Quantum effective action in spacetimes with branes and boundaries

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We construct quantum effective action in spacetime with branes/boundaries. This construction is based on the reduction of the underlying Neumann type boundary value problem for the propagator of the theory to that of the much more manageable Dirichlet problem. In its turn, this reduction follows from the recently suggested Neumann-Dirichlet duality which we extend beyond the tree-level approximation. In the one-loop approximation this duality suggests that the functional determinant of the differential operator subject to Neumann boundary conditions factorizes into the product of its Dirichlet counterpart and the functional determinant of a special operator on the brane—the inverse of the brane-to-brane propagator. As a byproduct of this relation we suggest a new method for surface terms of the heat kernel expansion. This method allows one to circumvent well-known difficulties in the heat kernel theory on manifolds with boundaries for a wide class of generalized Neumann boundary conditions. In particular, we easily recover several lowest-order surface terms in the case of Robin and oblique boundary onditions. We briefly discuss multiloop applications of the suggested Dirichlet reduction and the prospects of constructing the universal background-field method for systems with branes/boundaries, analogous to the Schwinger-DeWitt technique.

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I. INTRODUCTION

Prospective goal of the present paper is to develop background-field method [1,2] for brane models in gravity/string theory and cosmology. Current status of brane concept essentially relies on the analysis of quantum properties beyond the tree-level approximation. This is especially important in the problem of low strong coupling scale in brane induced gravity models which incorporate an infinite sequence of strongly coupled higher dimensional operators. Their calculation is most efficient in the language of quantum brane effective action. This, in its turn, requires application of the covariant method of background field [1,2] in which the propagators of the theory within certain approximation are calculable in the external (mean) fields of a generic form.

The peculiarity of brane models is that their bulk propagators, rather than being defined in infinite spacetime with simple falloff conditions, are subject to nontrivial boundary conditions on branes. Since the fields are subject to dynamical quantum fluctuations on timelike branes, these boundary conditions belong to the class of generalized Neumann boundary conditions involving on branes the values of fields together with their normal and tangential derivatives. Finding such propagators (usually based on the method of images) is a very hard task, especially for fields with spins, when their Green's functions have numerous spin-tensor indices.

On the contrary, Green's functions with Dirichlet boundary conditions are much easier to obtain—the method of images gives them as relatively simple linear combinations of known propagators in spacetime without boundaries (defined by mirror image continuation across the original boundary [3]). It turns out that Feynman diagrammatic technique based on Neumann type Green's function can be systematically reduced to that of the Dirichlet type, and the goal of the present paper is to develop the needed technique. This technique is based on the duality between the Dirichlet and Neumann boundary value problems, recently discovered in [4] at the tree level. Here it will be extended to the one-loop and multiloop levels.

The action of a (free field) brane model generally contains the bulk and the brane parts

$$S[\phi] = \frac{1}{2}\int_B dX\,\phi(X)\overleftrightarrow{F}(\nabla)\phi(X) + \int_b dx\Big(\frac{1}{2}\varphi(x)\kappa(\partial)\varphi(x) + j(x)\varphi(x)\Big), \quad (1.1)$$

where the bulk $(d+1)$-dimensional and the brane d-dimensional coordinates are labeled, respectively, by $X = X^A$ and $x = x^\mu$, and the boundary values of bulk fields $\phi(X)$ on the brane/boundary are denoted by $\varphi(x)$

$$\phi(X)|_b \equiv \phi| = \varphi(x). \quad (1.2)$$

The kernel of the bulk Lagrangian in given by the second order differential operator $F(\nabla)$, whose derivatives $\nabla \equiv \partial_X$ are integrated by parts in such a way that they form bilinear combinations of first-order derivatives acting on two different fields (this is denoted by $\overleftrightarrow{F}(\nabla)$). Integration by parts in the bulk gives nontrivial surface terms on the brane/boundary. In particular, this operation results for a *symmetric* operator $F(\nabla)$ in the Wronskian relation for generic test functions $\phi_{1,2}(X)$

$$\int_B d^{d+1}X(\phi_1\vec{F}(\nabla)\phi_2 - \phi_1\overleftarrow{F}(\nabla)\phi_2) = -\int_{\partial B} d^dx(\phi_1\vec{W}\phi_2 - \phi_1\overleftarrow{W}\phi_2). \tag{1.3}$$

This relation can be regarded as a definition of the first-order *Wronskian operator* $W = W(\nabla)$ on the boundary/brane $b = \partial B$ of spacetime bulk domain B. Arrows everywhere here indicate the direction of action of derivatives either on ϕ_1 or ϕ_2.

The brane part of the action contains as a kernel some local differential operator $\kappa(\partial)$, $\partial = \partial_x$. Integration by parts here is irrelevant for our purposes, because b is considered to be either closed compact or having trivial vanishing boundary conditions at infinitely remote boundary ∂b. $j(x)$ plays the role of sources located on the boundary. The order of the operator $\kappa(\partial)$ in derivatives depends on the model in question. In the Randall-Sundrum model [5], for example, for certain gauges it is just an ultralocal multiplication operator generated by the tension term on the brane. In the Dvali-Gabadadze-Porrati model [6] this is a second order operator induced by the brane Einstein term, $\kappa(\partial) \sim \Box/\mu$ where μ is a very low DGP scale of the order of magnitude of the horizon scale, responsible for the cosmological acceleration [7]. In context of Born-Infeld action in D-brane string theory with vector gauge fields $\kappa(\partial)$ is a first-order operator [8].

In all these cases the action (1.1) with dynamical (not fixed) boundary conditions $\varphi(x)$ for bulk fields naturally gives rise to generalized Neumann boundary conditions of the form

$$(\vec{W}(\nabla) + \kappa(\partial))\phi_N| = 0, \tag{1.4}$$

which involve normal derivatives of $\phi(X)$ contained in $\vec{W}(\nabla)$ and generically also the tangential to the boundary derivatives contained in $\kappa(\partial)$ (and possibly in $\vec{W}(\nabla)$). These boundary conditions when imposed on all brane boundaries of the bulk along with regularity requirements at the bulk infinity uniquely define the Neumann type bulk propagator of the theory, brane-to-brane propagator, etc. and, therefore, uniquely specify all orders of perturbation theory for both tree-level and quantum effective action.

Main result which we want to advocate here is that all the Neumann type ingredients of the perturbation theory can be systematically reduced to those of the Dirichlet boundary conditions

$$\phi_D| = 0. \tag{1.5}$$

In particular, the tree-level brane effective action, obtained from (1.1) by integrating out the bulk fields subject to boundary conditions (1.2) reads as

$$\boldsymbol{S}_{\rm brane}[\varphi] = \frac{1}{2}\int_b dxdy\varphi(x)\boldsymbol{F}^{\rm brane}(x,y)\varphi(y) + \int_b dxj(x)\varphi(x), \tag{1.6}$$

$$\boldsymbol{F}^{\rm brane}(x,y) = -\vec{W}G_D\overleftarrow{W}||(x,y) + \kappa(\partial)\delta(x,y) \tag{1.7}$$

with the brane-to-brane operator $\boldsymbol{F}^{\rm brane}(x,y)$ expressed in terms of the Dirichlet Green's function $G_D(X,Y)$ of the operator $F(\nabla)$ in the bulk. This expression implies that the kernel of the Dirichlet Green's function is being acted upon both arguments by the Wronskian operators with a subsequent restriction to the brane. Double vertical bar indicates that both points of the operator kernel are restricted to the brane and labeled by corresponding low case letters. That is, if the embedding of the boundary/brane in the bulk is denoted by $X = e(x)$, then this explicitly means:

$$\vec{W}G_D\overleftarrow{W}||(x,y) \equiv \vec{W}(\nabla_X)G_D(X,Y)\overleftarrow{W}(\nabla_Y)|_{X=e(x),Y=e(y)}. \tag{1.8}$$

It is obvious that $\boldsymbol{F}^{\rm brane}(x,y)$ is essentially nonlocal, its local part being presented by the last term of (1.7)—the contribution from the brane. The Green's function $\boldsymbol{G}_{\rm brane}(x,y)$ of the brane operator,

$$\int_b dz\boldsymbol{F}^{\rm brane}(x,z)\boldsymbol{G}_{\rm brane}(z,y) = \delta(x,y), \tag{1.9}$$

is the brane-to-brane propagator of the bulk theory, and with the conventions of the above type this reads as the following expression for the brane restriction of the Neumann Green's function $G_N(X,Y)$ of $F(\nabla)$

$$G_N||(x,y) = \boldsymbol{G}_{\rm brane}(x,y). \tag{1.10}$$

The duality relations (1.7) and (1.10) were derived in [4] for a simplest case of $\kappa(\partial) = 0$. Below we generalize them to the case of a nontrivial $\kappa(\partial)$ and, moreover, extend them beyond the tree level. In particular, in the one-loop approximation we show that the functional determinant of the bulk operator $F(\nabla)$ subject to the generalized Neumann boundary conditions (1.4) factorizes into the product of the Dirichlet type determinant of $F(\nabla)$ and the functional determinant of the brane-to-brane operator $\boldsymbol{F}^{\rm brane}$ of the *boundary* (*d-dimensional*) theory—the fact briefly reported in [9]. This implies the following additive property for the one-loop effective action

$$\boldsymbol{\Gamma}_{\text{1-loop}} \equiv \frac{1}{2}{\rm Tr}_N\ln F = \frac{1}{2}{\rm Tr}_D\ln F + \frac{1}{2}{\rm tr}\ln\boldsymbol{F}^{\rm brane}, \tag{1.11}$$

where ${\rm Tr}_{D,N}$ denotes functional traces of the bulk theory subject to Dirichlet and Neumann boundary conditions, while tr is a functional trace in the boundary d-dimensional theory.

Certainly, beyond tree level the effective action contains ultraviolet divergences, so that this one-loop Dirichlet-

Neumann reduction property (1.11) should be understood within certain regularization. Below we will use the dimensional regularization with the dimension d continued to the complex plane playing the role of regularization parameter. In principle, other regularizations are possible, and their admissible types will be discussed in Sec. II below.

As a byproduct of (1.11) we suggest a new technique for surface terms of the local heat kernel expansion in spacetimes with boundaries. Heat kernel gives a proper-time representation for the functional determinant of (pseudo)-differential operators and, therefore, serves as a basic tool for the calculation of effective action in background-field formalism [1,2,10–12]. In the presence of boundaries its local expansion is modified by additional surface terms which amend easily calculable bulk terms well known in physics context as Schwinger-DeWitt coefficients. Calculation of these surface terms [3] presents a strong challenge of both technical and sometimes conceptual (nonperturbative) nature, especially for the so-called oblique boundary conditions [13] which contain derivatives tangential to the brane and arise, in particular, in Born-Infeld context [8]. Interestingly, Neumann-Dirichlet duality relations suggest an alternative method of their calculation, which in view of simplicity and universality has essential advantages as compared to the conventional approach of [12–15].

The paper is organized as follows. In Sec. II we derive the algorithms (1.7), (1.10), and (1.11). In Sec. III we demonstrate them on several simple examples. In Sec. IV we calculate several lowest-order surface terms of the heat kernel for a wide class of generalized Neumann boundary conditions including, in particular, the case of oblique ones. In the concluding Sec. V we briefly discuss the extension of the technique to multiloop orders, its peculiarities in gauge theories and other problems of the curvature expansion in spacetimes with boundaries, which will be fully considered in the forthcoming papers [16,17]. In Appendix A the classical Feynman derivation of the gaussian functional integral in spacetime with boundaries is briefly revisited, and Appendix B is devoted to the derivation of the heat kernel for a special case of the brane-to-brane operator.

II. NEUMANN VS DIRICHLET PROBLEMS

We begin this section with specifying in more detail the structure of the second order bulk operator $F(\nabla)$. As a kernel of the bulk action in (1.1) it should be symmetric and have the following general form

$$F(\nabla) = -\partial_A a^{AB} \partial_B - b^A \partial_A + \partial_A (b^A)^T + c. \tag{2.1}$$

Its coefficients are some general coordinate dependent matrices acting in the vector space of $\phi(X)$ labeled by some spin-tensor indices which we do not specify here. With respect to these indices the coefficients a^{AB} and c are symmetric $(a^{AB})^T = a^{AB}$, $c^T = c$.

The Lagrangian of the bulk part of (1.1) for this operator, containing the first-order derivatives, is of the form

$$\frac{1}{2}\phi \overleftrightarrow{F}(\nabla)\phi = \frac{1}{2}\partial_A \phi a^{AB} \partial_B \phi - \phi b^A \partial_A \phi + \frac{1}{2}\phi c \phi. \tag{2.2}$$

With one integration by parts, this Lagrangian differs by the total derivative term from the expression in which the operator $F(\nabla)$ acts entirely to the right. For two different test functions $\phi_{1,2}$ this reads as

$$\phi_1 \overleftrightarrow{F} \phi_2 = \phi_1 (\vec{F} \phi_2) + \partial_A (\phi_1 \vec{W}^A \phi_2) \tag{2.3}$$

in terms of the *local* Wronskian operator

$$\vec{W}^A(\nabla) = a^{AB} \partial_B + b^A \tag{2.4}$$

and can also be rewritten as a *local* Wronskian relation

$$\phi_1 \vec{F}(\nabla)\phi_2 - \phi_1 \overleftarrow{F}(\nabla)\phi_2 = -\partial_A (\phi_1 \vec{W}^A \phi_2 - \phi_1 \overleftarrow{W}^A \phi_2). \tag{2.5}$$

Integrating (2.3) over the bulk we have the equation

$$\int_B d^{d+1}X \phi_1 \overleftrightarrow{F} \phi_2 = \int_B d^{d+1}X \phi_1 (\vec{F}\phi_2) + \int_b d^d x \phi_1 \vec{W} \phi_2 \bigg|. \tag{2.6}$$

It determines the *boundary/brane* Wronskian operator $\vec{W}$ which is given by the normal projection of the local operator (2.4) at the boundary (up to the measure factor involving the ratio of determinants of the bulk metric G_{AB} and induced on the brane metric $g_{\mu\nu}$), $\vec{W} = (\sqrt{g}/\sqrt{G})\vec{W}^{\perp}$. Similar integration of (2.5) yields Eq. (1.3) of Introduction.[1]

Now consider the functional integral in the brane model with the action (1.1)

$$Z = \int D\phi \exp(-S[\phi]), \tag{2.7}$$

where the integration runs over the bulk fields $\phi(X)$ and also over its boundary values $\varphi(x)$, (1.2), on the timelike branes. Integration over the latter follows from the dynamical nature of $\varphi(x)$ which are subject to independent quantum fluctuations. This gaussian path integral equals

$$Z = (\mathrm{Det} G_N)^{1/2} \exp(-S[\phi_N]), \tag{2.8}$$

where ϕ_N is a stationary point of the action (1.1) satisfying the following problem with the inhomogeneous Neumann boundary conditions

$$F(\nabla)\phi_N(X) = 0, \qquad (\vec{W} + \kappa)\phi_N| + j(x) = 0, \tag{2.9}$$

[1] Note that the Wronskian relation (1.3) specifies $W(\nabla)$ only up to arbitrary symmetric operator acting on the boundary (like $\kappa(\partial)$), while Eq. (2.6) fixes it uniquely.

and G_N is the Neumann Green's function of the bulk operator—the solution of the following problem

$$F(\nabla)G_N(X, Y) = \delta(X, Y), \qquad (\vec{W} + \kappa)G_N(X, Y)|_b = 0. \tag{2.10}$$

For completeness in Appendix A we present the derivation of the gaussian integral (2.8) by Feynman's method [18] which clearly shows how the boundary conditions enter the calculation of the preexponential part of this algorithm. This derivation, in particular, gives the variational definition of the corresponding functional determinant which goes far beyond its matrix (finite-dimensional) analogue. It is important that the boundary value problem (2.9) naturally follows from the action (1.1) and Wronskian relations for $F(\nabla)$, because the variation of the action is given by the sum of bulk and brane terms

$$\delta S[\phi] = \int_B dX \delta\phi(\vec{F}\phi) + \int_b dx \delta\varphi(\vec{W}\phi + \kappa\phi + j)| \tag{2.11}$$

which separately should vanish (remember that the action should be stationary also with respect to arbitrary variations of the boundary fields $\delta\varphi$).

The solution of (2.9) has the following form in terms of the Neumann Green's function

$$\phi_N(X) = -\int_b dy G_N(X, y) j(y) \equiv -G_N| j, \qquad G_N(X, y) \equiv G_N(X, Y)|_{Y=e(y)}, \tag{2.12}$$

and the stationary action as a functional of the boundary source $j(x)$ equals

$$S[\phi_N] = \frac{1}{2}\int_B dX \phi(\vec{F}\phi) + \int_b dx \left(\frac{1}{2}\phi(\vec{W} + \kappa)\phi + j\varphi\right) = -\frac{1}{2}\int_b dx dy\, j(x) G_N(x, y) j(y) \equiv -\frac{1}{2} j G_N|| j \tag{2.13}$$

$$G_N(x, y) \equiv G_N(X, Y)|_{X=e(x), Y=e(y)} \equiv G_N||. \tag{2.14}$$

Here to simplify the formalism we used condensed notations by omitting the sign of integration over *boundary/brane coordinates*.[2] Thus finally we have

$$Z = (\mathrm{Det} G_N)^{1/2} \exp\left(\frac{1}{2} j G_N || j\right). \tag{2.15}$$

[2]We will never use this rule for bulk integration which will always be explicitly indicated together with the corresponding integration measure. It is useful to apply this rule for integral operations on the brane, though, because these operations never lead to surface terms and in our context have properties of formal matrix contraction and multiplication. In the case of the one-dimensional bulk this rule applies literally, and it can be extended to higher dimensions without any risk of confusion.

Alternatively one can calculate the same integral by splitting the integration procedure into two steps—first integrating over bulk fields with fixed boundary values followed by the integration over the latter

$$\int D\phi(\ldots) = \int d\varphi \int_{\phi|=\varphi} D\phi(\ldots). \tag{2.16}$$

This allows one to rewrite the same result in the form

$$Z = \int d\varphi Z(\varphi), \tag{2.17}$$

$$Z(\varphi) = \int_{\phi|=\varphi} D\phi \exp(-S[\phi]), \tag{2.18}$$

where the inner integral over bulk fields in view of gaussianity is again given by the contribution of a saddle point ϕ_D

$$Z(\varphi) = (\mathrm{Det} G_D)^{1/2} \exp(-S[\phi_D]). \tag{2.19}$$

This saddle point configuration satisfies the problem with the inhomogeneous Dirichlet boundary conditions

$$F(\nabla)\phi_D(X) = 0, \qquad \phi_D| = \varphi(x), \tag{2.20}$$

and the preexponential factor of (2.19) is given by the functional determinant of the Dirichlet Green's function subject to

$$F(\nabla)G_D(X, Y) = \delta(X, Y), \qquad G_D(X, Y)|_X = 0. \tag{2.21}$$

In terms of this Green's function and using condensed notations we have

$$\phi_D(X) = -\int_b dy G_D(X, Y)\overleftarrow{W}\Big|_{Y=e(y)} \varphi(y) \equiv -G_D \overleftarrow{W}|\varphi, \tag{2.22}$$

$$S[\phi_D] = \frac{1}{2}\int_b dx dy \varphi(x)[-\vec{W} G_D \overleftarrow{W}(x, y) + \kappa(x, y)]\varphi(y) + \int_b dx\, j(x)\varphi(x) \equiv \frac{1}{2}\varphi[-\vec{W} G_D \overleftarrow{W}|| + \kappa]\varphi + j\varphi, \tag{2.23}$$

where $\vec{W} G_D \overleftarrow{W}||$ is defined by Eq. (1.8) in Introduction. Note that the last expression is exactly the tree-level brane effective action obtained from the original action (1.1) by integrating out the bulk fields subject to boundary conditions $\varphi(x)$,

$$\boldsymbol{S}_{\mathrm{brane}}[\varphi] = S[\phi_D[\varphi]]. \tag{2.24}$$

Substituting (2.19) with (2.23) into (2.17) we again obtain the gaussian integral which is saturated by the saddle point φ_0 of the above brane action (2.23)

$$\varphi_0 = -[-\vec{W} G_D \overleftarrow{W}|| + \kappa]^{-1} j, \tag{2.25}$$

and the final result reads

$$Z = (\mathrm{Det}G_D)^{1/2}(\mathrm{Det}[-\vec{W}G_D\overleftarrow{W}|| + \kappa])^{-1/2} \times \exp\left(\frac{1}{2}j[-\vec{W}G_D\overleftarrow{W}|| + \kappa]^{-1}j\right), \tag{2.26}$$

where det denotes the functional determinants in the d-dimensional boundary theory.

Comparison of its tree-level and one-loop (preexponential) parts with those of (2.15) then immediately yields two relations

$$G_N|| = [-\vec{W}G_D\overleftarrow{W}|| + \kappa]^{-1} \equiv \boldsymbol{G}_{\mathrm{brane}}, \tag{2.27}$$

$$(\mathrm{Det}G_N)^{1/2} = (\mathrm{Det}G_D)^{1/2}(\det[-\vec{W}G_D\overleftarrow{W}|| + \kappa])^{-1/2}. \tag{2.28}$$

They underlie the algorithms (1.6), (1.7), (1.10), and (1.11) advocated in Introduction and give the possibility in a systematic way to express all the quantities in brane theory in terms of the objects subject to Dirichlet boundary conditions.

As mentioned in Introduction, the relation between functional determinants (2.28) should be understood within some ultraviolet regularization. It should regulate the both bulk functional integrals (2.7), (2.8), (2.18), and (2.19) as well as the boundary functional integral over $\varphi = \varphi(x)$ (of d-dimensional theory) in (2.16) and (2.17). The simplest procedure which regularizes all three integrations without violating the basic relation (2.16) consists in the analytic continuation in d to the domain of convergence of relevant Feynman integrals. Other types of regularization do not seem to violate (2.16) too, however they can qualitatively change the setting of the boundary value problem underlying the obtained algorithms. For example, the regularization by higher derivatives, as well as certain versions of the Pauli-Villars regularizations, increases the order of differential operators. This changes the order of the normal derivative in the boundary conditions (1.4) and even the number of the latter, so that the Dirichlet-Neumann reduction stops working or has to be essentially modified. For this reason, in what follows we will use dimensional regularization as the simplest and most efficient scheme. As we shall see in Sec. IV, it correctly generates the ultraviolet finite heat kernel underlying the calculation of functional determinants by the Schwinger proper-time method and, thus, confirms the validity of the chosen regularization technique.

III. SIMPLE EXAMPLES

A. One-dimensional problem

The simplest case of the Dirichlet-Neumann duality relation can be demonstrated on the example of the one-dimensional Sturm-Liouville problem on the segment of finite length $y_+ - y_- = l$

$$F(\nabla) = m^2 - \frac{\partial^2}{\partial y^2}, \qquad y_- \le y \le y_+. \tag{3.1}$$

Its Wronskian operator on the two boundaries of this segment is given by

$$\vec{W}|_\pm = \pm\partial_y, \tag{3.2}$$

and the Dirichlet and Neumann Green's functions are correspondingly

$$G_D(y, y') = -\frac{\sinh m(y' - y_+)\sinh m(y - y_-)}{m\sinh ml} \times \theta(y' - y) + (y \leftrightarrow y'), \tag{3.3}$$

$$G_N(y, y') = \frac{\cosh m(y' - y_+)\cosh m(y - y_-)}{m\sinh ml} \times \theta(y' - y) + (y \leftrightarrow y'). \tag{3.4}$$

Since the boundary of the one-dimensional bulk consists of two points $y_\pm$, the full brane operator (1.7) has the form of the two-dimensional matrix with the elements corresponding to the $\pm$ entries on the two zero-dimensional "branes" (for simplicity we take the case of $\kappa = 0$)

$$\boldsymbol{F}^{\mathrm{brane}} = \frac{m}{\sinh ml}\begin{pmatrix} \cosh ml & -1 \\ -1 & \cosh ml \end{pmatrix}. \tag{3.5}$$

On the other hand, the restriction of the Neumann Green's function (3.4) to the boundary is given by

$$G_N|| = \frac{1}{m\sinh ml}\begin{pmatrix} \cosh ml & 1 \\ 1 & \cosh ml \end{pmatrix}. \tag{3.6}$$

This is a matter of a simple verification to check that these two matrices are inverse to one another which is just the relation (1.9).

To check the one-loop duality relation one can write the Dirichlet and Neumann functional determinants of $F(\nabla)$ as products of eigenvalues of the corresponding spectra. Interestingly, for this simple problem the Dirichlet spectrum

$$F(\nabla)\phi_k^D = \lambda_k^D\phi_k^D, \qquad \phi_k^D(y_\pm) = 0, \tag{3.7}$$

$$\lambda_k^D = \frac{\pi^2 k^2}{l^2} + m^2, \qquad k = 1, 2, 3, \ldots \tag{3.8}$$

coincides with the Neumann spectrum

$$F(\nabla)\phi_k^N = \lambda_k^N\phi_k^N, \qquad \partial_y\phi_k^N(y_\pm) = 0, \tag{3.9}$$

$$\lambda_k^N = \frac{\pi^2 k^2}{l^2} + m^2, \qquad k = 0, 1, 2, 3, \ldots \tag{3.10}$$

except one eigenmode $k = 0$. This constant mode, $\phi_0^N(y) = \mathrm{const}$, is just absent in the spectrum of the Dirichlet problem. Therefore

$$\det{}_N F(\nabla) = \det{}_D F(\nabla)\lambda_0, \tag{3.11}$$

and this immediately confirms the relation (2.28) in view of the fact that

$$\det \boldsymbol{F}^{\rm brane} = m^2 = \lambda_0. \tag{3.12}$$

B. Half-space with the Killing symmetry in extra dimension

Another example is of field-theoretic nature. It corresponds to $(d+1)$-dimensional half-space with the d-dimensional boundary plane. Let the operator be given by the $(d+1)$-dimensional d'Alembertian with mass and let the boundary be located at the position $y = 0$ of the extra-dimensional coordinate $y = X^{d+1}$,

$$F(\nabla) = m^2 - \Box^{(d+1)} = m^2 - \Box - \partial_y^2, \qquad \Box \equiv \Box^{(d)}, \tag{3.13}$$

$$X^A = (x^\mu, y), \qquad y \geq 0, \tag{3.14}$$

$$X|_b = (x, 0) \tag{3.15}$$

The d-dimensional part of the full d'Alembertian can in principle be curved and nontrivially depending on x-coordinates. We only assume the possibility of separation of variables, so that y is a Killing direction in the bulk.

For such a setting the Wronskian operator is given by the normal derivative with respect to the outward-pointing normal and equals

$$\vec{W} = -\partial_y, \tag{3.16}$$

while the exact Dirichlet and Neumann Green's functions are

$$G_{D,N}(y, y') = \frac{1}{\Delta}(e^{-\Delta|y-y'|} \mp e^{-\Delta(y+y')}), \tag{3.17}$$

$$\Delta \equiv \sqrt{m^2 - \Box}, \tag{3.18}$$

where the minus and plus signs refer, respectively, to the Dirichlet and Neumann cases.

According to (1.7) the brane-to-brane operator equals

$$\boldsymbol{F}^{\rm brane} = -\vec{\partial}_y G_D(y, y') \overleftarrow{\partial}_y \big|_{y=y'=0} = \Delta, \tag{3.19}$$

while from (3.17)

$$G_N|| = G_N(0, 0) = \frac{1}{\Delta}, \tag{3.20}$$

which confirms the relations (1.7) and (1.10).

To check (1.11) let us write a variational definition of the functional determinants for both Dirichlet and Neumann boundary conditions with respect to general variations of the d-dimensional part of the full operator $\delta F = -\delta\Box$. We have

$$\delta \ln \det G_{D,N} = -{\rm Tr} G_{D,N}\delta F = {\rm tr} \int_0^\infty dy G_{D,N}(y, y)\delta\Box, \tag{3.21}$$

where we decomposed the $(d+1)$-dimensional functional trace into the operation of integrating over y the coincidence limit of the corresponding y-dependent kernel and the d-dimensional functional trace tr. Then, substituting (3.17) we have

$$\begin{aligned} \delta(\ln \det G_N - \ln \det G_D) &= {\rm tr} \int_0^\infty dy \frac{1}{\Delta} e^{-2\Delta y}\delta\Box \\ &= -\delta \ln \det\Delta, \end{aligned} \tag{3.22}$$

which, in view of (3.19), in the infinitesimal variational form fully confirms (1.11).

IV. BOUNDARY TERMS OF THE LOCAL HEAT KERNEL EXPANSION

Neumann-Dirichlet duality can be used for the calculation of the boundary terms in the local expansion of the heat kernel. In the absence of boundaries with trivial falloff conditions at infinity, the heat kernel

$$K(s|x, y) = e^{-sF(\nabla)}\delta(x, y) \tag{4.1}$$

turns out to be a very efficient tool of the covariant diagrammatic technique for quantum effective action in curved spacetime and in external fields of a very generic form [1,10–12]. Its efficiency is based on the possibility of expanding this kernel in asymptotic series in integer powers of $s \to 0$ with the coefficients $\hat{a}_n(X, Y)$ which satisfy simple recurrent equations. These coefficients, often called in the physics context the Schwinger-DeWitt or HAMIDEW coefficients, can be explicitly found in the coincidence limit $y = x$ as local invariants built in terms of spacetime curvature of the bulk metric G_{AB}, fibre bundle connection and other background fields. Thus, they give rise to local low energy expansion of the effective action in inverse powers of the mass parameter $m^2 \to \infty$, when the inverse propagator of the theory is supplied with the mass term, $F(\nabla) \to F(\nabla) + m^2$.

In spacetime with boundaries the situation becomes more complicated, because, similarly to Green's functions, the heat kernel should be obtained from that of the infinite spacetime by the method of images. This leads to the expansion of the functional trace of the heat kernel in half-integer powers of the proper-time parameter [3,12,19,20]

$$\begin{aligned} {\rm Tr}_{(d+1)} e^{-sF(\nabla) - sm^2} &= \frac{1}{(4\pi s)^{(d+1)/2}} e^{-sm^2} \\ &\quad \times \sum_{n=0}^\infty (s^n A_n + s^{n/2} B_{n/2}). \end{aligned} \tag{4.2}$$

Together with the volume (bulk) terms of the Schwinger-DeWitt type,[3]

[3]In this section we denote the functional trace in the $(d+1)$-dimensional bulk and on the d-dimensional boundary, respectively, by ${\rm Tr}_{(d+1)}$ and ${\rm Tr}_{(d)}$, while the notation tr is reserved for the trace over spin-tensor indices of matrices. The latter are denoted by hats.

$$A_n = \int_{\mathcal{M}} d^{d+1}X G^{1/2}(X) \mathrm{tr}\hat{a}_n(X,X), \tag{4.3}$$

this expansion acquires surface integrals at the boundary $B_{n/2}$ which are built of local invariants incorporating also such local characteristics of the surface as its extrinsic curvature $K_{\mu\nu}$ and the curvature of the induced metric $g_{\mu\nu}$,

$$B_{n/2} = \int_{\partial\mathcal{M}} d^d x g^{1/2}(x) b_{n/2}(x). \tag{4.4}$$

Here $G(X)$ and $g(x)$ denote the determinants of the bulk and brane metrics, so that $\mathrm{tr}\hat{a}_n(X,X)$ and $b_{n/2}(x)$ turn out to be the bulk and boundary scalars.

In contrast to the bulk Schwinger-DeWitt coefficients $\hat{a}_n(X,X)$ which are universal and independent of the type of boundary conditions, the surface terms essentially depend on the latter, and their calculation is much less universal and often very cumbersome. For the operator of the form

$$F(\nabla) = -G^{AB}\nabla_A\nabla_B - \hat{P} + \frac{1}{6}R\hat{1}, \tag{4.5}$$

with second order covariant derivatives ∇_A forming a covariant $(d+1)$-dimensional D'Alembertian in the metric G_{AB} and $\hat{P}$ denoting some matrix-valued potential term, few lowest Schwinger-DeWitt coefficients read as

$$\hat{a}_0(X,X) = \hat{1}, \qquad \hat{a}_1(X,X) = \hat{P}, \ldots. \tag{4.6}$$

The corresponding surface terms for the Dirichlet and simple Neumann boundary conditions

$$b_0^{D,N}(x) = 0, \tag{4.7}$$

$$b_{1/2}^D(x) = -\frac{\sqrt{\pi}}{2}\mathrm{tr}\hat{1}, \qquad b_{1/2}^N(x) = \frac{\sqrt{\pi}}{2}\mathrm{tr}\hat{1}, \tag{4.8}$$

$$b_1^{D,N}(x) = \frac{1}{3}K\,\mathrm{tr}\hat{1}, \quad \ldots \tag{4.9}$$

involve the trace of the extrinsic curvature of the boundary $K = g^{\mu\nu}K_{\mu\nu}$ [3,12].

For the generalized Neumann (Robin) boundary conditions

$$(\nabla_n - \hat{S})\phi(X)|_{\partial\mathcal{M}} = 0 \tag{4.10}$$

the last of the above coefficients is modified by the matrix-valued potential term $\hat{S}$ [12]

$$b_1^R(x) = \mathrm{tr}\left[2\hat{S} + \frac{1}{3}K\hat{1}\right]. \tag{4.11}$$

This modification is even more sophisticated in the case of the so-called oblique boundary conditions [13], which include the tangential to the boundary (d-dimensional) covariant derivatives D_μ

$$\left(\nabla_n - \hat{\Gamma}^\mu D_\mu - \frac{1}{2}(D_\mu\hat{\Gamma}^\mu) - \hat{S}\right)\phi(X)\Big|_{\partial\mathcal{M}} = 0. \tag{4.12}$$

These derivatives enter the boundary conditions with the *dimensionless* matrix-valued vector coefficients $\hat{\Gamma}^\mu$. For the generic case their contribution to $b_{n/2}$ is not known, but in the case of commuting matrices

$$[\hat{\Gamma}^\mu, \hat{\Gamma}^\nu] = 0 \tag{4.13}$$

lengthy calculations of [13–15] lead to the following expressions for few lowest-order surface densities (see also [12,14] for higher order $b_{n/2}$)

$$b_{1/2}^O(x) = \frac{\sqrt{\pi}}{2}\mathrm{tr}\left[\frac{2}{\sqrt{1+\hat{\Gamma}^2}} - \hat{1}\right], \tag{4.14}$$

$$b_1^O(x) = \mathrm{tr}\left[\frac{2}{1+\hat{\Gamma}^2}\hat{S} + \frac{1}{3}K\hat{1} + \left(\frac{1}{1+\hat{\Gamma}^2} - \frac{\mathrm{arctanh}\sqrt{-\hat{\Gamma}^2}}{\sqrt{-\hat{\Gamma}^2}}\right) \times \left(K - K^{\mu\nu}\frac{\hat{\Gamma}_\mu\hat{\Gamma}_\nu}{\hat{\Gamma}^2}\right)\right], \tag{4.15}$$

where

$$\hat{\Gamma}^2 = \hat{\Gamma}^\mu\hat{\Gamma}_\mu. \tag{4.16}$$

It is important to note that these matrix functions are nonpolynomial in $\hat{\Gamma}^\mu$ because of the dimensionless nature of these matrices. This means that their contribution to any given surface coefficient $b_{n/2}$ cannot be obtained by the perturbation theory in $\hat{\Gamma}^\mu$, which is the main reason of difficulties in their derivation.

Let us now give a simple derivation of these coefficients by the technique of the previous section. To begin with, note that the Wronskian operator for (4.5) is defined by the normal derivative with respect to outward-pointing normal n^A to the boundary

$$\vec{W} = \nabla_n, \qquad \nabla_n = n^A\nabla_A. \tag{4.17}$$

Introduce the covariant operator $\kappa(D)$ corresponding to the generalized Neumann boundary condition (4.12) and the d-dimensional brane action, from which these boundary condition can be obtained by the variational procedure of Sec. II. Obviously, they read

$$\kappa = \kappa(D) = -\hat{\Gamma}^\mu D_\mu - \frac{1}{2}(D_\mu\hat{\Gamma}^\mu) - \hat{S}, \tag{4.18}$$

$$S^{(d)}[\varphi] = \frac{1}{2}\varphi\kappa\varphi = -\frac{1}{2}\int_b dx g^{1/2}\varphi^T(\hat{\Gamma}^\mu D_\mu + \hat{S})\varphi(x), \tag{4.19}$$

provided the following symmetry property of matrices $\hat{\Gamma}^\mu$ and $\hat{S}$ holds (which we assume in what follows)

$$\hat{\Gamma}^{\mu T} = -\hat{\Gamma}^\mu, \qquad \hat{S}^T = \hat{S}. \tag{4.20}$$

According to the Shwinger-DeWitt proper-time method [1,10] the functional determinants of massive operators are given by the proper-time integrals of the corresponding heat kernels. In view of the representation (4.2) this gives the following inverse-mass expansions in the Dirichlet and generalized Neumann cases

$$\begin{aligned}\mathrm{Tr}_{D,N}\ln[F(\nabla)+m^2] &= -\mathrm{Tr}_{D,N}\int_0^\infty \frac{ds}{s}e^{-sF(\nabla)-sm^2}\\ &= -\left(\frac{m^2}{4\pi}\right)^{(d+1)/2}\left[\sum_{n=0}^\infty \frac{\Gamma(n-\frac{d+1}{2})}{m^{2n}}A_n\right.\\ &\quad\left.+\sum_{n=1}^\infty \frac{\Gamma(\frac{n-d-1}{2})}{m^n}B^{D,N}_{n/2}\right]. \end{aligned}\tag{4.21}$$

The heat kernel trace (4.2) is always ultraviolet finite, and the one-loop divergences originate from the proper-time integration diverging at $s=0$. In dimensional regularization (with d analytically continued to the complex plane) they arise here as gamma-function poles for first few terms of nonnegative powers in the mass parameter (in even dimension $d+1$ for bulk Schwinger-DeWitt coefficients and both even and odd $d+1$ for surface terms).

The Dirichlet and Neumann expressions (4.21) differ only by the contributions of surface integrals, their total difference, on the other hand, being defined from the duality relation (1.11). Subtracting the Dirichlet version of (4.21) from the Neumann one, one therefore obtains

$$\begin{aligned}\mathrm{Tr}_{(d)}\ln\boldsymbol{F}^{\rm brane} &= -\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\sum_{n=1}^\infty\frac{\Gamma(\frac{n-d-1}{2})}{m^{n-1}}\\ &\quad\times\int d^dx g^{1/2}\frac{b^N_{n/2}-b^D_{n/2}}{\sqrt{\pi}},\end{aligned}\tag{4.22}$$

where the brane-to-brane operator $\boldsymbol{F}^{\rm brane}$ is defined by Eq. (1.7) in Introduction for a particular case of the boundary operator (4.19). Thus, the difference of the boundary terms for Neumann and Dirichlet cases can be disentangled from the functional determinant of $\boldsymbol{F}^{\rm brane}$ in relevant orders of the $1/m$-expansion. In view of the ultraviolet finiteness of the heat kernel (finiteness of the coefficients $b^{N,D}_{n/2}$) the divergences of this functional determinant should have the structure of gamma-function coefficients in the right hand side of (4.22), which serves as a consistency check of the whole procedure.

Though the operator $\boldsymbol{F}^{\rm brane}$ is itself not exactly known, all we need is its inverse-mass expansion which is equivalent to the bulk-brane curvature expansion of the Dirichlet Green's function G_D of $F(\nabla)$. Moreover, it is defined on the brane manifold without a boundary (or with trivial regularity conditions at infinity). All this essentially facilitates the solution of the problem. As we show in the following examples, choosing the brane operator (1.7) for $\kappa(D)$ of the form (4.18) we easily calculate the lowest-order surface terms, and this procedure can undoubtedly be extended to all $b_{n/2}$.

A. Simple Neumann boundary conditions

For simplest Neumann boundary conditions with $\kappa(D)=0$ the brane-to-brane operator (1.7) was obtained in the leading order approximation in Sect. III B, Eq. (3.19). It reads

$$\boldsymbol{F}^{\rm brane}=\Delta\equiv\sqrt{m^2-\Box}+O[R,K],\tag{4.23}$$

where $O[R,K]$ denotes corrections due to the bulk curvature and extrinsic curvature of the boundary. Therefore, the inverse-mass expansion for its determinant is dominated by

$$\begin{aligned}\mathrm{Tr}_{(d)}\ln\boldsymbol{F}^{\rm brane} &= \frac{1}{2}\mathrm{Tr}_{(d)}\ln(m^2-\Box)+O[R,K]\\ &= -\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\Gamma(-d/2)\int d^dx g^{1/2}\,\mathrm{tr}\hat{1}\\ &\quad+O[R,K]\end{aligned}\tag{4.24}$$

with $O[R,K]=O[m^{d-1}]$. From the $n=1$ term of (4.22) it follows then that

$$b^N_{1/2}-b^D_{1/2}=\sqrt{\pi}\,\mathrm{tr}\hat{1},\tag{4.25}$$

which fully agrees with (4.8). The dependence of (4.24) on dimensionality (yielding the logarithmic divergence for even d) is exactly the same as in the $n=1$ term of (4.22) which, as expected, guarantees the ultraviolet finiteness of the obtained difference $b^N_{1/2}-b^D_{1/2}$.

B. Robin boundary conditions

For Robin boundary conditions (4.10) with $\kappa(D)=-\hat{S}$

$$\boldsymbol{F}^{\rm brane}=\sqrt{m^2-\Box}-\hat{S}+O[R,K].\tag{4.26}$$

The functional determinant of this operator can be obtained by perturbation theory in the *dimensional* quantity $\hat{S}$

$$\begin{aligned}\mathrm{Tr}_{(d)}\ln\boldsymbol{F}^{\rm brane} &= \frac{1}{2}\mathrm{Tr}_{(d)}\ln(m^2-\Box)-\mathrm{Tr}_{(d)}\frac{\hat{S}}{\sqrt{m^2-\Box}}\\ &\quad+O[R,K,\hat{S}^2].\end{aligned}\tag{4.27}$$

Only the first order in $\hat{S}$ contributes to b^R_1 in the Robin case, and to zeroth order in the curvature this term equals

$$\begin{aligned}-\mathrm{Tr}_{(d)}\frac{\hat{S}}{\sqrt{m^2-\Box}} &= -\frac{1}{\Gamma(1/2)}\mathrm{Tr}_{(d)}\int_0^\infty ds s^{-1/2}e^{-s(m^2-\Box)}\hat{S}\\ &= -\int_0^\infty\frac{ds}{\sqrt{\pi s}}\frac{e^{-sm^2}}{(4\pi s)^{d/2}}\int d^dx g^{1/2}\,\mathrm{tr}\hat{S}\\ &\quad+O[R,K]\\ &= -\left(\frac{m^2}{4\pi}\right)^{d/2}\frac{\Gamma(\frac{1-d}{2})}{m\sqrt{\pi}}\int d^dx g^{1/2}\,\mathrm{tr}\hat{S}\\ &\quad+O[R,K].\end{aligned}\tag{4.28}$$

Therefore, from the $n=2$ term of (4.22) it follows that

$$b_1^R - b_1^D = 2\,\mathrm{tr}\hat{S} + O[K], \tag{4.29}$$

which fully agrees with (4.9) and (4.11).

C. Oblique boundary conditions

For simplicity consider the case of oblique boundary conditions (4.12) with $\hat{S} = 0$. Then the surface operator (4.19) is $\kappa(D) = -\hat{\Gamma}^\mu D_\mu + O[D\hat{\Gamma}]$, and the brane-to-brane operator reads

$$\boldsymbol{F}^{\,\mathrm{brane}} = \sqrt{m^2 - \Box} - \hat{\Gamma}^\mu D_\mu + O[R, K, D\hat{\Gamma}], \tag{4.30}$$

where together with curvature terms we disregard terms with covariant derivatives of $\hat{\Gamma}^\mu$. With the same accuracy

$$\mathrm{Tr}\ln \boldsymbol{F}^{\mathrm{brane}} = -\int d^d x\,\mathrm{tr}\int_0^\infty \frac{d\tau}{\tau} e^{-\tau\sqrt{m^2-\Box}+\tau\hat{\Gamma}^\mu D_\mu}\delta(x,y)\Big|_{y=x}. \tag{4.31}$$

Here we denote the proper-time parameter by τ to emphasize that it has the dimensionality different from s (length rather than length squared), which in its turn is explained by the dimensionality of (4.30).

In contrast to the Robin case further calculations cannot be performed by perturbations in powers of the term $\hat{\Gamma}^\mu D_\mu$, because the vector coefficient $\hat{\Gamma}^\mu$ is dimensionless, and the perturbation theory in this term will not generate asymptotic expansion in inverse mass.[4] Instead, with the same accuracy of zeroth order in the curvature one can disentangle this term in the exponential as a matrix-valued shift operator

$$\exp[-\tau\sqrt{m^2 - \Box} + \tau\hat{\Gamma}^\mu \nabla_\mu]\delta(x,y)|_{y=x} = e^{\tau\hat{\Gamma}^\mu\nabla_\mu} K(\tau|x,y)|_{y=x} = K(\tau|x + \tau\hat{\Gamma}, x) \tag{4.32}$$

acting on the heat kernel of the operator $\sqrt{m^2 - \Box}$,

$$K(\tau|x,y) = \exp[-\tau\sqrt{m^2 - \Box}]\delta(x,y). \tag{4.33}$$

Note that in view of the commutativity assumption (4.13) this nontrivial matrix-valued function is uniquely defined without any matrix-ordering prescription.[5]

As shown in Appendix B, in flat $(d+1)$-dimensional space without boundaries

$$K(\tau|x,y) = 2\tau\left(\frac{m^2}{2\pi Z}\right)^{(d+1)/2} K_{(d+1)/2}(Z), \tag{4.34}$$

where

$$Z \equiv Z(\tau|x,y) = m\sqrt{|x-y|^2 + \tau^2} \tag{4.35}$$

and $K_\nu(Z)$ is the modified Bessel (MacDonald) function. Because of

$$Z(\tau|x + \tau\hat{\Gamma}, x) = m\tau\sqrt{1 + \hat{\Gamma}^2} \tag{4.36}$$

the τ-integration in (4.31) gives

$$\begin{aligned}\mathrm{Tr}\ln\boldsymbol{F}^{\mathrm{brane}} &= -\int d^d x g^{1/2}\,\mathrm{tr}\int_0^\infty \frac{d\tau}{\tau} K(\tau|x+\tau\hat{\Gamma}, x)\\ &= -\left(\frac{m^2}{2\pi}\right)^{(d+1)/2}\frac{2}{m}\int d^d x g^{1/2}\,\mathrm{tr}\frac{1}{\sqrt{1+\hat{\Gamma}^2}}\\ &\quad\times\int_0^\infty dZ\frac{K_{(d+1)/2}(Z)}{Z^{(d+1)/2}},\end{aligned} \tag{4.37}$$

whence in view of the integral (Eq. 6.561.16 of [21])

$$\int_0^\infty dx x^{-\nu} K_\nu(x) = 2^{-\nu-1}\sqrt{\pi}\,\Gamma\left(\frac{-2\nu+1}{2}\right) \tag{4.38}$$

we finally have

$$\begin{aligned}\mathrm{Tr}_{(d)}\ln\boldsymbol{F}^{\mathrm{brane}} &= -\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\Gamma\left(-\frac{d}{2}\right)\\ &\quad\times\int d^d x g^{1/2}\,\mathrm{tr}\frac{1}{\sqrt{1+\hat{\Gamma}^2}} + O[m^{d-1}].\end{aligned} \tag{4.39}$$

Therefore, from the $n = 1$ term of (4.22) it follows that

$$b_{1/2}^O - b_{1/2}^D = \sqrt{\pi}\,\mathrm{tr}\frac{1}{\sqrt{1+\hat{\Gamma}^2}} \tag{4.40}$$

which fully agrees with (4.14) for oblique boundary conditions with $\hat{S} = 0$. Similarly, one can check the $\hat{S}$-dependent term of (4.15) for the case of nonvanishing $\hat{S}$.

The approximation of zero bulk and brane curvatures in all the above examples can be used as a starting point of the perturbation theory in $O[R, K]$ (and in other dimensional background-field quantities like $D_\mu\hat{\Gamma}^\mu$ and $\hat{P}$). Then the Neumann-Dirichlet duality method of the above type will give extrinsic curvature terms of (4.15) and all higher order surface terms in the heat kernel expansion (4.2).

V. CONCLUSIONS

Thus, the algorithms (1.7), (1.10), and (1.11) allow one to reduce calculations of brane effective action to those of the Dirichlet boundary conditions. This reduction technique can obviously be extended to multiloop orders by applying the same trick of splitting the functional integration into two steps, as in (2.16), (2.17), and (2.18), also in the nonlinear case. The resulting Feynman diagrammatic technique from combinatorical viewpoint is not so simple as in (1.11), but still manageable. In addition to the bulk Dirichlet type propagator G_D it has the brane-to-brane propagator $\boldsymbol{G}_{\mathrm{brane}}$—the Green's function of $\boldsymbol{F}^{\mathrm{brane}}$. The

[4]In other words, the leading symbol of the pseudodifferential operator (4.30)—the highest (first) order in derivatives part of $\boldsymbol{F}^{\mathrm{brane}}$—is given by $\sqrt{-\partial^2} - \hat{\Gamma}^\mu\partial_\mu$, and it should be treated as a whole without breaking into pieces.

[5]Apparently, the commutativity assumption can be removed. For noncommuting matrices Eq. (4.32) will still be valid under the symmetric matrix-ordering prescription in the right hand side. This symmetrization follows from the symmetry of the Taylor series coefficients.

modification of Feynman diagrams, therefore, consists in the insertions into the bulk diagrams of the Dirichlet type the lines connecting bulk vertices to the brane by the brane-to-boundary propagators

$$\frac{\delta\phi_D(X)}{\delta\varphi(y)} = -G_D(X, Y)\overleftarrow{W}|_{Y=e(y)} \tag{5.1}$$

(cf. Eq. (2.22)) and also developing the d-dimensional diagrammatic technique on the brane with the propagator $\boldsymbol{G}_{\rm brane}$. This technique will be considered in more detail in [16]. In gravitational brane models the question of gauge invariance (especially with respect to general coordinate transformations) becomes very important, while at present even the details of Faddeev-Popov gauge fixing procedure in spacetimes with boundaries are not clearly studied [9]. Therefore, gauge properties of Neumann-Dirichlet duality will be a major aspect of [16], where gravitational Ward identities in brane models will be established (they are briefly reported in [9]).

As a byproduct, the Neumann-Dirichlet reduction technique suggests also a new method of calculating surface terms of the heat kernel expansion. Given the Dirichlet type terms, those of the generalized Neumann case can be obtained from (4.22). This allows one to circumvent the limitations of the conventional method for these terms. In particular, we were able to recover correct expressions for few lowest surface contributions to the Schwinger-DeWitt expansion and, in case of oblique boundary conditions, perhaps even extend their validity beyond the commutative case (4.13) (see footnote after Eq. (4.33) on the symmetric matrix-ordering prescription).

Of course, the success of Neumann-Dirichlet reduction programme depends on our ability to find the Dirichlet Green's function G_D and the corresponding brane-to-brane operator $\boldsymbol{F}^{\rm brane}$. The latter is a nonlocal pseudodifferential operator, so the problem of efficiently handling its nonlocality arises. For a wide class of problems $\boldsymbol{F}^{\rm brane}$ was found in the zero curvature approximation as

$$\boldsymbol{F}^{\rm brane}(D) = \sqrt{m^2 - \Box} + \kappa(D) + O[R, K]. \tag{5.2}$$

Despite nonlocality, local expansion of its functional determinant is still manageable. For ultralocal $\kappa(D) = -\hat{S}$, as in Robin case, it is easily available by perturbations. It is more complicated in the case of oblique boundary conditions, when the leading symbol of $\boldsymbol{F}^{\rm brane}(D)$ gives rise to the propagation off light cone on the brane (in the physical theory with the Lorentzian signature)—the phenomenon called *generalized causality* in [10].[6] Finally, in brane induced gravity models with the operator $\kappa(D) \sim \Box/\mu$ generated by the brane Einstein term (μ is the DGP scale [6,7]), the calculational technique still has to be worked out [17].

Final comment of this paper concerns the (bulk and boundary) curvature expansion of the Dirichlet Green's function. Classical method of images for this expansion is known [3], and this method for the Dirichlet case is much simpler than for the generalized Neumann boundary conditions. Still further efforts are necessary to convert it into a regular systematic calculational scheme comparable in its universality to the Schwinger-DeWitt technique in spacetime without branes/boundaries [10,11]. The progress in this direction will be reported in forthcoming papers [16,17], and here it remains to express a hope that at least the general shape of background-field method for quantum effective action in brane theory becomes visible.

[6]Related phenomenon of loss of strong ellipticity, which is the unboundedness of the operator spectrum from below in the Euclidean theory [22] or the presence of ghost modes in the Lorentzian case, takes place when $1 + \hat{\Gamma}^2$ acquires zero or negative eigenvalues leading to singularities in the algorithms (4.14) and (4.15)

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APPENDIX A: GAUSSIAN PATH INTEGRAL IN SPACETIME WITH BOUNDARIES

Feynman's calculation [18] of the gaussian functional integral (2.8) is based on the integral

$$Z[F, J] = \int D\phi \exp(-S[\phi, J]). \tag{A1}$$

Here instead of the source $j(x)$ at the boundary, as in (1.1), the action has the source $J(X)$ to the integration field $\phi(x)$ in the bulk

$$S[\phi, J] = \int_B dX\Big(\frac{1}{2}\phi(X)\overleftrightarrow{F}(\nabla)\phi(X) + J(X)\phi(X)\Big) + \frac{1}{2}\int_{\partial B} dx\varphi(x)\kappa(\partial)\varphi(x). \tag{A2}$$

To find the dependence of (A1) on $J(X)$ consider the stationary point of this action with respect to arbitrary variations of $\phi(X)$ both in the bulk and on the boundary. Similarly to (2.9) this field satisfies the generalized Neumann boundary value problem

$$F(\nabla)\phi_N(X) + J(X) = 0, \qquad (\vec{W} + \kappa)\phi_N| = 0. \tag{A3}$$

Now make the shift of the integration variable in (A1) by ϕ_N

$$\phi = \phi_N + \Delta. \tag{A4}$$

Under this replacement the action decomposes in the part $S[\Delta, 0]$ quadratic in Δ and the part independent of Δ. Linear in Δ term is absent (both in the bulk and on the boundary) in view of the stationarity of the action at ϕ_N, so that

$$S[\phi, J] = S[\Delta, 0] + S[\phi_N, J], \tag{A5}$$

$$S[\phi_N, J] = \frac{1}{2}\int_B dX dY J(X) G_N(X, Y) J(Y). \tag{A6}$$

Therefore

$$Z[F, J] = Z[F, 0] \exp(-S[\phi_N, J]), \tag{A7}$$

which justifies the exponential (tree-level) part of (2.8).

To find the prefactor, consider the variation of the integral (A1) at $J = 0$ with respect to the operator $F(\nabla)$ and make the following set of obvious identical transformations using the above Eqs. (A6) and (A7)

$$\begin{aligned}\delta_F Z[F, 0] &= -\int D\phi\Big(\frac{1}{2}\int_B dX \phi(X) \overset{\leftrightarrow}{\delta F}(\nabla)\phi(X)\Big)\exp(-S[\phi, 0]) \\ &= -\int D\phi\Big(\frac{1}{2}\int_B dX \frac{\delta}{\delta J(X)} \overset{\leftrightarrow}{\delta F}(\nabla) \frac{\delta}{\delta J(X)}\Big) Z[F, J]\Big|_{J=0} = -\frac{1}{2}\int_B dX \overset{\leftrightarrow}{\delta F}(\nabla) G_N(X, Y)\Big|_{Y=X} Z[F, 0].\end{aligned} \tag{A8}$$

Here $\overset{\leftrightarrow}{\delta F}(\nabla)$ means arbitrary variations of the coefficients of the operator, $\delta a^{AB}(X)$, $\delta b^A(X)$, $\delta c(X)$, and the double arrow implies symmetric action of two first-order derivatives of $F(\nabla)$ on both arguments of $G_N(X, Y)$ similar to Eq. (2.2)

$$\begin{aligned}\int_B dX \overset{\leftrightarrow}{\delta F}(\nabla) G_N(X, Y)\Big|_{Y=X} &\equiv \int_B dX[\delta a^{AB}(X)\partial_A^Y \partial_B^X \\ &\quad - 2\delta b^A(X)\partial_A^X \\ &\quad + \delta c(X)] G_N(X, Y)\Big|_{Y=X} \\ &= \mathrm{Tr}\overset{\leftrightarrow}{\delta F}(\nabla) G_N.\end{aligned} \tag{A9}$$

Therefore, from (A8) one gets

$$\begin{aligned}\delta_F \ln Z[F, 0] &= -\frac{1}{2}\mathrm{Tr}\overset{\leftrightarrow}{\delta F}(\nabla) G_N = -\frac{1}{2}\delta \ln \mathrm{Det} F \\ &\equiv \frac{1}{2}\delta \ln \mathrm{Det} G_N.\end{aligned} \tag{A10}$$

This expression in the variational form justifies the representation of the prefactor in (2.8) in terms of the functional determinant of the Neumann Green's function (or the inverse of the determinant of $F(\nabla)$ subject to Neumann boundary conditions). Simultaneously, it gives the variational definition of these functional determinants which specifies how the Neumann boundary conditions enter them and how the action of differential operators should be understood in the sense of integration by parts. The derivation of the gaussian integral (2.19) with fixed fields at the boundary can be done along the same lines [18] and leads to a similar variational definition with G_D replacing G_N.

APPENDIX B: HEAT KERNEL OF THE SQUARE-ROOT TYPE OPERATOR

The heat kernel (4.33) obviously satisfies the following Dirichlet problem $K(\tau|x, y)$:

$$(-\partial_\tau^2 - \Box + m^2) K(\tau|x, y) = 0, \tag{B1}$$

$$K(0|x, y) = \delta(x - y). \tag{B2}$$

Similarly to (2.22) its solution is given by the "brane-to-bulk propagator"

$$K(\tau|x, y) = -G_D^{(d+1)}(x, \tau|y, \tau')\overleftarrow{W}|_{\tau'=0}, \qquad \overleftarrow{W} = -\overleftarrow{\partial}_{\tau'}, \tag{B3}$$

where $G_D^{(d+1)}(x, \tau|y, \tau')$ is the $(d+1)$-dimensional Dirichlet Green's function of the operator $\Box^{(d+1)} = \partial_\tau^2 + \Box$ on half-space $\tau \geq 0$

$$(m^2 - \Box^{(d+1)}) G_D^{(d+1)}(x, \tau|y, \tau') = \delta(\tau - \tau')\delta(x, y), \tag{B4}$$

$$G_D^{(d+1)}(x, 0|y, \tau') = 0. \tag{B5}$$

By the method of images one can construct it in terms of $G^{(d+1)}(x, \tau|y, \tau')$—the Green's function in full space without boundary,

$$\begin{aligned}G_D^{(d+1)}(x, \tau|y, \tau') &= G^{(d+1)}(x, \tau|y, \tau') \\ &\quad - G^{(d+1)}(x, \tau|y, -\tau').\end{aligned} \tag{B6}$$

For the massive case it reads

$$\begin{aligned}G^{(d+1)}(x, \tau|y, \tau') &= \frac{1}{2\pi}\Big(\frac{m^2}{2\pi Z}\Big)^{(d-1)/2} K_{(d-1)/2}(Z), \\ Z &= m\sqrt{|x - y|^2 + (\tau - \tau')^2}.\end{aligned} \tag{B7}$$

Substituting (B6) and (B7) into (B3) and using recurrent relation between the modified Bessel functions of different orders one obtains (4.34).

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