PHYSICAL REVIEW D **73,** 066001 (2006)

Fuzzy spacetime with $SU(3)$ isometry in the IIB matrix model

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(Received 28 November 2005; published 6 March 2006)

A group of fuzzy spacetime with $SU(3)$ isometry is studied at the two-loop level in a IIB matrix model. It consists of spacetime from four to six dimensions, namely, from CP^2 to $SU(3)/U(1) \times U(1)$. The effective action scales in a universal manner in the large N limit as N and $N^{4/3}$ on four- and six-dimensional manifolds, respectively. The four-dimensional spacetime CP^2 possesses the smallest effective action in this class.

DOI: 10.1103/PhysRevD.73.066001 PACS numbers: 11.25.Sq, 11.30.Pb, 11.10.Lm

I. INTRODUCTION

The investigations of the properties of the spacetime at the microscopic level have become an important physical subject since we now have a clear picture where the Universe comes from and is going. At the current stage, the space is found to be almost flat and accelerating its expansion rate. It is therefore approaching a four-dimensional de Sitter spacetime. Furthermore, the scale independent fluctuation of the cosmic microwave background at a long distance scale suggests that the Universe also started as a de Sitter spacetime. In order to explain why the Universe evolves in such a peculiar way, we need to obtain a deeper understanding of the spacetime. It is expected that string theory plays a crucial role in understanding the spacetime at the microscopic level. In order to address a time-dependent issue, it is likely that we need a nonperturbative formulation of string theory such as the IIB matrix model [1,2].

In this model, Euclidean spacetime is expected to emerge out of the distributions of the eigenvalues of the 10 matrices. We can certainly imagine that the eigenvalues are homogeneously distributed on S^4 in 10 dimensions. Since a de Sitter space becomes a S^4 after the Euclidean continuation, we may interpret Euclidean spacetime a la Hartle and Hawking [3]. If we divide a S^4 into the two halves, we obtain a S^3 at the boundary. With the identification of the S^3 as a space, the effective action for S^4 in a IIB matrix model determines the relative probability of the emergence of a S^3 out of nothing. We find it remarkable that the matrix models can accommodate a realistic spacetime in a nonperturbative way. In this sense our studies of homogeneous spacetime in a IIB matrix model may shed light on the origin of the Universe.

A fuzzy homogeneous spacetime G/H can be embedded in matrix models by choosing background matrices as the generators of a group G [4]. G has to be a subgroup of $SO(10)$ and H has to be a closed subgroup of G. We obtain noncommutative (NC) gauge theory on the fuzzy spacetime in this construction [5] and can calculate an effective action on this background and investigate the large N scaling behavior of it.

In this paper, we choose G to be $SU(3)$ and investigate the class of the manifolds with $SU(3)$ isometry in the IIB matrix model. They include $CP^2 = SU(3)/U(2)$ and $SU(3)/U(1) \times U(1)$. Each manifold is labeled by an irreducible representation of $SU(3)$. Note that CP^2 is a four-dimensional manifold, while $SU(3)/U(1) \times U(1)$ is six dimensional. Therefore, we can investigate the large N scaling behavior of the effective action for the both four- and six-dimensional manifolds.

In a series of papers [6], we investigated the manifolds with $SU(2) \times SU(2)$ isometry and found certain instabilities associated with fuzzy $S^2 \times S^2$. Each fuzzy S^2 can be parameterized by l, the spin of a representation, and f, a scale factor. We recall that the radius of S^2 is lf while the NC length scale is $\sqrt{lf}$. Thus both the spin and scale factor specify the overall size of each S^2. In this construction $S^2 \times S^2$ can be characterized by the ratios of the spins and scale factors between the two S^2's. The instability has been found under the variation of both ratios. However, it does not take place if we are constrained to have the identical scale factor for both S^2's. We thus expect that a more symmetric manifold will be stable.

In this respect CP^2 backgrounds are interesting. CP^2 can be embedded in Hermitian matrices as

$$A_i = fT_i, \tag{1.1}$$

where T_i are the generators of $SU(3)$ in a particular class of representations. As CP^2 can have only one scale factor, it may not suffer from such an instability. The irreducible representations of $SU(3)$ from which CP^2 can be constructed as $SU(3)/U(2)$ and are relatively well studied [7–9]. Therefore, it is interesting to investigate the large N scaling behavior of the effective action of CP^2 and other manifolds with $SU(3)$ isometry and to see which manifold is most stable among them. We emphasize that investigat-

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ing the large N limit is physically required. Since the IIB matrix model possesses $SO(10)$ symmetry, the symmetry breaking down to $SU(3)$ can take place only in the large N limit. Because superselection rules can arise in such a limit, we need not average over degenerate vacua with respect to their different orientations in R^{10}.

The organization of this paper is as follows: In Sec. II, we construct a IIB matrix model on fuzzy CP^2. We find a universal expression for the two-loop effective action on a homogeneous space. In Sec. III, we derive the effective actions on the manifolds with $SU(3)$ isometry and investigate the large N scaling behavior of them. We find that they scale in a universal fashion which depends only on the dimensionality of the manifold. We argue that there is indeed a universality in the large N scaling of the effective action on G/H. We conclude in Sec. IV with discussions. In Appendix A, we construct the generators and eigenmatrices of $SU(3)$. In Appendix B, we derive the two-loop effective action on the manifolds constructed from $SU(3)$ algebra. In Appendix C, we numerically evaluate the two-loop effective action.

II. IIB MATRIX MODEL ON FUZZY CP^2

A. Group theoretic construction

Let us recall a construction of fuzzy homogeneous spacetime G/H and a gauge field on them [4]. We pick a state $|0\rangle$ in a definite representation G which is invariant under H. The set of all states which can be reached by multiplying elements of G to $|0\rangle$ is called the orbit of $|0\rangle$. A fuzzy homogeneous spacetime G/H is constructed as the orbit of $|0\rangle$. It is represented by the irreducible representation that is descended from $|0\rangle$. The basic degrees of freedom in NC gauge theory are bilocal fields. We construct the NC gauge field as the bilocal field by forming the tensor product of the relevant irreducible representation and its complex conjugate.

We take a Lie group G to be $SU(3)$ in the present investigation. An irreducible representation of $SU(3)$ is labeled by a set of two integers (p, q). An invariant subgroup H depends on the irreducible representation. We have $U(2)$ as the invariant subgroup for a $(p, 0)$ representation. It gives rise to a four-dimensional fuzzy $CP^2 = SU(3)/U(2)$. On the other hand H is $U(1) \times U(1)$ for a generic (p, q) representation. In this case we obtain a fuzzy flag manifold $SU(3)/U(1) \times U(1)$. It is a six-dimensional NC spacetime which locally looks like $CP^2 \times S^2$. The representation (p, p) may give the most symmetric six-dimensional manifold.

In the large N limit, the extension of the manifold becomes infinite with respect to the NC scale. In such a situation, we expect that the effective action scales in a definite way. As we find such a scaling exhibits a universality which depends only on the dimensionality of the manifold, a group of the representations represents a universal class. We are thus interested in identifying such a universal manifold in the large N limit.

We introduce a fuzzy homogeneous spacetime as a background of the IIB matrix model and calculate the effective action in a background field method. For this purpose, we expand the matrices around the background with a scale factor f:

$$A_\mu = f(p_\mu + a_\mu), \tag{2.1}$$

where p_μ is the background and a_μ represents the NC gauge field. The background is taken as

$$p_\mu = \begin{cases} \mathbf{1}_{n\times n} \otimes T_\mu^{(p,q)} & \mu = 1, \ldots, 8 \\ 0 & \mu = 9, 10, \end{cases} \tag{2.2}$$

where $T_\mu^{(p,q)}$ are the $SU(3)$ generators of a (p, q) representation. Here we have taken a simple reducible representation. We obtain $U(n)$ gauge theory on a fuzzy homogeneous spacetime in this way. This background can be realized by the matrices whose dimension is

$$N = n \cdot \dim(p, q) = n \cdot \tfrac{1}{2}(p + 1)(q + 1)(p + q + 2). \tag{2.3}$$

One could consider a more general background such as

$$\sum_i \oplus (\mathbf{1}_{n_i\times n_i} \otimes T_\mu^{(p_i,q_i)}). \tag{2.4}$$

However, we consider a simple case (2.2) only in the present paper.

The gauge field is expanded by harmonic functions on the (p, q) background

$$a_\mu = \sum_A a_\mu^{(A)} Y^{(A)}, \tag{2.5}$$

where the harmonic function matrices $Y^{(A)}$ are the eigenfunctions of $[T_3,]$, $[T_8,]$, and $[T_\mu, [T_\mu,]]$. The quantum numbers (A) are determined by decomposing the gauge field into the irreducible representations. An explicit construction procedure of them is explained in Appendix A. We obtain the propagators and vertices by using the expansion (2.5). By a perturbative calculation, we obtain the effective action $\Gamma = \Gamma(p, q, \lambda^2, n)$. Here λ^2 is a natural expansion parameter which is proportional to $1/f^4$. It is a 't Hooft coupling constant which should be kept fixed in the large N limit. We can determine the parameters $\{p, q, \lambda, n\}$ by requiring that the effective action is stationary with respect to the change of them $\delta\Gamma = 0$. Such a set constitutes a solution of the IIB matrix model. Dynamical generation of fuzzy homogeneous spacetime can be investigated in this way. We can compare the extremal values of the effective action for these (stable) solutions to find the most favored one.

In this paper we carry out the loop expansion up to the two-loop level. The tree-level action does not admit a nontrivial solution. Such a solution appears when the two-loop quantum correction is included in the effective action. The situation is the same with the backgrounds based on $SU(2)$ algebras [6] and, as we discuss later, a common aspect for backgrounds based on Lie algebras $G \subset SO(10)$.

In what follows, we explain the details of our evaluation of the effective action.

1. Universal properties of the two-loop effective action

We can draw some common features of the effective action in homogeneous spacetime from a series of our studies. Here we assume the expansion (2.1) and p denotes a set of generators of a Lie algebra $G \subset SO(10)$ of the form (2.2). We also assume that one can find a set of harmonic functions which are eigenfunctions of the adjoint operators $P = [p,]$. In the large N limit, the leading terms of the effective action of the IIB matrix model up to the two-loop level can be summarized as the following universal expression[1]:

$$\Gamma = \frac{f^4}{4} C_G C_2(G,R) N + n^2 O\left(\mathrm{tr}\frac{1}{P^4}\right) + 2n^3 \frac{C_G}{f^4}\left\langle \frac{1}{P_1^2 P_2^2 P_3^2}\right\rangle, \tag{2.6}$$

where R denotes an irreducible representation of a Lie algebra and

$$C_G \delta_{\rho\sigma} = f_{\mu\nu\rho} f_{\mu\nu\sigma}, \qquad C_2(G,R)N = \mathrm{tr} p_\mu p^\mu. \tag{2.7}$$

$f_{\mu\nu\rho}$ is the structure constant of the Lie algebra. The first, second, and third terms in (2.6) are the tree, one-loop, and two-loop contributions, respectively.

The two-loop contributions consist of the planer and nonplanar contributions. In NC theory, the nonplanar contributions are suppressed due to the NC phase. We argue that the upper cutoff becomes $\sqrt{l}$ instead of l in the nonplanar sector since the NC scale is $\sqrt{l}$. As the two-loop contributions are quadratically divergent in the large N limit for a four-dimensional background, we argue that the nonplanar contributions are suppressed by $\sqrt{N}$ in that case. The analogous suppressions should take place in higher dimensions. The two-loop nonplanar contributions will be suppressed by N in comparison to the planar contributions for six-dimensional backgrounds. We thus argue that the two-loop contributions are always positive since the nonplanar contributions can be neglected in the large N limit.

The two-loop level effective action can be bounded as

$$\Gamma \geq (1 - \mathrm{loop}) + 2C_G \sqrt{\frac{C_2(G,R)Nn^3}{2}\left\langle \frac{1}{P_1^2 P_2^2 P_3^2}\right\rangle}. \tag{2.8}$$

[1] $G = SU(2)$ is the exception since the two-loop amplitude is finite in the large N limit. We must use the exact propagators for gauge bosons and fermions to evaluate the two-loop contributions in such a case.

after we minimize it with respect to f. Without the two-loop contributions, we can obtain only trivial solutions as $f = 0$ is required to minimize the action. Therefore higher loop, at least two-loop, corrections are necessary to obtain a fuzzy homogeneous spacetime in a IIB matrix model.

III. THE EFFECTIVE ACTION ON FUZZY SPACETIME WITH $SU(3)$ ISOMETRY

In this section, we evaluate the effective action on the fuzzy manifolds with $SU(3)$ isometry. We set $n = 1$ for simplicity since we can recover easily the n dependence as (2.6).

The tree level effective action of a (p, q) representation is

$$\begin{aligned}\Gamma_{\mathrm{tree}} &= -\frac{1}{4}\mathrm{tr}[p_\mu, p_\nu]^2 \\ &= \frac{3f^4}{4} N \frac{1}{3}[p(p+3) + q(q+3) + pq].\end{aligned} \tag{3.1}$$

When the background is CP^2 [$(p, 0)$ representation], the leading term of (3.1) in the large N limit becomes

$$\Gamma_{\mathrm{tree}} \simeq \frac{f^4}{2} N^2, \qquad N \simeq \frac{p^2}{2}. \tag{3.2}$$

On a 6d manifold [(p, p) representation], it becomes

$$\Gamma_{\mathrm{tree}} \simeq \frac{3f^4}{4} N^{5/3}, \qquad N \simeq p^3. \tag{3.3}$$

The leading term of the one-loop effective action in the large N limit can be estimated as

$$\Gamma_{1-\mathrm{loop}} \propto \mathrm{tr}\left(\frac{1}{P^2}\right)^2 \sim \begin{cases} O(\log N) & CP^2 \\ O(N^{1/3}) & 6\mathrm{d}. \end{cases}$$

We can neglect this term in the effective action because we shortly find that the effective action scales as $O(N)$ on CP^2 or $O(N^{4/3})$ on a six-dimensional manifold.

The leading term of the two-loop effective action in the large N limit is evaluated as

$$\Gamma_{2-\mathrm{loop}} = \frac{6}{f^4} F_3 \equiv \frac{6}{f^4}\left\langle \frac{1}{P_1^2 P_2^2 P_3^2}\right\rangle, \tag{3.4}$$

where the detailed calculations are explained in Appendix B. In this way, we obtain the effective action in the large N limit as

$$\begin{aligned}\Gamma &= \Gamma_{\mathrm{tree}} + \Gamma_{2-\mathrm{loop}} \\ &= \frac{f^4 N}{4}[p(p+3) + q(q+3) + pq] + \frac{6}{f^4} F_3.\end{aligned} \tag{3.5}$$

We now can explore the behavior of the effective action. First, we investigate F_3 of (3.4) to determine the scaling behavior for various representations. We have numerically estimated F_3 in Appendix C. Figure 1 shows F_3 against N. We first observe that F_3 of the $(p, 0)$ representations ap-

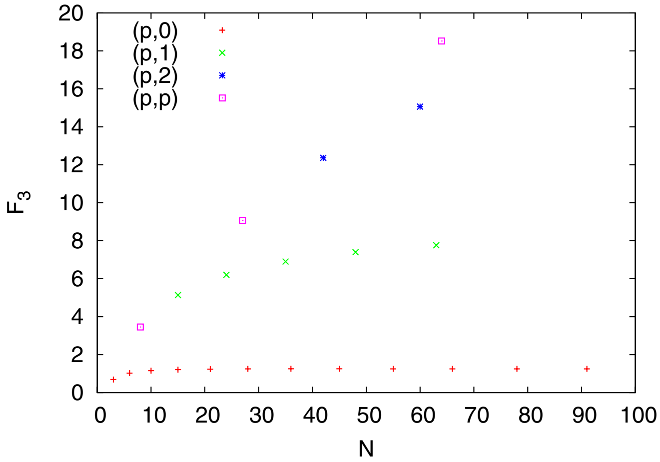


FIG. 1 (color online). F_3 against N.

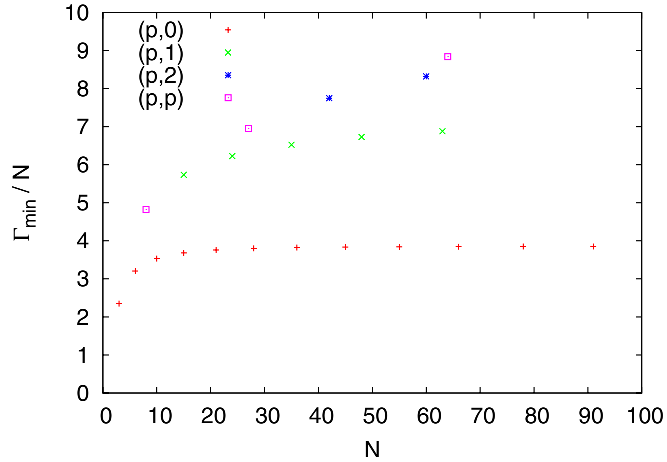


FIG. 2 (color online). $\Gamma_{\rm min}/N$ against N.

proaches a constant in the large N limit. This value is estimated as

$$F_3 \sim 1.197 + \frac{1.03}{p} - \frac{5.4}{p^2} + \frac{6.8}{p^3} - \frac{2.9}{p^4}. \qquad (3.6)$$

Second, we observe that F_3 of the (p, p) representations behaves as $O(N)$. Third, we find that F_3 of the (p, q) representations where $0 < q < p$ behaves like that of $U(q + 1)$ gauge theory in the large N limit when q is fixed. This is because it approaches a constant which is consistent with the two-loop effective action of $U(q + 1)$ gauge theory on CP^2:

$$(q + 1)^3 F_3. \qquad (3.7)$$

By assuming that we have identified correctly the large N scaling behavior of F_3 for various representations, we can obtain the large N limit of the effective actions after identifying the suitable 't Hooft couplings for CP^2 and 6d manifolds. In the CP^2 case, the action in the large N limit is

$$\Gamma = N\left[\frac{1}{2\lambda^2} + 6\lambda^2 F_3\right], \qquad \lambda^2 = \frac{1}{f^4 N}. \qquad (3.8)$$

In a 6d manifold of the (p, p) representations, it is

$$\Gamma = N^{4/3}\left[\frac{3}{4\lambda^2} + 6\lambda^2 \frac{F_3}{N}\right], \qquad \lambda^2 = \frac{1}{f^4 N^{1/3}}. \qquad (3.9)$$

Because of the different large N scaling behaviors of the effective actions, we find that the CP^2 background is preferable to the 6d manifold.

After identifying the 't Hooft coupling, we can minimize the effective action with respect to it. We can use (2.8) to determine the minimum of the effective action:

$$\Gamma \geq \Gamma_{\rm min} \equiv 2\sqrt{\Gamma_{\rm tree}\Gamma_{2-{\rm loop}}}. \qquad (3.10)$$

Figure 2 shows $\Gamma_{\rm min}/N$ against N. We can observe that the effective action on the fuzzy CP^2 in the large N limit is the smallest in this class with $SU(3)$ symmetry as it approaches a constant. This value can be estimated by using (3.6) as

$$\frac{\Gamma}{N} \simeq 3.79. \qquad (3.11)$$

The 't Hooft coupling at this minimum is

$$\lambda^2 \simeq 0.26. \qquad (3.12)$$

We remark here that (3.11) is comparable to the minimum of the effective action of the fuzzy $S^2 \times S^2$ background at the most symmetric point [6]:

$$\frac{\Gamma_{S^2\times S^2}}{N} \simeq 3.61. \qquad (3.13)$$

Although we believe that the estimate (3.13) is accurate, our estimate (3.11) suffers considerable uncertainty since it is derived from our numerical investigation up to $N \sim 100$. As we observe in Table I that F_3 is gradually decreasing, we cannot determine the lower bound of the effective action of CP^2 yet. Within these limitations, we can still conclude that the fuzzy CP^2 background is stable in its class and its effective action is comparable to that of fuzzy $S^2 \times S^2$.

Here we summarize our findings for the backgrounds with $SU(3)$ symmetry. The effective action becomes $O(N)$ for the $(p, 0)$ representations in the large N limit. On the other hand the (p, p) representations give the effective action $O(N^{3/4})$. We recall here that the $(p, 0)$ representations give a four-dimensional NC spacetime while the (p, p) representations give a six-dimensional one in the large N limit. Since both effective actions are positive, the $(p, 0)$ representations are favored over the (p, p) representations in the large N limit. We also have an observation for the (p, q) representations with $q \ll p$. In this case the (p, q) representations behave like a direct product of the $(p, 0)$ representations and the $(q + 1) \times (q + 1)$ identity matrix. In such a case, we effectively obtain $U(q + 1)$ gauge theory on CP^2 and the effective action is propor-

TABLE I. The results of F_3 using Monte Calro simulation.

$SU(3)$ representation	N	F_3
(1, 0)	3	0.691 52 + / − 0.000 56
(2, 0)	6	1.027 63 + / − 0.000 64
(3, 0)	10	1.156 20 + / − 0.000 69
(4, 0)	15	1.211 68 + / − 0.000 72
(5, 0)	21	1.236 53 + / − 0.000 72
(6, 0)	28	1.248 58 + / − 0.000 71
(7, 0)	36	1.253 57 + / − 0.000 73
(8, 0)	45	1.254 74 + / − 0.000 86
(9, 0)	55	1.252 22 + / − 0.000 91
(10, 0)	66	1.252 01 + / − 0.000 88
(11, 0)	78	1.251 88 + / − 0.000 91
(12, 0)	91	1.249 59 + / − 0.000 91
(1, 1)	8	3.4551 + / − 0.0020
(2, 1)	15	5.1412 + / − 0.0031
(3, 1)	24	6.2030 + / − 0.0043
(4, 1)	35	6.9072 + / − 0.0048
(5, 1)	48	7.3973 + / − 0.0051
(6, 1)	63	7.7632 + / − 0.0054
(2, 2)	27	9.0688 + / − 0.0051
(3, 2)	42	12.366 + / − 0.0086
(4, 2)	60	15.064 + / − 0.011
(3, 3)	64	18.522 + / − 0.013

tional to $(q + 1)^3 N$. We thus argue that the effective action is minimized for $q = 0$. Therefore, the $(p, 0)$ representations are a solution of the IIB matrix model as long as $SU(3)$ symmetry is not broken. We conclude that a four-dimensional fuzzy CP^2 is singled out by a IIB matrix model within the manifolds with $SU(3)$ symmetry.

One of our goals of this paper is to investigate the scaling behavior of the effective action of this class of spacetime in the large N limit. Let us recall the situation for the manifolds constructed from $SU(2)$ algebras [6]. The four-dimensional fuzzy $S^2 \times S^2$ makes the effective action to be $O(N)$, and a six-dimensional spacetime $S^2 \times S^2 \times S^2$ gives $O(N^{4/3})$ action. These scaling behaviors can be derived from the power counting of the higher-loop contributions. We also assumed that the leading quantum corrections cancel due to supersymmetry. Such an assumption can be justified since the quantum corrections do cancel for commuting backgrounds and the commutators of the backgrounds reduce the degrees of divergences. In our identification of the 't Hooft couplings, we used the fact that the three point vertices scale as $1/\sqrt{N}$ in the large N limit.

We argue that the same scaling rule holds in general. In fact our reasoning to identify the scaling behavior of the effective action does not depend on the details of a particular Lie algebra. In particular, the large N scaling rule of the three point vertices are the consequence of our normalization of the two point vertices to be $O(1)$. Therefore, it must hold in generic Lie algebra. In fact, we have numerically found, at the two-loop level, that a four-dimensional fuzzy CP^2, namely, the $(p, 0)$ representation, gives $O(N)$ effective action, and a six-dimensional fuzzy flag manifold, namely, the (p, p) representation, gives $O(N^{4/3})$ behavior. These findings support our argument that any four-dimensional fuzzy homogeneous spacetime gives $O(N)$ effective action and six-dimensional one gives $O(N^{4/3})$ action.

We investigated whether the IIB matrix model had a fuzzy $S^2 \times S^2$ solution at the two-loop level previously. The most symmetric $S^2 \times S^2$ solution turns out to be unstable along some directions of their moduli parameters. They describe the relative sizes of the two spheres. The instability drives the symmetric $S^2 \times S^2$ to the asymmetric one. Fortunately we find fuzzy CP^2 has no such instability. The extremal value of the effective action is comparable to that of the symmetric $S^2 \times S^2$. We thus obtain a new evidence for the existence of a symmetric stable four-dimensional spacetime in a IIB matrix model.

IV. CONCLUSIONS

In this paper we have investigated the effective action of a IIB matrix model on fuzzy CP^2 and the related manifold with $SU(3)$ isometry at the two-loop level. Since the backgrounds constructed by using $SU(3)$ algebra contain the manifolds with different dimensionality such as CP^2 (4d) and a 6d manifold, we can compare the minimum of the effective action of the 4- and six-dimensional backgrounds like [6] in our investigation of the stability of CP^2.

We have investigated the large N scaling behavior of the effective action. The action scales as N on CP^2 and $N^{4/3}$ on a 6d manifold, respectively. The effective action of the (p, q) representations where $p > q$ with fixed q also scales as N, since it behaves like $U(q + 1)$ gauge theory of CP^2. From these results, we have found that CP^2 minimizes the effective action among the backgrounds which are constructed by $SU(3)$ algebra. We conclude that the fuzzy CP^2 background is a solution in a IIB matrix model and stable as long as $SU(3)$ symmetry is not broken.

These scaling behaviors are in accord with other 4d manifolds like $S^2 \times S^2$ and $T^2 \times T^2$ and also a 6d manifold $S^2 \times S^2 \times S^2$ [6,10]. These facts support our contention that the effective action of a compact manifold embedded in a IIB matrix model has the universal scaling behavior: it scales as N and $N^{4/3}$ on a 4d and 6d manifold, respectively.

We also have compared the minimum of the effective actions of CP^2 with that of $S^2 \times S^2$. We have observed that the effective action of CP^2 is comparable to that of $S^2 \times S^2$. Although we have observed in Table I that the two-loop effective action on CP^2 is gradually decreasing, we cannot determine the lower bound of it yet. Therefore, we cannot say which is smaller even at the two-loop level. To answer this question, it is desirable to obtain an asymptotic expression of the two-loop effective action on CP^2 like such an expression on S^2 which is obtained from the Wigner's $6j$

symbols. Such an effort may be useful to determine whether higher symmetry of the background may lower the effective action or not.

ACKNOWLEDGMENTS

This work is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan. We thank T. Azuma, J. Nishimura, and N. Kawahara for discussions.

APPENDIX A

1. Construction of background

A fundamental representation of $SU(3)$ is three dimensional. The Lie group generators can be written by Gell-Mann matrices λ_μ as $t_\mu = \lambda_\mu/2$. We take Gell-Mann matrices as the following form:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A1}$$

We denote state vectors on which these generators act as $|a\rangle, |b\rangle, \ldots$; here indices $a, b, \ldots$ run from 1 to 3. These vectors have the following components:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A2}$$

The Cartan matrices are t_3 and t_8. They act on $|a\rangle$ in the following way:

$$t_3|1\rangle = \frac{1}{2}|1\rangle, \qquad t_3|2\rangle = \frac{-1}{2}|2\rangle, \qquad t_3|3\rangle = 0\cdot|3\rangle,$$
$$t_8|1\rangle = \frac{1}{2\sqrt{3}}|1\rangle, \qquad t_8|2\rangle = \frac{1}{2\sqrt{3}}|2\rangle, \qquad t_8|3\rangle = \frac{-1}{\sqrt{3}}|3\rangle. \tag{A3}$$

The raising/lowering operators are

$$j_1^\pm = t_4 \pm it_5, \qquad j_2^\pm = t_6 \pm it_7, \tag{A4}$$

and they act on the state vectors as

$$j_1^\pm : |3\rangle \leftrightarrow |1\rangle, \qquad j_2^\pm : |2\rangle \leftrightarrow |3\rangle, \quad \text{otherwise gives zero.} \tag{A5}$$

A general $SU(3)$ representation is labeled by a set of two integers (p, q) and have the dimension $\dim(p, q) = (p+1)(q+1)(p+q+2)/2$. The fundamental representation is denoted as $(1, 0)$. The (p, q) representation can be constructed from $(1, 0)$ by forming tensor products.

As the first example, we construct the $(2, 0)$ representation. The $(2, 0)$ state vectors are constructed from the tensor products of the two sets of the $(1, 0)$ vectors:

$$|v^{(2,0)}\rangle = |a\rangle|b\rangle + |b\rangle|a\rangle. \tag{A6}$$

We should take an appropriate normalization factor in the above expression. The symmetric property of this tensor product is represented by a Young tableau □□. A single box □ denotes the $(1, 0)$ vector. The $(2, 0)$ generators which act on the state vectors are the tensor products of $(1, 0)$ generators t_μ and the 3×3 unit matrix $\mathbf{1_3}$:

$$T_\mu^{(2,0)} = t_\mu \otimes \mathbf{1_3} + \mathbf{1_3} \otimes t_\mu. \tag{A7}$$

To obtain the explicit matrix representation of the generators, we need to calculate the matrix elements

$$\langle v^{(2,0)}|T^{(2,0)}|v^{(2,0)}\rangle. \tag{A8}$$

In this way, we can write down the generators as 6×6 matrices. An extension to the $(p, 0)$ representation is obtained easily by tensoring p sets of the fundamental representations. The $(p, 0)$ state vectors up to the normalization factor are given by totally symmetrized tensor products of the $(1, 0)$ vectors

$$|v^{(p,0)}\rangle = \prod_{i=1}^{p} |a_i\rangle + \text{ permutations for } \{a_i\}. \tag{A9}$$

Its symmetric property is represented by the Young tableau: | 1 | 2 | ⋯ | p |.

The representations of the generators which act on these $(p, 0)$ state vectors are

$$T_\mu^{(p,0)} = \sum_{i=0}^{p-1} (\mathbf{1_3}\otimes)^i t_\mu (\otimes\mathbf{1_3})^{p-1-i}. \tag{A10}$$

To obtain an explicit matrix representation of the generators, we need to calculate the matrix elements

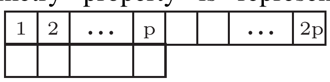
$$\langle v^{(p,0)}|T^{(p,0)}|v^{(p,0)}\rangle. \tag{A11}$$

In this way, we can write down the generators as $(p+1)\times(p+2)/2\times(p+1)(p+2)/2$ matrices.

Next we consider an extension of our construction to the (p, p) representations. It is obtained by $(2p + p)$-fold tensor products. The state vectors of the (p, p) representation up to the normalization factors can be written as

$$|v^{(p,p)}\rangle = \prod_{i=1}^{p}(|a_i\rangle|b_i\rangle - |b_i\rangle|a_i\rangle)\prod_{j=1}^{p}|a_j\rangle + \text{ permutations of } \{a_i, a_j\}. \tag{A12}$$

Here the permutations between a_i and a_j also should be included. Indices a_i and b_i are antisymmetrized. Its symmetry property is represented by a Young tableau:



. The representations of the generators which act on these (p, p) state vectors, up to normalization factor, are

$$T_\mu^{(p,p)} = \sum_{i=0}^{3p-1}(\mathbf{1_3}\otimes)^i t_\mu(\otimes\mathbf{1_3})^{3p-1-i}. \tag{A13}$$

To obtain explicit form of the generators, we need to calculate the matrix elements

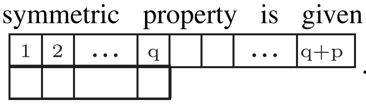
$$\langle v^{(p,p)}|T^{(p,p)}|v^{(p,p)}\rangle. \tag{A14}$$

In this way, we can write down the generators as $(p+1)^3 \times (p+1)^3$ matrices.

An extension to an arbitrary (p, q) representation is easily obtained by forming the $(p + 2q)$-fold tensor products. The state vectors of (p, q) type can be written as

$$|v^{(p,q)}\rangle = \mathcal{C}_{(p,q)}\prod_{i=1}^{q}(|a_i\rangle|b_i\rangle - |b_i\rangle|a_i\rangle)\prod_{j=1}^{p}|a_j\rangle + \text{ permutations of } \{a_i, a_j\}. \tag{A15}$$

Here the permutations between a_i and a_j should be included also. Indices a_i and b_i are antisymmetrized. The symmetric property is given by a Young tableau:



. Here $\mathcal{C}_{(p,q)}$ is a normalization constant. The representations of the generators which act on these (p, q) state vectors are

$$T_\mu^{(p,q)} = \sum_{i=0}^{2p+q-1}(\mathbf{1_3}\otimes)^i t_\mu(\otimes\mathbf{1_3})^{2p+q-1-i}. \tag{A16}$$

To obtain an explicit matrix form of the generators, we need to calculate the matrix elements

$$\langle v^{(p,q)}|T^{(p,q)}|v^{(p,q)}\rangle. \tag{A17}$$

In this way, we can write down the generators as $N^{(p,q)} \times N^{(p,q)}$ matrices where

$$N^{(p,q)} = \frac{(p+1)(q+1)(p+q+2)}{2}. \tag{A18}$$

2. Construction of matrix harmonics in $SU(3)$ background

Suppose that we take a matrix model background to be a (p, q) representation. The gauge (and adjoint fermion) fields are expanded by harmonic matrices as follows:

$$\phi = \sum_{(A)}\sum_{ms}\phi_{ms}^{(A)}Y_{ms}^{(A)}, \tag{A19}$$

where $Y_{ms}^{(A)}$ are the matrix harmonics. The index (A) denotes the sets of two integers (p_A, q_A) which label the irreducible representations. They are $N^{(p,q)} \times N^{(p,q)}$ matrices which satisfy

$$\begin{aligned} P_3 Y_{ms}^{(A)} &\equiv [p_3, Y_{ms}^{(A)}] = mY_{ms}^{(A)}, \\ P_8 Y_{ms}^{(A)} &\equiv [p_8, Y_{ms}^{(A)}] = sY_{ms}^{(A)}, \\ P^2 Y_{ms}^{(A)} &\equiv [p_\mu[p_\mu, Y_{ms}^{(A)}]] \\ &= \left(\frac{1}{2}p_A^2 + p_A + \frac{1}{2}q_A^2 + q_A\right)Y_{ms}^{(A)}. \end{aligned} \tag{A20}$$

The gauge fields are constructed as bilocal fields. When the background is a (p, q) representation, the bilocal state has a tensor structure $(p, q) \otimes (q, p)$. They can be decomposed into the irreducible representations, and the decomposition may have the following form:

$$(p, q) \otimes (q, p) = \sum_{n=0}^{p+q} D_n(n, n) + \sum_{l\neq m}^{p+2q} E_{ml}((l, m) + (m, l)), \tag{A21}$$

where D_n and E_{lm} are multiplicity factors. If we take $q = 0$, the decomposition becomes a simple form as

$$(p, 0) \otimes (0, p) = \sum_{n=0}^{p}(n, n). \tag{A22}$$

Here we give the $p = q = 1$ case for another simple example

$$(1, 1) \otimes (1, 1) = (2, 2) + 2(1, 1) + (0, 0) + (3, 0) + (0, 3). \tag{A23}$$

Thus, in expansion (A19), the sets of the integers (p_A, q_A) run over the irreducible representations which appear in the decomposition, and m and s take the value of these irreducible representations (p_A, q_A).

Now we explain how to construct such matrices in a given background. Let us describe a background [i.e. $SU(3)$ generator of a (p, q) representation] in terms of a $SU(N^{(p,q)})$ basis

$$T_\mu^{(p,q)} = \sum_\alpha (\mathcal{A}_\alpha E_\alpha + \mathcal{B}_{-\alpha} E_{-\alpha}) + \sum_a \mathcal{C}_a H_a, \qquad \text{(A24)}$$

where $\{E_\alpha, E_{-\alpha}, H_\alpha\}$ are Cartan's basis which satisfy the following relations:

$$\begin{aligned} &[H_a, H_b] = 0, \\ &[H_a, E_{\pm\alpha}] = \pm\alpha_a E_{\pm\alpha}, \\ &[E_\alpha, E_{-\alpha}] = \alpha^a H^a, \\ &[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \qquad (E_\alpha^\dagger = E_{-\alpha}). \end{aligned} \qquad \text{(A25)}$$

One can take a representation of $T_3^{(p,q)}$ and $T_8^{(p,q)}$ as diagonal matrices

$$T_3^{(p,q)} = \sum_a \mathcal{C}_a H_a, \qquad T_8^{(p,q)} = \sum_b \mathcal{C}'_b H_b. \qquad \text{(A26)}$$

Each E_α can be assigned to an off-diagonal matrix which has only one nonzero component:

$$(E_\alpha)_{ij} = \begin{cases} 1 & \text{for } (i, j) = (i_\alpha, j_\alpha), \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(A27)}$$

Then we have

$$[T_3^{(p,q)}, E_\alpha] = \sum_a \mathcal{C}_a \alpha^a E_\alpha, \qquad [T_8^{(p,q)}, E_\alpha] = \sum_b \mathcal{C}'_b \alpha^b E_\alpha. \qquad \text{(A28)}$$

It implies that $Y_{ms}^{(A)}$ with $(m, s) = (\sum_a \mathcal{C}_a, \sum_b \mathcal{C}'_b)$ can be written as linear combinations of E_αs which have the same eigenvalues of (m, s). On the other hand, Cartan subalgebra $[H, H] = 0$ implies that $Y_{m=s=0}^{(A)}$ can be obtained by linear combinations of H.

Following the above observation, we first take all commutators $[T^3, E]$ and $[T^8, E]$ to find quantum numbers m and s of each E. Next we determine suitable linear combinations in the matrix basis which possess the same m and s. Then we obtain matrix harmonics which correspond to the irreducible representations in the decomposition (A21).

One way to determine such linear combinations is to use the raising/lowering operators. The decomposition (A21) contains the irreducible representation $(p_A, q_A) = (p + 2q, p - q)$. The value $p + 2q$ is the maximum value of p_A in this decomposition. The highest weight state is unique in each irreducible representation, and $p + 2q$ is the largest number in the decomposition. Then there should be only one matrix base corresponding to such a state whose eigenvalues are $m = \frac{1}{2}(p + 2q + p - q) = 2p + q/2$ and $s = \frac{1}{2\sqrt{3}}(p + 2q + p - q - 2(p - q)) = 3q/2\sqrt{3}$. Therefore, a matrix base with the eigenvalues $m_0 \equiv 2p + q/2$ and $s_0 \equiv 3q/2\sqrt{3}$ is uniquely identified with the highest weight state of $(p + 2q, p - q)$. Next we carry out the operations of the lowering operators and generate sets of independent combinations of the matrix basis with $m'(<m_0)$ and $s'(\neq s_0)$. After suitable orthogonalizations, they form the state vectors with quantum number m' and s'. Some of these belong to the $(p + 2q, p - q)$ representation and form $Y_{m's'}^{(p+2q,p-q)}$. Others belong to different irreducible representations and form $Y_{m's'}^{(A')}$. In this way, we can identify all $(A') \neq (p + 2q, p - q)$ which appear in the decomposition (A21).

There is another way to obtain suitable combinations of the matrix basis more straightforwardly. First we collect matrix basis with the same quantum numbers m and s and denote this set of basis as $\{w_i\}$. Next we diagonalize the Casimir operator P^2 whose matrix elements are

$$P_{ij}^2 = \mathrm{tr}(w_i^\dagger P^2 w_j). \qquad \text{(A29)}$$

A different eigenvalue of P^2 corresponds to a different (A) of $Y_{ms}^{(A)}$, and $Y_{ms}^{(A)}$ themselves are obtained as the eigenvectors. This method is useful if one has automatic computation tools for linear algebra, like MATHEMATICA or MAPLE.

3. An explicit example

We give an explicit construction of a background (generators) and the matrix harmonics in a simple case. We consider the (2, 0) representation.

An expression of the state vectors of the (2, 0) representation is the following

$$\begin{aligned} &|1\rangle^{(2,0)} = |a\rangle|a\rangle, \qquad |2\rangle^{(2,0)} = \frac{|a\rangle|b\rangle + |b\rangle|a\rangle}{\sqrt{2}}, \\ &|3\rangle^{(2,0)} = |b\rangle|b\rangle, \qquad |4\rangle^{(2,0)} = \frac{|a\rangle|c\rangle + |c\rangle|a\rangle}{\sqrt{2}}, \\ &|5\rangle^{(2,0)} = \frac{|b\rangle|c\rangle + |c\rangle|b\rangle}{\sqrt{2}}, \qquad |6\rangle^{(2,0)} = |c\rangle|c\rangle, \end{aligned} \qquad \text{(A30)}$$

where $|a\rangle$, $|b\rangle$, and $|c\rangle$ are the state vectors of the fundamental representation.

The $SU(3)$ generators of the (2, 0) representation are

$$T_3^{(2,0)} = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & -1 & & & \\ & & & \frac{1}{2} & & \\ & & & & -\frac{1}{2} & \\ & & & & & 0 \end{pmatrix}, \qquad T_8^{(2,0)} = \begin{pmatrix} \frac{1}{\sqrt{3}} & & & & & \\ & \frac{1}{\sqrt{3}} & & & & \\ & & \frac{1}{\sqrt{3}} & & & \\ & & & -\frac{1}{2\sqrt{3}} & & \\ & & & & -\frac{1}{2\sqrt{3}} & \\ & & & & & -\frac{2}{\sqrt{3}} \end{pmatrix},$$

$$T_1^{(2,0)} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad T_2^{(2,0)} = \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{2} & 0 \\ 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad T_4^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix},$$

$$T_5^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \qquad T_6^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \qquad T_7^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 \\ 0 & \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & \frac{-i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}. \tag{A31}$$

To construct matrix harmonics, we define off-diagonal matrix basis as

$$\begin{pmatrix} 0 & E_{\alpha_1} & E_{\alpha_2} & E_{\alpha_3} & E_{\alpha_4} & E_{\alpha_5} \\ E_{-\alpha_1} & 0 & E_{\alpha_6} & E_{\alpha_7} & E_{\alpha_8} & E_{\alpha_9} \\ E_{-\alpha_2} & E_{-\alpha_6} & 0 & E_{\alpha_{10}} & E_{\alpha_{11}} & E_{\alpha_{12}} \\ E_{-\alpha_3} & E_{-\alpha_7} & E_{-\alpha_{10}} & 0 & E_{\alpha_{13}} & E_{\alpha_{14}} \\ E_{-\alpha_4} & E_{-\alpha_8} & E_{-\alpha_{11}} & E_{-\alpha_{13}} & 0 & E_{\alpha_{15}} \\ E_{-\alpha_5} & E_{-\alpha_9} & E_{-\alpha_{12}} & E_{-\alpha_{14}} & E_{-\alpha_{15}} & 0 \end{pmatrix}. \tag{A32}$$

This notation means that E_{α_1} is given by the form

$$E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A33}$$

and so on.

Following the decomposition

$$(2, 0) \otimes (0, 2) = (2, 2) + (1, 1) + (0, 0), \tag{A34}$$

we construct $Y^{(2,2)}$, $Y^{(1,1)}$, and $Y^{(0,0)}$ using the above matrix basis and diagonal matrices.

Here is a result. (2, 2) is 27 dimensional:

$$
\begin{aligned}
&Y^{(2,2)}_{1,\sqrt{3}} = E_{\alpha_5}, \qquad Y^{(2,2)}_{0,\sqrt{3}} = E_{\alpha_9}, \qquad Y^{(2,2)}_{-1,\sqrt{3}} = E_{\alpha_{12}}, \qquad Y^{(2,2)}_{3/2,\sqrt{3}/2} = -E_{\alpha_4}, \qquad Y^{(2,2)}_{1/2,\sqrt{3}/2} = \begin{cases} \frac{-E_{\alpha_3}+E_{\alpha_{14}}}{\sqrt{2}} \\ \frac{E_{\alpha_3}+E_{\alpha_{14}}-2\sqrt{2}E_{\alpha_8}}{\sqrt{10}}, \end{cases} \\
&Y^{(2,2)}_{-1/2,\sqrt{3}/2} = \begin{cases} \frac{-E_{\alpha_{11}}+E_{\alpha_{15}}}{\sqrt{2}} \\ \frac{-E_{\alpha_{11}}-E_{\alpha_{15}}+2\sqrt{2}E_{\alpha_7}}{\sqrt{10}}, \end{cases} \qquad Y^{(2,2)}_{-3/2,\sqrt{3}/2} = -E_{\alpha_{10}}, \qquad Y^{(2,2)}_{2,0} = E_{\alpha_2}, \qquad Y^{(2,2)}_{1,0} = \begin{cases} \frac{E_{\alpha_1}-\sqrt{2}E_{\alpha_{13}}}{\sqrt{3}} \\ \frac{-2E_{\alpha_1}-\sqrt{2}E_{\alpha_{13}}+3E_{\alpha_6}}{\sqrt{15}}, \end{cases} \\
&Y^{(2,2)}_{0,0} = \begin{cases} \mathrm{diag}(0,0,1,0,-2,1)/\sqrt{6} \\ \mathrm{diag}(0,2,-1,-2,0,1)/\sqrt{10} \\ \mathrm{diag}(3,-3,1,-3,1,1)/\sqrt{30}, \end{cases} \qquad Y^{(2,2)}_{-1,0} = \begin{cases} \frac{E_{-\alpha_1}-\sqrt{2}E_{-\alpha_{13}}}{\sqrt{3}} \\ \frac{-2E_{-\alpha_1}-\sqrt{2}E_{-\alpha_{13}}+3E_{-\alpha_6}}{\sqrt{15}}, \end{cases} \qquad Y^{(2,2)}_{-2,0} = E_{-\alpha_2}, \\
&Y^{(2,2)}_{3/2,-\sqrt{3}/2} = E_{-\alpha_{10}}, \qquad Y^{(2,2)}_{1/2,-\sqrt{3}/2} = \begin{cases} \frac{E_{-\alpha_{11}}-E_{-\alpha_{15}}}{\sqrt{2}} \\ \frac{-E_{-\alpha_{11}}-E_{-\alpha_{15}}+2\sqrt{2}E_{-\alpha_7}}{\sqrt{10}}, \end{cases} \qquad Y^{(2,2)}_{-1/2,-\sqrt{3}/2} = \begin{cases} \frac{E_{-\alpha_3}-E_{-\alpha_{14}}}{\sqrt{2}} \\ \frac{-E_{-\alpha_3}-E_{-\alpha_{14}}+2\sqrt{2}E_{-\alpha_8}}{\sqrt{10}}, \end{cases} \\
&Y^{(2,2)}_{-3/2,-\sqrt{3}/2} = E_{-\alpha_4}, \qquad Y^{(2,2)}_{1,-\sqrt{3}} = E_{-\alpha_{12}}, \qquad Y^{(2,2)}_{0,-\sqrt{3}} = E_{-\alpha_9}, \qquad Y^{(2,2)}_{-1,-\sqrt{3}} = E_{-\alpha_5}.
\end{aligned} \tag{A35}
$$

(1, 1) is eight dimensional:

$$
\begin{aligned}
&Y^{(1,1)}_{1/2,\sqrt{3}/2} = -\frac{\sqrt{2}E_{\alpha_3} + \sqrt{2}E_{\alpha_{14}} + E_{\alpha_8}}{\sqrt{5}}, \qquad Y^{(1,1)}_{-1/2,\sqrt{3}/2} = -\frac{E_{\alpha_7} + \sqrt{2}E_{\alpha_{11}} + \sqrt{2}E_{\alpha_{15}}}{\sqrt{5}}, \\
&Y^{(1,1)}_{1,0} = \frac{2E_{\alpha_1} + \sqrt{2}E_{\alpha_{13}} + 2E_{\alpha_6}}{\sqrt{10}}, \qquad Y^{(1,1)}_{(0,0)} = \begin{cases} \frac{\mathrm{diag}(0,1,2,-1,0,-2)}{\sqrt{10}} \\ \frac{\mathrm{diag}(4,1,-2,1,-2,-2)}{\sqrt{30}} \end{cases}, \qquad Y^{(1,1)}_{-1,0} = \frac{2E_{-\alpha_1} + \sqrt{2}E_{-\alpha_{13}} + 2E_{-\alpha_6}}{\sqrt{10}}, \\
&Y^{(1,1)}_{1/2,-\sqrt{3}/2} = \frac{E_{-\alpha_7} + \sqrt{2}E_{-\alpha_{11}} + \sqrt{2}E_{-\alpha_{15}}}{\sqrt{5}}, \qquad Y^{(1,1)}_{-1/2,-\sqrt{3}/2} = \frac{\sqrt{2}E_{-\alpha_3} + \sqrt{2}E_{-\alpha_{14}} + E_{-\alpha_8}}{\sqrt{5}}.
\end{aligned} \tag{A36}
$$

Finally there is the singlet which corresponds to (0, 0):

$$
Y^{(0,0)}_{(0,0)} = \frac{1}{\sqrt{6}}\mathbf{1}_6. \tag{A37}
$$

The two-loop contribution to the effective action is calculated with these harmonics. The planar contribution is

$$
6n^3 \sum_{(n_1,n_2,n_3)=1}^{2} \sum_{m_1,s_1} \sum_{m_2,s_2} \sum_{m_3,s_3} \frac{\mathrm{tr}(Y^{(n_1,n_1)}_{m_1,s_1} Y^{(n_2,n_2)}_{m_2,s_2} Y^{(n_3,n_3)}_{m_3,s_3})\, \mathrm{tr}(Y^{(n_3,n_3)\dagger}_{m_3,s_3} Y^{(n_2,n_2)\dagger}_{m_2,s_2} Y^{(n_1,n_1)\dagger}_{m_1,s_1})}{n_1(n_1+1)n_2(n_2+1)n_3(n_3+1)} \tag{A38}
$$

in $U(n)$ gauge theory. On the other hand, the nonplanar contribution is

$$
-6n \sum_{(n_1,n_2,n_3)=1}^{2} \sum_{m_1,s_1} \sum_{m_2,s_2} \sum_{m_3,s_3} \frac{\mathrm{tr}(Y^{(n_1,n_1)}_{m_1,s_1} Y^{(n_2,n_2)}_{m_2,s_2} Y^{(n_3,n_3)}_{m_3,s_3})\, \mathrm{tr}(Y^{(n_1,n_1)\dagger}_{m_1,s_1} Y^{(n_2,n_2)\dagger}_{m_2,s_2} Y^{(n_3,n_3)\dagger}_{m_3,s_3})}{n_1(n_1+1)n_2(n_2+1)n_3(n_3+1)}. \tag{A39}
$$

By substituting the explicit form of $Y^{(n,n)}_{ms}$, we obtain the planar contribution

$$
6n^3 \frac{42\,605}{41\,472}, \tag{A40}
$$

and the nonplanar contribution

$$
-6n \frac{1115}{41\,472}. \tag{A41}
$$

APPENDIX B

Here we evaluate the two-loop effective action of the IIB matrix model in a fuzzy background which is made from a (p, q) representation of the $SU(3)$ generators (3.4).

In this calculation, we make use of the following relation:

$$
\sum_m P_\mu Y^{(r,s)\dagger}_m P_\nu Y^{(r,s)}_m = -\sum_m Y^{(r,s)\dagger}_m P_\nu P_\mu Y^{(r,s)}_m, \tag{B1}
$$

where the superscript (r, s) denotes an irreducible representation of $SU(3)$ and m denotes the eigenvalues of the Cartan subalgebra in the (r, s) representation. We first note that the harmonic matrices of $SU(3)$ obey the orthogonal relations

$$\mathrm{tr}\,(Y_m^{(r,s)\dagger} Y_{m'}^{(r',s')}) = \delta_{(r,s),(r',s')}\delta_{m,m'}. \tag{B2}$$

Let us perform a unitary transformation on $Y_m^{(r,s)}$:

$$\begin{aligned} Y_m^{(r,s)} &\to UY_m^{(r,s)}U^\dagger = \sum_n u_{mn} Y_n^{(r,s)}, \\ Y_m^{(r,s)\dagger} &\to (UY_m^{(r,s)}U^\dagger)^\dagger = \sum_n u^*_{mn} Y_n^{(r,s)\dagger}, \end{aligned} \tag{B3}$$

where U is a $N \times N$ unitary matrix and u_{mn} is the unitary transformation represented in the m basis. Under (B3), (B2) is transformed as

$$\begin{aligned} \mathrm{tr}\,(Y_m^{(r,s)\dagger} Y_{m'}^{(r',s')}) &\to \sum_{n,n'} u^*_{mn} u_{m'n'}\, \mathrm{tr}(Y_n^{(r,s)\dagger} Y_{n'}^{(r',s')}) \\ &= \sum_{n,n'} u^*_{mn} u_{m'n'} \delta_{nn'} = (uu^\dagger)_{m'm}\delta_{(r,s),(r',s')}. \end{aligned} \tag{B4}$$

Since (B2) is apparently invariant under (B3), we can obtain

$$(uu^\dagger)_{m'm} = \delta_{m'm}. \tag{B5}$$

Using this relation, we can show that $\sum_m Y_m^{(r,s)\dagger} Y_m^{(r,s)}$ is invariant under (B3):

$$\begin{aligned} \sum_m Y_m^{(r,s)\dagger} Y_m^{(r,s)} &\to \sum_{m,n,n'} u^*_{mn} u_{mn'} Y_n^{(r,s)\dagger} Y_{n'}^{(r,s)} \\ &= \sum_n Y_n^{(r,s)\dagger} Y_n^{(r,s)}. \end{aligned} \tag{B6}$$

Since P_μ are the generators of $SU(3)$ transformation, (B6) is equivalent to

$$P_\mu\left(\sum_m Y_m^{(r,s)\dagger} Y_m^{(r,s)}\right) = 0. \tag{B7}$$

From this formula, we can obtain (B1).

We introduce the wave functions and averages as

$$\begin{aligned} &\Psi_{123} \equiv \mathrm{tr}(Y_{m_1}^{(r_1,s_1)} Y_{m_2}^{(r_2,s_2)} Y_{m_3}^{(r_3,s_3)}), \\ &\langle X\rangle_P \equiv \sum_{(r_i,s_i),m_i} \Psi^*_{123} X \Psi_{123}, \qquad P_i^\mu Y_{m_1}^{(r_1,s_1)} \equiv [p_\mu, Y_{m_1}^{(r_1,s_1)}], \end{aligned} \tag{B8}$$

where the sum of (r_i, s_i) runs over the representations which are made from the product of (p, q) and (q, p). We introduce the following quantity:

$$f_{\mu\nu\rho} f_{\mu\nu\sigma} = C_G \delta_{\rho\sigma}, \tag{B9}$$

where C_G is a constant which assumes $C_G = 2$ for $SU(2)$ and $C_G = 3$ for $SU(3)$. With these preparations, we can calculate the two-loop effective action almost the same way as the fuzzy sphere case.

We expand quantum fluctuations in terms of the harmonic matrices:

$$\begin{aligned} \text{gauge boson } a^\mu &= \sum_{(r,s),m} a_m^{(r,s)\mu} Y_m^{(r,s)}, \\ \text{fermion } \varphi &= \sum_{(r,s),m} \varphi_m^{(r,s)} Y_m^{(r,s)}, \\ \text{antighost } b &= \sum_{(r,s),m} b_m^{(r,s)} Y_m^{(r,s)}, \\ \text{ghost } c &= \sum_{(r,s),m} c_m^{(r,s)} Y_m^{(r,s)}. \end{aligned} \tag{B10}$$

Then the propagators are derived from the kinematic terms:

$$\begin{aligned} \langle a^\mu a^\nu\rangle &= \sum_{(r,s),m} (P^2\delta_{\mu\nu} + 2if_{\mu\nu\rho}P^\rho)^{-1} Y_m^{(r,s)} Y_m^{(r,s)\dagger}, \\ \langle \varphi\bar\varphi\rangle &= \sum_{(r,s),m} (-\Gamma_\mu P_\mu)^{-1} Y_m^{(r,s)} Y_m^{(r,s)\dagger}, \\ \langle cb\rangle &= \sum_{(r,s),m} \frac{1}{P^2} Y_m^{(r,s)} Y_m^{(r,s)\dagger}. \end{aligned} \tag{B11}$$

We exclude the singlet state (0, 0) in the propagator. To calculate the leading contributions in the large N limit, we expand the boson and the fermion propagators as

$$\begin{aligned} (P^2\delta_{\mu\nu} + 2if_{\mu\nu\rho}P^\rho)^{-1} &\simeq \frac{\delta_{\mu\nu}}{P^2} - 2i\frac{f_{\mu\nu\rho}P^\rho}{P^4} + 4\frac{I_{\mu\nu}(P)}{P^6}, \\ (-\Gamma_\mu P_\mu)^{-1} &\simeq \frac{\Gamma^\mu P_\mu}{P^2} + \frac{i}{2}\frac{f_{\mu\nu\sigma}\Gamma^{\mu\nu\rho}P_\sigma P_\rho}{P^4}. \end{aligned} \tag{B12}$$

We have introduced the following tensor:

$$I_{\mu\nu} \equiv f_{\tau\mu\rho} f_{\tau\nu\sigma} P_\rho P_\sigma. \tag{B13}$$

Using these propagators, we can calculate the contributions to the two-loop effective action from various interaction vertices as follows:

Four-gauge boson vertex is

$$V_4 = -\frac{1}{4}\mathrm{tr}[a_\mu, a_\nu]^2. \tag{B14}$$

The leading contribution to the two-loop effective action is

$$\langle -V_4\rangle = -45F_1 - 42C_G G_1 + 3C_G G_2. \tag{B15}$$

Here

$$\begin{aligned} F_1 = \left\langle \frac{1}{P_1^4 P_2^4}\right\rangle_P, \qquad G_1 = \left\langle \frac{1}{P_1^4 P_2^2}\right\rangle_P, \\ G_2 = \left\langle \frac{P_3^2}{P_1^4 P_2^4}\right\rangle_P. \end{aligned} \tag{B16}$$

Ghost vertex is

$$V_g = \mathrm{tr} b[p_\mu, [a_\mu, c]]. \tag{B17}$$

Their contribution is

$$\frac{1}{2}\langle V_g V_g \rangle = F_2 + 4H_2. \tag{B18}$$

Here

$$F_2 = \left\langle \frac{P_2 \cdot P_3}{P_1^2 P_2^2 P_3^2} \right\rangle_P, \qquad H_2 = \left\langle \frac{P_2 \cdot I(1) \cdot P_3}{P_1^6 P_2^2 P_3^2} \right\rangle_P, \tag{B19}$$

and

$$P_i \cdot I(j) \cdot P_k \equiv P_i^\mu I_{\mu\nu}(P_j) P_k^\nu. \tag{B20}$$

Three-gauge boson vertex is

$$V_3 = -\mathrm{tr} P_\mu a_\nu [a_\mu, a_\nu]. \tag{B21}$$

Their contribution is

$$\frac{1}{2}\langle V_3 V_3 \rangle = 9F_1 - 9F_2 + C_G(6F_3 + 2G_1 + G_2) + 32H_1 - 36H_2 - 16H_3 + 12H_4 - 4H_5. \tag{B22}$$

Newly introduced functions are defined as

$$F_3 = \left\langle \frac{1}{P_1^2 P_2^2 P_3^2} \right\rangle_P, \qquad G_1' = G_1 - \frac{1}{N} \mathrm{tr}\left[\left(\frac{1}{P^2}\right)^3\right],$$
$$H_1 = \left\langle \frac{P_1 \cdot I(2) \cdot P_1}{P_1^2 P_2^6 P_3^2} \right\rangle_P, \qquad H_3 = \left\langle \frac{P_2 \cdot I(1) \cdot P_3}{P_1^4 P_2^4 P_3^2} \right\rangle_P,$$
$$H_4 = \left\langle \frac{P_1 \cdot I(2) \cdot P_1}{P_1^4 P_2^4 P_3^2} \right\rangle_P, \qquad H_5 = \left\langle \frac{P_2 \cdot I(1) \cdot P_3}{P_1^2 P_2^4 P_3^4} \right\rangle_P. \tag{B23}$$

In $SU(3)$, we can evaluate the following quantity as

$$\frac{1}{N} \mathrm{tr}\left[\left(\frac{1}{P^2}\right)^3\right] = \frac{1}{N} \sum_{(r,s),m} \frac{\frac{1}{2}(r+1)(s+1)(r+s+2)}{[\frac{1}{2}(r(r+2) + s(s+2))]^3}. \tag{B24}$$

Fermion vertex is

$$V_f = -\frac{1}{2} \mathrm{tr} \bar{\varphi} \Gamma_\mu [a_\mu, \varphi]. \tag{B25}$$

Their contribution is

$$\frac{1}{2}\langle V_f V_f \rangle = -64F_2 + (-8C_G G_1' + 4C_G G_2 + 8C_G F_3 + 32H_4) - 16C_G F_3 + 48C_G G_1' + -8C_G G_2 + 64H_2 + 64H_3. \tag{B26}$$

After summing up (B15), (B18), (B22), and (B26), we find the two-loop effective action:

$$\Gamma_{2-\mathrm{loop}} = 2C_G F_3 + 32H_1 + 32H_2 + 48H_3 + (12 + 32)H_4 - 4H_5 = 2C_G F_3. \tag{B27}$$

It is because

$$H_1 + H_2 = 0, \qquad H_3 + H_4 = 0, \qquad H_3 - H_5 = 0. \tag{B28}$$

Since we have used the common properties of $SU(2)$ and $SU(3)$, the result (B27) is valid for $SU(2)$ and $SU(3)$ and consistent with the fuzzy sphere's results.

APPENDIX C

In this appendix, we calculate F_3 in (B27) numerically. A practical way to calculate F_3 is to use the Monte Carlo simulation [11]. Our strategy is to construct a Gaussian matrix model to calculate it:

$$F_3 = \left\langle \frac{1}{P_1^2 P_2^2 P_3^2} \right\rangle_P = \int dadbdc\, \mathrm{tr}(abc)\, \mathrm{tr}(cba) \times \exp\left(-\frac{1}{2}[a, p^\mu]^2 - \frac{1}{2}[b, p^\mu]^2 - \frac{1}{2}[c, p^\mu]^2\right). \tag{C1}$$

We can use the heat-bath algorithm to calculate this correlator. The result is shown in Table I. We estimate the statical errors using a jackknife method [11,12].

The other way to calculate F_3 is to use the harmonic matrices. We can obtain these matrices on the computer using the method explained in Appendix A. The result is shown in Table II.

Since Table II shows the exact results, this calculation is preferable to the Monte Carlo. But we have used the Monte Carlo method, because the exact evaluation requires more computer power than the Monte Carlo. Nevertheless, we can use Table II to check Table I. We can thus claim that the Monte Carlo method gives the correct results.

TABLE II. The results of F_3 using the harmonic matrices.

$SU(3)$ representation	N	F_3
(1, 0)	3	0.691 358
(2, 0)	6	1.027 320
(3, 0)	10	1.156 321
(4, 0)	15	1.211 689
(5, 0)	21	1.236 921
(6, 0)	28	1.248 420

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