

Clustering of primordial black holes: Basic results

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We investigate the spatial clustering properties of primordial black holes (PBHs). With minimal assumptions, we show that PBHs created in the radiation era are highly clustered. Using the peaks theory model of bias, we compute the PBH two-point correlation function and power spectrum. For creation from an initially adiabatic power spectrum of perturbations, the PBH power spectrum contains both isocurvature and adiabatic components. The absence of observed isocurvature fluctuations today constrains the mass range in which PBHs may serve as dark matter. We briefly discuss other consequences of PBH clustering.

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I. INTRODUCTION

Primordial black holes (PBHs) are a unique probe of cosmology, general relativity, and quantum gravity. Formed by high concentrations of energy density in the early universe, PBHs are distinguished from other (astrophysical) black holes by not being formed through stellar collapse. In this paper we concentrate on PBHs formed from the direct gravitational collapse of density perturbations that are of order unity on the scale of the cosmological horizon [1,2] upon horizon entry, though there are other mechanisms for their creation, e.g. collapse of cosmic strings [3] or domain walls [4], or from bubble collisions [5] in the early universe.

Measurements of the cosmic microwave background (CMB) anisotropy [6] imply that density perturbations at the time of decoupling are much smaller ($\delta_H \approx 10^{-5}$). As such, PBH formation will be cosmologically negligible during and beyond this era. Less constrained are the conditions in the early universe before decoupling, and we cannot preclude the existence of much larger density contrasts which could have formed PBHs.

The theory of inflation [7] has been successful in describing both the large-scale homogeneity of the Universe and the formation of small-scale structure through the creation of a spectrum of cosmological perturbations. It predicts an era of accelerated expansion dominated by the energy of a slowly rolling scalar field, ending in a period of reheating where the energy density is transferred into (more or less) the particles we observe today and the radiation dominated epoch begins. The period of reheating is important for PBH production in two ways. First, it is the highest energy scale at which one would expect PBH to take place. Gravitational collapse is inhibited by the accelerated expansion, and the number density of any PBHs that do form would be drastically diluted. Second, several models of inflation exhibit an increase in the amplitude

of perturbations at the end of inflation (at the epoch of reheating), which increases the probability of PBH formation.

One topic of interest is the feasibility of PBHs as dark matter (DM) [8]. PBHs appear to be an *a priori* good cold dark matter (CDM) candidate. Formed purely by gravity, they require no special extensions to the standard model of particle physics (such as supersymmetry) and are predicted on quite generic grounds to form in the early universe [2]. While the smaller masses of PBHs (compared to astrophysical black holes) mean that Hawking radiation is non-negligible, PBHs that are still in the present universe are still “dark” like other BHs.

Because of this, there have been a number of studies of PBHs as CDM in the literature. We can split them roughly into three categories:

QCD PBHs.—These are PBHs formed during the QCD phase transition, being a fraction of a solar mass [9,10]. This was initially attractive as evidence from microlensing events suggested a population of massive compact halo objects (MACHOs) in just this mass range. However, in order to produce the correct Ω_m , one needs to invoke a “blue” spectrum ($n > 1$) of perturbations, which is highly disfavored by CMB observations. Further, the evidence that these MACHOs compromise a substantial fraction of DM halos is lessening [11].

“Spiky” PBHs.—These are PBHs formed due to the enhancement of power below a certain scale due to features (such as spikes) in the radiation power spectrum [12–16]. Such a spiky power spectrum (a generalization of a blue spectrum, where just the power-law slope is changed) can be produced in inflationary models with “plateaus” in the inflationary potential. PBHs created in this manner can exist over a larger range of masses, given the increased freedom in choosing an inflationary model. Included in this class are PBHs created due to perturbation amplification due to preheating [17–20].

Relic PBHs.—These are PBHs of around a Planck mass that exist in some theories of quantum gravity as the end

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result of PBH evaporation [21–23]. As all PBHs with initial masses less than $\sim 10^{15}$ g would have evaporated by the present day, any model that produces a number of light PBHs will leave behind relic PBHs.

The only limits on PBHs with masses above 10^{15} g derive from the requirement that they do not overclose the Universe ($\Omega_{\text{PBH}} < 1$), so there is a range of PBH masses over which they may serve as DM. Knowing the PBH abundance is necessary, but not sufficient, to fully gauge their feasibility as DM. Also important are their spatial clustering properties, as that too is constrained by CMB and large-scale structure data, though to date discussions of PBH clustering have been sparse in the literature. A recent general review of PBHs can be found in [24].

A. PBH clustering

The first discussions of PBH clustering came soon after their “discovery.” A theory was posited by Mészáros [25] where galaxy formation proceeds from the fluctuations in PBH number density. The model does not address how the PBHs are created but assumes they are around a solar mass and created at or before the QCD phase transition. It claimed that for PBH fluctuations that are uncorrelated on scales greater than the horizon scale (i.e., Poisson fluctuations only), it would be sufficient to be able to allow for galaxy formation. This model was refuted in [26] (and later expanded upon in [27]) where it was pointed out that the PBH creation process cannot create the “extra” density fluctuations on superhorizon scales that was claimed.¹

Kotok and Naselsky [29] posit a theory where an initial stage (1st generation) of PBH formation leads to an early stage of matter (PBH) domination. PBH clustering then enhances a second stage (2nd generation) of PBH formation due to collapse in this (pressureless) era—specifically, due to the coagulation of PBHs during matter domination. Provided this coagulation is not complete, the remainder of the 1st generation PBHs evaporate (thus, reheating the Universe) leaving behind the 2nd generation of PBHs. They claim that with a blue spectrum of initial perturbations ($n \geq 1.2$), PBHs of the 2nd generation are overproduced with respect to observational constraints.

While PBH reheating has been considered [20,30], it can be shown that [31–33] the period of PBH domination necessary would lead to the overproduction of (supersymmetric) moduli fields and gravitinos upon their evaporation that contradict the predictions of big bang nucleosynthesis (BBN). While the authors of [29] seem to confuse the distinction between radiation perturbations and PBH perturbations [see their Eq. (11)], we show later that PBH merging could be a natural consequence of clustering.

Assuming PBHs comprise the bulk of the CDM, Afshordi, McDonald, and Spergel [34] study how the dis-

creteness of their population affects the CDM power spectrum. They note that PBH perturbations on large scales (superhorizon sized at creation) are a mixture of adiabatic (as with other forms of CDM) and isocurvature (due to Poisson fluctuations alone). Using $\text{Ly}\alpha$ forest observations, they use this to constrain the mass of PBHs to be less than $10^4 M_\odot$. They are also the first to investigate PBH cluster dynamics; estimating the lifetime due to “evaporation” (different from Hawking evaporation) to show that PBH clusters with $N \lesssim 3000$ objects will evaporate by the current day. We expand on this analysis later.

Results from microlensing experiments indicate a population of MACHOs in our galaxy. A possibility is that this population is made up of PBHs of around a half a solar mass—the right mass range for QCD PBHs. In such a population, gravitational attraction between PBHs would induce the formation of PBH-PBH binaries. As such, such objects have been studied as sources of gravitational waves [35–38], though to date no such signals have been detected [39].

PBHs would be the first gravitationally collapsed objects in the Universe. As clustering is ubiquitous in other, observed gravitationally collapsed systems (galaxies, clusters of galaxies, superclusters, etc.), it will be no different for PBHs. The aim of this work is to compute the spatial clustering properties of PBHs and see what impact that has for PBHs in cosmology. We will be particularly interested in the viability of PBHs as DM. In Sec. II we describe general properties of PBHs we will use throughout the paper. In Sec. III we derive the initial clustering properties of PBHs after their formation, computing the PBH two-point correlation function and power spectrum. We conclude in Sec. IV with a discussion of observational constraints and avenues for further research.

II. PBH BASICS

A black hole of mass M has a Schwarzschild radius $R_S = 2GM = \frac{2M}{M_P^2}$. Throughout we assume that any PBHs have negligible angular momentum and electric charge.

PBHs form from large perturbations in the radiation density field that are able to overcome the resistance of radiation pressure and collapse directly to black holes. For a perturbation of a fixed comoving size, it cannot begin to collapse until it passes within the cosmological horizon. The size of a PBH when it forms, therefore, is related to the horizon size when the collapsing perturbation enters the horizon.² In the radiation dominated regime where $a \propto t^{1/2}$ and assuming a top-hat window function, the horizon mass is simply

$$M_H(t) = M_P \left(\frac{t}{t_P} \right) = (2 \times 10^5 M_\odot) \left(\frac{t}{1 \text{ s}} \right), \quad (1)$$

¹Though see [28] for a refutation of some of the refutations of [26,27].

²Which is to be expected, being the only characteristic length scale involved.

where t_P is the Planck time. Assuming radiation domination, we can rewrite this in terms of temperature as

$$M_H(T) = \left(\frac{3\sqrt{5}}{4\pi^{3/2}g_*^{1/2}} \right) \left(\frac{T}{M_P} \right)^{-2} M_P \approx 10^{18} \text{ g} \left(\frac{T}{10^7 \text{ GeV}} \right)^{-2}, \quad (2)$$

where g_* is the effective number of relativistic degrees of freedom.

The Hubble scale is then determined by the Friedmann equation

$$H^2 = \frac{8\pi G}{3} \rho. \quad (3)$$

The fluctuation of the (radiation) density field is defined as

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}} \quad (4)$$

and is characterized by its variance on a comoving scale χ at a time t as

$$\sigma^2(\chi, t) \equiv \frac{V}{2\pi^2} \int dk k^2 P(k) T(k, t)^2 |W_k(k\chi)|^2, \quad (5)$$

where $P(k)$ is the (primordial) power spectrum, W_k is the Fourier transform of the window function, and $T(k, t)$ is the transfer function appropriate for the type of perturbation (adiabatic or isocurvature). We further assume that the perturbations are Gaussian. It is known that the perturbations cannot be completely Gaussian, as that would predict perturbations with $\delta < -1$, implying negative energy densities. This non-Gaussianity is especially important in the production of PBHs [10], as they derive from the high end (tail) of the probability distribution. Nevertheless, we focus here on the case of underlying Gaussian perturbations for computational ease.

Consider a perturbation $\delta(r_H)$ smoothed over the comoving Hubble radius $r_H = R_H/a = (aH)^{-1}$. As the underlying perturbations δ_k are assumed Gaussian, the smoothed perturbation will be as well (central limit theorem). As the perturbation must have enough mass to overcome pressure, there is a threshold value δ_c below which a PBH will not form. Further, the horizon sized perturbation cannot be larger than unity, or it will pinch off and form a separate universe [40]. Therefore the range that forms PBHs is $\delta \in [\delta_c, 1]$. The exact value of δ_c is not known precisely. Analytically, $\delta_c = w$, where w is the equation of state parameter of the background universe defined through $p = w\rho$. The PBH mass was then estimated to be $M = w^{3/2}M_H$. Numerical simulations of PBH formation [41–43] have shown a more complex relation, where

$$M = \kappa M_H (\delta - \delta_c)^\gamma, \quad (6)$$

in accordance with other critical phenomena, where $\kappa \approx 3$,

$\gamma \approx 0.7$, and $\delta_c \approx 2w$. The values of these parameters vary depending on the shape of the initial perturbation (Gaussian, polynomial, etc.). This formally allows for PBHs of an arbitrarily small mass compared to the horizon size, though some numerical simulations [41] have showed that there is a minimum value of $\approx 10^{-3.5}M_H$ as $\delta \rightarrow \delta_c$. Rather than focus on one particular formula, we can encapsulate our uncertainty in the PBH mass-horizon mass relation with a parameter f :

$$M_{\text{PBH}} = f(w, \delta_c, t, \dots) M_H. \quad (7)$$

While the study of single PBH creation is numerically tractable, the same is not true for studying the PBH population as a whole due to their incredible rarity. As such we resort to analytical estimates. Given a creation threshold δ_c and the value of the radiation fluctuation size at horizon crossing $\sigma_{\text{rad}}(r_H)$, the probability of forming a PBH within a given horizon volume is simply the probability of having a perturbation with $\delta_c < \delta < 1$, or

$$\beta = \int_{\delta_c}^1 (2\pi\sigma_{\text{rad}}^2(r_H))^{-1/2} \exp\left(-\frac{\delta^2}{2\sigma_{\text{rad}}^2(r_H)}\right) d\delta. \quad (8)$$

Introducing $\nu = \delta_c/\sigma_{\text{rad}}(r_H)$ (the threshold in ‘‘sigma’’ units) in the limit where $\nu = \delta_c/\sigma_{\text{rad}}(r_H) \gg 1$, the upper limit can be taken to infinity, so that the expression can be written in terms of the complementary error function (erfc) as

$$\beta = \text{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \approx \sqrt{\frac{2}{\pi}} \frac{e^{-\nu^2/2}}{\nu}. \quad (9)$$

Note that this can be used to determine the initial PBH density³

$$\Omega_{\text{PBH}}(\nu, M) = \frac{\rho_{\text{PBH}}}{\rho_c} = \beta \left(\frac{M}{M_H} \right) = f\beta \equiv B, \quad (10)$$

where we have used $n_{\text{PBH}} \equiv \beta/V_H$.

Without an observed population to compare calculations to, the value of the (physical) PBH number density varies in the literature. While [44] did not address PBH formation *per se*, knowing that PBHs form at peaks in the density field implies

$$n_{\text{PBH}} = \frac{(n+3)^{3/2}}{(2\pi)^2 6^{3/2}} (\nu^2 - 1) e^{-\nu^2/2} R_H^{-3}, \quad (11)$$

where n is the index of the power spectrum ($n = 1$ for a scale-invariant spectrum). Whereas [44] use peaks in the density field, [45] uses peaks in the metric perturbation to compute a density which is identical to the [44] result, but with $(n+3)$ replaced with $(n-1)$. This latter calculation only holds for $n > 1$ [46]. Generically, we can write the

³Sometimes quoted in the literature instead of β is $\alpha = \beta/(1-\beta)$. In the limit where the PBH mass $M \approx M_H$, the initial $\rho_{\text{PBH}}/\rho_{\text{rad}} = \alpha$.

initial PBH density as

$$n_{\text{PBH}} = \frac{N_*(\nu)e^{-\nu^2/2}}{V_W}, \quad (12)$$

where $N_*(\nu)$ encapsulates the nonexponential dependence upon ν . Equivalently, the initial horizon fraction going into PBHs is

$$\beta = N_*(\nu)e^{-\nu^2/2}. \quad (13)$$

The values of $N_*(\nu)$ for the different models is summarized below:

$$N_*(\nu) = \left\{ \begin{array}{ll} \sqrt{\frac{2}{\pi}}\nu^{-1} & \text{erfc approximation} \\ \frac{1}{\sqrt{2\pi}}\left(\frac{n+3}{6}\right)(\nu^2 - 1) & \text{BBKS} \\ \frac{1}{\sqrt{2\pi}}\left(\frac{n-1}{6}\right)(\nu^2 - 1) & \text{GLMS} \end{array} \right\}. \quad (14)$$

Having determined the initial PBH density, their abundance at subsequent times is simple to calculate. PBHs are nonrelativistic matter, so $\rho_{\text{PBH}} \propto a^{-3}$. Because radiation redshifts as $\rho_{\text{rad}} \propto a^{-4}$, the PBH to radiation ratio grows until the epoch of matter-radiation equality:

$$\frac{\rho_{\text{PBH}}}{\rho_{\text{rad}}}(t_{\text{eq}}) = \frac{B(t)}{1 - B(t)} \left(\frac{t_{\text{eq}}}{t}\right)^{1/2}. \quad (15)$$

After the epoch of equality, Ω_{PBH} remains constant during matter domination up until the era of vacuum energy domination. The condition that PBHs do not overclose the Universe⁴ is $\Omega_{\text{PBH}}(t_{\text{eq}}) < 1/2$, or

$$B(t) < \frac{1}{2} \left(\frac{t}{t_{\text{eq}}}\right)^{1/2}. \quad (16)$$

Throughout, we will assume a monochromatic mass function such that $\rho_{\text{PBH}} = Mn_{\text{PBH}}$. We can then write

$$B(t) = fN_*e^{-\nu^2/2}. \quad (17)$$

Figures 1–3 show the lower limit on ν derived from the latter equations. This exponential dependence of the PBH abundance upon ν means we must now turn to a discussion of the form of the underlying power spectrum $P(k)$.

A given mode crosses within the horizon at a time t given by $k_H = a(t)H(t)$. The radiation fluctuation on the horizon scale (crossing during radiation domination) is computed using Eq. (5)

$$\sigma_{\text{rad}}^2(r_H, t) = \frac{V}{2\pi^2} \int dk k^2 P_{\text{rad}}(k) T_{\text{ad}}^2(k, t) W_k^2(kr_H). \quad (18)$$

For adiabatic perturbations (which we are assuming for radiation field), $T_{\text{ad}}(k, t) \propto k_H^{-2}$ (up to horizon crossing) and for a power-law spectrum $P_{\text{rad}}(k) \propto k^n$,

⁴This is also the condition that PBHs do not induce an early matter-dominated phase.

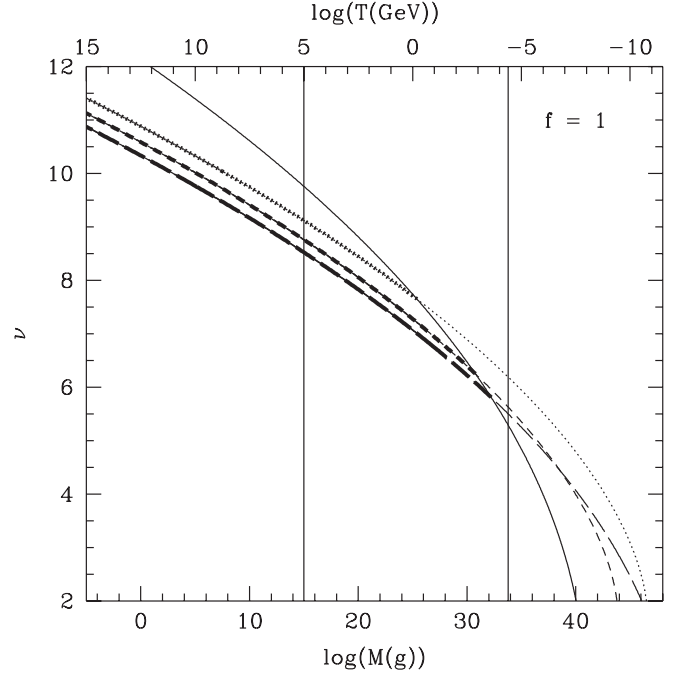


FIG. 1. Allowed region in $\nu - M_{\text{PBH}}$ parameter space for PBHs, assuming $f = 1$. Solid curve is the upper limit on ν due to isocurvature perturbations [from Eq. (59)]. Other curves are lower limits on ν due to number density [Eq. (14)]: long dashed line uses the erfc approximation; dotted line uses the BBKS formula with $n = 1$; short dashed line uses the GLMS formula with $n = 1.5$. Heavy lines show where PBH dark matter is allowed by the isocurvature constraint. Shown also is the temperature of the Universe T when PBHs form. The line at $M = M_* \sim 10^{15}$ g is mass below which PBHs would have Hawking evaporated by the current day (assuming no accretion or merging). The line at $M \sim 3M_{\odot}$ is the mass above which PBHs would be confused with astrophysical BHs.

$$\sigma_{\text{rad}}^2(r_H) \propto k_H^{(n-1)}, \quad (19)$$

where $k_H = 1/r_H$. The spectrum for $n = 1$ is known as the Harrison-Zel'dovich [47,48] spectrum (also called a scale-invariant spectrum) and corresponds to fluctuations of different physical sizes having identical power when they enter the horizon. Spectra with $n > 1$ are known as blue spectra and correspond to models having more power at smaller scales (larger k).

During radiation domination, the horizon mass $M_H \propto t \propto a^2 \propto k_H^{-2}$, or $k_H \propto M_H^{-1/2}$, so that $\sigma_{\text{rad}}^2(r_H) \propto M_H^{(1-n)/2}$. During matter domination the scaling is different, $M_H \sim \rho R_H^3 \propto a^{-3} H^{-3} = k_H^{-3}$, or $k_H \propto M_H^{-1/3}$, so that $\sigma_{\text{rad}}^2 \propto M_H^{(1-n)/3}$. For a pure power-law spectrum then, we can relate the power at any earlier time to the power today:

$$\sigma^2(r_H) = \sigma^2(H_0^{-1}) \left(\frac{M_{\text{eq}}}{M_0}\right)^{(1-n)/3} \left(\frac{M_H}{M_{\text{eq}}}\right)^{(1-n)/2}, \quad (20)$$

where 0 subscripts refer to current values and eq refers to the epoch of matter-radiation equality. From this, a value of

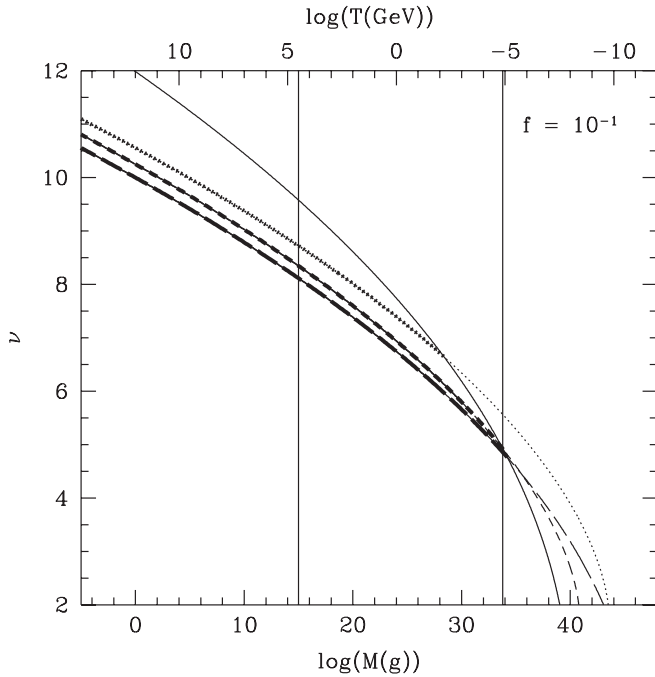


FIG. 2. The same as Fig. 1 except with $f = 0.1$.

$n > 1$ can produce sufficient power at small scales to produce significant black holes. Our understanding of the physics at these scales in the early universe is only theoretical, and thus there may be significant deviations from pure power-law behavior then.

Because of quantum effects [49], a BH of mass M will emit particles as a blackbody with temperature T_h given by

$$T_h(M) = \frac{1}{8\pi GM} = \frac{M_p^2}{8\pi M} \approx 10^{22} \left(\frac{M}{1 \text{ g}}\right)^{-1} \text{ eV.} \quad (21)$$

As the temperature is inversely proportional to the mass, this is unobservable for a one solar mass (and higher) BH ($T_h(M_\odot) \approx 62 \text{ nK}$) but cannot be neglected in the mass range of PBHs. This emission also corresponds to a mass loss for the PBH,

$$\dot{M} = -L_h = -\sigma_{\text{SB}}^* T_h^4 (4\pi R_s^2) = -\frac{\alpha(M)}{M^2}, \quad (22)$$

where σ_{SB}^* is the effective Stefan-Boltzmann constant and is related to the effective number of relativistic degrees of freedom in the emitted particles. PBHs therefore have a finite lifetime, after which they would have emitted their entire rest mass, given by

$$\tau = \frac{M_0^3}{3\alpha(M_0)} \approx (10^{-26} \text{ s}) \left(\frac{M}{1 \text{ g}}\right)^3. \quad (23)$$

The variation of the parameter α with mass is not great, changing by a factor of 10 over at least 7 decades of mass [50]. As the lifetime scales with M^3 , there is a threshold mass above which holes will not have evaporated by the present day (t_0). This threshold mass M_* is given by

$$M_* \approx (4 \times 10^{14} \text{ g}) \left[\left(\frac{\alpha(M_*)}{6.94 \times 10^{25} \text{ g}^3/\text{s}} \right) \times \left(\frac{t_0}{4.4 \times 10^7 \text{ s}} \right) \right]^{1/3}. \quad (24)$$

Given the uncertainties in α and t_0 , a threshold mass of $M_* \sim 10^{15} \text{ g}$ is typically quoted in the literature.

A large enough abundance of PBHs with $M \approx M_*$ will produce a number of observable effects through their evaporation in the current day. They would contribute to cosmic rays [51], the gamma ray background [50,52], 511 keV emission due to positron annihilation in the galactic center [53], or be the cause of short duration gamma ray bursts [54,55]. Observations (or the lack thereof) of PBHs evaporating today depend critically upon not only the number density of PBHs present today $n_{\text{PBH}}(t_0)$, but also upon how clustered they are within the galaxy. Assuming an isothermal halo model, the effective number density is $\zeta n_{\text{PBH}}(t_0)$ where ζ is the local density enhancement factor [50–52] and ranges from 10^5 – 10^7 .

PBHs with $M < M_*$ would have evaporated by the present day. The main mechanism for “observing” PBHs in cosmology is through their Hawking radiation. In the absence of a direct detection, the main utility of PBHs is to set limits of PBH abundance at various times given a nondetection—although PBHs also have been invoked to explain baryogenesis [56], reionization [57], and as a solution to the magnetic monopole problem [58,59].

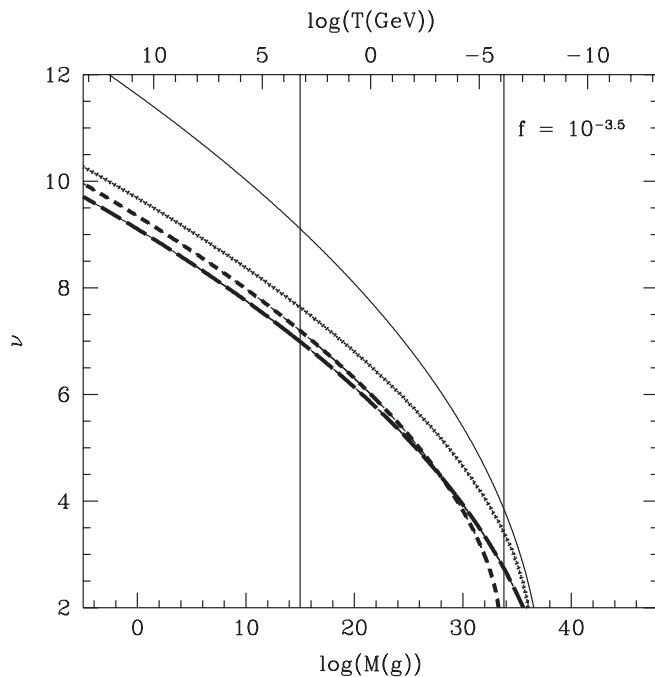


FIG. 3. The same as Fig. 1 except with $f = 10^{-3.5}$.

Evaporating PBHs have their most dramatic effect during the era of BBN, where Hawking radiation can alter light element abundances [60]. Therefore, the success of BBN implies an upper limit to the number of PBHs evaporating at that time.

Combining Eqs. (1), (7), and (23) gives the relation

$$\tau(t) = \frac{f^3 M_P^3}{3\alpha} \left(\frac{t}{t_P}\right)^3, \quad (25)$$

the lifetime τ of a PBH created at a time t . What this allows one to do is use information from a “late epoch” (time τ) to examine conditions at an “early epoch” (time $t \ll \tau$). In the above example, $\tau \sim t_{\text{BBN}}$, and the limits on initial PBH abundance from BBN imply $\beta < 10^{-16}$ for M_{PBH} between 10^9 g and 10^{15} g (see, i.e., [61]).

This relation depends critically upon the PBH mass monotonically decreasing due to evaporation and not gaining mass in any way (accretion or merging). Should this not be the case, the lifetime τ is no longer given by the initial PBH mass, and the link between late epoch and early epoch is broken. Instead, the energy in PBHs that would have evaporated away can now linger for longer periods of time. It was shown in [40] that PBHs will not appreciably increase their mass through radiation accretion. PBH merging then would be the dominant mechanism for (significant) mass growth in the radiation dominated epoch. Since $\tau \propto M^3$, the merging of two equal mass BHs will result in a BH with a lifetime 8 times as long. If this merging can continue, then there is a greater chance of PBHs produced in the early universe still existing today.

Depending on the epoch of PBH formation, there is reason to believe there would be merging occurring before, say, the epoch of nucleosynthesis, which could skew limits obtained from using Eq. (25). Assuming an unclustered population, PBH binaries can form in the radiation era and be a source of gravitational waves today [38]. Any PBH clustering will only enhance the formation of close PBH binaries (and possibly of larger bound structures), and orbital decay will cause merging before evaporation can occur.

III. BIAS MODEL

Measuring the two-point function (or its Fourier transform, the power spectrum) of astrophysical objects is a powerful tool in studying their clustering properties. The physical interpretation of $\xi(r)$ is as follows: The differential probability of finding two objects (galaxies, clusters, PBHs, etc.) in volume dV_1 and dV_2 , a distance r apart is given by

$$dP = \rho^2(1 + \xi(r))dV_1dV_2. \quad (26)$$

The two-point function then measures the excess probability (over random) of finding pairs with a separation r (here and throughout we use comoving distances). A large (positive) value of ξ implies a large amount of clustering

(objects are preferentially close to each other), a negative value of ξ implies anticlustering (objects are preferentially far away).

It is important to note that the galaxy-galaxy correlation function ξ_{gg} is not identical to the underlying mass correlation function ξ_m ; in other words, galaxies are not a perfect tracer of mass. Further, different types of objects which may act as tracers (quasar, galaxy clusters) have different clustering properties. Measurements of ξ for clusters of galaxies showed that they were more clustered than galaxies themselves by a factor of 10. Kaiser [62] showed that this may be explained using what is now known as the peak-background split model of bias: as clusters of galaxies form from higher peaks in the density field than galaxies, it is natural that they be more clustered. In the limit of large separation and large peaks, the bias is given by

$$\xi_{\text{peak}}(r) = \frac{v^2}{\sigma^2} \xi(r). \quad (27)$$

This can be roughly understood as follows: Split the density field into a long wavelength and a short wavelength component. Next, consider a peak in just the long wavelength component (“background”); the physical density field will consist of this component modulated by the short wavelength portion. If the threshold for gravitational collapse is close to the value of the background peak value, the physical field will cross this threshold a number of times in the vicinity of the peak. The regions above threshold, therefore, are preferentially found near the background peak.

The assumptions used follow:

PBH creation is rare.—PBH formation occurs during radiation domination ($w = 1/3$); and the radiation perturbations are Gaussian. At creation, there will be at most one PBH per horizon volume and PBH formation at around the horizon mass.

Peaks theory bias.—Since PBH formation is a threshold process, we can use peaks theory [44] to determine the number density and correlation statistics. While we only consider the two-point function and power spectrum here, all higher order correlation functions can be derived in a similar manner.

We now derive the bias for a population of PBHs formed at a single mass scale, compared to the underlying radiation field. For the overdensities of PBHs and radiation δ_{PBH} and δ_r , we define their two-point correlation functions

$$\xi_{\text{PBH}}(r) = \langle \delta_{\text{PBH}}(x)\delta_{\text{PBH}}(x+r) \rangle, \quad (28)$$

$$\xi_{\text{rad}}(r) = \langle \delta_{\text{rad}}(x)\delta_{\text{rad}}(x+r) \rangle, \quad (29)$$

and the bias parameter

$$\xi_{\text{PBH}}(r) = b(r)^2 \xi_{\text{rad}}(r), \quad (30)$$

where, in general, $b(r)$ is not a constant. The averaging

done in the definition of the correlation functions includes a window function on the scale of the horizon for smoothing. Thus, the size of the fluctuations⁵ in either the radiation and PBHs is characterized by

$$\sigma_{X,0}^2 = \xi_X(0). \quad (31)$$

From the definition of the radiation and PBH correlation functions, this is given by

$$\sigma_{\text{PBH},0}^2 = b(0)\sigma_{\text{rad},0}^2. \quad (32)$$

The power spectrum $P(k)$ is defined as

$$P(k) = \left(\frac{4\pi}{V}\right) \int dr r^2 \xi(r) \left(\frac{\sin(kr)}{kr}\right). \quad (33)$$

As PBHs form in regions above a certain threshold density, it is straightforward to compute the number density and bias assuming PBHs form at a single mass only. The bias is given by an integral over a bivariate Gaussian distribution; using the notation of Jensen and Szalay [63], the full expression is given by

$$1 + \xi_{\text{PBH}}(r) = \left[\frac{1}{2} \operatorname{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\right]^{-2} \int_{\nu}^{\infty} dy_1 \times \int_{\nu}^{\infty} dy_2 (2\pi)^{-1} (1 - w(r)^2)^{-1/2} \times \exp\left[-\frac{y_1^2 + y_2^2 - 2y_1 y_2 w(r)}{2(1 - w(r)^2)}\right], \quad (34)$$

where $w(r) = \xi_{\text{rad}}(r)/\sigma_{\text{rad},0}^2$ is the normalized radiation correlation function and $\nu = \delta_c/\sigma_{\text{rad},0}$. It is possible to write this as a power series (the so-called tetrachoric series) in $w(r)$ [63],

$$\xi_{\text{PBH}}(r) = \sum_{m=1}^{\infty} \frac{A_m^2}{m!} w(r)^m, \quad (35)$$

where the coefficients are given by

$$A_m = \frac{2H_{m-1}\left(\frac{\nu}{\sqrt{2}}\right)2^{-m/2}}{\sqrt{\pi}e^{\nu^2/2}\operatorname{erfc}\left(\frac{\nu}{\sqrt{2}}\right)}, \quad (36)$$

where H_n are the Hermite polynomials.

The result of Kaiser [62] is obtained by assuming $w(r) \ll 1$ and $\nu \gg 1$, so that only the first term in the series need be used to obtain $\xi_{\text{PBH}}(r) \approx \nu^2 w(r)$. Relaxing the condition on $w(r)$ (but not on ν), the coefficients $A_m \rightarrow$

⁵While the terms *perturbation* and *fluctuation* are sometimes used interchangeably in the literature to refer to an inhomogeneity, we will make a distinction in the usage for radiation and PBHs. The word *perturbation* typically implies smallness in the context of (cosmological) perturbation theory, and we use it to describe the (initial) radiation field, as they will be no larger than order unity. As we will show, this will not be the case for PBHs, and therefore we use the word *fluctuation* for their case.

ν^m , obtaining the result of Politzer and Wise [64],

$$1 + \xi_{\text{PBH}}(r) = 1 + \sum_{m=1}^{\infty} \frac{(\nu^m)^2}{m!} w(r)^m = \exp(\nu^2 w(r)). \quad (37)$$

As $r \rightarrow 0$, $w \rightarrow 1$ by definition, so that in the case of arbitrary ν ,

$$\xi_{\text{PBH}}(0) = \left[\sqrt{\pi}e^{\nu^2/2}\operatorname{erfc}\left(\frac{\nu}{\sqrt{2}}\right)\right]^{-2} \sum_{m=1}^{\infty} \frac{(2H_{m-1}\left(\frac{\nu}{\sqrt{2}}\right))^2}{2^m m!}. \quad (38)$$

Recall that $\sigma_{\text{PBH},0}^2 = \xi_{\text{PBH}}(0)$. For large ν , it follows that $\sigma_{\text{PBH},0}^2 = e^{\nu^2}$. In other words, PBHs start with a large fluctuation amplitude (compared to radiation) and their evolution begins in the nonlinear regime. However, the number density goes as $e^{-\nu^2/2}$, so the fewer PBHs are formed, the more clustered they will be.

Note this bias is independent of the PBH mass–horizon mass relation [Eq. (7)]. Specifically, we have computed the correlation function of horizon sized regions that contain at most one PBH. As such $P_{\text{PBH}}(k)$ will have an initial upper cutoff at k_H .

While we can compute exactly ξ_{PBH} from the peak-background split model, it is customary in large-scale structure studies to measure the power spectrum P_{PBH} instead. Inserting Eq. (37) into Eq. (33) we obtain the integral expression

$$P_{\text{PBH}}(k) = \frac{4\pi}{V} \int dr r^2 \xi_{\text{PBH}}(r) \left(\frac{\sin(kr)}{kr}\right) = \frac{4\pi}{V} \int_{r_H}^{\infty} dr r^2 \left[\exp\left(\frac{\nu^2}{\sigma_{\text{rad},0}^2} \xi_{\text{rad}}(r)\right) - 1\right] \times \left(\frac{\sin(kr)}{kr}\right). \quad (39)$$

The lower cutoff at $r_H = R_H/a = k_H^{-1}$, the comoving horizon length at PBH formation, is due to the finite size of the PBHs. This will translate into an upper cutoff in $P_{\text{PBH}}(k)$ at k_H . Generically, the above integral can be done numerically, but we can say more about the nature of the PBH fluctuations without it.

By expanding the exponential, we can rewrite Eq. (39) as

$$P_{\text{PBH}}(k) = \frac{\nu^2}{\sigma_{\text{rad},0}^2} P_{\text{rad}}(k) + \sum_{m=2}^{\infty} \frac{4\pi}{V} \int_{r_H}^{\infty} dr r^2 \left(\frac{\sin(kr)}{kr}\right) \frac{1}{m!} \times \left[\frac{\nu^2}{\sigma_{\text{rad},0}^2} \xi_{\text{rad}}(r)\right]^m. \quad (40)$$

The higher order terms in the above expansion show the nonlinear dependence of P_{PBH} upon P_{rad} .

Because of the discrete nature of the PBHs, the normalization condition for P_{PBH} is that as $k \rightarrow 0$, P_{PBH} ap-

proaches a spectrum for pure Poisson noise, i.e., a constant value. This is manifest in our above expression in the first term, where the PBH power spectrum is simply $b^2 P_{\text{rad}}$, with the bias b given by the Kaiser value of ν/σ . We can Taylor expand the sine term in the integrals such that $\sin(kr)/(kr) \rightarrow 1$, and those integrals evaluate to constants:

$$P_{\text{Poisson}} = \frac{1}{V} \sum_{m=2}^{\infty} \frac{4\pi}{m!} \frac{\nu^{2m}}{\sigma_{\text{rad},0}^{2m}} \int_{r_H}^{\infty} dr r^2 \xi_r(r)^m. \quad (41)$$

The total PBH power spectrum then can be written as

$$P_{\text{PBH}}(k) = P_{\text{Poisson}} + \frac{\nu^2}{\sigma_{\text{rad},0}^2} P_{\text{rad}}(k) + P_{\text{SS}}(k), \quad (42)$$

where

$$P_{\text{SS}}(k) = \frac{1}{V} \sum_{l=1}^{\infty} \sum_{m=2}^{\infty} \int_{r_H}^{\infty} dr r^2 \left(\frac{(-1)^l (kr)^{2l}}{(2l+1)!} \right) \frac{4\pi}{m!} \times \left[\frac{\nu^2}{\sigma_{\text{rad},0}^2} \xi_{\text{rad}}(r) \right]^m \quad (43)$$

represents the small-scale power when kr is not small.

To see the behavior of P_{PBH} at small k , we numerically integrate Eq. (39). The underlying radiation power spectrum P_{rad} is a $n = 1$ spectrum normalized to the four-year Cosmic Background Explorer (COBE) value along with a Gaussian spike at the horizon scale (the latter being normalized to unity). For a fixed $\delta_c = 2/3$, varying the spike

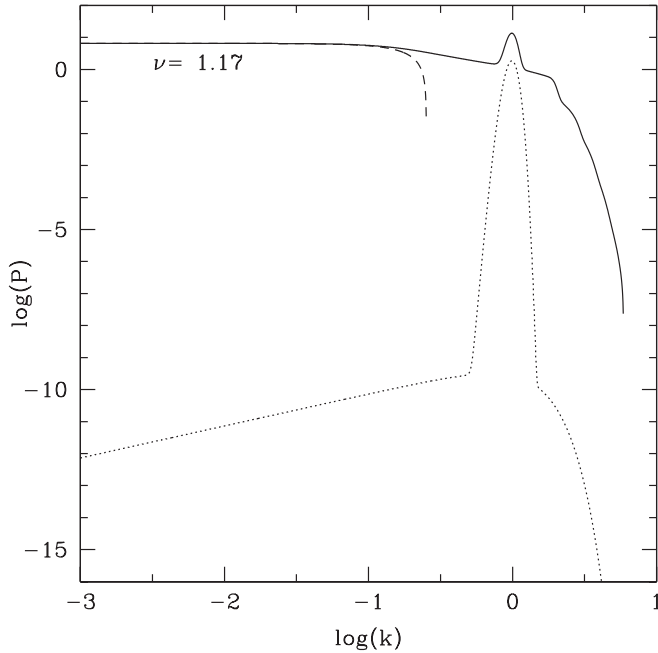


FIG. 4. The PBH power spectrum for $\nu = 1.17$. Dotted line is the radiation power spectrum, consisting of a $n = 1$ spectrum with COBE normalization, along with a Gaussian spike in power at $k = 1$. Solid line is the PBH power spectrum; dashed line is the quadratic estimate of the PBH power spectrum.

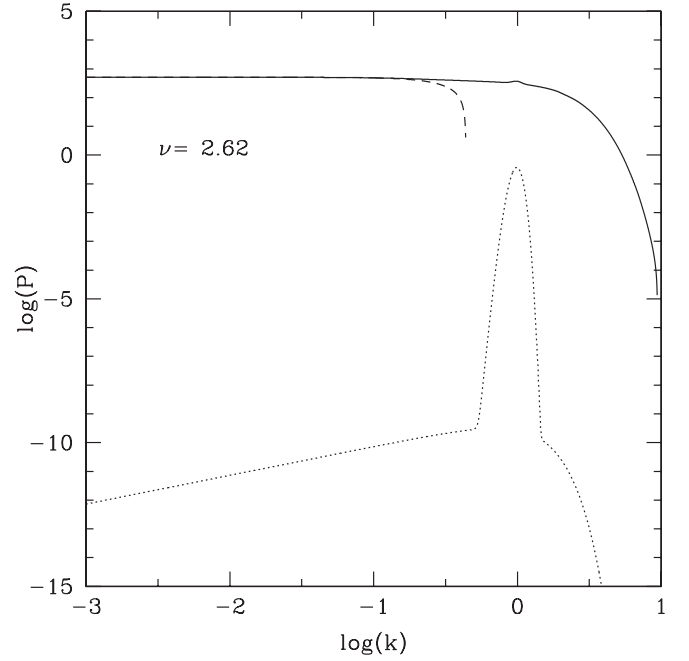


FIG. 5. The PBH power spectrum for $\nu = 2.62$. Dotted line is the radiation power spectrum, consisting of a $n = 1$ spectrum with COBE normalization, along with a Gaussian spike in power at $k = 1$. Solid line is the PBH power spectrum; dashed line is the quadratic estimate of the PBH power spectrum.

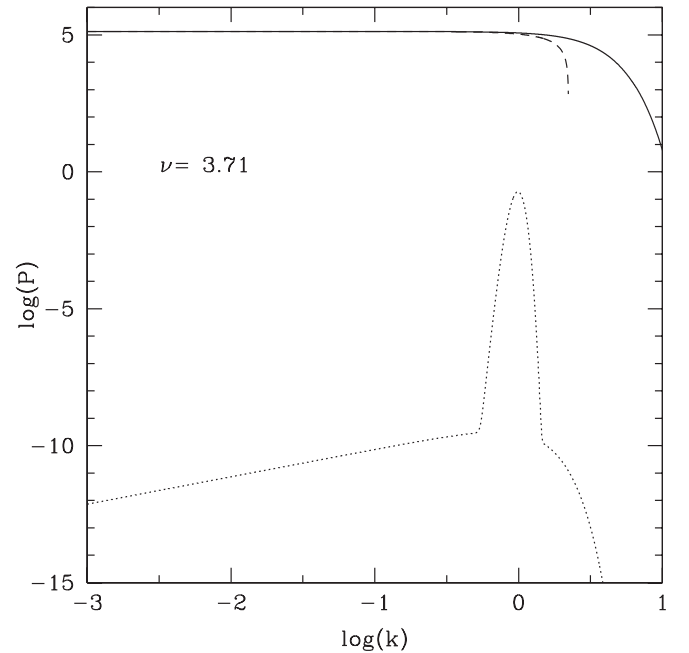


FIG. 6. The PBH power spectrum for $\nu = 3.71$. Dotted line is the radiation power spectrum, consisting of a $n = 1$ spectrum with COBE normalization, along with a Gaussian spike in power at $k = 1$. Solid line is the PBH power spectrum; dashed line is the quadratic estimate of the PBH power spectrum.

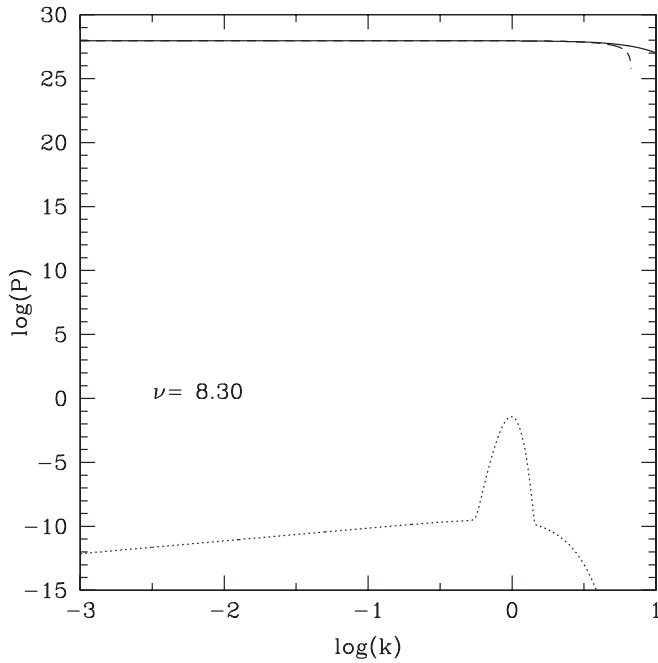


FIG. 7. The PBH power spectrum for $\nu = 8.30$. Dotted line is the radiation power spectrum, consisting of a $n = 1$ spectrum with COBE normalization, along with a Gaussian spike in power at $k = 1$. Solid line is the PBH power spectrum; dashed line is the quadratic estimate of the PBH power spectrum.

amplitude will vary the value of ν . Figures 4–8 show $P_{\text{PBH}}(k)$ for four values of ν . We see that as ν increases, the constant (Poisson) power quickly damps out the linear (Kaiser) term. The $l = 2$ terms of $P_{\text{SS}}(k)$ survive for inter-

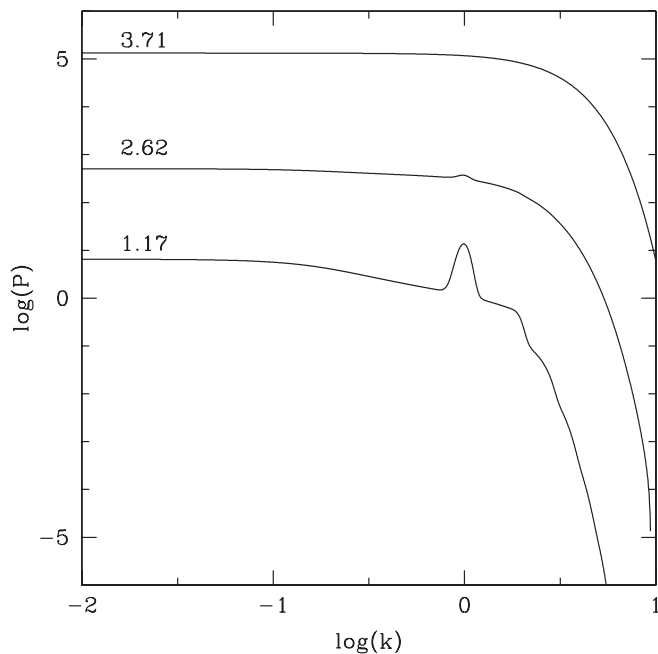


FIG. 8. The PBH power spectrum for $\nu = 1.17, 2.62, 3.71$.

mediate values of k as a small negative quadratic contribution ($\propto -k^2$).

Note that the power spectrum given in Eq. (42) is the initial spectrum immediately after PBH creation. Because of the different k dependence of each of the terms, the power at later times ($k \ll k_H$) will not be dominated by the Poisson term. We return to this in Sec. III C where we compute the power at horizon crossing at later times.

From Eq. (39), we expect $P_{\text{Poisson}} \sim e^{\nu^2}$; a better fit for $\nu \gtrsim 4$ yields

$$P_{\text{Poisson}} \approx \frac{10}{7} \exp\left(\frac{3}{4} \nu^{2.1}\right). \quad (44)$$

The power spectrum for a group of N objects randomly distributed (with a uniform distribution) is $1/N = (nV)^{-1} = \beta^{-1}$. Note that our above expression for $P_{\text{Poisson}} \neq \beta^{-1}$, indicating the PBHs are distributed as clusters of objects with mean occupation number

$$\begin{aligned} N_c &= P_{\text{Poisson}} \beta \\ &= \frac{10}{7} N_*(\nu) \exp\left(\frac{3}{4} \nu^{2.1} - \frac{1}{2} \nu^2\right) \sim N_*(\nu) e^{\nu^2/4}. \end{aligned} \quad (45)$$

A. Adiabatic vs isocurvature

We now take an aside to further consider the nature of the PBH fluctuations. That PBHs correspond to isocurvature perturbations has been noted in the literature [26,44,65,66], though it has not received a lot of attention in the recent PBH publications. In models where PBHs constitute the dark matter, it was assumed that their perturbations would be purely adiabatic, as with other types of dark matter. We point out that this is not the case; a large isocurvature component exists at shorter scales in addition to the adiabatic component at longer scales.

To demonstrate this, assume that radiation is the only component in the Universe; there is, therefore, no distinction between adiabatic or isocurvature type perturbations. The radiation perturbation corresponds to a perturbation in the spatial curvature.⁶ Once PBHs are created from gravitational collapse, they will evolve as a matter ($w = 0$) field in the Universe. As such, we can examine the fluctuations in the PBH density. At the time of PBH creation, *their* fluctuations can be classified as either adiabatic or isocurvature. By assumption, PBHs form from the collapse of a density perturbation once it enters the horizon. In the radiation dominated era, the PBH mass is close to the horizon mass, so that at most one PBH forms per horizon volume. Each PBH is separated by at least a horizon distance. The population cannot have correlations on scales

⁶Whether the perturbation is Gaussian or non-Gaussian is largely irrelevant at this point; perturbations of order unity must be non-Gaussian to some degree, and we will show in the next section that the perturbations of PBHs are generically non-Gaussian.

smaller than the horizon, so that the perturbations only exist for superhorizon scales. Any superhorizon perturbation can be written as a sum of adiabatic and isocurvature modes.

Note that, in our setup, only after PBH creation does the distinction between adiabatic and isocurvature perturbations exist. We intend to prove that the PBH fluctuations have an isocurvature component. This can be generalized to the case where there are additional fields and the initial perturbation is wholly adiabatic.

Isocurvature perturbations correspond to perturbations in the local equation of state $w = p/\rho$, while adiabatic perturbations correspond to perturbations in the local energy density, and thus the local curvature. Consider a volume of space greater than the horizon volume at PBH creation. The formation of PBHs cannot change the energy density within this space: the gravitational collapse corresponds to a “shuffling” of energy density from one form (radiation) into another (matter). The decrement in the radiation energy density is exactly balanced by the creation of PBH energy density. Therefore, the curvature is unchanged on superhorizon scales. The total perturbation will only become adiabatic if this shuffling takes place as to satisfy the adiabatic condition. Further, by the second law of black hole thermodynamics, a black hole will always have a higher entropy than the material that formed it. PBH formation thus corresponds to an increase in entropy, and should this process occur nonuniformly, this will result in entropy perturbations, i.e. isocurvature perturbations.

The proof that the PBH fluctuations are isocurvature, then, derives from the fact that PBH formation is highly nonuniform and equivalently, that PBHs are created highly clustered, which was shown earlier in this section. Using the notation of [7], we write the entropy perturbation as

$$S_{\text{PBH}} = \delta_{\text{PBH}} - \frac{3}{4} \delta'_{\text{rad}}, \quad (46)$$

where δ'_{rad} is the radiation perturbation *after* PBH formation, and $\delta'_{\text{rad}} \neq \delta_{\text{rad}}$. Using the parameter B from Eq. (10), we can trivially write

$$\rho_{\text{rad}} = \rho'_{\text{rad}} + \rho_{\text{PBH}} = (1 - B)\rho_{\text{rad}} + B\rho_{\text{rad}}, \quad (47)$$

which allows us to relate the perturbations as

$$\delta_{\text{rad}} = (1 - B)\delta'_{\text{rad}} + B\delta_{\text{PBH}}. \quad (48)$$

We can use this latter equation to rewrite Eq. (46) as

$$S_{\text{PBH}} = \delta_{\text{PBH}} \left(1 + \frac{3}{4} \frac{B}{1 - B} \right) - \frac{3}{4} \frac{1}{1 - B} \delta_{\text{rad}} \quad (49)$$

$$\approx \delta_{\text{PBH}} - \frac{3}{4} \delta_{\text{rad}}, \quad (50)$$

which is now a function of the *initial* radiation perturbation and the (final) PBH fluctuation, and the approximation holds as long as $B \ll 1$. While $\delta_{\text{rad}} < 1$ by assumption,

we know from Eq. (32) that δ_{PBH} typically will not be due to clustering. We see that the entropy perturbation is simply a function of the bias parameter:

$$S_{\text{PBH}} \approx \left(b - \frac{3}{4} \right) \delta_{\text{rad}}. \quad (51)$$

It is apparent that the isocurvature perturbation is almost inevitable for realistic (rare) PBH production. The bias parameter b will be dependent on scale; in Fourier space b is given roughly by $\sqrt{P_{\text{PBH}}/P_{\text{rad}}}$. For a given k , the bias is dominated by the term domination of the power spectrum as given in Eq. (42). At the smallest scales (close to PBH creation scales), the bias is largest and using Eq. (37) gives $b \sim \exp(\nu^2/2)/\sigma \gg 1$. For larger scales, the linear (Kaiser) bias gives $b = \nu/\sigma$. In either case, the parameters (σ, ν) would have to be finely tuned in order to produce a purely adiabatic PBH perturbation.

We note that this mechanism for generating an isocurvature perturbation is independent of the process that created the initial (adiabatic) perturbation, though we assume throughout that it is done through an epoch of cosmological inflation. This mechanism then is an exception to the generally held thought that an isocurvature perturbation cannot be produced from single field inflation [67]. The reason this occurs is that PBH creation (i.e. gravitational collapse) is an inherently nonlinear and non-perturbative process that is not bound by this restriction from perturbation theory. PBH dark matter is not like particulate dark matter. Further, for PBHs lighter than M_* this isocurvature fluctuation is transferred to the products of Hawking evaporation. Thus, the absence of an observed isocurvature perturbation implies a limit on the number of PBHs that have evaporated in the past. We plan to further explore this topic in a future paper.

B. Gaussian vs non-Gaussian

In our derivation of the PBH number density and clustering properties, we assumed the underlying radiation perturbation was Gaussian. As PBHs form only at the peaks of the density field, and the initial size of the fluctuation is greater than unity, the PBH fluctuations cannot be Gaussian. They appear instead to be lognormal (LN) in character. Roughly, a LN distribution is the exponentiation of a Gaussian distribution. The two-point correlation function of a LN field is given by [68]

$$1 + \xi_{\text{LN}}(r) = \exp(\Xi(r)), \quad (52)$$

where $\Xi(r)$ is the correlation function of a Gaussian field with variance $\Xi(0) = S^2$. From Eq. (37), this is exactly the correlation function for the PBH population assuming $\Xi(r) = \nu^2 w(r)$ and $\nu \gg 1$.

An isocurvature perturbation necessarily is defined between two components, here radiation and PBHs. While we have been focusing on the PBHs, there is of course a change in the radiation perturbations; the increase in PBH

density is exactly cancelled by a decrease in radiation density. From the perspective of the radiation field, not only is there an isocurvature component along with the (initial) adiabatic component, but there is now a non-Gaussian fluctuation along with the (initial) Gaussian one.

C. PBH fluctuation evolution

The evolution of the PBH population after creation is a complex problem, outside the bounds of perturbation theory due to the size of the initial PBH fluctuations, and better addressed as a N -body problem [69]. However, we can make a rough estimate of the power at horizon crossing of other scales using the results from cosmological perturbation theory. We can break the PBH power spectrum into its isocurvature and adiabatic components:

$$\begin{aligned} P_{\text{PBH}}(k) &= \left(P_{\text{PBH}}(k) - \frac{9}{16} P_{\text{rad}}(k) \right) + \frac{9}{16} P_{\text{rad}}(k) \\ &= P_{\text{iso}}(k) + P_{\text{ad}}(k). \end{aligned} \quad (53)$$

We can then write the variance at horizon crossing as

$$\begin{aligned} \sigma_{\text{PBH}}^2(r_H, t) &= \frac{V}{2\pi^2} \int dk k^2 (P_{\text{iso}}(k) T_{\text{iso}}^2(k, t) \\ &\quad + P_{\text{ad}}(k) T_{\text{ad}}^2(k, t)) W_k^2(kr_H). \end{aligned} \quad (54)$$

Rather than computing this explicitly, we will note that for power-law spectra where $P_{\text{iso}}(k) \propto k^{n_{\text{iso}}}$ and $P_{\text{ad}}(k) \propto k^n$, their contributions to the variance at horizon crossing can be written as

$$\sigma_{\text{ad}}^2(r_H) = \sigma_{\text{ad}}^2(H_0^{-1}) \left(\frac{M_{\text{eq}}}{M_0} \right)^{(1-n)/3} \left(\frac{M_H}{M_{\text{eq}}} \right)^{(1-n)/2}, \quad (55)$$

$$\sigma_{\text{iso}}^2(r_H) = \sigma_{\text{iso}}^2(H_0^{-1}) \left(\frac{M_{\text{eq}}}{M_0} \right)^{(n_{\text{iso}}+3)/3} \left(\frac{M_H}{M_{\text{eq}}} \right)^{(n_{\text{iso}}+3)/2}, \quad (56)$$

while the total variance is their sum:

$$\sigma^2 = \sigma_{\text{iso}}^2 + \sigma_{\text{ad}}^2. \quad (57)$$

The condition for scale invariance is no scaling with mass; for adiabatic perturbations this is $n = 1$ and for isocurvature perturbations this is $n_{\text{iso}} = -3$. While the adiabatic portion of $P_{\text{PBH}}(k)$ is scale invariant by assumption, for scales larger than the horizon size at their creation, the isocurvature component has a flat spectrum ($n_{\text{iso}} \approx 0$) and diminishes at longer scales. Thus while the isocurvature portion dominates initially, there is a crossover scale where the spectrum becomes adiabatic. Given the lack of measured isocurvature component at the time of the CMB (upper limit on isocurvature fraction is $f_{\text{iso}} < 0.33$, from [70]), we can put a limit on the PBH population so that it does not violate this bound. Roughly, at the scale of matter-radiation equality ($M_{\text{EQ}} \sim 10^{48}$ g),

$$\sigma^2(r_{\text{EQ}}) = \sigma_{\text{iso}}^2 + \sigma_{\text{ad}}^2 = \delta_H^2, \quad (58)$$

and the bound is

$$\sigma_{\text{iso}}^2(r_{\text{EQ}}) < f_{\text{iso}}^2 \delta_H^2, \quad (59)$$

where $\delta_H = 1.91 \times 10^{-5}$. To compute $\sigma_{\text{iso}}^2(r_{\text{EQ}})$, we assume $P_{\text{iso}}(k) \approx P_{\text{Poisson}}$, which, as shown in Figs. 4–7, is valid for $k \lesssim k_H/10$. The upper limit is plotted in Figs. 1–3 for three different values of f . This constraint becomes an upper limit on the (initial) PBH mass if it is to serve as dark matter. For $f = 1$, allowed regions for PBH dark matter all have $M_{\text{PBH}} < M_{\odot}$, so that there would be no confusion with astrophysical BHs. As we decrease f , the upper limit increases: for $f = 10^{-3.5}$ it is pushed above the confusion limit.

IV. CONCLUSIONS

We have shown that for PBHs to serve as dark matter, clustering constrains them to lie in a particular mass range. Further, PBHs will preferentially be found in clusters.

As shown in the previous section, PBH fluctuations enter the horizon with a very large amplitude ($\sigma_{\text{PBH}} \sim e^{v^2/2}$). It is therefore no longer of value to treat their evolution using linear perturbation theory, as one is able to do for other forms of CDM. Instead, we examine the subhorizon evolution of the PBH population as an N -body problem. Being nonrelativistic, PBHs will cluster hierarchically (just as CDM)—creating smaller bound systems that get incorporated into larger ones. The internal dynamics of these systems are determined solely by gravitational clustering, analogous to other gravitationally bound systems such as star clusters and galaxies. For this, we are aided by the work done in the context of studying more massive black holes in globular clusters [71] and galaxies [72]. In those cases, gravitational interactions tend to either produce bound pairs or ejections, rather than BH coalescence [73].

What occurs in the case of PBHs depends upon how many form in a ‘‘PBH cluster’’ and what their initial separations are. The estimate of cluster population in Eq. (45) is likely an overestimate since our approximation for $\xi_{\text{PBH}}(r)$ breaks down for small r . The initial separations should be on order of the horizon size at formation, being the only length scale involved in PBH formation. This would seem to indicate rather compact clusters (initial separation on order the size of the PBHs themselves), though more work (e.g., higher order statistics, numerical simulations) is needed to verify this.

Frequent merging due to clustering could have a profound impact upon cosmology. Since their lifetime $\tau \propto M^3$, PBHs, due to merging, exist longer than they would have initially. This could feasibly lead to a PBH population in the present universe that was formed in the earliest moments of the early universe, opening up a new and unique observational window into that time. At the very least, PBH merging in clusters dramatically changes the limits on initial PBH abundance, such as those used to put limits on models of inflation [61, 74–81]. The issue of PBH

clusters and merging is discussed more fully in a companion paper [69]

Limits on the current number density of PBHs depend critically upon how clustered PBHs are in our galaxy. Naïvely, from our work in this paper, we might expect a local clustering enhancement $\zeta \sim e^{\nu^2/2}$, or $\zeta \sim 10^{22}$ for $\nu = 10$. This is many orders of magnitude larger than the factors of 10^7 computed in the literature. This ignores the effect of PBH merging though; sufficient merging might concentrate all galactic PBHs into the center SMBH. This will have implications for models where PBHs are used to be the “seed” BHs needed for the growth of SMBHs in the centers of galaxies [82–84].

This PBH merging scenario we have discussed has other predictions. One prediction is more gravitational wave emission than originally assumed for a uniform PBH population. This is due to the increased probability of PBH binary formation and emission from resonant bound states.

The other prediction is related to dark matter. Suppose now that PBHs are not the only component of the dark matter, and that there also exists a “standard” CDM candidate with adiabatic perturbations (in accordance with CMB measurements), in which case the CDM perturbation amplitude is related to the radiation perturbation amplitude by $\delta_{\text{CDM}} = (3/4)\delta_{\text{rad}}$.

Perturbations in the radiation density can only collapse (into PBHs) if they are of sufficient amplitude on the scale of the horizon. Perturbations smaller than this, in accordance with linear perturbation theory, will simply oscillate but not collapse. This implies that there will be scales slightly larger than those where PBH formation took place where δ_r is below the threshold for PBH formation but still large compared to, say, the amplitude at the time of the CMB (10^{-5}). There is, accordingly, a similarly large perturbation in the CDM density assuming adiabaticity. While the linear growth of matter perturbations is delayed until after matter-radiation equality, they still grow logarithmically in the radiation dominated era. This leads to the possibility that they will become nonlinear before equality and form bound dark matter structures along with PBHs. In which case, one would have to include the interaction between these two populations of primordial bound objects.

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