

Factorization of finite temperature graphs in thermal QED

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(Received 30 January 2006; published 30 March 2006)

We extend our previous analysis of gauge and Dirac fields in the presence of a chemical potential. We consider an alternative thermal operator which relates in a simple way the Feynman graphs in QED at finite temperature and charge density to those at zero temperature but nonzero chemical potential. Several interesting features of such a factorization are discussed in the context of the thermal photon and fermion self-energies.

DOI: [10.1103/PhysRevD.73.067702](https://doi.org/10.1103/PhysRevD.73.067702)

PACS numbers: 11.10.Wx

In earlier papers [1,2], we gave a simple derivation of an interesting relation [3–5] between finite temperature Feynman graphs and the corresponding zero-temperature diagrams in the imaginary as well as in real time formulations of thermal field theories. We showed that in the absence of a chemical potential, the finite temperature diagrams involving scalar, fermions or gauge fields can be related to the zero-temperature graphs through a simple thermal operator. On the other hand, in the presence of a chemical potential μ , the thermal operator is more complicated as it involves also time derivatives. In a theory involving fermionic fields with a chemical potential, we have shown that a complete factorization seems to be violated by the presence of an infrared singular contact term [2]. As we have pointed out, such a behavior may be due to the simplest possible choice of generalizing the basic thermal operator in the presence of a chemical potential, which involved only a single reflection operator $S(E)$ that changes the energy as $E \rightarrow -E$.

In the present note, we propose an alternate thermal operator without time derivatives, but which involves an additional fermionic distribution operator $\hat{N}_F^{(T,\mu)}(E)$. We will show that such a thermal operator representation naturally leads in QED to a simple factorization of the thermal amplitudes in the presence of a chemical potential. A rather similar approach has been recently proposed in the context of complex scalar fields at finite temperature and charge density [6].

For brevity, we will discuss here the theory in the imaginary time formalism [7–9], although everything also holds in the real time formalism. Using this formulation and the mixed space representation, the fermion propagator at zero temperature but finite chemical potential can be written in the form

$$S^{(0,\mu)}(\tau, E) = \frac{1}{2E} [\theta(\tau)A(E)e^{-(E-\mu)\tau} + \theta(-\tau)B(E)e^{(E+\mu)\tau}], \quad (1)$$

for $E > |\mu|$, where $E = \sqrt{\vec{p}^2 + m^2}$ and $A(E)$, $B(E)$ are

Euclidean matrices given by

$$\begin{aligned} A(E) &= i\gamma^0 E - \vec{\gamma} \cdot \vec{p} + m, \\ B(E) &= -i\gamma^0 E - \vec{\gamma} \cdot \vec{p} + m. \end{aligned} \quad (2)$$

The action of the operator $\hat{N}_F^{(T,\mu)}(E)$ on the above time dependent exponential functions is defined as follows:

$$\begin{aligned} \hat{N}_F^{(T,\mu)}(E)e^{s(E\pm\mu)\tau} &= n_F(E \pm \mu)e^{s(E\pm\mu)\tau} \\ &\equiv n_F^\pm(E)e^{s(E\pm\mu)\tau}, \end{aligned} \quad (3)$$

where T is the temperature, n_F denotes the Fermi-Dirac distribution function and s is a sign. As shown below, this prescription enforces the antiperiodicity condition of the thermal fermion propagator at nonzero chemical potential.

In terms of the operators $S(E)$ and $\hat{N}_F^{(T,\mu)}(E)$, our alternate basic fermion operator has the simple form

$$\hat{O}_F^{(T,\mu)}(E) = 1 - \hat{N}_F^{(T,\mu)}(E)[1 - S(E)]. \quad (4)$$

Then, acting on the propagator $S^{(0,\mu)}(\tau, E)$, this fermion operator will naturally reproduce the fermion propagator at finite temperature and chemical potential

$$\begin{aligned} S^{(T,\mu)}(\tau, E) &= \hat{O}_F^{(T,\mu)}(E)S^{(0,\mu)}(\tau, E) \\ &= \frac{1}{2E} \{A(E)[\theta(\tau) - n_F^-(E)]e^{-(E-\mu)\tau} \\ &\quad + B(E)[\theta(-\tau) - n_F^+(E)]e^{(E+\mu)\tau}\}, \end{aligned} \quad (5)$$

which satisfies, for $-1/T < \tau < 0$, the antiperiodicity condition [10]

$$S^{(T,\mu)}(\tau, E) = -S^{(T,\mu)}\left(\tau + \frac{1}{T}, E\right). \quad (6)$$

The above factorization of the thermal fermion propagator is of fundamental importance for the factorization of arbitrary graphs in QED at finite temperature. Before we present a general proof, let us consider first some simple amplitudes at one-loop order. For example, the integrand of the fermion self-energy (in the Feynman gauge) may be

written with the help of Eq. (5) in a completely factorized form as

$$\begin{aligned}\Sigma^{(T,\mu)}(E_1, E_2, \tau) &= -e^2 D^{(T)}(\tau, E_1) \gamma_\alpha S^{(T,\mu)}(\tau, E_2) \gamma_\alpha \\ &= \mathcal{O}_B^{(T)}(E_1) \hat{\mathcal{O}}_F^{(T,\mu)}(E_2) \Sigma^{(0,\mu)}(E_1, E_2, \tau),\end{aligned}\quad (7)$$

where $\mathcal{O}_B^{(T)}$ denotes the basic photon projection operator

$$\mathcal{O}_B^{(T)}(E_1) = 1 + n_B(E_1)[1 - S(E_1)] \quad (8)$$

and n_B is the Bose-Einstein distribution function.

If we Fourier transform Eq. (7) in the time variable, we readily obtain that

$$\begin{aligned}\Sigma^{(T,\mu)}(E_1, E_2, p_0) &= -\frac{e^2}{8E_1 E_2} \left\{ \left[\frac{1 + n_B(E_1) - n_F^-(E_2)}{E_1 + E_2 - \mu - ip_0} \right. \right. \\ &\quad \left. \left. - \frac{n_B(E_1) + n_F^-(E_2)}{E_1 - E_2 + \mu + ip_0} \right] \gamma_\alpha A(E_2) \gamma_\alpha \right. \\ &\quad \left. + \left[\frac{1 + n_B(E_1) - n_F^+(E_2)}{E_1 + E_2 + \mu + ip_0} \right. \right. \\ &\quad \left. \left. - \frac{n_B(E_1) + n_F^+(E_2)}{E_1 - E_2 - \mu - ip_0} \right] \gamma_\alpha B(E_2) \gamma_\alpha \right\},\end{aligned}\quad (9)$$

which agrees with the result obtained by a direct evaluation. As shown in [2], apart from a contribution to the thermal mass of the fermion, the above radiative correc-

tions also lead to a finite renormalization of the chemical potential which is given by

$$\mu_R = \left(1 - \frac{e^2}{16\pi^2} \right) \mu. \quad (10)$$

As a second example, which exhibits a subtlety associated with the alternate fermion operator in Eq. (4), let us consider the thermal photon self-energy at finite chemical potential. Using the basic property given in Eq. (5), the integrand of this amplitude at one-loop order may also be written in a manifestly factorized form as

$$\begin{aligned}\Pi_{\lambda\rho}^{(T,\mu)}(E_1, E_2, \tau) &= e^2 \text{Tr}[\gamma_\lambda S^{(T,\mu)}(\tau, E_1) \gamma_\rho S^{(T,\mu)}(-\tau, E_2)] \\ &= \hat{\mathcal{O}}_F^{(T,\mu)}(E_1) \hat{\mathcal{O}}_F^{(T,\mu)}(E_2) \Pi_{\lambda\rho}^{(0,\mu)}(E_1, E_2, \tau).\end{aligned}\quad (11)$$

Using Eq. (1), we see that the factors $e^{\pm\mu\tau}$ actually cancel in the zero-temperature amplitude at finite chemical potential. However, this would then make the action of the operator $\hat{N}_F^{(T,\mu)}(E)$ introduced in Eq. (3) ambiguous. In order to define these operators unambiguously, one may, in the intermediate calculations, associate with each fermion propagator of energy E_i a chemical potential μ_i . At the end of the calculation, after the thermal operators have acted, one can set $\mu_i = \mu$.

Using this procedure and Fourier transforming Eq. (11) in the time variable, we then readily get the result

$$\begin{aligned}\Pi_{\lambda\rho}^{(T,\mu)}(E_1, E_2, p_0) &= \frac{e^2}{8E_1 E_2} \left[\frac{n_F^-(E_2) - n_F^-(E_1)}{E_2 - E_1 + ip_0} \text{Tr} \gamma_\lambda A(E_1) \gamma_\rho A(E_2) + \frac{n_F^+(E_1) - n_F^+(E_2)}{E_1 - E_2 + ip_0} \text{Tr} \gamma_\lambda B(E_1) \gamma_\rho B(E_2) \right. \\ &\quad \left. + \frac{1 - n_F^+(E_2) - n_F^-(E_1)}{E_1 + E_2 - ip_0} \text{Tr} \gamma_\lambda A(E_1) \gamma_\rho B(E_2) + \frac{1 - n_F^+(E_1) - n_F^-(E_2)}{E_1 + E_2 + ip_0} \text{Tr} \gamma_\lambda B(E_1) \gamma_\rho A(E_2) \right],\end{aligned}\quad (12)$$

which agrees with the expression for the thermal photon self-energy obtained by a direct evaluation. In this case, the only effect of the chemical potential is to yield a correction to the thermal mass of the photon [8]. In consequence of the symmetry of thermal QED [2], the above radiative corrections do not renormalize the bare zero chemical potential of the photon.

The complete factorization in the above examples, which occurs in consequence of the basic relation (5), can be immediately extended to any one-loop graph. The difficulty in establishing such a factorization for an arbitrary higher-loop graph arises when there are internal vertices for which the time coordinate τ has to be integrated over. At finite temperature, τ as well as the external times τ_i lie in the interval $[0, 1/T]$. On the other hand, in the zero-temperature graphs, the internal time needs to be integrated over the interval $[-\infty, \infty]$. Since the basic thermal operators are independent of the time coordinates, these may be taken out of the time integral. Then, using

the procedure outlined after Eq. (11), the essential step in proving the factorization of an arbitrary graph consists in showing that the function

$$\begin{aligned}V_\alpha &= \left[\int_{-\infty}^0 d\tau + \int_{1/T}^{\infty} d\tau \right] S^{(0,\mu_1)}(\tau_1 - \tau, E_1) \\ &\quad \times \gamma_\alpha S^{(0,\mu_2)}(\tau - \tau_2, E_2) D^{(0)}(\tau - \tau_3, E_3)\end{aligned}\quad (13)$$

which appears in the basic electron-photon vertex, is annihilated by the thermal operator

$$\mathcal{O}^{(T,\mu)}(E_1, E_2, E_3) = \hat{\mathcal{O}}_F^{(T,\mu_1)}(E_1) \hat{\mathcal{O}}_F^{(T,\mu_2)}(E_2) \mathcal{O}_B^{(T)}(E_3). \quad (14)$$

To show this, we evaluate the τ -integrals in Eq. (13) and note that the result may be written in the form

$$V_\alpha = [1 + e^{-(E_1 + \mu_1 + E_2 - \mu_2 + E_3)/T} S(E_1)S(E_2)S(E_3)] \\ \times \left[\frac{A(E_1)\gamma_\alpha B(E_2)}{E_1 - \mu_1 + E_2 + \mu_2 + E_3} \left(\frac{e^{-(E_1 - \mu_1)\tau_1}}{2E_1} \right. \right. \\ \left. \left. \times \frac{e^{-(E_2 + \mu_2)\tau_2}}{2E_2} \frac{e^{-E_3\tau_3}}{2E_3} \right) \right] \quad (15)$$

Using the relation (3) together with the identities

$$e^{-(E \pm \mu)/T} = \frac{n_F^\pm(E)}{1 - n_F^\pm(E)}; \quad e^{-E_3/T} = \frac{n_B(E_3)}{1 + n_B(E_3)}, \quad (16)$$

it is now straightforward to show that the thermal operator given in Eq. (14) annihilates the quantity V_α in Eq. (15). (For more details on such an annihilation mechanism, see, for example, Ref. [1].) This establishes that for the product of propagators in the basic electron-photon vertex, which are integrated over the common time τ , we can extend the range of integration to the interval $[-\infty, \infty]$.

With the help of this property, one can prove by using a procedure similar to the one given in [1] that in the presence of a chemical potential, an arbitrary N -point thermal diagram can be factorized in the form

$$\Gamma_N^{(T)} = \int \prod_{i=1}^I \frac{d^3 k_i}{(2\pi)^3} \prod_{v=1}^V (2\pi)^3 \delta_v^3(k, p) \mathcal{O}^{(T, \mu)} \gamma_N^{(0, \mu)}, \quad (17)$$

where we denote the internal and external three momenta of the graph generically by k and p , respectively, and $\delta_v^3(k, p)$ enforces the three-momentum conservation at the vertex v . The thermal operator for the graph is given by

$$\mathcal{O}^{(T, \mu)} = \prod_{i=1}^{I_F} \hat{\mathcal{O}}_F^{(T, \mu_i)}(E_i) \prod_{j=I_F+1}^I \mathcal{O}_B^{(T)}(E_j), \quad (18)$$

with I_F , I being, respectively, the number of internal fermion propagators and the total number of propagators. Furthermore, $\gamma_N^{(0, \mu)}$ represents the integrand of the zero-temperature graph at finite chemical potential, which involves, apart from a product of photon propagators, also a product of fermion propagators $S^{(0, \mu_i)}(\tau_i, E_i)$. We note from (1) that, since $S^{(0, \mu_i)}(\tau_i, E_i) = \exp(\tau_i \mu_i) S^{(0, 0)}(\tau_i, E_i)$, we can moreover directly relate $\gamma_N^{(0, \mu)}$ to the integrand of the zero-temperature and chemical potential graph $\gamma_N^{(0, 0)}$. As we have pointed out, the limit $\mu_i \rightarrow \mu$ is assumed to be taken only after the action of the thermal operator.

Thus, we have obtained a thermal operator representation for QED at finite temperature and chemical potential, which leads to a simple factorization of the thermal amplitudes. This interesting result can be also extended, by following in a straightforward way the analysis presented in [2], to a non-Abelian gauge theory. The factorization property is computationally quite useful and allows us to study in a direct and transparent way many questions of interest at finite temperature. For example, since the thermal operator is independent of the external energies of the diagram, one can more easily relate the analytic structure of the thermal graphs to that of the analytically continued zero-temperature diagrams. This would lead to a simpler determination of the cutting rules at finite temperature which will be reported in a future communication.

This work was supported in part by the US DOE Grant number DE-FG 02-91ER40685, by MCT/CNPq as well as by FAPESP, Brazil and by CONICYT, Chile under grant Fondecyt 1030363 and 7040057 (Int. Coop.).

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