

Spacetime structure of an evaporating black hole in quantum gravity

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The impact of the leading quantum gravity effects on the dynamics of the Hawking evaporation process of a black hole is investigated. Its spacetime structure is described by a renormalization group improved Vaidya metric. Its event horizon, apparent horizon, and timelike limit surface are obtained by taking the scale dependence of Newton's constant into account. The emergence of a quantum ergosphere is discussed. The final state of the evaporation process is a cold, Planck size remnant.

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I. INTRODUCTION

One of the very remarkable features of black hole radiance [1] is the observation that the global spacetime structure of a black hole losing mass by the evaporation process is far more complicated than that of its static counterpart [2–4]. Even a Schwarzschild black hole, when it radiates, does not have a single horizon that fully characterizes its structure, and one must distinguish at least three important horizonlike loci. The future event horizon (EH) is the boundary of the causal past of future null infinity, and it represents the locus of outgoing future-directed null geodesic rays that never manage to reach arbitrarily large distances from the hole. The apparent horizon (AH) is defined as the outermost marginally trapped surface for the outgoing photons. Classically it can be null or spacelike, in presence of quantum radiance it can be timelike also, when regarded as a three-dimensional surface. The third important locus is the timelike limit surface (TLS) or “quasistatic limit” which is defined as the locus where static observers become lightlike. The TLS can be null, timelike, or spacelike [2]. For a classical Schwarzschild black hole (which does not radiate), the three surfaces EH, AH, and TLS are all identical. Upon “switching on” the Hawking evaporation this degeneracy is partially lifted. According to the analysis by York [2,3] the AH continues to coincide with the TLS for a spherically symmetric emission, but the EH is different from $AH = TLS$.

In particular, if we approximate the stress-energy tensor near the horizon as a radial influx of negative energy which balances the outward Hawking flux at infinity, the event horizon is located *inside* the AH [5], the portion of spacetime between the two surfaces forming the so-called “quantum ergosphere.” This name stems from the analogy with the classical (stationary) Kerr black hole for which $AH = TLS \neq EH$. Here the ergosphere is the space between “the” horizon $AH = TLS$ and the EH, usually called the “static limit.” In both cases particles and light signals can escape from within the ergosphere and reach infinity.

The definition of the EH via the locus of outgoing photons that can never reach large distances from the hole has the unfortunate “teleological” property of requiring knowledge of the entire future history of the hole [2,6]. In particular when the black hole radiance is described semiclassically (quantized matter in a classical geometry), the Bekenstein-Hawking temperature and the luminosity diverge for $M \rightarrow 0$, as $T_{BH} \propto 1/M$ and $L \propto 1/M^2$, respectively. As a result, this approximation breaks down for very light holes, and in order to determine the final state of the evaporation process a much more precise treatment, including backreaction and quantum gravitational effects, is required.

In York's work [2,3], which is strictly within the semiclassical approximation, the teleological problem is circumvented by relaxing the definition of the EH in the following way. Rather than demanding that the photons “never” reach infinity he demands only that they are imprisoned by the event horizon for times which are very long compared to the dynamical time scale of the hole. Using this working definition of the EH he is then able to determine its location to first order in the luminosity L . In this manner the difficult question about the real final state of the evaporation is not touched upon.

It is the purpose of the present paper to analyze the dynamical evaporation process and the corresponding spacetime structure of a radiating Schwarzschild black hole. We include the leading quantum gravitational corrections of the geometry which, as we shall discuss, seem to lead to a termination of the evaporation process and the formation of a cold, Planck size remnant. Our main tool will be the “renormalization group improvement” of classical solutions, a technique which is very popular in conventional field theory.

In fact, recently a lot of work went into the investigation of the nonperturbative renormalization group (RG) behavior of quantum Einstein gravity [7–26] and its possible manifestations [27–37]. In particular, in [28], a “RG improvement” of the Schwarzschild metric has been per-

formed and the properties of the corresponding “quantum black hole” have been explored. The improvement was based upon the scale-dependent (“running”) Newton constant $G(k)$ obtained from the exact RG equation for gravity [7] describing the scale dependence of the effective average action [38–42]. Here k denotes the mass scale of the infrared cutoff which is built into the effective average action $\Gamma_k[g_{\mu\nu}]$ in such a way that it generates the field equations for a metric which has been averaged over a spacetime volume of linear dimension k^{-1} . The running of G is approximately given by

$$G(k) = \frac{G_0}{1 + \omega G_0 k^2}, \quad (1.1)$$

where G_0 denotes the laboratory value of Newton’s constant, and ω is a constant. At large distances ($k \rightarrow 0$), $G(k)$ approaches G_0 , and in the ultraviolet limit ($k \rightarrow \infty$), it decreases as $G(k) \propto 1/k^2$. This is the fixed point behavior responsible for the conjectured nonperturbative renormalizability of quantum Einstein gravity [7,9,11,12].

While the result (1.1) employs the nonperturbative renormalization group theory, the gist of it can be understood in intuitive and physical terms. If we assume that in the large distance limit the leading quantum effects are described by quantizing the linear fluctuations of the metric, $g_{\mu\nu}$, one obtains a free field theory in a curved background spacetime whose elementary quanta, the gravitons, carry energy and momentum. The vacuum of this theory will be populated by virtual graviton pairs, and the central question is how these virtual gravitons respond to the perturbation by an external test body which we immerse in the vacuum. Assuming that also in this situation gravity is universally attractive, the gravitons will be attracted towards the test body. Hence it will become “dressed” by a cloud of virtual gravitons surrounding it so that its effective mass seen by a distant observer is larger than it would be in absence of any quantum effects. This means that while in QED the quantum fluctuations *screen* external charges; in quantum gravity they have an *antiscreening* effect on external test masses. This entails Newton’s constant becoming a scale-dependent quantity $G(k)$ which is small at small distances $1/k$, and which becomes large at larger distances. This behavior is similar to the running of the non-Abelian gauge coupling in Yang-Mills theory.

In the RG improvement scheme of [28] the information about the k dependence of G is exploited in the following way. The starting point is the classical Schwarzschild metric (in Schwarzschild coordinates)

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2, \quad (1.2)$$

with $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ and the classical lapse function $f(r) = 1 - 2G_0M/r \equiv f_{\text{class}}(r)$. The RG improvement is affected by substituting, in $f_{\text{class}}(r)$, G_0 by the r -dependent Newton constant $G(r) \equiv G(k = k(r))$ which obtains from $G(k)$ via an appropriate “cutoff identifica-

tion” $k = k(r)$. In flat space the natural choice would be $k \propto 1/r$. In [28] we argued that in the Schwarzschild background the correct choice, in leading order at least, is $k(r) = \xi/d(r)$ where ξ is a constant of the order of unity, and $d(r) \equiv \int_0^r dr' |f_{\text{class}}(r')|^{-1/2}$ is the proper distance from a point with coordinate r to the center of the black hole. (We refer to [28] for a detailed physical justification of this choice.) While the integral defining $d(r)$ can be evaluated exactly, it is sufficient to use the following approximation which becomes exact for both $r \rightarrow \infty$ and $r \rightarrow 0$:

$$d(r) = \left(\frac{r^3}{r + \gamma G_0 M} \right)^{1/2}. \quad (1.3)$$

The resulting $G(r) \equiv G(k = \xi/d(r))$ reads

$$G(r) = \frac{G_0 r^3}{r^3 + \tilde{\omega} G_0 [r + \gamma G_0 M]}, \quad (1.4)$$

where $\tilde{\omega} \equiv \omega \xi^2$. In these equations the parameter γ has the value $\gamma = 9/2$ if one sets $k = \xi/d(r)$ as above. It turns out, however, that most of the qualitative properties of the improved metric, in particular, all those related to the structure of its horizons, are fairly insensitive to the precise value of γ . Specifically, $\gamma = 0$ (corresponding to $k = \xi/r$) and $\gamma = 9/2$ where found [28] to lead to rather similar results throughout. For this reason we shall adopt the choice $\gamma = 0$ in the present paper. It has the advantage that with this choice many calculations can be performed analytically which require a numerical treatment otherwise.

The metric of the RG improved Schwarzschild black hole is given by the line element (1.2) with

$$f(r) = 1 - \frac{2G(r)M}{r}. \quad (1.5)$$

Let us briefly list its essential features.¹

(a) There exists a critical mass value²

$$M_{\text{cr}} = \sqrt{\tilde{\omega}/G_0} = \sqrt{\tilde{\omega}} m_{\text{Pl}}, \quad (1.6)$$

such that $f(r)$ has two simple zeros at r_- and $r_+ > r_-$ if $M > M_{\text{cr}}$, one double zero at $r_+ = r_- = \sqrt{\tilde{\omega} G_0}$ if $M = M_{\text{cr}}$, and no zero at all if $M < M_{\text{cr}}$. For $M > M_{\text{cr}}$ the zeros are at

$$r_{\pm} = G_0 M [1 \pm \sqrt{1 - \Omega}], \quad (1.7)$$

with the convenient abbreviation

$$\Omega \equiv \frac{M_{\text{cr}}^2}{M^2} = \tilde{\omega} \left(\frac{m_{\text{Pl}}}{M} \right)^2. \quad (1.8)$$

The spacetime has an outer horizon at r_+ and inner

¹All formulas quoted refer to $\gamma = 0$, but the qualitative features are the same for $\gamma = 9/2$; see [28] for details.

²We define the (standard) Planck mass and length in terms of the laboratory value G_0 : $m_{\text{Pl}} = \ell_{\text{Pl}}^{-1} = 1/\sqrt{G_0}$.

(Cauchy) horizon at r_- . At M_{cr} , the black hole is extremal, the two horizons coincide, and the spacetime is free from any horizon if the mass is sufficiently small, $M < M_{\text{cr}}$.

- (b) The Bekenstein-Hawking temperature $T_{\text{BH}} = \kappa/2\pi$ is given by the surface gravity at the outer horizon, $\kappa = \frac{1}{2}f'(r_+)$. Explicitly,

$$\begin{aligned} T_{\text{BH}}(M) &= \frac{1}{4\pi G_0 M} \frac{\sqrt{1-\Omega}}{1+\sqrt{1-\Omega}} \\ &= \frac{1}{4\pi G_0 M_{\text{cr}}} \frac{\sqrt{\Omega(1-\Omega)}}{1+\sqrt{1-\Omega}} \\ &= \frac{M_{\text{cr}}}{4\pi\tilde{\omega}} \frac{\sqrt{\Omega(1-\Omega)}}{1+\sqrt{1-\Omega}}. \end{aligned} \quad (1.9)$$

This temperature vanishes for $M \searrow M_{\text{cr}}$, i.e. $\Omega \nearrow 1$, thus motivating the interpretation of the improved Schwarzschild metric with $M = M_{\text{cr}}$ as describing a ‘‘cold’’ remnant of the evaporation process.

- (c) The energy flux from the black hole, its luminosity L , can be estimated using Stefan’s law. It is given by $L = \sigma \mathcal{A}(M)T_{\text{BH}}(M)^4$ where σ is a constant and $\mathcal{A} \equiv 4\pi r_+^2$ denotes the area of the outer horizon. With (1.7) and (1.9) we obtain

$$L(M) = \frac{\sigma M_{\text{cr}}^2}{(4\pi)^3 \tilde{\omega}^2} \frac{\Omega(1-\Omega)^2}{[1+\sqrt{1-\Omega}]^2}. \quad (1.10)$$

For a single massless field with 2 degrees of freedom one has $\sigma = \pi^2/60$. (We use units such that $\hbar = c = k_B = 1$.)

The remaining sections of this paper are organized as follows: In Sec. II we discuss the essential properties of the quantum-corrected Vaidya metric. In Sec. III we analyze the properties of the various horizonlike loci of the improved Vaidya metric. In Sec. IV we discuss the dynamics of the event horizon and the global structure of the spacetime, while Sec. V contains the conclusions.

II. THE QUANTUM-CORRECTED VAIDYA METRIC

Our aim is to find a metric which describes the history of an evaporating Schwarzschild black hole and its gravitational field. In the small luminosity limit ($L \rightarrow 0$) this metric is supposed to reduce to the static metric of the RG improved Schwarzschild spacetime.

We begin by reexpressing the metric (1.2) with the improved lapse function (1.5) in terms of ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ) . We trade the Schwarzschild time t for the advanced time coordinate

$$v = t + r^*, \quad r^* \equiv \int^r dr'/f(r'). \quad (2.1)$$

Here r^* is a generalization of the familiar ‘‘tortoise’’ radial coordinate to which it reduces if $G(r) = \text{const}$. For $G(r) \neq$

const the function $r^* = r^*(r)$ is more complicated, but its explicit form will not be needed here. Equation (2.1) implies $dv = dt + dr/f(r)$, turning (1.2) with (1.5) into

$$ds^2 = -[1 - 2G(r)M/r]dv^2 + 2dvdr + r^2d\Omega^2. \quad (2.2)$$

Equation (2.2) is exactly the Schwarzschild metric in Eddington-Finkelstein coordinates, with G_0 replaced by $G(r)$. It is thus reassuring to see that the two operations, the RG improvement $G_0 \rightarrow G(r)$ and the change of the coordinate system, can be performed in either order—they ‘‘commute.’’

The thermodynamical properties derived in [28] and summarized in the previous section refer to the metric (2.2). In the exterior of the hole the spacetime is static, and while we can deduce a temperature and a corresponding luminosity from its periodicity in imaginary time (or by computing the surface gravity at r_+ directly) the backreaction of the mass loss due to the evaporation is not described by (2.2). From the static metric we obtained the mass dependence of the luminosity, $L = L(M)$. Using this information we can compute the mass of the hole as seen by a distant observer at time v , $M(v)$, by solving the differential equation

$$-\frac{d}{dv}M(v) = L(M(v)). \quad (2.3)$$

In our case $L(M)$ is given by Eq. (1.10). To first order in the luminosity, the metric which incorporates the effect of the decreasing mass is obtained by replacing the constant M in (2.2) with the $M(v)$ obtained from Eq. (2.3):

$$ds^2 = -[1 - 2G(r)M(v)/r]dv^2 + 2dvdr + r^2d\Omega^2. \quad (2.4)$$

For $G(r) = \text{const}$, Eq. (2.4) is the Vaidya metric which frequently had been used to explore the influence of the Hawking radiation on the geometry [43–45]. It is a solution of Einstein’s equation $G_{\mu\nu} = 8\pi G_0 T_{\mu\nu}$ where $T_{\mu\nu}$ describes an inward moving null fluid. In this picture the decrease of M is due to the inflow of negative energy, as it is appropriate if the field whose quanta are radiated off is in the Unruh vacuum [46].

The metric (2.4) can be regarded as a RG improved Vaidya metric. It encapsulates two different mechanisms whose combined effect can be studied here: the black hole radiance, and the modifications of the spacetime structure due to the quantum gravity effects, the running of G in particular.

It is instructive to ask which energy-momentum tensor $T_{\mu\nu}$ would give rise to the improved Vaidya metric (2.4) according to the classical equation $G_{\mu\nu} = 8\pi G_0 T_{\mu\nu}$. Computing the Einstein tensor of (2.4) one finds that its only nonzero components are

$$T^v_v = T^r_r = -\frac{G'(r)M(v)}{8\pi G_0 r^2}, \quad (2.5a)$$

$$T^r_v = \frac{G(r)\dot{M}(v)}{8\pi G_0 r^2}, \quad (2.5b)$$

$$T^\theta_\theta = T^\phi_\phi = -\frac{G''(r)M(v)}{16\pi G_0 r}. \quad (2.5c)$$

Here the prime (dot) denotes a derivative with respect to $r(v)$. The nonzero components (2.5) contain either r or v derivatives but no mixed terms. The terms with r derivatives of G , also present for $M(v) = \text{const}$, describe the vacuum energy density and pressure of the improved Schwarzschild spacetime in absence of radiation effects. They have been discussed in [28] already. Allowing for $M(v) \neq \text{const}$, the new feature is a nonzero component $T^r_v \neq 0$ which, for $\dot{M} < 0$, describes the inflow of negative energy into the black hole.

Taking advantage of the luminosity function $L(M)$, Eq. (1.10), we can solve the differential Eq. (2.3) numerically and obtain the mass function $M = M(v)$. (We have set $\sigma/(4\pi)^3 \tilde{\omega} = 1$ in the numerical calculations in order to reach the almost complete evaporation for $v \approx 200$ in units of r_{cr} .) The result is shown in Fig. 1 for various initial masses, in the domain $v > 0$. In fact, for definiteness we assume that the black hole is formed at $v = 0$ by the implosion of a spherical null shell [44]. Hence $M(v)$ is given by Fig. 1 together with $M = 0$ for $v < 0$. We observe that, for any initial mass, $M(v)$ approaches the critical mass M_{cr} for $v \rightarrow \infty$. This behavior is the most important mani-

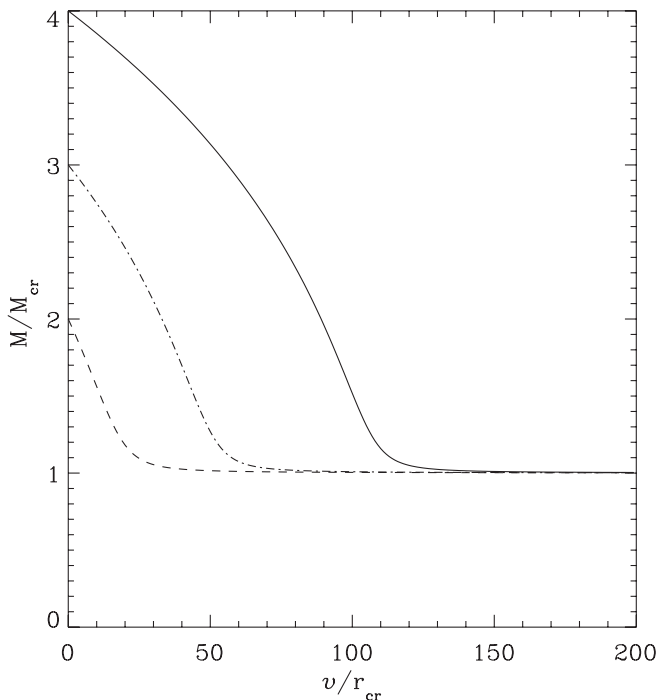


FIG. 1. The ratio M/M_{cr} as a function of v/r_{cr} for various initial masses.

festation of the quantum gravity effects: according to Eq. (1.9), the temperature $T_{\text{BH}}(M)$ goes to zero when M approaches M_{cr} from above. Hence the luminosity vanishes, too, the evaporation process stops, and $M(v) \approx M_{\text{cr}}$ remains approximately constant at very late times, $v \gg M_{\text{cr}}^{-1}$.

In Figs. 2 and 3 we plot the advanced time dependence of the temperature $T_{\text{BH}}(v) \equiv T_{\text{BH}}(M(v))$ and the luminosity $L(v) \equiv L(M(v))$, respectively. They are obtained by inserting the numerical solution of Eq. (2.3) into (1.9) and (1.10).

Both the very early and the very late stages of the evaporation process can be described analytically. A black hole with $M(v = 0) \gg M_{\text{cr}}$ starts in what we call the ‘‘Hawking regime.’’ It is defined by the approximation $\Omega \approx 0$ which is realized if the hole is very heavy ($M \gg M_{\text{cr}}$) or if $\tilde{\omega} = 0$ in the semiclassical limit where the quantum gravity corrections are ‘‘switched off.’’ In the Hawking regime, (1.9) and (1.10) reproduce the familiar results

$$T_{\text{BH}}(M) = \frac{1}{8\pi G_0 M}, \quad (2.6a)$$

$$L(M) = \frac{B}{G_0^2 M^2}, \quad B \equiv \frac{\sigma}{4(4\pi)^3}. \quad (2.6b)$$

It is easy to solve the differential equation $-\dot{M} = L(M)$ for the luminosity (2.6b). With the initial condition $M(v = 0) = M_0$ the solution reads

$$M(v) = [M_0^3 - 3(B/G_0^2)v]^{1/3}. \quad (2.7)$$

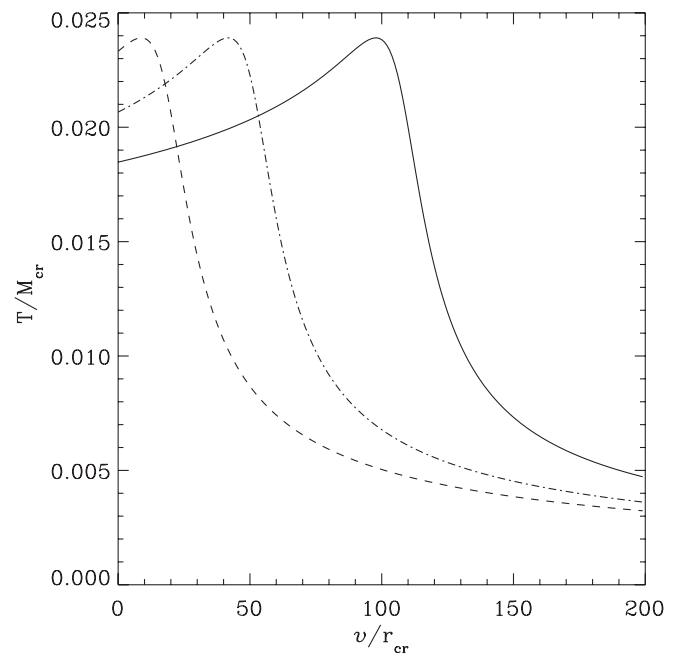


FIG. 2. Time dependence of the Bekenstein-Hawking temperature during the evaporation process for the same initial masses as in Fig. 1.

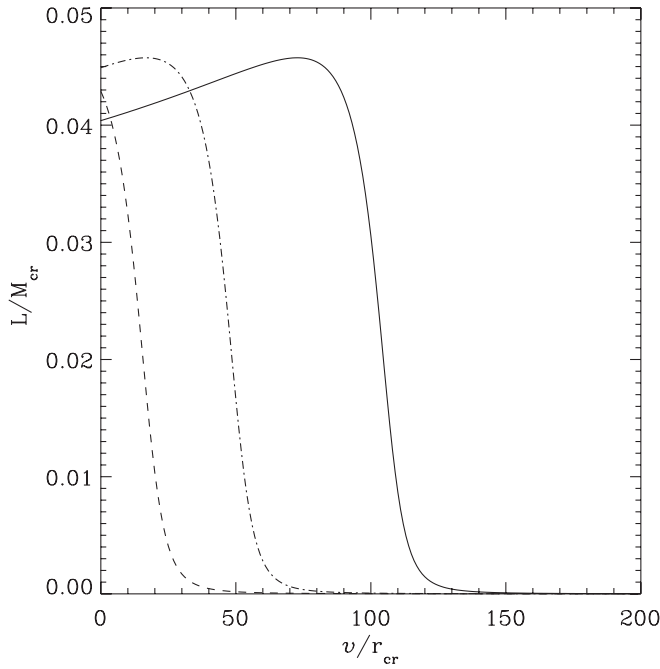


FIG. 3. The black hole's luminosity as a function of v/r_{cr} , for the same initial masses as in Figs. 1 and 2.

This is the mass function during the early stages of the evaporation process, valid as long as $M(v)$ is well above the critical mass. If one naïvely extrapolates (2.7) to small masses one finds $M(v_0) = 0$, implying a final “explosion” with $T \rightarrow \infty$ and $L \rightarrow \infty$, after a finite time $v_0 = G_0^2 M_0^3 / (3B)$. As a consequence of the quantum gravity effects, this is not what really happens, however.

The final part of the evaporation process, where the cold remnant forms, is in the “critical regime.” It is described by those terms in the above expressions which are dominant for $M \searrow M_{\text{cr}}$, or $\Omega \nearrow 1$. From (1.9) and (1.10) we obtain in this approximation:

$$T_{\text{BH}}(M) = \frac{1}{4\pi\tilde{\omega}} \sqrt{M^2 - M_{\text{cr}}^2}, \quad (2.8a)$$

$$L(M) = \frac{\sigma G_0}{(4\pi\tilde{\omega})^3} (M^2 - M_{\text{cr}}^2)^2. \quad (2.8b)$$

Solving $-\dot{M} = L(M)$ with (2.8b) one finds

$$M(v) = M_{\text{cr}} + \frac{M_1 - M_{\text{cr}}}{1 + \alpha(M_1 - M_{\text{cr}})(v - v_1)}. \quad (2.9)$$

Here $\alpha \equiv \sigma / (16\pi^3 \tilde{\omega}^2)$, and v_1 is a time, already in the critical regime, where $M(v_1) = M_1$ is imposed. For $v \rightarrow \infty$, the difference $M(v) - M_{\text{cr}}$ vanishes proportional to $1/v$, as a consequence of which $T_{\text{BH}}(v) \rightarrow 0$ and $L(v) \rightarrow 0$.

We mentioned already that the RG improved Vaidya metric (2.4) can be a correct description only to first order in L . In fact, deriving the surface gravity and luminosity from (2.4) the results differ from those for the improved

Schwarzschild metric by terms due to the v dependence of M . In our approximation those terms are neglected as they would contain additional factors of $\dot{M} = -L$.

III. APPARENT HORIZON AND TIMELIKE LIMIT SURFACE

Next we turn to the various horizonlike loci of the improved Vaidya metric. Regarded as 3-surfaces, all of them are histories of spherical 2-surfaces.

The apparent horizon is a marginally trapped surface. We determine it from the condition that one of the congruences of radial null geodesics, in affine parametrization, has vanishing expansion scalar there, $\Theta = 0$. The improved Vaidya metric (2.4) has the structure

$$ds^2 = [-f(r, v)dv + 2dr]dv + r^2 d\Omega^2. \quad (3.1)$$

Along outgoing radial null geodesics we have $f dv = 2dr$. Hence, parametrizing them as $r = r(v)$, they obey the differential equation $\dot{r}(v) = f(r(v), v)/2$ where the dot denotes a derivative with respect to v . We can rewrite this equation in the autonomous form

$$\frac{d}{d\lambda} x^\mu(\lambda) = u^\mu(x(\lambda)) \quad (3.2)$$

with the null vector field

$$u^\mu \equiv (u^v, u^r, u^\theta, u^\phi) = \left(1, \frac{1}{2}f, 0, 0\right). \quad (3.3)$$

A short calculation reveals that the geodesic equation holds in the form $u^\nu D_\nu u^\mu = \mathcal{K}u^\mu$ with a nonzero function

$$\mathcal{K} = \frac{1}{2}\partial_r f. \quad (3.4)$$

Hence the parameter λ in (3.2) is not an affine one. In order for the standard discussion [6,47] to be applicable we must reexpress the null geodesics in terms of an affine parameter λ_* . Given a solution $x(\lambda)$ of (3.2) we compute the function $\lambda_*(\lambda)$ by integrating

$$\frac{d}{d\lambda} \lambda_*(\lambda) = \exp \int^\lambda d\lambda' \mathcal{K}(x(\lambda')) \quad (3.5)$$

and determine its inverse $\lambda = \lambda(\lambda_*)$. Then we define $x_*^\mu(\lambda_*) \equiv x^\mu(\lambda(\lambda_*))$ which satisfies

$$\frac{d}{d\lambda_*} x_*^\mu(\lambda_*) = n^\mu(x_*(\lambda_*)). \quad (3.6)$$

Here

$$n^\mu(x) = e^{-\Gamma(x)} u^\mu(x) \quad (3.7)$$

is a new vector field, with $\Gamma(x)$ satisfying

$$u^\mu \partial_\mu \Gamma = \mathcal{K}. \quad (3.8)$$

Using (3.8) one easily verifies that $n^\nu D_\nu n^\mu = 0$, implying that λ_* is an affine parameter [6].

The expansion scalar Θ which determines the location of the AH is the divergence of n^μ . Using (3.8) we find

$$\Theta \equiv D_\mu n^\mu = e^{-\Gamma} [D_\mu u^\mu - \mathcal{K}]. \quad (3.9)$$

For $f(r, v) = 1 - 2G(r)M(v)/r$ we have

$$D_\mu u^\mu = \frac{1}{r}f + \frac{1}{2}\partial_r f. \quad (3.10)$$

Together with (3.4) this yields the expansion scalar

$$\Theta = \frac{1}{r}e^{-\Gamma}f. \quad (3.11)$$

This is the same result as for the classical Vaidya metric; the r dependence of G did not lead to extra terms.

Equation (3.11) tells us that Θ vanishes if, and only if, f vanishes. According to point (a) of the introduction this is the case at $r = r_+$ and $r = r_-$ with r_\pm defined by (1.7) with (1.8). [We assume that the r dependence of G is given by (1.4) with $\gamma = 0$.] Since r_\pm depends on M , it has become a function of the advanced time v now: $r_\pm(v) \equiv r_\pm(M(v))$. Defined as the outermost trapped surface, the AH is characterized by the implicit equation $r = r_+(v)$ where, explicitly,

$$r_+(v) = \frac{\tilde{\omega}}{M_{\text{cr}}} \left[\frac{M(v)}{M_{\text{cr}}} + \sqrt{\left(\frac{M(v)}{M_{\text{cr}}}\right)^2 - 1} \right]. \quad (3.12)$$

The second horizon surface, the TLS, is defined as the locus where the 4-velocity of static observers $u^\alpha \propto \delta_\tau^\alpha$ becomes lightlike, with $\partial/\partial\tau$ a vector orthogonal to the $r = \text{const}$ hypersurfaces. Since we consider a spherically symmetric spacetime, this vector is precisely $\partial/\partial v$ and AH and TLS coincide in this case, being $u^\alpha u^\beta g_{\alpha\beta} = g_{vv} = -f$.

Strictly speaking, apart from the outer AH $r = r_+(v)$ there exists also an inner TLS = AH $r = r_-(v)$, at which the vector field $\partial/\partial v$ switches back from spacelike to timelike.

IV. THE EVENT HORIZON

The radial light rays $r = r(v)$ of the outgoing null congruence are to be found by solving the differential equation $\dot{r}(v) = f(r(v), v)/2$, or explicitly,

$$\frac{dr(v)}{dv} = \frac{1}{2} \left(1 - \frac{2G(r(v))M(v)}{r(v)} \right). \quad (4.1)$$

Depending on their initial points, light rays can, or cannot, escape to infinity for $v \rightarrow \infty$. By definition, the ‘‘separatrix’’ separating those two classes of solutions is the event horizon—the outermost locus traced by outgoing photons that can never reach arbitrarily large distances.

It is easy to understand why for a radiating (as opposed to an accreting) black hole the EH is *inside* the TLS. Inside (outside) the TLS, f is negative (positive), implying that the light ray’s $r(v)$ decreases (increases) if $r(v) < r_{\text{TLS}}(v)$

[$r(v) > r_{\text{TLS}}(v)$]. Under certain conditions it can happen that due to the hole’s mass loss the radius of the TLS decreases faster than $r(v)$. As a result, the light ray intersects the TLS, with $\dot{r} = 0$ there, and then escapes from the hole with $\dot{r} > 0$.

The situation is illustrated in Fig. 4. The region between r_{EH} and r_{TLS} forms the hole’s ‘‘quantum ergosphere’’ which owes its existence entirely to the evaporation process.

Clearly a determination of the EH’s $r_{\text{EH}}(v)$ requires knowledge of $M(v)$ for arbitrarily late times v even. Therefore the semiclassical approximation is not sufficient to find the EH since it breaks down for small M . As a way out, York [2,3] proposed to replace the above rigorous definition of the EH by an approximate criterion which is local in v and does not require the teleological information about $M(v)$ at late v . It is supposed to be valid for small L . In the following we analyze (4.1) both using the approximate criterion and the exact definition of the EH. Since we have an explicit prediction for the final stages of the evaporation process we are in a position to determine the EH exactly, and thus to assess the validity and precision of York’s approximate ‘‘working definition.’’

Let us begin with this working definition, applied to the RG improved spacetime. While according to Eq. (4.1), the ‘‘velocity’’ $\dot{r}(v)$ is strictly negative inside the TLS, the acceleration $\ddot{r}(v)$ can have either sign there. York proposes

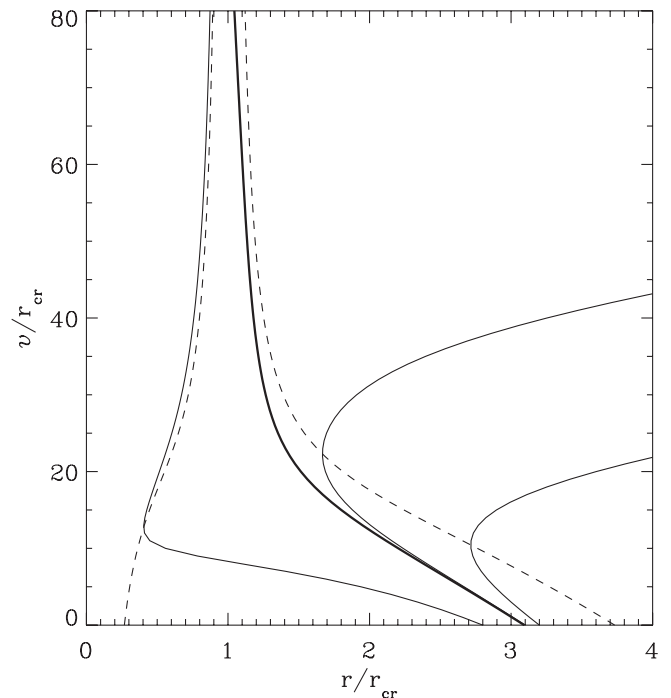


FIG. 4. Light rays of the outgoing null congruence for $M/M_{\text{cr}} = 2$. The thick solid line is the EH determined numerically, the dashed lines are the outer and inner TLS = AH, and the thin solid lines are generic null rays, inside and outside the EH.

to approximately identify $r_{\text{EH}}(v)$ with the radius where the light ray's "acceleration" vanishes: $\ddot{r}(\text{EH}) = 0$. In pictorial terms one could think of this condition as separating the light rays curving downward ($\ddot{r} < 0$) from those curving upward ($\ddot{r} > 0$). Clearly the former (latter) are very unlikely (likely) to subsequently cross the TLS, even if this is not a rigorous criterion, of course. (See Fig. 5 for a schematic sketch.) At least in the Hawking regime, photons are imprisoned by the approximate horizon for times long compared to the dynamical time scale of the evaporating hole [2,3].

Taking a second v derivative of (4.1) we obtain the acceleration

$$\ddot{r}(v) = L(v) \frac{G(r)}{r} + G(r)M(v) \frac{\dot{r}}{r^2} - G'(r)M(v) \frac{\dot{r}}{r}. \quad (4.2)$$

Here we assumed that $G(r)$ does not have a parametric dependence on M , which is actually true for $\gamma = 0$:

$$G(r) = \frac{G_0 r^2}{r^2 + \tilde{\omega} G_0}. \quad (4.3)$$

[For general γ there appears an additional term $\propto dG/dM$ in (4.2) which is irrelevant qualitatively.] Equation (4.2) tells us that, when $\ddot{r} = 0$, the radius r , the velocity \dot{r} , and the time v are related by

$$\dot{r} = -LG(r) \left[\frac{G(r)M(v)}{r} - M(v)G'(r) \right]^{-1}. \quad (4.4)$$

Since $r_{\text{EH}} = r_{\text{TLS}}$ if the hole would not radiate, and since higher orders in the luminosity are neglected, the difference $r_{\text{TLS}} - r_{\text{EH}}$ which we would like to compute is of order L . For this reason we may replace $r \equiv r_{\text{EH}}$ on the right-hand side of (4.4) with $r_{\text{TLS}} = r_+(v)$, the error being of order L^2 :

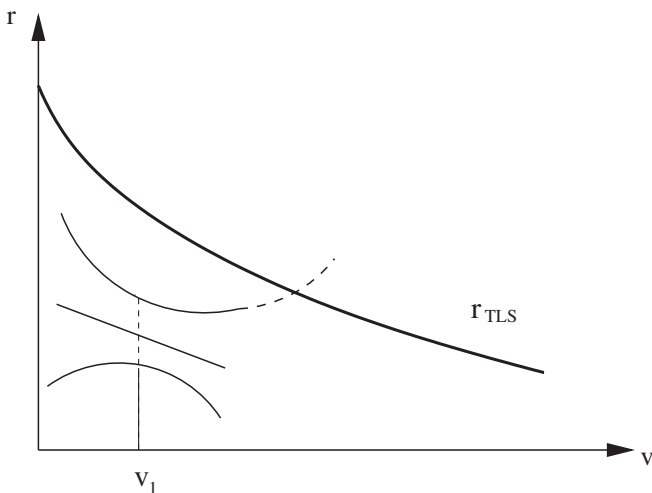


FIG. 5. Light rays of the outgoing null congruence with $\ddot{r} > 0$, $\ddot{r} = 0$, and $\ddot{r} < 0$ at v_1 . Those with $\ddot{r} > 0$ are most likely to eventually cross the TLS.

$$\begin{aligned} \dot{r} &= -2LG(r_+) \left[\frac{2G(r_+)M(v)}{r_+} - 2M(v)G'(r_+) \right]^{-1} \\ &= \frac{-2LG(r_+)}{[1 - 2M(v)G'(r_+)]}. \end{aligned} \quad (4.5)$$

In the second equality of (4.5) we used that $2G(r_+)M = r_+$ which follows from $f(r_+) = 0$. Equation (4.5) provides us with the velocity \dot{r} at the point where the acceleration vanishes. The corresponding coordinate, r_{EH} , is the approximate location of the EH. We obtain it by using $\dot{r} = f(r)/2$ in order to rewrite the left-hand side of (4.5) in the form $f(r_{\text{EH}})/2$, and then inverting the function f . This inversion is easy to perform since, again, we may expand in $r_{\text{EH}} - r_+ = O(L)$:

$$f(r_{\text{EH}}) = f(r_+) + (r_{\text{EH}} - r_+)f'(r_+) + O(L^2).$$

Hence, to order L , $\dot{r}(\text{EH}) = (r_{\text{EH}} - r_+)f'(r_+)/2$ which yields, together with (4.5),

$$r_{\text{EH}} = r_+ - \frac{4LG(r_+)}{f'(r_+)[1 - 2MG'(r_+)]}. \quad (4.6)$$

Differentiating $f(r) = 1 - 2G(r)M/r$ and using $2G(r_+)M = r_+$ one obtains the relation $f'(r_+) = [1 - 2MG'(r_+)]/r_+$. This leads to the following explicit formula for the v dependence of r_{EH} :

$$r_{\text{EH}}(v) = r_+(v) \left\{ 1 - \frac{4L(v)G(r_+(v))}{[1 - 2M(v)G'(r_+(v))]^2} \right\}. \quad (4.7)$$

Equation (4.7) is valid for an arbitrary r dependence of G , still. If $G = \text{const}$ we recover York's result $r_{\text{EH}} = r_+[1 - 4G_0L]$ with $r_+ = 2G_0M$.

Let us now specialize for the function $G(r)$ motivated by quantum gravity, Eq. (4.3). This r dependence of Newton's constant implies that, at the TLS, $G(r_+) = r_+/(2M)$ and $G'(r_+) = \tilde{\omega}/(2M^2r_+)$. These equations were simplified using the relation $r_+^2 + \tilde{\omega}G_0 = 2G_0Mr_+$ which is equivalent to $f(r_+) = 0$ with (4.3). They lead to the following result for the position of the EH:

$$r_{\text{EH}} = r_+[1 - 4G_0L(M)Y(\Omega)], \quad (4.8a)$$

$$Y(\Omega) \equiv \frac{1 + \sqrt{1 - \Omega}}{2(1 - \Omega)}. \quad (4.8b)$$

The radius r_+ is given by Eq. (1.7) or (3.12), respectively, and M and $\Omega \equiv M_{\text{cr}}^2/M(v)^2$ are understood to be functions of v , of course. The correction factor $Y(\Omega)$ measures the deviations from the semiclassical result; we have $Y(\Omega) \approx 1$ everywhere in the Hawking regime ($\Omega \approx 0$). For $\Omega \nearrow 1$ the corrections become large; in fact, $Y(\Omega)$ diverges at $\Omega = 1$. Nevertheless, the product LY vanishes for $\Omega \nearrow 1$, $M \searrow M_{\text{cr}}$, which becomes clear when we insert (1.10) into (4.8a):

$$r_{\text{EH}} = r_+ \left[1 - \frac{2\sigma}{(4\pi)^3 \tilde{\omega}} \frac{\Omega(1-\Omega)}{[1 + \sqrt{1-\Omega}]} \right] \quad (4.9a)$$

$$= r_+ - \frac{2\sigma}{(4\pi)^3} \frac{1}{M} \left(1 - \frac{M_{\text{cr}}^2}{M^2} \right). \quad (4.9b)$$

This is our final result for the radius of the event horizon, as given by York's approximate local criterion. In the early stages of the evaporation we recover the semiclassical result. For $\nu \rightarrow \infty$, however, $M(\nu) \rightarrow M_{\text{cr}}$, and both r_{EH} and $r_+(\nu)$ approach the same limiting value asymptotically: $r_{\text{cr}} \equiv r_{\pm}(M_{\text{cr}}) = \sqrt{\tilde{\omega} G_0} = \sqrt{\tilde{\omega}} \ell_{\text{pl}}$. This behavior can be seen in Fig. 6.

As we anticipated, we now see that r_{EH} is indeed smaller than r_+ during the entire evaporation process; the EH is *inside* the (outer) TLS, thus giving rise to a quantum ergosphere.

It is an important observation that, while r_{EH} is smaller than r_+ , it is always larger than r_- , provided the radiation effects are small compared to the quantum gravity effects in an appropriate sense. Comparing the radius (4.9) to r_- as given by (1.7) we find that $r_{\text{EH}}(M) > r_-(M)$ provided

$$\frac{\sigma}{(4\pi)^3 \tilde{\omega}} < \frac{1}{\Omega \sqrt{1-\Omega}}. \quad (4.10)$$

For $0 \leq \Omega \leq 1$ the right-hand side of (4.10) is bounded below by the constant $3\sqrt{3}/2$. As a result, r_{EH} is larger than r_- during the entire evaporation process provided

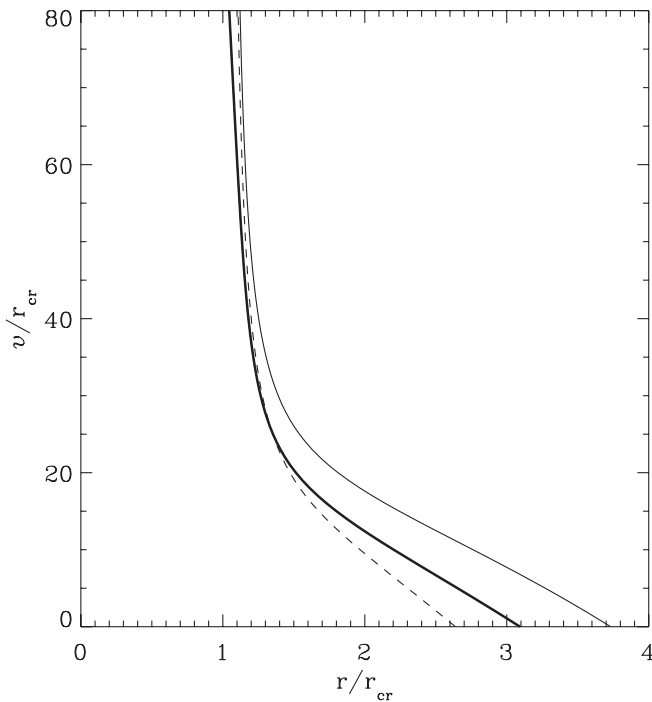


FIG. 6. The EH determined numerically (solid thick line) and by York's approximate criterion (dashed line) for $M/M_{\text{cr}} = 2$. The solid thin line is the AH.

$$\sigma < \frac{3}{2}\sqrt{3}(4\pi)^3 \tilde{\omega}. \quad (4.11)$$

If (4.11) is satisfied, we have $r_+(\nu) > r_{\text{EH}}(\nu) > r_-(\nu)$ for any finite ν , and the EH touches both the outer and the inner TLS only in the limit $\nu \rightarrow \infty$ where $r_+, r_{\text{EH}}, r_- \rightarrow r_{\text{cr}}$.

If (4.11) is violated our method is inapplicable, most probably, and the improved Vaidya metric is not a reliable description of the spacetime structure. This metric is valid to first order in L only, which means that the dimensionless luminosity has to be much smaller than unity, $G_0 L \ll 1$. If so, Eq. (1.10) implies $\sigma/[(4\pi)^3 \tilde{\omega}] \ll 1$, and (4.11) is indeed satisfied. (Note that $M_{\text{cr}}^2/\tilde{\omega}^2 = G_0^{-1}/\tilde{\omega}$.) It is reassuring to see that, for pure gravity, Eq. (4.11) is fulfilled with a very broad margin; the expansion parameter $\sigma/[(4\pi)^3 \tilde{\omega}]$ assumes the tiny value $8.3 \cdot 10^{-5}/\tilde{\omega}$ in this case, with $\tilde{\omega} = O(1)$.

Let us return to the exact definition of the EH now. We compute it for our model by numerically solving (4.1) for a set of initial conditions $r(0)$. For a certain range of $r(0)$'s the trajectories will ultimately cross the TLS and escape to infinity, while the others remain at radii below r_{TLS} for arbitrarily late advanced times. The light ray separating those two classes of solutions defines the EH, a null hypersurface by construction.

In Fig. 4 we show various trajectories of either class, as well as the EH, determined numerically from its *exact* definition. We have numerically evolved several initial conditions $r(0)$ in order to determine the boundary between trapped and escaping null geodesics in the limit $\nu \rightarrow \infty$. In Fig. 6 the true EH is compared to the prediction of York's criterion. Obviously the latter provides a rather accurate approximation to the true horizon.

The global structure of the spacetime is depicted in the conformal diagram in Fig. 7. Region I is a flat spacetime, while at $V = V_0$ (V is the Kruskal advanced time coordinate, defined as $V = -\exp(-\kappa\nu)$ being κ the surface gravity of the outer horizon) an imploding null shell is present (strictly speaking it must have a negative tension in order to balance the flux of negative energy on its future side [48]). Region II is the evaporating black hole spacetime. The AH is a timelike hypersurface which "meets" the EH at future null infinity in the conformal diagram. The null ray which is tangent to the earliest portion of the apparent horizon A would have been the EH if the hole were not radiating. The final state of the black hole is an extremal black hole whose inner and outer horizons have the same radius ($r = r_{\text{cr}}$) and are located at the event horizon EH and the inner (Cauchy) horizon CH in Fig. 7.

It is instructive to compare the areas \mathcal{A} of the various horizons. They are defined by intersecting the EH, AH, and TLS with the incoming null surfaces $\nu = \text{const}$. Thus $\mathcal{A}_{\text{TLS}}(\nu) = 4\pi r_+(\nu)^2$ and $\mathcal{A}_{\text{EH}}(\nu) = 4\pi r_{\text{EH}}(\nu)^2$. From Eq. (1.7) we obtain for $\mathcal{A}_{\text{TLS}} \equiv \mathcal{A}_{\text{AH}}$

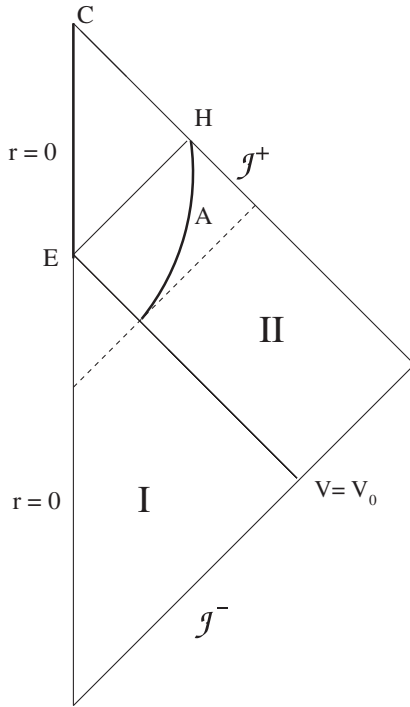


FIG. 7. The conformal diagram of the evaporating quantum black hole: region I is a flat spacetime, and region II is the evaporating BH spacetime. EH is the event horizon, CH is the inner (Cauchy) horizon, and A is the apparent horizon.

$$\mathcal{A}_{\text{TLS}} = 4\pi G_0^2 M^2 \left[1 + \sqrt{1 - (M_{\text{cr}}/M)^2} \right]^2, \quad (4.12)$$

and the approximate result (4.9a) for the event horizon implies

$$\mathcal{A}_{\text{EH}} = \mathcal{A}_{\text{TLS}} \left[1 - \frac{4\sigma}{(4\pi)^3 \tilde{\omega}} \frac{\Omega(1-\Omega)}{1 + \sqrt{1-\Omega}} \right], \quad (4.13)$$

where a term of second order in $\sigma/(4\pi)^3 \tilde{\omega}$ has been neglected. The difference $\delta\mathcal{A} \equiv \mathcal{A}_{\text{TLS}} - \mathcal{A}_{\text{EH}}$ is given by

$$\delta\mathcal{A} = \frac{\sigma}{4\pi^2} \ell_{\text{Pl}}^2 (1-\Omega) [1 + \sqrt{1-\Omega}]. \quad (4.14)$$

During the early stages of the evaporation process, $\delta\mathcal{A} \approx \sigma \ell_{\text{Pl}}^2 / (2\pi^2) = 128\pi B \ell_{\text{Pl}}^2$ which coincides with the known result [2] for the Hawking regime, while $\delta\mathcal{A}$ vanishes proportional to $(M^2 - M_{\text{cr}}^2) \rightarrow 0$ for $v \rightarrow \infty$. It had been emphasized by York [2,3] that in the Hawking regime he considered, $\delta\mathcal{A}$ is a universal (i.e. M -independent) quan-

tity which depends only on σ , thus counting the degrees of freedom of the field quanta which can be evaporated off. Looking at Eq. (4.14) we see that this universality does not persist beyond the semiclassical approximation.

V. CONCLUSION

The renormalization group improvement of black hole spacetimes according to quantum Einstein gravity leads to concrete predictions on the final state of the evaporation process. Unlike previous studies based on *ad hoc* modifications of the equation of state of matter at very high (Planckian) densities [49–51], or models based on loop quantum gravity [52], the mass of the remnant can be calculated explicitly: $M_{\text{cr}} = \sqrt{\tilde{\omega}} \ell_{\text{Pl}}$. Its precise value is determined by the value of $\tilde{\omega}$ which is a measurable quantity in principle [53]. No naked singularity forms, at variance with the paradigm proposed in [44,51], so that the remnant is a mini-black hole of Planckian size. On the other hand, it is intriguing to note that exactly solvable semiclassical gravity-dilaton models predict a final state described by an extremal configuration which is reached in an infinite amount of time [54]. (See also [55,56] for analogous semiclassical models, and [57,58] for an approach based on special resummations of higher order graviton loops.)

It would be interesting to investigate the possible astrophysical implications of a population of stable Planck size mini-black holes produced in the early universe or by the interaction of cosmic rays with the interstellar medium [59,60]. We hope to address these issues in a subsequent publication.

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