm NL}$ and $f_{\rm NL}$ even in the equal momentum limit, caused partly by the difference between ζ and the gravitational potential Φ which was used in the original definition.

[8]Note, however, that in [5] a different normalization factor was used.

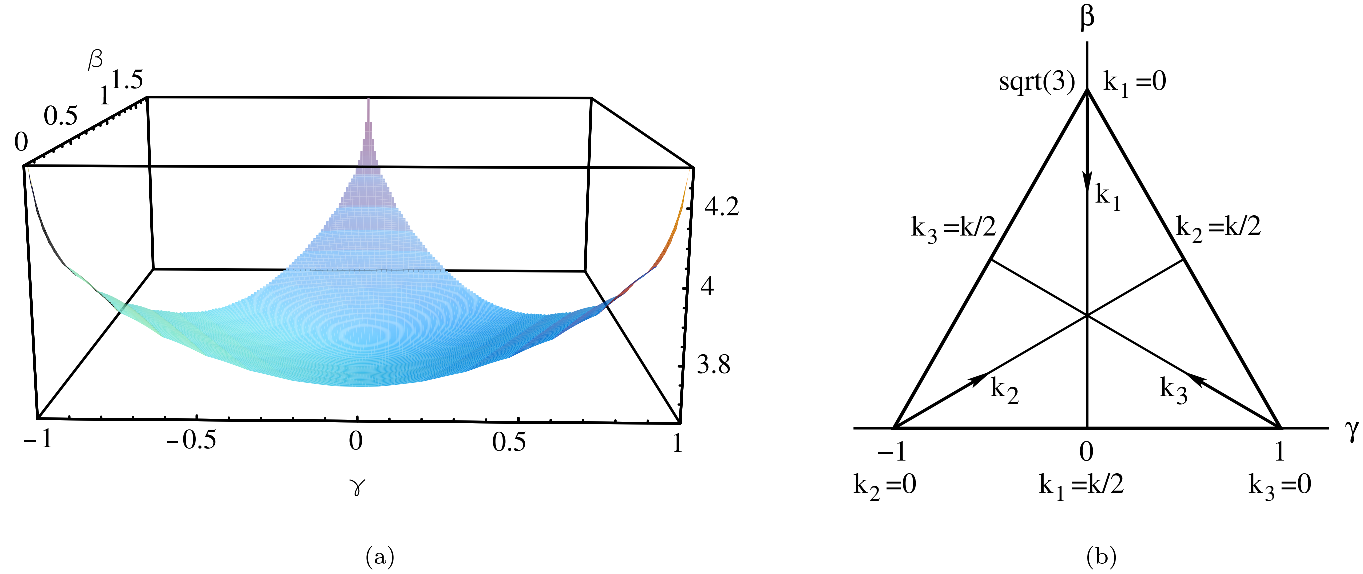

FIG. 4 (color online). (a) The bispectrum (69) and (70), without the overall $(2\pi)^3\delta^3(\sum_s \boldsymbol{k}_s)$ factor and divided by the square of the power spectrum (54) as defined in (75), plotted as a function of the relative size of the three momenta. The sum of the momenta is chosen as $(k_1 + k_2 + k_3) = 3k_*$ with k_* fixed by choosing $\Delta t_* = 50$. The values for the parameters are $\tilde{\epsilon} = \tilde{\eta}^{\|} = 0.05$, $\tilde{\eta}^{\perp} = 0.2$, $\chi = 0.01$, $\tilde{\xi}^{\|} = \tilde{\xi}^{\perp} = (\sqrt{2\tilde{\epsilon}}/\kappa)V_{221}/(3H^2) = (\sqrt{2\tilde{\epsilon}}/\kappa)V_{222}/(3H^2) = 0.003$ (as well as $c = 3$, although the dependence on c is negligible). (b) An explanation of the triangular domain used, defined in (76), with $k \equiv k_1 + k_2 + k_3$.

$$\gamma \equiv 2\frac{k_2 - k_3}{k_1 + k_2 + k_3}, \qquad \beta \equiv -\sqrt{3}\frac{k_1 - k_2 - k_3}{k_1 + k_2 + k_3}. \quad (76)$$

At the vertices of the triangle one of the three momenta is equal to zero. Lines of constant k_s are parallel to the sides of the triangle (a different side for each $s = 1, 2, 3$) and k_s increases linearly perpendicular to these. At the side itself the corresponding momentum is equal to half the total sum, $(k_1 + k_2 + k_3)/2$. In the center of the triangle all momenta have equal length.

The plot in Fig. 4(a) has been made for all first-order slow-roll parameters equal to 0.05, except $\tilde{\eta}^{\perp} = 0.2$ and $\chi = 0.01$, and all second-order slow-roll parameters equal to 0.003. We see that there is a dependence on the relative magnitude of the momenta. Though not visible in the figure, this dependence is strongest very near the vertices of the triangle, which is the limit of (71), where for this specific example the value 9.0 is reached. Of course logarithmically the region near the vertices covers an infinite range of magnitudes in momentum ratios. (The fact that the result is largest in the squeezed momentum limit agrees with the findings of [18].) The value at the center is 3.7. Assuming that a naive extrapolation of this result at the end of inflation to the time of recombination is allowed, so that the quantity plotted is indeed comparable to the observable $f_{\rm NL}$, we see that this model does produce sufficient non-Gaussianity to be detectable with the Planck satellite. To compare this plot with the one for the single-field case in [5] one should keep in mind that there an additional factor of $(2\tilde{\epsilon} + \tilde{\eta}^{\|})$ was left out (and there are some differences in the momentum normalization factor, but that does not change the magnitude much), so that the multiple-field result is indeed about 2 orders of magnitude larger.

D. Comparison with semianalytic calculation using quadratic potential

Of course it may be argued that the approximate model considered here is not very realistic, with all slow-roll parameters constant with time (in particular $\tilde{\eta}^{\perp}$ and χ). We should also stress that the calculation here was made under the assumption that $\chi > 0$, which is not true for all models. While we will study more realistic models in great detail in a future publication [6], both semianalytically and purely numerically without any approximations, for direct comparison here we present a bispectrum calculation using the Green's function formalism outlined in Sec. III. We have investigated a simple two-field model with a quadratic potential $V = \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}m_2^2\phi_2^2$ with $m_1 = 1 \cdot 10^{-5}\kappa^{-1}$ (the overall mass magnitude can be freely adjusted to fix the amplitude of the power spectrum). The analytic solution (30) is used as the linear source term in (23), the superhorizon Green's function is then calculated from (22) and (33), and the bispectrum computed from these using (29) and (37). We find that for a mass ratio $m_2/m_1 = 9$ and initial conditions $\phi_1 = \phi_2 = 13\kappa^{-1}$ we get relatively large non-Gaussianity: with all momenta equal, that is, at the center of the triangle, the ratio of the bispectrum to the square of the power spectrum is, in the slow roll limit,

$$\tilde{f}_{\rm NL} = 1.8, \quad (77)$$

where we have taken horizon crossing to be 58 e folds before the end of inflation. The ratio of the contribution from the I-integrals to that of the J- integrals is 74. This confirms our assertion that the integrated secular terms (the $\bar{A}$ term in (29)) subsequent to horizon crossing dominate the contributions to the bispectrum. We note also that the spectral index in this model is 0.93, which is observationally acceptable. While the investigation of the quadratic model is preliminary at this stage, it is clear that large non-Gaussianity ($f_{\rm NL}$ greater than unity) can be obtained in a real multiple-field inflation model.

Even though the slow-roll parameters are definitely not constant in the quadratic model, we find that the expressions in the previous subsection can be used to make a useful approximation of the numerical result. In this case, for example, to estimate $\tilde{\eta}^{\perp}$ we use a representative value and adjust Δt to reflect the region of support where its value is significant. To compare to the quadratic potential result, here we have taken $(\tilde{\eta}^{\perp})^2 = 0.73$ (its maximum value) and $\Delta t = 1.0$ (full width at half maximum), and use the limit $2(\tilde{\eta}^{\perp})^2\Delta t$ given below (73). From this we estimate the value 1.5, which is relatively close to the numerical value. This seems to indicate that the constant slow-roll, analytic results of the previous subsection[9], while in principle unrealistic, can be used to get a first estimate of the amount of non-Gaussianity even in real models with varying slow-roll parameters.

E. Discussion

Before we conclude, a couple of points regarding the consistency of our approach need to be discussed. The first is an inherent limitation of our method in capturing all possible sources of non-Gaussianity, since, by using the linear perturbation solutions to source our nonlinear equations, we are implicitly neglecting all nonlinear interactions up to horizon crossing. We believe that this accounts for the small discrepancy between the momentum dependence of our single-field three-point correlator [5] and that obtained from the tree-level action calculations of [7] when $k_1 \approx k_2 \approx k_3$, whereas in the limit k_1, $k_2 \gg k_3$ the two correlators agree exactly. We surmise that the superhorizon nonlinear effects described by our method and the horizon-crossing effects we are missing are of comparable magnitude for single-field inflation in the equal momentum limit.

For multiple-field inflation, however, the situation is very different. We can see this by using the quantitative results above to interpret our key integral expression for the three-point correlator (29). In the case where multiple-field effects are large (as indicated by the behavior of $\tilde{\eta}^{\perp}$) it is the perturbation of the long-wavelength evolution term in the integrand of (29) (represented by $\bar{A}^{(0)}_{abc}$ and absent in the single-field case) that dominates over perturbations of the stochastic source term which contains the linear perturbations (represented by $\bar{X}^{(1)}_{amc}$; the term that would, in principle, be influenced by these horizon-crossing effects). For example, in the case considered in Fig. 4(a) the contribution of the $\bar{X}^{(1)}_{amc}$ term, which is given by the J-integrals in (70), is 20 times smaller than the contribution from the $\bar{A}^{(0)}_{abc}$ term at the vertices of the triangle, and 200 times smaller at the center. But would the $\bar{X}^{(1)}_{amc}$ term be similarly enhanced when taking into account nonlinear effects at horizon crossing in the multiple-field case? This seems unlikely, given that horizon crossing is only a short transition, while the large effects of the other term are caused by a buildup over a significant time interval. Moreover, this question appears to have been answered definitively in the negative by recent work [19]. Generalising [7] for multiple-field inflation, though only up to horizon crossing, it shows that these extra contributions remain of the order of small slow-roll parameters, just as in the single-field case. In that sense, the papers [7,19] are important null results which clarify that our approach focusing on nonlinear superhorizon effects will indeed capture the main non-Gaussian contributions from multiple-field inflation models.

The second point regards the possible influence of loop corrections to the stochastic picture for generating and evolving inflationary perturbations. It is generally accepted within the cosmological community that quantum fluctuations can be considered classical for modes which have crossed the horizon, and we explicitly make such an assumption here by using classical random fields to set up initial conditions for long wavelengths via the source terms in (10). The long-wavelength evolution is then followed by using the classical equations of motion. A natural question to be asked is whether loop corrections might play a role in the superhorizon evolution. Recently, the question was addressed in [20] for a single inflaton field plus a number of noninteracting massless scalar fields. A theorem was proved about the growth of loop effects and it was shown that for the theories mentioned loop effects were determined at horizon crossing and were subdominant. Since the conditions of the theorem imply $\tilde{\eta}^{\perp} = 0$, these results are not directly applicable to the kind of models considered in this paper. However, even if loop effects were to grow with time in such models, they would still need to dominate over the classical growth that these models can exhibit in order to interfere with the classical picture for the evolution of the perturbations we have developed here. Nevertheless, a definitive answer to such matters requires further investigation.

VI. CONCLUSIONS

In this paper we have investigated non-Gaussianity in multiple-field inflation using the formalism of [2,3], em-

[9]Except expression (74) for the spectral index, which is a poor approximation in this case.

phasizing analytic calculations. That formalism is based on fully nonlinear equations for long wavelengths, with stochastic source terms taking into account the short-wavelength quantum fluctuations. For analytic calculations an expansion of the relevant equations in perturbation orders is necessary. However, it is much easier to derive the perturbed equations at second order directly from the nonlinear equation of motion for ζ_i than from perturbing the original Einstein equations. Of course, in a fully numerical investigation no expansion in perturbation orders has to be made; this will be explored in future work.

We derived two main results in this paper. The first is the general solution for the bispectrum, (27) with (28) or (29). Even though this is an integral expression, it will be relatively simple to evaluate in a semianalytic calculation and it yields the full momentum dependence. To achieve this one only needs solutions for the homogeneous background quantities in the inflation model, for the linear perturbation variable $Q^{\rm lin}$ around horizon crossing, and for the homogeneous Green's function, as well as expressions for the spatial derivatives of the various coefficient functions. The latter can all be computed analytically from the constraint Eqs. (14)–(16); for the general two-field case all relevant expressions were given explicitly in [3] and this paper. Computing the bispectrum is then just a question of performing a few time integrals. An accurate semianalytic treatment will be the subject of a forthcoming paper [6], though we do provide the results of a slow-roll calculation for a quadratic potential here. In the present paper, however, we have emphasized an example where we could proceed purely analytically.

In the second part of the paper we studied two-field slow-roll inflation, with the strong leading-order approximation that all slow-roll parameters are constant. In this case we could work out the bispectrum explicitly analytically (apart from a few integrals that had to be done numerically, although we found analytic approximations in certain limits), which is the other main result of this paper, Eq. (70). We found that in this two-field case the bispectrum can easily be 2 orders of magnitude larger than in the single-field case, due to the continued buildup of non-Gaussianity on superhorizon scales caused by the influence of the isocurvature mode on the adiabatic perturbation. We note that even though the presence of isocurvature perturbations during inflation is crucial, it is not mandatory that they survive afterwards. In fact they feed into the adiabatic perturbation and can disappear by the end of inflation (as in the cases studied here). On the other hand, if any isocurvature modes do persist at the end of inflation, their fate will depend on the details of reheating and further evolution.

The bispectrum divided by the square of the power spectrum, which can be seen as a momentum-dependent generalization of the $f_{\rm NL}$ observable, can be $f_{\rm NL} \approx \mathcal{O}(1)$–$\mathcal{O}(10)$, or even larger in extreme cases. If a straightforward extrapolation of this result at the end of inflation to the time of recombination is justified, a subject which still needs to be studied in more detail, this means that the Planck satellite, and to a lesser extent even the WMAP satellite, should be able to confirm or rule out certain classes of multiple-field inflation models. Finally we want to stress that beyond estimating the amplitude of the bispectrum, we also give its explicit momentum dependence. While this dependence is rather flat for momenta of comparable size, there is a significant difference between more extreme momentum limits.

We believe this paper is a significant step towards providing quantitative and testable predictions of non-Gaussianity from multiple-field inflation. Nevertheless there are still a number of issues that remain to be investigated in more detail, and on which we are working for future publications. In the first place, we will apply our general solution for the bispectrum to more realistic inflation models, particularly those strongly motivated by fundamental theory. This will require a semianalytic treatment, but because we are dealing with integral equations, we believe that the strong slow-roll approximations presented here will actually provide reasonable analytic estimates of the exact results. As a first step we presented here the results of the semianalytic slow-roll treatment of an explicit two-field model with a quadratic potential. This will be investigated in more detail in [6], but the results confirm the fact that non-Gaussianity can be large in multiple-field inflation models, and that our analytic approximations provide a good estimate. Next, it is of course important to study the further evolution of non-Gaussianities after inflation through recombination to the present day (see [4,21] for some work in this direction). In this paper we restricted ourselves to computing only the bispectrum of the adiabatic component of ζ, even though we have the solution for all components. In future work we will investigate isocurvature and mixed bispectra as well. Finally, we will test our results with a purely numerical implementation of our formalism, which can also be applied to non-slow-roll models where our analytic approximations fail. In this case, the real-space realizations for ζ that result allow for other measures of non-Gaussianity to be determined, not just the three-point (or higher) correlator. After the disappointing results for single-field inflation, primordial non-Gaussianity is now back as an important quantitative tool for confirming or ruling out multiple-field inflation models, offering an exciting new window on the early universe as observations continue to improve over the next 5–10 years.

ACKNOWLEDGMENTS

We thank the organisers of the ''The Origin of the Primordial Density Perturbation'' workshop in Lancaster, U.K., in March 2005 for organizing such a stimulating meeting, especially regarding non-Gaussianity, and where we presented the first version of the results of Sec. V. This research is supported by PPARC Grant No. PP/C501676/1.

Note added.—After our investigation of the example of Sec. V D in an earlier version of this paper, the non-Gaussianity produced by a quadratic potential was also considered by the authors of [22]. They conclude that $f_{NL} \ll 1$ using the "δN formalism" and extrapolating results from a potential with (almost) equal masses. They have not computed the general case of unequal masses and assume deviations from the (almost) equal mass case to be small. In the case of almost equal masses, which is effectively single-field, we agree that non-Gaussianity is small. However, already in [13] it was quantitatively shown that even at linear order additional leading-order effects arise in the case of unequal masses from the effective coupling between the fields caused by the bending of the trajectory in field space. Moreover, in the model we consider here, the dominant non-Gaussianity is caused by a relatively large $\tilde{\eta}^{\perp}$, so that a naive slow-roll order counting is not valid, a situation where the δN formalism as we understand it is not applicable.

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