Preparation of Schrödinger cat states in noncommutative space

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We show that there exist a large class of new nontrivial exactly solvable models with their potentials depending only on one spatial variable on a 2D noncommutative plane and show that their general solutions are Schrödinger cat states. This feature not only establishes an interesting connection between quantum mechanics on noncommutative spaces and quantum information science but also shed light on proposing new schemes based on the quantum entanglement to test the noncommutativity.

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Over the last decade, there has been considerable interest in the issues of noncommutative geometry since the discovery in string theory that the low energy effective theory of a D-brane in the background of a NS-NS B field lives on a noncommutative space [1–3]. It is pointed out that noncommutative spaces may provide a natural background for a possible regularization of quantum field theories [4,5]. Although the effects of noncommutativity should presumably become significant at very high energy scales close, for instance, to the string scale, it is expected that there should be some relics of the effects of spatial noncommutativity because of the incomplete decoupling mechanism between the high and low energy sectors [5,6]. The quantum mechanics on noncommutative spaces (NCQM) seems to be able to reveal such low energy relics and has thus been investigated intensively in the previous literature [4–27].

In this paper, we show that all the models with their potentials depending only on one spatial variable on a two-dimensional (2D) noncommutative plane are exactly solvable models if the corresponding one-dimensional models in the commutative space with the same form of potentials are exactly solvable models. These exactly solvable models on a 2D noncommutative plane include in fact a large class of nontrivial models as will be demonstrated shortly. We further show that the general solutions to these exactly solvable models on a 2D noncommutative plane are Schrödinger cat states, which establishes an interesting connection between NCQM and quantum information science. This connection not only broadens our views of NCQM but also may shed light on proposing promising new schemes based on the quantum entanglement to test the noncommutativity within the framework of NCQM.

We begin with the commutative relations and the Hamiltonian on a 2D noncommutative plane of the following forms [26],

$$[\hat{x}_j, \hat{x}_k] = i\theta\epsilon_{jk}, \qquad [\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}, \qquad [\hat{p}_j, \hat{p}_k] = 0, \tag{1}$$

$$\hat{H} = \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} \otimes I + f(\hat{x}_1) \otimes S + g(\hat{x}_2) \otimes Q, \tag{2}$$

where $j, k = 1, 2$, $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$, $\delta_{11} = \delta_{22} = 1$, $\delta_{12} = \delta_{21} = 0$, θ is the parameter characterizing the spatial noncommutativity, $\hat{x}_j$ and $\hat{p}_j$ are coordinate and momentum operators, respectively, I, S and Q are $n \times n$ matrices (n may be any positive integers) or the direct product of several such matrices independent of coordinate and momentum operators of the noncommutative plane, and they can describe the internal freedoms of an atom such as its spin freedom and/or its energy levels (for instance, these $n \times n$ matrices describe atomic energy levels while $\hat{x}_j$ and $\hat{p}_j$ describe the center of mass motion of the atom), etc.

We introduce the standard transformation [26]

$$\hat{x}_1 = x_1 - \frac{\theta}{2\hbar} p_2, \qquad \hat{x}_2 = x_2 + \frac{\theta}{2\hbar} p_1, \\ \hat{p}_1 = p_1, \qquad \hat{p}_2 = p_2, \tag{3}$$

where x_j and p_j are the coordinate and momentum operators in commutative space, respectively, satisfying the commutation relations $[x_j, x_k] = [p_j, p_k] = 0$ and $[x_j, p_k] = i\hbar\delta_{jk}$ ($j, k = 1, 2$). The transformation (3) can be put into another form

$$\hat{x}_1 = U x_1 U^\dagger, \qquad \hat{p}_1 = U p_1 U^\dagger \equiv p_1, \\ \hat{x}_2 = U^\dagger x_2 U, \qquad \hat{p}_2 = U^\dagger p_2 U \equiv p_2, \tag{4}$$

where $U = \exp(-i\theta p_1 p_2 / 2\hbar^2)$ is a unitary operator ($U^\dagger = U^{-1}$) and $U p_j = p_j U$, and it generates a twist [28].

It is then straightforward to show that $f(U x_1 U^\dagger) = U f(x_1) U^\dagger$, $g(U^\dagger x_2 U) = U g(U^{\dagger 2} x_2 U^2) U^\dagger = U g(x_2 + \theta p_1/\hbar) U^\dagger$ and hence $\hat{H} = U H U^\dagger \Longleftrightarrow H = U^\dagger \hat{H} U$ with the Hamiltonian in commutative space

$$H = \frac{p_1^2 + p_2^2}{2m} \otimes I + f(x_1) \otimes S + g(x_2 + \theta p_1/\hbar) \otimes Q, \tag{5}$$

denoted as the function of the coordinate and momentum operators x_j and p_j in commutative space.

The fact of $H|\Psi_c\rangle = E|\Psi_c\rangle \Longleftrightarrow \hat{H} U|\Psi_c\rangle = E U|\Psi_c\rangle$ leads to the conclusion: that any model (5) in commutative space is exactly solvable implies that the corresponding model (2) in noncommutative space is also exactly solvable, and vice versa. Specifically, the models (2) and (5) in noncommutative and commutative spaces, respectively, share the same set of eigen-energies and their eigenstates relate to each other by the relation $|\Psi_{nc}\rangle = U|\Psi_c\rangle$, and they either are both exactly solvable or both are not.

Let us illustrate that this conclusion in fact leads to a large class of new nontrivial exactly solvable models of the noncommutative plane and some important novel physical implications even for the simplest $g \equiv 0$ cases.

In the special situations where $g \equiv 0$ and all the matrices I, S and Q are 1×1 unit matrices (i.e., they are simply the unit c-number or 1), The Hamiltonian in commutative space given by Eq. (5) becomes $H = p_2^2/2m + H_1$ with $H_1 = p_1^2/2m + f(x_1)$. Let $E^{(1)}$ and $|\psi\rangle$ denote the eigen-energy and the eigenstate of the one-dimensional Hamiltonian H_1 in commutative space, respectively, i.e.,

$$\left[\frac{p_1^2}{2m} + f(x_1)\right]|\psi\rangle = E^{(1)}|\psi\rangle, \tag{6}$$

Obviously, for any given $E^{(1)}$ and $|\psi\rangle$, the general forms for the eigen-energy and the eigenstate of $H = p_2^2/2m + H_1$ are

$$E = E^{(1)} + \frac{\mathcal{P}^2}{2m}, \tag{7a}$$

$$|\Psi_c\rangle = |\psi\rangle(c_+|\mathcal{P}\rangle + c_-|-\mathcal{P}\rangle), \tag{7b}$$

respectively. Here the c-number $\mathcal{P}$ and $|\mathcal{P}\rangle$ are the eigenvalue and eigenket of the operator p_2 respectively (i.e., $p_2|\mathcal{P}\rangle = \mathcal{P}|\mathcal{P}\rangle$) and $c_\pm$ are two constants satisfying normalization condition $|c_+|^2 + |c_-|^2 = 1$.

Consequently, the eigenstates and the eigen-energies of the corresponding model (2) in noncommutative space in the special situations of $g \equiv 0$ and $I = S = Q = 1$ are

$$E = E^{(1)} + \frac{\mathcal{P}^2}{2m}, \tag{8a}$$

$$|\Psi_{nc}\rangle = c_+|\psi_+\rangle|\mathcal{P}\rangle + c_-|\psi_-\rangle|-\mathcal{P}\rangle, \tag{8b}$$

respectively. Here $|\psi_\pm\rangle = \exp(\mp i\tilde{\mathcal{P}}p_1)|\psi\rangle$ with the constant $\tilde{\mathcal{P}} = \theta\mathcal{P}/2\hbar^2$.

The above discussions demonstrate that as long as the eigenvalue problem, Eq. (6), of the one-dimensional Hamiltonian H_1 in commutative space can exactly be solved, the corresponding model (2) in noncommutative space in the special situations of $g \equiv 0$ and $I = S = Q = 1$ can also be exactly solved with its eigenstates and the eigen-energies explicitly given by Eq. (8). Apart from the exactly solvable models with the potentials of the form $f(x_1) = c_2x_1^2 + c_1x_1 + c_0$, there obviously exist a vast number of other one-dimensional exactly solvable models in commutative space that can be found in the standard text books of quantum mechanics and/or in the previous relevant literature. All these vast number of one-dimensional exactly solvable models in commutative space correspond to the nontrivial exactly solvable models in noncommutative space. We only list two examples of such kinds of nontrivial exactly solvable models below.

Example 1: For $f(x_1) = -(\hbar^2/2m)n(n+1)/\cosh^2 x_1$ with n denoting an arbitrary positive integer, Eq. (6) in Schrödinger representation becomes,

$$\left[-\frac{d^2}{dx_1^2} - \frac{n(n+1)}{\cosh^2 x_1}\right]\psi(x_1) = \frac{2mE^{(1)}}{\hbar^2}\psi(x_1), \tag{9}$$

which has n and only n bound states followed by a continuum band [29]. The n bound states and the corresponding eigenenergies are readily shown to have the forms,

$$\begin{aligned} \psi_k(x_1) &= q_k\left[-\frac{d}{dx_1} + n\tanh(x_1)\right]^k \frac{1}{\cosh^{n-k}(x_1)}, \\ E_k^{(1)} &= -\frac{\hbar^2}{2m}(n-k)^2, \qquad k = 0, 1, 2, \cdots, (n-1) \end{aligned} \tag{10}$$

where q_k are the normalization constants such that $\int_{-\infty}^{\infty}|\psi_k(x_1)|^2 dx_1 = 1$. To prove Eq. (10), we introduce operators $A_n^\dagger = -d/dx_1 + n\tanh x_1$ and $A_n = d/dx_1 + n\tanh x_1$ so that $A_n^\dagger A_n = -\frac{d^2}{dx_1^2} + n^2 - n(n+1)/\cosh^2(x_1)$, $A_nA_n^\dagger = -\frac{d^2}{dx_1^2} + n^2 - n(n-1)/\cosh^2(x_1)$ and $A_{n+1}A_{n+1}^\dagger = A_n^\dagger A_n + (n+1)^2 - n^2$. If $\psi(n, x_1)$ is the eigenstate of $A_n^\dagger A_n$ with eigenvalue β_n, i.e., $A_n^\dagger A_n\psi(n, x_1) = \beta_n\psi(n, x_1)$, we obtain $A_{n+1}A_{n+1}^\dagger\psi(n, x_1) = \beta_n\psi(n, x_1) + (2n+1)\psi(n, x_1) = (\beta_n + 2n + 1)\psi(n, x_1)$ or $A_nA_n^\dagger\psi(n-1, x_1) = (\beta_{n-1} + 2n - 1)\psi(n-1, x_1)$. Consequently $\gamma(n, x_1) = A_n^\dagger\psi(n-1, x_1)$ satisfies $A_n^\dagger A_n\gamma(n, x_1) = A_n^\dagger[A_nA_n^\dagger\psi(n-1, x_1)] = (\beta_{n-1} + 2n - 1)\gamma(n, x_1)$. In other words, if $A_n^\dagger A_n\psi(n, x_1) = \beta_n\psi(n, x_1)$, then $\gamma(n, x_1) = A_n^\dagger\psi(n-1, x_1)$ is also the eigenstate of $A_n^\dagger A_n$ with $A_n^\dagger A_n\gamma(n, x_1) = (\beta_{n-1} + 2n - 1)\gamma(n, x_1)$. Using iteratively this fact and noting $A_n\psi_0(n, x_1) = 0$ with $\psi_0(n, x_1) = 1/\cosh^n(x_1)$, it is then straightforward to show Eq. (10). Then Eq. (8) in Schrödinger representation for this case becomes

$$E_{k,\mathcal{P}} = \frac{\mathcal{P}^2 - \hbar^2(n-k)^2}{2m}, \qquad k = 0, 1, \cdots, (n-1) \tag{11a}$$

$$\Psi_{nc}^{(k,\mathcal{P})} = \sum_{j=\pm} c_j\psi_k\left(x_1 - \frac{j\theta\mathcal{P}}{2\hbar}\right)\exp\left(ij\frac{\mathcal{P}x_2}{\hbar}\right), \tag{11b}$$

where $\mathcal{P}$ is an arbitrary real number, $\psi_k(x_1)$ is given by Eq. (10), and the constants $c_\pm$ are complex constants satisfying $|c_+|^2 + |c_-|^2 = 1$.

Equation (11) gives the exact analytical solutions to the Schrödinger equation

$$\hat{H}\Psi_{nc}^{(k,\mathcal{P})} = E_{k,\mathcal{P}}\Psi_{nc}^{(k,\mathcal{P})}, \tag{12}$$

for the following Hamiltonian in the noncommutative space

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - \frac{\hbar^2}{2m}\frac{n(n+1)}{\cosh^2(\hat{x}_1)}, \tag{13}$$

with $\hat{x}_1 = x_1 - \frac{\theta}{2\hbar}p_2 = x_1 + i(\theta/2)\partial/\partial x_2$.

Example 2: For the Schrödinger equation $\hat{H}\Psi_{nc} = E\Psi_{nc}$ with the following Hamiltonian in the noncommutative space

$$\hat{H} = \frac{\hbar^2}{2m}\left[-\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + n(n+1)Msn^2(\hat{x}_1)\right], \quad (14)$$

where $\hat{x}_1 = x_1 + i(\theta/2)\partial/\partial x_2$, n is an arbitrary positive integer, and $sn^2(\hat{x}_1) \equiv sn^2(\hat{x}_1, M)$ is a Jacobi elliptic function with a modulus parameter $M \in [0, 1]$ [29]. From the discussions leading to Eqs. (7) and (8), we know that

$$E = E^{(1)} + \frac{\mathcal{P}^2}{2m}, \quad (15a)$$

$$\Psi_{nc} = \sum_{j=\pm} c_j \psi\left(x_1 - \frac{j\theta\mathcal{P}}{2\hbar}\right)\exp\left(ij\frac{\mathcal{P}x_2}{\hbar}\right), \quad (15b)$$

with $\psi(x_1)$ determined by the following one-dimensional Schrödinger equation called also as Lamé's equation [29],

$$-\frac{d^2\psi(x_1)}{dx_1^2} + n(n+1)Msn^2(x_1)\psi(x_1) = \frac{2mE^{(1)}}{\hbar^2}\psi(x_1). \quad (16)$$

It is well known that this equation has the solutions of n bound bands followed by a continuum band and all band edge energies and wave functions are analytically known [29]. For instance, all the band edge energies and wave functions for $n = 2$ are as follows [29]

$$E_0^{(1)} = \frac{\hbar^2(2M - 2 - 2\delta)}{2m}, \quad (17a)$$

$$\psi_0(x_1) = M + 1 + \delta - 3Msn^2(x_1), \quad (17b)$$

$$E_1^{(1)} = \frac{\hbar^2(M-3)}{2m}, \qquad \psi_1(x_1) = cn(x_1)dn(x_1), \quad (17c)$$

$$E_2^{(1)} = \frac{\hbar^2(4M-3)}{2m}, \qquad \psi_2(x_1) = sn(x_1)dn(x_1), \quad (17d)$$

$$E_3^{(1)} = \frac{\hbar^2 M}{2m}, \qquad \psi_1(x_1) = sn(x_1)cn(x_1), \quad (17e)$$

$$E_4^{(1)} = \frac{\hbar^2(2M - 2 + 2\delta)}{2m}, \quad (17f)$$

$$\psi_4(x_1) = M + 1 - \delta - 3Msn^2(x_1), \quad (17g)$$

where $\delta = \sqrt{1 - M + M^2}$, $E^{(1)}$ in the ranges of $E_1^{(1)} - E_0^{(1)}$, $E_3^{(1)} - E_2^{(1)}$ and $\geq E_4^{(1)}$ corresponds to the bands, while $E^{(1)}$ in the ranges of $E_2^{(1)} - E_1^{(1)}$ and $E_4^{(1)} - E_3^{(1)}$ corresponds to the gaps. Obviously, Eq. (16) with $E^{(1)} = E_j^{(1)}$ and $\psi(x_1) = \psi_j(x_1)$ $(j = 0, 1, 2, 3, 4)$ corresponds to the exact analytical solutions to the Schrödinger equation $\hat{H}\Psi_{nc} = E\Psi_{nc}$ with the Hamiltonian in the noncommutative space given by Eq. (14).

It is interesting to note that the energy eigenstate (7b) in commutative space is a direct-product state or an unentangled state while the corresponding energy eigenstate (8b) in noncommutative space is an entangled state, the Schrödinger cat state, so long as $c_+c_- \neq 0$ and $|\psi_-\rangle \neq c|\psi_+\rangle$ ($|\psi_-\rangle$ is not proportional to $|\psi_+\rangle$). Obviously $|\psi_-\rangle \neq c|\psi_+\rangle$ is generally true for a nonconstant potential f because $|\psi_-\rangle = c|\psi_+\rangle \longrightarrow \exp(2i\tilde{\mathcal{P}}p_1)|\psi\rangle = c|\psi\rangle$ implying that H_1's eigenstate $|\psi\rangle$ is also the eigenstate of the momentum operator p_1 which is obviously false for a nonconstant potential f. In other words, the spatial noncommutativity can cause an unentangled state to become an entangled state (to be more specific, a Schrödinger cat state).

It is worthwhile to mention another interesting and important feature in producing entangled states by the spatial noncommutativity. The entangled states thus produced are all the eigenstates of the system's Hamiltonian $\hat{H}$ in noncommutative space and hence any one of them remains the same entangled state at all the subsequent times under the unitary evolution described by the evolution operator $\exp(-i\hat{H}t/\hbar)$. While the states are usually entangled only at some specific time(s) in other schemes [30–33] and hence they require a careful and sophisticated time control of often technically difficulty.

All the above conclusions concerning the special situations of $g \equiv 0$ and $I = S = Q = 1$ can be readily generalized to the broader situations of $g \equiv 0$ but I, S and Q are $n \times n$ matrices with $n > 1$ in order to describe the internal freedoms of an atom such as its spin freedom and/or its energy levels (for instance, these $n \times n$ matrices describe atomic energy levels while $\hat{x}_j$ and $\hat{p}_j$ describe the center of mass motion of the atom), etc. Below gives such as an example.

Example 3: Considering the Schrödinger equation $\hat{H}\Psi_{nc} = E\Psi_{nc}$ with the following Hamiltonian in the noncommutative space

$$\hat{H} = \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_1^2 + \frac{\hbar\omega_a}{2}\sigma_z + \frac{m\omega\Omega}{2}\left[\left(\hat{x}_1 + i\frac{i}{m\omega}\hat{p}_1\right)\sigma_+ + \text{h.c.}\right], \quad (18)$$

where [34] $\sigma_+ = |+\rangle\langle-|$, $\sigma_- = |-\rangle\langle+|$, $|+\rangle$ and $|-\rangle$ describe the excited and ground levels of an atomic internal energy degree of a two-level atom, $\hbar\omega_a$ is the energy difference of the two internal levels, and $\sigma_z = |+\rangle\langle+| - |-\rangle\langle-|$. Following the same routine as we have done, we readily obtain again Eq. (8) but with $|\psi\rangle$ and $E^{(1)}$ determined by $H_1|\psi\rangle = E^{(1)}|\psi\rangle$ with $H_1 = \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2x_1^2 + \frac{\hbar\omega_a}{2}\sigma_z + \frac{m\omega\Omega}{2\hbar}[(x_1 + i\frac{i}{m\omega}p_1)\sigma_+ + \text{h.c.}]$ or

$$\frac{H_1}{\hbar} = \omega\left(a^\dagger a + \frac{1+\sigma_z}{2}\right) + \frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}(a\sigma_+ + a^\dagger\sigma_-), \quad (19)$$

where $\Delta = \omega_a - \omega$ is the detuning [34]. Here we have introduced annihilation and creation operators a and $a^\dagger$ satisfying $[a, a^\dagger] = 1$ by the usual relations $a = \sqrt{m\omega/2\hbar}(x_1 + ip_1/m\omega)$ and $a^\dagger = \sqrt{m\omega/2\hbar}(x_1 - ip_1/m\omega)$. The Schrödinger equation $H_1|\psi\rangle = E^{(1)}|\psi\rangle$ with the Hamiltonian (19) are readily be solved to give the results [34]

$$E^{(1)}_{N,M} = \hbar\omega N + \frac{\hbar M(1-\delta_{N,0})}{2}\sqrt{\Omega^2 N + \Delta^2}, \quad (20a)$$

$$|\psi\rangle_{N,M} = \alpha_N|N, M\rangle + M\beta_N|N, -M\rangle, \quad (20b)$$

where $N = 0, 1, 2, \cdots$, $M = \pm$, $|N, +\rangle = |n = N-1\rangle_f|+\rangle$ and $|N, -\rangle = |n = N\rangle_f|-\rangle$ for $N \neq 0$, $|N = 0, -\rangle = |n = 0\rangle_f|-\rangle$, $|n\rangle_f$ is the Fock state for $a^\dagger a$, i.e., $a^\dagger a|n\rangle_f = n|n\rangle_f$, $|\pm\rangle$ satisfy $\sigma_z|\pm\rangle = \pm|\pm\rangle$, $\alpha_N = \sqrt{(\sqrt{\Omega^2 N + \Delta^2} - \Delta)/2\sqrt{\Omega^2 N + \Delta^2}}$ and $\beta_N = \sqrt{(\sqrt{\Omega^2 N + \Delta^2} + \Delta)/2\sqrt{\Omega^2 N + \Delta^2}}$. Substituting Eq. (20) into Eq. (8) and noting that $p_1 = i\sqrt{\hbar m\omega/2}(a^\dagger - a)$, we know that the Schrödinger equation $\hat{H}\Psi_{nc} = E\Psi_{nc}$ with the Hamiltonian (18) has the following exact analytical solutions

$$E_{N,M,\mathcal{P}} = E^{(1)}_{N,M} + \frac{\mathcal{P}^2}{2m}, \quad (21a)$$

$$|\Psi_{nc}\rangle_{N,M,\mathcal{P}} = c_+|\psi_+\rangle|\mathcal{P}\rangle + c_-|\psi_-\rangle|-\mathcal{P}\rangle, \quad (21b)$$

where $|\psi_\pm\rangle = \exp[\pm i\eta(a^\dagger - a)]|\psi\rangle_{N,M}$ and $\eta = \theta\mathcal{P}\sqrt{m\omega/8\hbar^3}$.

In summary, we have shown that there exist a large class of new nontrivial exactly solvable models with their potentials depending only on one spatial variable on a 2D noncommutative plane and show that their general solutions are Schrödinger cat states. We have shown that the spatial noncommutativity can produce Schrödinger cat states in a natural way with some advantages over the previous schemes for realizing entangled states and hence establishes an interesting connection between quantum mechanics in noncommutative spaces and quantum information science.

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[1] A. Connes, M.R. Douglas, and C. Hull, J. High Energy Phys. 02 (1998) 008.
[2] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.
[3] M.R. Douglas and M.A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).
[4] P.-M. Ho and H.-C. Kao, Phys. Rev. Lett. **88**, 151602 (2002).
[5] O. Bertolami, J.G. Rosa, C.M.L. de Aragão, P. Castorina, and D. Zappalà, Phys. Rev. D **72**, 025010 (2005).
[6] J.-Z. Zhang, Phys. Lett. B **597**, 362 (2004).
[7] B. Muthukumar and P. Mitra, Phys. Rev. D **66**, 027701 (2002).
[8] A. Jellal, E.H. EI kinani, and M. Schreiber, Int. J. Mod. Phys. A **20**, 1515 (2005).
[9] J.-Z. Zhang, Phys. Rev. Lett. **93**, 043002 (2004).
[10] Ö.F. Dayi and A. Jellal, Phys. Lett. A **287**, 349 (2001);
[11] M. Chaichian, P. Presnajder, and A. Tureanu, Phys. Rev. Lett. **94**, 151602 (2005).
[12] G.D. Barbosa and N. Pinto-Neto, Phys. Rev. D **69**, 065014 (2004).
[13] J. Gamboa, M. Loewe, and J.C. Rojas, Phys. Rev. D **64**, 067901 (2001).
[14] V.P. Nair and A.P. Polychronakos, Phys. Rev. Lett. **87**, 030403 (2001).
[15] A.I. Frank1 and V.G. Nosov, JETP Lett. **79**, 313 (2004).
[16] S. Jing, F. Zuo, and T. Heng, J. High Energy Phys. 10 (2004) 049.
[17] C. Acatrinei, J. High Energy Phys. 09 (2001) 007.
[18] D. Bini, C. Cherubini, and B. Mashhoon, Phys. Rev. D **70**, 044020 (2004).
[19] A. Kokado, T. Okamura, and T. Saito, Phys. Rev. D **69**, 125007 (2004).
[20] R.J. Szabo, Phys. Rep. **378**, 207 (2003).
[21] L. Alvarez-Gaume and S.R. Wadia, Phys. Lett. B **501**, 319 (2001).
[22] A. Micu and M.M. Sheikh-Jabbari, J. High Energy Phys. 01 (2001) 025.
[23] I.F. Riad and M.M. Sheikh-Jabbari, J. High Energy Phys. 01 (2001) 025.
[24] M. Chaichian, M.M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001).
[25] H. Falomir, J. Gamboa, M. Loewe, F. Méndez, and J.C. Rojas, Phys. Rev. D **66**, 045018 (2002).
[26] A. Kijanka and P. Kosinski, Phys. Rev. D **70**, 127702 (2004).
[27] M.A. Anacleto, M. Gomes, A.J. da Silva, and D. Spehler, Phys. Rev. D **70**, 085005 (2004).
[28] M. Chaichian, P. Kulish, K. Nishijima, and A. Tureanu, Phys. Lett. B **604**, 98 (2004); **604**, 98 (2004).
[29] A. Khare and U. Sukhatme, J. Math. Phys. (N.Y.) **40**, 5473 (1999); **42**, 5652 (2001).
[30] Y. Wu and L. Deng, Opt. Lett. **29**, 1144 (2004).
[31] X. Yang and Y. Wu, J. Opt. B **7**, 54 (2005).
[32] G.-X. Li, S.-P. Wu, and G.-M. Huang, Phys. Rev. A **71**, 063817 (2005).
[33] Shi-Biao Zheng, Phys. Rev. A **69**, 055801 (2004).
[34] X. Yang, Y. Wu, and Y. Li, Phys. Rev. A **55**, 4545 (1997);