PHYSICAL REVIEW D **73,** 065027 (2006)

Gauge theory of gravity and supergravity

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(Received 17 January 2006; published 29 March 2006)

We present a formulation of gravity in terms of a theory based on complex $SU(2)$ gauge fields with a general coordinate invariant action functional quadratic in the field strength. Self-duality or anti-self-duality of the field strength emerges as a constraint from the equations of motion of this theory. This in turn leads to Einstein gravity equations for a dilaton and an axion conformally coupled to gravity for the self-dual constraint. The analysis has also been extended to $N = 1$ and 2 super Yang-Mills theory of complex $SU(2)$ gauge fields. This leads to, besides other equations of motion, self-duality/anti-self-duality of generalized supercovariant field strengths. The self-dual case is then shown to yield as its solutions $N = 1$, 2 supergravity equations, respectively.

DOI: 10.1103/PhysRevD.73.065027 PACS numbers: 04.65.+e, 04.50.+h, 04.60.-m

I. INTRODUCTION

Quest for a gauge theory description of Einstein's General Theory of Relativity (GTR) has a long history. Pioneering attempts made by Utiyama, Kibble, and Mandelstam are now about five decades old [1]. There is also another more recent and famous formulation of Ashtekar where gravity is described in terms of a new variable, the complex $SU(2)$ Sen-Ashtekar gauge connection [2]. This gauge field is the self-dual part of the spin connection ω_μ^{ab} where self-duality is with respect to the Lorentz indices $[ab]$. The action used is complex. It is linear in the field strength much in the same manner as the standard Einstein-Hilbert or Hilbert-Palatini action. The formulation is entirely chiral in that it deals with the local Lorentz representations involving only the chiral part of $SL(2, C)$ and not its conjugate. A Hamiltonian formulation is set up in terms of phase space consisting of the spatial part of the Lorentz self-dual spin connection and its canonically conjugate density-weighted spatial triad. A related formulation is that of Plebanski [3] where again we have a first-order Palatini-type action for complex general relativity described in terms of a spinor-valued two-form Σ^{AB}, an $SL(2, C)$ one-form ω_{AB} which is identified with the Lorentz-self-dual part of the spin connection, and a totally symmetric Lagrange multiplier field Ψ_{ABCD} where the Latin letters A, B, C, D denote two-component spinor indices. Ashtekar canonical formulation may be viewed as the $(3 + 1)$ decomposition of the first-order formalism of Plebanski [4].

There have also been other attempts to set up a gauge theory description of gravity. For example, there is an $SL(3)$ and diffeomorphism invariant Euclidean space action, still linear in field strength, presented by 't Hooft [5].

As is well known, it has been a long standing challenge to set up a quantum theory of gravity. It is generally believed that perturbative quantum general relativity set up in terms of quantized corrections to a background metric and as a perturbative expansion in dimensionful Newton's constant is not renormalizable. Choice of a background metric fixes the coordinate system and thus breaks general covariance. It is possible that difficulties faced are due to the tools and methods used so far. A consistent perturbative quantum description may be possible if it is set up in terms of a quantization based on some other, more suitable set of fields with an appropriate action functional and as a pertubation in terms of a dimensionless coupling instead of dimensionful Newton's constant. For this we have to first develop a classical description of Einstein's general relativity in terms of these fields. Newton's constant should emerge as a parameter in the space of solutions of such a theory. It is, therefore, worthwhile to explore various possible action principles involving only dimensionless couplings which yield Einstein's equations of the classical gravity as solutions of their equations of motion.

Gravity actions with quadratic curvature terms have been discussed for many decades now. For example, DeWitt in 1960 had hoped that such terms would provide a cure for the divergence problem [6]. One of the earliest studies of gravity theory with action made of only quadratic curvature terms, $R_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}$, was the parallel displacement gauge theory of Yang [7]. There are two types of variational principles that can be adopted. In the Einstein-Hilbert variational picture (also known as second-order formalism) where spacetime is Riemannian, the quadratic action is to be varied with respect to the metric $g_{\mu\nu}$. This yields a fourth-order differential equation of motion for the graviton field $h^{\mu\nu}$ defined as $\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}$. In fact, a general higher derivative theory of gravity with R^2, R, and cosmological constant terms can be shown to be renormalizable [8]. But such theories generically imply nonunitarity due to a negative residue spin-two pole (i.e. a ghost) in the bare propagator of the graviton field. On the other hand, in the Einstein-Palatini variational picture (also called first-order formalism), the connection (not the Riemann-Christoffel connection) and the metric are inde-

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pendently varied. This leads, in general, to space-times with torsion [9]. The equation of motion obtained by varying the action, quadratic in curvature tensor, with respect to the metric leads to a constraint expressing the gravitational stress-energy tensor to be zero. This equation is solved by a double self-dual or anti-self-dual curvature tensor [10]. These are only second-order differential equations for the metric. In the next section we shall recast this theory in terms of a complex $SU(2)$ gauge field theory. Unlike Ashtekar theory which is described in terms of an action linear in complex $SU(2)$ field strength $F^i_{\mu\nu}$, here we shall deal with an action functional quadratic in this field strength. This is much in line with gauge theories used to describe other fundamental interactions of particle physics. Both the complex gauge field and the metric are taken to be independent variables in the action as in the Einstein-Palatini variational principle. Solutions of the equations of motion fall into two classes: those with self-dual and those with anti-self-dual field strengths. These constraints can be solved to write the metric in terms of the field strength of gauge fields. Thus, geometric quantities are related to the gauge fields. Finally we shall be led to standard Einstein equations of motion for gravity conformally coupled to a dilaton and an axion as a solution to the self-dual constraint. There is no dimensionful parameter in the definition of the gauge theory. However, a dimensionful parameter, to be identified with Newton's constant, will emerge as a modulus of the space of solutions of the equations of motion of this theory.

In Sec. III, we shall extend the discussion to $N = 1$ complex $SU(2)$ super Yang-Mills theory. The equations of motion imply self-duality or anti-self-duality of the supercovariantized field strength. Supergravity theory emerges as a solution to the self-dual case. The same structure gets carried over to the case of $N = 2$ super Yang-Mills theory, where self-duality of a generalized covariant field strength, obtained as a solution to the equation of motion, leads to $N = 2$ supergravity equations. This we discuss in Sec. IV. Some concluding remarks will follow in Sec. V.

II. COMPLEX $SU(2)$ GAUGE THEORY AS A THEORY OF GRAVITY

Consider a complex $SU(2)$ gauge field A^i_μ ($i = 1, 2, 3$) and metric $g_{\mu\nu}$ as independent variable fields in the action

$$S = \frac{\tau}{4}\int d^4x e g^{\mu\alpha} g^{\nu\beta} F^i_{\mu\nu} F^i_{\alpha\beta}, \tag{1}$$

where $e^2 = g = \det g_{\mu\nu} < 0$ and the complex field strength is

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \epsilon^{ijk} A^j_\mu A^k_\nu.$$

Here τ is a dimensionless complex coupling constant. The action is complex; we may wish to make it real by adding to it a conjugate action given as a functional of $\bar{A}^i_\mu$ which is a complex conjugate of the gauge field A^i_μ:

$$\bar{S} = \frac{\bar{\tau}}{4}\int d^4x \bar{e} g^{\mu\alpha} g^{\nu\beta} \bar{F}^i_{\mu\nu} \bar{F}^i_{\alpha\beta},$$

where $\bar{F}^i_{\mu\nu} = \partial_\mu \bar{A}^i_\nu - \partial_\nu \bar{A}^i_\mu - \epsilon^{ijk} \bar{A}^j_\mu \bar{A}^k_\nu$ and $\bar{e} = -e$. However, in the following we shall work with the complex action S.

The action S is invariant under complex $SU(2)$ gauge transformations and also under general coordinate transformations. It contains no kinetic energy term for the metric $g_{\mu\nu}$. In contrast to Ashtekar theory [2] where the action functional is linear in field strength, it is quadratic here.

A. Equations of motion

The equations of motion are obtained by varying the action S above with respect to the independent fields. Variation with respect to the gauge field A^i_μ yields the Yang-Mills equation of motion

$$D^\mu(eF^i_{\mu\nu}) = 0, \tag{2}$$

where the gauge covariant derivative is $D_\mu \Phi^i = \partial_\mu \Phi^i - \epsilon^{ijk} A^j_\mu \Phi^k$. Next, variation of the action with respect to the metric $g_{\mu\nu}$ gives the second equation of motion, which is in fact a constraint equation:

$$T_{\mu\nu} \equiv F^i{}_\mu{}^\alpha F^i_{\nu\alpha} - \tfrac{1}{4} g_{\mu\nu} F^i_{\alpha\beta} F^{i\alpha\beta} = 0. \tag{3}$$

Notice that the gauge field stress-energy tensor $T_{\mu\nu}$ is traceless, $g^{\mu\nu} T_{\mu\nu} = 0$, and also conserved, $\nabla^\mu T_{\mu\nu} = 0$, by the first of the two equations of motion, the Yang-Mills equation (2). Here the derivative ∇_μ is covariant with respect to the general coordinate transformations: $\nabla_\mu T^{\alpha\beta} = \partial_\mu T^{\alpha\beta} + \Gamma_{\mu\lambda}{}^\alpha T^{\lambda\beta} + \Gamma_{\mu\lambda}{}^\beta T^{\alpha\lambda}$, where the Riemann-Christoffel connection is given in terms of the metric through the condition $\nabla_\mu g_{\alpha\beta} \equiv \partial_\mu g_{\alpha\beta} - \Gamma_{\mu\alpha}{}^\lambda g_{\lambda\beta} - \Gamma_{\mu\beta}{}^\lambda g_{\alpha\lambda} = 0$.

We need to solve these equations of motion. To solve the constraint equation (3), we introduce the dual field strength

$$^*F^{i\mu\nu} \equiv \frac{1}{2e} \epsilon^{\mu\nu\alpha\beta} F^i_{\alpha\beta}, \tag{4}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the usual completely antisymmetric Levi-Civita density of weight 1 with values ± 1 or 0. Notice this duality operation is involutive: $^*(^*F^i_{\mu\nu}) = F^i_{\mu\nu}$. With this definition and the identity (in four dimensions) $\delta^{[\nu}_\mu \epsilon^{\alpha\beta\rho\sigma]} = 0$ (where square brackets indicate antisymmetrization of the contained indices), it is straightforward to check that the gauge field stress-energy tensor can be rewritten as

$$T_{\mu\nu} \equiv \tfrac{1}{2}(F^i_{\mu\alpha} + {}^*F^i_{\mu\alpha})(F^i{}_\nu{}^\alpha - {}^*F^i{}_\nu{}^\alpha). \tag{5}$$

Thus the constraint equation $T_{\mu\nu} = 0$ is solved by self-dual or anti-self-dual field strength:

$$F^i_{\mu\nu} = \pm {}^*F^i_{\mu\nu}. \tag{6}$$

It is important to notice that such field strengths satisfy the Yang-Mills equation of motion (2) identically. Also, for these field strengths, the Lagrangian density becomes a total divergence:

$$\begin{aligned}\tfrac{1}{4}eF^{i\mu\nu}F^i_{\mu\nu} &= \pm\tfrac{1}{4}e\,{}^*F^{i\mu\nu}F^i_{\mu\nu} = \pm\tfrac{1}{8}\epsilon^{\mu\nu\alpha\beta}F^i_{\mu\nu}F^i_{\alpha\beta}\\ &= \pm\partial_\mu J^\mu,\end{aligned}$$

where $J^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}(A^i_\nu\partial_\alpha A^i_\beta - \frac{1}{3}\epsilon^{ijk}A^i_\nu A^j_\alpha A^k_\beta)$.

The self-duality or anti-self-duality constraint implies that the metric is not an independent field, but can be solved for as a function of the gauge field A^i_μ (or more exactly as a function of the gauge field strength $F^i_{\mu\nu}$). In fact, it can be shown that the metric for (nonzero) self-dual or anti-self-dual field strength is given by Urbantke type formulas [11]:

$$\begin{aligned}g^{-(1/4)}g_{\mu\nu} &= (\det\phi_{ij})^{-(1/2)}X_{\mu\nu},\\ g^{1/4}g^{\mu\nu} &= (\det\phi_{ij})^{-(1/2)}Y^{\mu\nu},\end{aligned} \tag{7}$$

where quantities ϕ_{ij}, $X_{\mu\nu}$, and $Y^{\mu\nu}$ are given in terms of the self-dual or anti-self-dual field strength as

$$\phi_{ij} = \pm\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F^i_{\mu\nu}F^j_{\alpha\beta}$$

and

$$\begin{aligned}X_{\mu\nu} &= \pm\tfrac{2}{3}\epsilon^{\alpha\beta\sigma\delta}\epsilon^{ijk}F^i_{\mu\alpha}F^j_{\beta\sigma}F^k_{\delta\nu},\\ Y^{\mu\nu} &= \tfrac{1}{3}\epsilon^{\mu\alpha\beta\gamma}\epsilon^{\lambda\nu\rho\sigma}\epsilon^{ijk}F^i_{\beta\gamma}F^j_{\alpha\lambda}F^k_{\rho\sigma}.\end{aligned}$$

Thus (7) gives the metric in terms of the self-dual or anti-self-dual field strength, but only up to a conformal factor. This is so because self-duality and anti-self-duality constraints are not sensitive to the conformal factor of the metric. Under a conformal transformation $g_{\mu\nu} \to \Omega^{-2}g_{\mu\nu}$ ($e \to \Omega^{-4}e$, $g^{\mu\nu} \to \Omega^2 g^{\mu\nu}$),

$$\left(F^{i\mu\nu} \mp \frac{1}{2e}\epsilon^{\mu\nu\alpha\beta}F^i_{\alpha\beta}\right) \to \Omega^4\left(F^{i\mu\nu} \mp \frac{1}{2e}\epsilon^{\mu\nu\alpha\beta}F^i_{\alpha\beta}\right).$$

To analyze the self-duality or anti-self-duality constraint further, we trade the three complex two-tensors $F^i_{\mu\nu}$ by six real two-tensors $\mathcal{R}_{\mu\nu}{}^{\alpha\beta}$ for six values of the antisymmetric pair of indices $(\alpha\beta)$ through the definition

$$F^i_{\mu\nu} = \mathcal{R}_{\mu\nu}{}^{\alpha\beta}\Sigma^i_{\alpha\beta}, \tag{8}$$

where $\Sigma^i_{\mu\nu}$ is a self-dual two-form, ${}^*\Sigma^i_{\mu\nu} = \Sigma^i_{\mu\nu}$, which can be viewed as a curved space generalization of the flat Minkowski space $\eta^i_{\mu\nu}$ symbol of 't Hooft [5,12]. This is constructed from the tetrads e^a_α defined as the square root of the metric through $g_{\mu\nu} = e^a_\mu e^b_\nu\eta_{ab}$, $\eta_{ab} = \delta_{ab} = \eta^{ab}$, and a and b take values 1, 2, 3, 4. (In our notation, e^4_α is pure imaginary.) And $\Sigma^i_{\alpha\beta} = \Sigma_{\alpha\beta}{}^{i4} + \frac{1}{2}\epsilon^{ijk}\Sigma_{\alpha\beta}{}^{jk}$ (each of i, j, k takes values 1, 2, 3) where $\Sigma_{\alpha\beta}{}^{ab}$ is the antisymmetrized product of tetrads e^a_α: $\Sigma_{\alpha\beta}{}^{ab} \equiv \frac{1}{2}e^a_{[\alpha}e^b_{\beta]}$.

Self-duality or anti-self-duality of the field strength (6) then implies the constraint

$${}^*\mathcal{R}_{\mu\nu}{}^{ab} = \pm\tilde{\mathcal{R}}_{\mu\nu}{}^{ab}, \tag{9}$$

where * is the duality with respect to the first pair of indices $[\mu\nu]$ and the tilde $\tilde{}$ is the duality in the second pair of flat internal space indices $[ab]$ defined as

$$\tilde{X}^{ab} = \tfrac{1}{2}\epsilon^{abcd}X_{cd}.$$

Here ϵ^{abcd} is completely antisymmetric with $\epsilon^{1234} = +1$. The condition (9) is the double self-duality/anti-self-duality condition studied in Refs. [10].

Next we use the Lanczos identity

$${}^*\tilde{\mathcal{R}}_{\mu\nu}{}^{ab} \equiv \mathcal{R}^{ab}{}_{\mu\nu} + \Sigma_{\mu\nu}{}^{ab}\mathcal{R} + 2\Sigma_{\mu\nu}{}^{c[a}\mathcal{R}^{b]}{}_c, \tag{10}$$

where $\mathcal{R}_\mu{}^a = \mathcal{R}_{\mu\nu}{}^{ab}e^\nu_b$ and $\mathcal{R} = \mathcal{R}_\mu{}^a e^\mu_a$. This identity and self-duality or anti-self-duality of the field strength imply

$$\pm\mathcal{R}_{\mu\nu}{}^{ab} - \mathcal{R}^{ab}{}_{\mu\nu} = \Sigma_{\mu\nu}{}^{ab}\mathcal{R} - e^{[a}_{[\mu}\mathcal{R}^{b]}{}_{\nu]}. \tag{11}$$

We need to solve these constraints. To develop such solutions we write

$$A^i_\mu = a^i_\mu + b^i_\mu, \tag{12}$$

where a^i_μ is such that $D_{[\mu}(a)\Sigma^i_{\nu\alpha]} \equiv \partial_{[\mu}\Sigma^i_{\nu\alpha]} - \epsilon^{ijk}a^j_{[\mu}\Sigma^k_{\nu\alpha]} = 0$. This constraint can be solved for a^i_μ in terms of the tetrads through $\Sigma^i_{\mu\nu}$ to obtain

$$a^i_\mu = \omega_{\mu\alpha\beta}(e)\Sigma^{i\alpha\beta}, \tag{13}$$

where $\omega(e)$ is the usual spin connection given in terms of the tetrads e^a_μ:

$$\omega_{\mu\alpha\beta}(e) = \tfrac{1}{2}(e_{a\alpha}\partial_{[\beta}e^a_{\mu]} + e_{a\beta}\partial_{[\mu}e^a_{\alpha]} - e_{a\mu}\partial_{[\alpha}e^a_{\beta]}).$$

Notice a^i_μ is the Sen-Ashtekar gauge field.

Next, we write

$$F^i_{\mu\nu} = f^i_{\mu\nu} + \ell^i_{\mu\nu} \tag{14}$$

where $f^i_{\mu\nu}$ is the field strength for the gauge field a^i_μ:

$$f^i_{\mu\nu} = \partial_{[\mu}a^i_{\nu]} - \epsilon^{ijk}a^j_\mu a^k_\nu \equiv R_{\mu\nu}{}^{\alpha\beta}(\omega(e))\Sigma^i_{\alpha\beta}$$

and

$$\ell^i_{\mu\nu} = D_{[\mu}(a)b^i_{\nu]} - \epsilon^{ijk}b^j_\mu b^k_\mu \equiv r_{\mu\nu}{}^{\alpha\beta}\Sigma^i_{\alpha\beta}.$$

Here the derivative $D_\mu(a)$ is a gauge covariant derivative involving the gauge field a^i_μ, $D_\mu(a)b^i_\nu = \partial_\mu b^i_\nu - \epsilon^{ijk}a^j_\mu b^k_\nu$; $R_{\mu\nu}{}^{\alpha\beta}(\omega(e))$ is the usual Riemann tensor and

$$\mathcal{R}_{\mu\nu}{}^{\alpha\beta} = R_{\mu\nu}{}^{\alpha\beta}(\omega(e)) + r_{\mu\nu}{}^{\alpha\beta}, \tag{15}$$

where the tensor $r_{\mu\nu}{}^{\alpha\beta}$ is to be determined. Writing

$b^i_\mu = h_\mu{}^{\alpha\beta}\Sigma^i_{\alpha\beta}$, this r tensor is

$$r_{\mu\nu\alpha\beta} = \nabla_{[\mu}h_{\nu]\alpha\beta} + h_{\mu\alpha}{}^{\lambda}h_{\nu\lambda\beta} - h_{\nu\alpha}{}^{\lambda}h_{\mu\lambda\beta}. \tag{16}$$

Notice we have traded three complex vectors b^i_μ with six real vectors $h_\mu{}^{\alpha\beta}$ with six values of the antisymmetric pair $(\alpha\beta)$. These can be viewed as contortion. The 24 dimensional space of real contortion $h_{\mu\alpha\beta}$ can be decomposed into three irreducible subspaces: a trace part $h_\alpha = g^{\mu\beta}h_{\mu\alpha\beta}$, a completely antisymmetric part $K_{\mu\alpha\beta}$, and a tensor part $J_{\mu\alpha\beta}$ with $g^{\mu\beta}J_{\mu\alpha\beta} = 0$ and $J_{[\mu\alpha\beta]} = 0$. These subspaces are, respectively, 4, 4, and 16 dimensional. In the following we shall take the tensor part to be zero. Thus we parametrize $h_{\mu\alpha\beta}$ as

$$h_{\mu\alpha\beta} = K_{\mu\alpha\beta} - \tfrac{1}{3}(g_{\mu\alpha}h_\beta - g_{\mu\beta}h_\alpha). \tag{17}$$

The four-tensor $r_{\mu\nu\alpha\beta}$ is given by

$$\begin{aligned} r_{\mu\nu}{}^{\alpha\beta} = {} & \nabla_{[\mu}K_{\nu]}{}^{\alpha\beta} + \tfrac{1}{3}\delta^{[\alpha}_{[\mu}\nabla_{\nu]}h^{\beta]} + K_{[\mu}{}^{\alpha\lambda}K_{\nu]\lambda}{}^{\beta} \\ & - \tfrac{1}{3}\delta^{[\alpha}_{[\mu}K_{\nu]\lambda}{}^{\beta]}h^\lambda + \tfrac{2}{3}K_{\mu\nu}{}^{[\alpha}h^{\beta]} + \tfrac{1}{9}(\delta^{[\alpha}_{[\mu}h_{\nu]}h^{\beta]} \\ & - \delta^\alpha_{[\mu}\delta^\beta_{\nu]}h^2). \end{aligned} \tag{18}$$

From this we construct the two-tensor $r_{\mu\nu} = r_{\mu\alpha\nu\beta}g^{\alpha\beta}$. The symmetric and antisymmetric parts of this tensor and its trace ($r = g^{\mu\nu}r_{\mu\nu}$) are

$$\begin{aligned} r_{\mu\nu} + r_{\nu\mu} = {} & \tfrac{2}{3}[\nabla_\mu h_\nu + \nabla_\nu h_\mu + g_{\mu\nu}\nabla\cdot h \\ & - \tfrac{2}{3}(g_{\mu\nu}h^2 - h_\mu h_\nu)] - 2K_{\mu\alpha\beta}K_\nu{}^{\alpha\beta}, \\ r_{\mu\nu} - r_{\nu\mu} = {} & \tfrac{2}{3}\nabla_{[\mu}h_{\nu]} - 2\nabla_\sigma K_{\mu\nu}{}^{\sigma}, \\ r = {} & 2\nabla\cdot h - \tfrac{2}{3}h^2 - K_{\mu\alpha\beta}K^{\mu\alpha\beta}. \end{aligned} \tag{19}$$

Let us now consider the two cases of self-duality and anti-self-duality separately.

B. Self-dual solution

Contracting the constraint equation (11) by e^ν_b for the self-dual case yields

$$\mathcal{R}_{\mu\nu} + \mathcal{R}_{\nu\mu} = \tfrac{1}{2}g_{\mu\nu}\mathcal{R}. \tag{20}$$

This is our master equation which we wish to solve. It fixes nine of the 21 independent components of $\mathcal{R}_{\mu\nu}{}^{ab} + \mathcal{R}^{ab}{}_{\mu\nu}$ leaving 12 independent components undetermined. This equation when substituted back into (11) yields the constraint

$$\mathcal{R}_{\mu\nu}{}^{ab} - \mathcal{R}^{ab}{}_{\mu\nu} = \tfrac{1}{2}e^{[a}_{[\mu}(\mathcal{R}_{\nu]}{}^{b]} - \mathcal{R}^{b]}{}_{\nu]}). \tag{21}$$

This equation fixes nine of the 15 independent components of $\mathcal{R}_{\mu\nu}{}^{ab} - \mathcal{R}^{ab}{}_{\mu\nu}$ leaving six undetermined. The two equations (20) and (21), which are equivalent to the self-dual constraint (11), then fix 18 of the independent components of $\mathcal{R}_{\mu\nu}{}^{ab}$, and the other 18 are undetermined. Solving these would be equivalent to solving the self-duality equation for our $SU(2)$ gauge field strength (6).

The constraint (21) further implies

$$\begin{aligned} (\nabla^{[\alpha} + \tfrac{2}{3}h^{[\alpha})K^{\beta]}{}_{\mu\nu} - (\nabla_{[\mu} + \tfrac{2}{3}h_{[\mu})K_{\nu]}{}^{\alpha\beta} \\ = (\nabla_\sigma + \tfrac{2}{3}h_\sigma)\delta^{[\alpha}_{[\mu}K_{\nu]}{}^{\beta]\sigma}, \end{aligned} \tag{22}$$

where we have used $\mathcal{R}_{\mu\nu}{}^{\alpha\beta} = R_{\mu\nu}{}^{\alpha\beta}(\omega(e)) + r_{\mu\nu}{}^{\alpha\beta}$ and the fact that the Riemann tensor is symmetric under the interchange of first and second pairs of indices, and therefore the Ricci tensor $R_{\mu\nu}(\omega(e))$ is also symmetric. Next from (20) we may write

$$R_{\mu\nu}(\omega(e)) = -\tfrac{1}{2}[r_{\mu\nu} + r_{\nu\mu}] + \tfrac{1}{4}g_{\mu\nu}[R(\omega(e)) + r],$$

or equivalently

$$\begin{aligned} R_{\mu\nu}(\omega(e)) - \tfrac{1}{2}g_{\mu\nu}R(\omega(e)) = {} & -t_{\mu\nu} \\ \equiv {} & -\tfrac{1}{2}[r_{\mu\nu} + r_{\nu\mu} - \tfrac{1}{2}g_{\mu\nu}r] \\ & - \tfrac{1}{4}g_{\mu\nu}R(\omega(e)). \end{aligned} \tag{23}$$

Now since $\nabla^\mu[R_{\mu\nu}(\omega(e)) - \frac{1}{2}g_{\mu\nu}R(\omega(e))] \equiv 0$, we need to solve $\nabla^\mu t_{\mu\nu} = 0$. This is what we shall attempt to do next.

Dilaton-axion gravity from the self-dual solution

We shall use the convenient ansatz for h_μ and $K_{\mu\alpha\beta}$ in (17):

$$h_\mu = -3\partial_\mu\phi, \qquad K_{\mu\alpha\beta} = \frac{\kappa}{2\sqrt{2}}e^{-2\phi}H_{\mu\alpha\beta}, \tag{24}$$

where κ is a constant and completely antisymmetric $H_{\mu\alpha\beta}$ is the field strength of an antisymmetric tensor gauge field $B_{\mu\nu}$: $H_{\mu\alpha\beta} = \partial_{[\mu}B_{\alpha\beta]}$. We shall take ϕ to be dimensionless (soon we shall see that it will represent a dilaton) and the antisymmetric gauge field $B_{\mu\nu}$ to have mass dimensions $+1$, and its field strength $H_{\mu\alpha\beta}$ then has dimensions $+2$. In order for the mass dimension of $h_{\mu\alpha\beta}$ of (17) to be $+1$ (mass dimension of the gauge field A^i_μ), the constant κ has to be of dimension -1. However, in the following discussion, we shall take $\kappa = 1$ for convenience; it can easily be restored whenever needed.

We use (24) to construct the tensors $r_{\mu\nu\alpha\beta}$ and $r_{\mu\nu}$ of (16) and (19). This leads us to

$$\begin{aligned} r_{\mu\nu} = {} & 2[\nabla_\mu\phi\nabla_\mu\phi - \nabla_\mu\nabla_\nu\phi] - g_{\mu\nu}[2(\nabla\phi)^2 + \nabla^2\phi] \\ & - \frac{1}{2\sqrt{2}}\nabla_\alpha(e^{-2\phi}H_{\mu\nu}{}^{\alpha}) - \frac{1}{8}e^{-4\phi}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta}, \\ r = {} & -6[(\nabla\phi)^2 + \nabla^2\phi] - \frac{1}{8}e^{-4\phi}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}. \end{aligned} \tag{25}$$

From (23) these, in turn, imply

$$t_{\mu\nu} \equiv -2[\nabla_\mu\phi\nabla_\nu\phi - \nabla_\mu\nabla_\nu\phi] - \tfrac{1}{2}g_{\mu\nu}[(\nabla\phi)^2 - \nabla^2\phi - \tfrac{1}{2}R(\omega(e))] - \tfrac{1}{8}e^{-4\phi}[H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} - \tfrac{1}{4}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}]. \quad (26)$$

Then $\nabla^\mu t_{\mu\nu} = 0$ is satisfied by the following solution:

$$R_{\mu\nu}(\omega(e)) = 2[\nabla_\mu\nabla_\nu\phi - \nabla_\mu\phi\nabla_\nu\phi] + g_{\mu\nu}[\nabla^2\phi + 2(\nabla\phi)^2 - \tfrac{1}{2}\Lambda e^{2\phi}] + \tfrac{1}{8}e^{-4\phi}[H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} - \tfrac{1}{3}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}],$$
$$\nabla^\mu[e^{-2\phi}H_{\mu\alpha\beta}] = 0. \quad (27)$$

To verify that these indeed provide a solution, substitute

$$R(\omega(e)) = 6[(\nabla\phi)^2 + \nabla^2\phi] - \tfrac{1}{24}e^{-4\phi}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} - 2\Lambda e^{2\phi}$$

obtained from the first equation into (26) to write

$$t_{\mu\nu} = 2[\nabla_\mu\phi\nabla_\nu\phi - \nabla_\mu\nabla_\nu\phi] + g_{\mu\nu}[(\nabla\phi)^2 + 2\nabla^2\phi - \tfrac{1}{2}\Lambda e^{2\phi}] - \tfrac{1}{8}e^{-4\phi}[H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} - \tfrac{1}{6}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}].$$

It is useful to notice that

$$H^{\mu\alpha\beta}\nabla_\mu H_{\nu\alpha\beta} - \tfrac{1}{6}\nabla_\nu(H^{\alpha\beta\gamma}H_{\alpha\beta\gamma}) = -\tfrac{1}{18}H^{\alpha\beta\gamma}\nabla_{[\nu}H_{\alpha\beta\gamma]} = 0,$$

where the last step is implied by the identity $\nabla_{[\nu}H_{\alpha\beta\gamma]} \equiv 0$. Then

$$\nabla^\mu t_{\mu\nu} = 2[\nabla^2\phi\nabla_\nu\phi + \nabla_\nu(\nabla\phi)^2 - \nabla^2\nabla_\nu\phi + \nabla_\nu\nabla^2\phi] - \partial_\nu\phi e^{2\phi}\Lambda + \tfrac{1}{2}e^{-4\phi}\nabla^\mu\phi[H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} - \tfrac{1}{6}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}] - \tfrac{1}{8}e^{-4\phi}\nabla^\mu H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta}. \quad (28)$$

Next use the identity $\nabla^2\nabla_\nu\phi - \nabla_\nu\nabla^2\phi = R_{\nu\lambda}(\omega(e))\nabla^\lambda\phi$ and the first equation of (27) to prove

$$2[\nabla^2\phi\nabla_\nu\phi + \nabla_\nu(\nabla\phi)^2 - \nabla^2\nabla_\nu\phi + \nabla_\nu\nabla^2\phi] - \partial_\nu\phi e^{2\phi}\Lambda = -\tfrac{1}{4}e^{-4\phi}\nabla^\mu\phi[H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} - \tfrac{1}{3}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}].$$

This, when substituted in (28), yields

$$\nabla^\mu t_{\mu\nu} = -\tfrac{1}{8}e^{-2\phi}\nabla^\mu(e^{-2\phi}H_{\mu\alpha\beta})H_\nu{}^{\alpha\beta} = 0$$

by the second equation in (27).

Thus the solution to the self-duality constraint is given in terms of Eqs. (27) along with the constraint (22) which may be rewritten as

$$\nabla^{[\alpha}H^{\beta]}{}_{\mu\nu} = 2\partial^{[\alpha}\phi H^{\beta]}{}_{\mu\nu} - 2\partial_{[\mu}\phi H_{\nu]}{}^{\alpha\beta} + \partial^\sigma\phi\delta^{[\alpha}_{[\mu}H^{\beta]}{}_{\nu]\sigma}. \quad (29)$$

This constraint is consistent with the second equation in (27).

Now notice that (27) are the equations of motion of a dilaton ϕ, an axion $B_{\mu\nu}$, and a cosmological constant Λ coupled to gravity in a conformally invariant manner. An effective action with linear R that yields these as its equations of motion is

$$S_{\rm eff} = \frac{1}{2}\int d^4xe\Big[e^{2\phi}(e^{2\phi}\Lambda + R(\omega(e)) + 6(\partial\phi)^2) - \frac{1}{24}e^{-2\phi}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}\Big]. \quad (30)$$

This action is conformally invariant. That is, this action is unchanged under transformations,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x),$$
$$\phi(x) \to \phi'(x) = \phi(x) - \ln\Omega(x),$$
$$B_{\mu\nu} \to B'_{\mu\nu} = B_{\mu\nu}.$$

That is not surprising, because our starting $SU(2)$ gauge theory action (1) is classically conformally invariant.

Notice that, while (27) are the equations of motion for the effective action (30), the constraint (29) has to be invoked additionally to describe the solution to the self-duality constraint ${}^*F^i_{\mu\nu} = F^i_{\mu\nu}$.

If we restore the constant κ of mass dimensions -1 introduced in (24), it shall appear in the gravity equations (27) and the effective action (30) in a way that κ^2 can be interpreted as Newton's constant of gravity. Though we started with a gauge theory (1) with no dimensionful parameter, Newton's constant emerges as a dimensionful modulus of the space of solutions in this theory.

The kinetic energy term for the scalar field ϕ in the effective action (30) has the wrong sign. It really is not a physical field, because it can be rotated away by a Weyl scaling of the metric by absorbing it into the conformal factor of a new scaled conformally invariant metric $g'_{\mu\nu}(x) = e^{-\phi(x)}g_{\mu\nu}(x)$ leading to Poincaré gravity from conformal action of a scalar field.

Thus we have demonstrated that the solution of the equations of motion, in particular the self-duality equation, of a complex $SU(2)$ gauge theory leads to the equations of motion of gravity conformally coupled to a dilaton and an axion and also a cosmological constant. It is worth pointing out that the axion field so obtained can also be viewed as propagating torsion.

C. Anti-self-dual solution

Next let us analyze the anti-self-dual solution of the equations of motion:

$$\mathcal{R}_{\mu\nu}{}^{ab} + \mathcal{R}^{ab}{}_{\mu\nu} = -\Sigma_{\mu\nu}{}^{ab}\mathcal{R} + e^{[a}_{[\mu}\mathcal{R}^{b]}{}_{\nu]} \quad (31)$$

which when contracted with e^ν_β yields

$$\mathcal{R} = 0, \qquad \mathcal{R}_{\mu\nu} = \mathcal{R}_{\nu\mu}. \quad (32)$$

This further, from (31), implies

$$\mathcal{R}_{\mu\nu}{}^{ab} + \mathcal{R}^{ab}{}_{\mu\nu} = e^{[a}_{[\mu}\mathcal{R}^{b]}{}_{\nu]}. \qquad (33)$$

This equation fixes 12 out of 21 independent components of $\mathcal{R}_{\mu\nu}{}^{ab} + \mathcal{R}^{ab}{}_{\mu\nu}$, and the remaining nine are undetermined. On the other hand, six components of $\mathcal{R}_{\mu\nu}{}^{ab} - \mathcal{R}^{ab}{}_{\mu\nu}$ are fixed leaving nine undetermined. Thus, the constraints (33), which are equivalent to the anti-self-dual constraint (31), fix 18 of the 36 independent components of $\mathcal{R}_{\mu\nu}{}^{ab}$. Notice that the self-dual constraints (20) and (21) and the anti-self-dual constraints (33) fix complimentary components of $\mathcal{R}_{\mu\nu}{}^{ab}$.

Next we define a (traceless) Weyl tensor associated with $\mathcal{R}_{\mu\nu}{}^{ab}$ as

$$\mathcal{C}_{\mu\nu}{}^{ab} = \mathcal{R}_{\mu\nu}{}^{ab} + \tfrac{1}{3}\Sigma_{\mu\nu}{}^{ab}\mathcal{R} - \tfrac{1}{2}e^{[a}_{[\mu}\mathcal{R}_{\nu]}{}^{b]}.$$

Then (32) and (33) imply

$$\mathcal{C}_{\mu\nu}{}^{ab} + \mathcal{C}^{ab}{}_{\mu\nu} = \mathcal{R}_{\mu\nu}{}^{ab} + \mathcal{R}^{ab}{}_{\mu\nu} - e^{[a}_{[\mu}\mathcal{R}_{\nu]}{}^{b]} = 0. \qquad (34)$$

Writing $\mathcal{R}_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab}(\omega(e)) + r_{\mu\nu}{}^{ab}$ as in (15), this constraint can be rewritten as

$$2R_{\mu\nu}{}^{\alpha\beta} - \delta^{[\alpha}_{[\mu}R_{\nu]}{}^{\beta]} = -(r_{\mu\nu}{}^{\alpha\beta} + r^{\alpha\beta}{}_{\mu\nu}) + \delta^{[\alpha}_{[\mu}r_{\nu]}{}^{\beta]}. \qquad (35)$$

Now for the trace and completely antisymmetric parts of $h_{\mu\alpha\beta}$ as defined in (24), the pairwise symmetric part of the four-tensor $r_{\mu\nu\alpha\beta}$ is

$$r_{\mu\nu}{}^{\alpha\beta} + r^{\alpha\beta}{}_{\mu\nu} = \tfrac{1}{4}e^{-4\phi}H_{[\mu}{}^{\alpha\lambda}H_{\nu]\lambda}{}^{\beta} - 2\delta^{[\alpha}_{[\mu}\nabla_{\nu]}\nabla^{\beta]}\phi + 2\delta^{[\alpha}_{[\mu}\nabla_{\nu]}\phi\nabla^{\beta]}\phi - 2\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}(\nabla\phi)^2.$$

The anti-self-dual constraint from (32) implies

$$r_{\mu\nu} - r_{\nu\mu} = \frac{1}{\sqrt{2}}\nabla_\sigma(e^{-2\phi}H_{\mu\nu}{}^{\sigma}) = 0 \qquad (36)$$

and hence

$$r_{\mu\nu} = 2[\nabla_\mu\phi\nabla_\nu\phi - \nabla_\mu\nabla_\nu\phi] - g_{\mu\nu}[2(\nabla\phi)^2 + \nabla^2\phi] - \tfrac{1}{8}e^{-4\phi}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} \qquad (37)$$

and

$$R(\omega(e)) = -r = 6[(\nabla\phi)^2 + \nabla^2\phi] + \tfrac{1}{8}e^{-4\phi}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}. \qquad (38)$$

Further (33) or (35) implies the constraint

$$\nabla^{[\alpha}R^{\beta]}_{\nu}(\omega(e)) = -\nabla^{\mu}(r_{\mu\nu}{}^{\alpha\beta} + r^{\alpha\beta}{}_{\mu\nu} - \delta^{[\alpha}_{[\mu}r^{\beta]}{}_{\nu]} + \tfrac{1}{2}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}r), \qquad (39)$$

where we have used the identity satisfied by the Riemann tensor: $\nabla^\mu R_{\mu\nu\alpha\beta} = \nabla_{[\alpha}R_{\beta]\nu}$. It can easily be checked that a solution of this constraint is given by

$$\begin{aligned} H_{\mu\alpha\beta} &= 0, \\ R_{\mu\nu}(\omega(e)) &= -r_{\mu\nu} + p_{\mu\nu} \\ &= 2[\nabla_\mu\nabla_\nu\phi - \nabla_\mu\phi\nabla_\nu\phi] + g_{\mu\nu}[\nabla^2\phi + 2(\nabla\phi)^2] + p_{\mu\nu}, \end{aligned} \qquad (40)$$

where $p_{\mu\nu}$ is symmetric ($p_{\mu\nu} = p_{\nu\mu}$) and traceless ($p \equiv p^\mu{}_\mu = 0$) and satisfies the equation

$$\nabla^{[\alpha}p^{\beta]}{}_\mu - \partial^{[\alpha}\phi p^{\beta]}{}_\mu + \delta^{[\alpha}_\mu p^{\beta]}{}_\nu\partial^\nu\phi = 0.$$

The Reimann tensor for this solution (40) is

$$\begin{aligned} R_{\mu\nu}{}^{\alpha\beta} &= -r_{\mu\nu}{}^{\alpha\beta} + \tfrac{1}{2}\delta^{[\alpha}_{[\mu}p^{\beta]}{}_{\nu]} \\ &= \delta^{[\alpha}_{[\mu}(\nabla_{\nu]}\nabla^{\beta]}\phi - \nabla_{\nu]}\phi\nabla^{\beta]}\phi) + \delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}(\partial\phi)^2 + \tfrac{1}{2}\delta^{[\alpha}_{[\mu}p^{\beta]}{}_{\nu]}. \end{aligned}$$

Thus this provides a solution to the anti-self-dual constraint of gauge theory. It is possible that there are other more general solutions for the anti-self-dual case.

So far we have discussed only pure complex $SU(2)$ gauge theory. Other matter can also be included in this formulation. This can be achieved by adding terms made of other representations of the complex $SU(2)$. For example, we can add Lorentz scalar fields in triplet representation Φ^i or fermions λ^i. In particular, we may add these extra fields in a supersymmetric manner. This would then lead to the equations of motion of supergravity. We do this in the next section.

III. $N = 1$ SUPERSYMMETRIC COMPLEX $SU(2)$ GAUGE THEORY

Supersymmetric generalization of Einstein gravity in its simplest form leads to $N = 1$ supergravity. This theory, first discovered about 30 years ago, is described in terms of, besides a set of auxiliary fields, the physical metric field $g_{\mu\nu}$ and its superpartner, spin 3/2 gravitino ψ_μ [13]. In the spirit of Sec. II, we wish to set up a locally supersymmetric Yang-Mills theory whose equations of motion admit $N = 1$ supergravity equations as a solution. General super Yang-Mills action coupled to a tetrad and gravitino, without kinetic terms for them, and the relevant supersymmetric and other transformation rules have also been known for a long time [14–16].

We need a supersymmetric generalization of the conformally invariant action of complex $SU(2)$ gauge theory of the previous section. For this purpose, we introduce a complex $SU(2)$ triplet vector $N = 1$ superconformal multiplet $(A^i_\mu, \lambda^i, \mathcal{D}^i)$ where complex $\mathcal{D}^i$ is the usual auxiliary field. Notice that, like complex A^i_μ, the fermion is also made of two Majorana triplets: $\lambda^i = \lambda^{(1)i} + i\lambda^{(2)i}$. We have nine complex off-shell degrees of freedom in A^i_μ, three in $\mathcal{D}^i$, with a total of 12 complex off-shell bosonic degrees of freedom, which is same as the number of off-shell degrees in the fermions λ^i. We couple this supermultiplet to off-shell fields of the background conformal supergravity Weyl multiplet $(e^a_\mu, \psi_\mu, B_\mu)$ where the last is an axial vector field. Here we have eight off-shell real degrees of freedom in the bosonic fields, e^a_μ and B_μ, and an equal number in the gravitino field ψ_μ. In terms of these fields, the Lagrangian density $\mathcal{L}$ for the $N = 1$ super(conformal) Yang-Mills theory is given by

$$e^{-1}\mathcal{L} = -\tfrac{1}{4}F^{i\mu\nu}F_{i\mu\nu} - \tfrac{1}{2}\bar{\lambda}^i\gamma^\mu D_\mu(\hat{\omega})\lambda^i + \tfrac{1}{2}\mathcal{D}^i\mathcal{D}^i - \tfrac{1}{4}\bar{\psi}_\mu\sigma^{\alpha\beta}\gamma^\mu\lambda^i[F^i_{\alpha\beta} + \hat{F}^i_{\alpha\beta}]. \tag{41}$$

Here the Majorana conjugate of the fermions is given by $(\bar{\lambda}^i)_A = (\lambda^i)^B\mathcal{C}_{BA}$ and $(\bar{\psi}_\mu)_A = (\psi_\mu)^B\mathcal{C}_{BA}$ where $\mathcal{C}$ is the charge conjugation matrix and (A, B) are four component Dirac spinor indices. Supercovariant spin connection $\hat{\omega}_\mu{}^{ab}$ contains ψ_μ torsion, but not λ^i torsion:

$$\begin{aligned} \hat{\omega}_\mu{}^{ab} &= \omega_\mu{}^{ab}(e) + \kappa_\mu{}^{ab}, \\ \kappa_{\mu ab} &= \tfrac{1}{4}(\bar{\psi}_a\gamma_\mu\psi_b + \bar{\psi}_\mu\gamma_a\psi_b - \bar{\psi}_\mu\gamma_b\psi_a). \end{aligned} \tag{42}$$

The covariant derivative acting on the fermion is

$$D_\mu(\hat{\omega})\lambda^i = \left(\partial_\mu + \frac{1}{2}\hat{\omega}_\mu{}^{ab}\sigma_{ab} - \frac{3i}{4}\gamma_5 B_\mu\right)\lambda^i - \epsilon^{ijk}A^j_\mu\lambda^k$$

and the supercovariant field strength is

$$\hat{F}^{\,i}_{\mu\nu} = F^i_{\mu\nu} - \tfrac{1}{2}\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^i. \tag{43}$$

The action (41), besides having complex $SU(2)$ gauge and general coordinate invariances, is invariant under local supersymmetric transformations:

$$\delta A^i_\mu = \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^i, \qquad \delta\lambda^i = -\frac{1}{2}(\sigma^{\alpha\beta}\hat{F}^i_{\alpha\beta} + i\gamma_5\mathcal{D}^i)\epsilon,$$
$$\delta\mathcal{D}^i = -\frac{i}{2}\bar{\epsilon}\gamma_5\gamma^\mu\left(\hat{D}_\mu(\hat{\omega})\lambda^i + \frac{i}{2}\gamma_5\mathcal{D}^i\psi_\mu\right),$$
$$\delta e^a_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \qquad \delta B_\mu = -i\bar{\epsilon}\gamma_5\phi_\mu,$$
$$\delta\psi_\mu = D_\mu(\hat{\omega})\epsilon,$$

where

$$D_\mu(\hat{\omega})\epsilon \equiv \left(\partial_\mu + \frac{1}{2}\sigma_{ab}\hat{\omega}^{ab}_\mu - \frac{3i}{4}\gamma_5 B_\mu\right)\epsilon,$$
$$\hat{D}_\mu(\hat{\omega})\lambda^i \equiv D_\mu(\hat{\omega})\lambda^i + \frac{1}{2}\sigma^{\alpha\beta}\hat{F}^i{}_{\alpha\beta}\psi_\mu,$$
$$\phi_\mu \equiv \frac{1}{3}\gamma^\nu\left(D_\nu(\hat{\omega})\psi_\mu - D_\mu(\hat{\omega})\psi_\nu + \frac{1}{2e}\gamma_5\epsilon_{\mu\nu\alpha\beta}D^\alpha(\hat{\omega})\psi^\beta\right),$$
$$D_\mu(\hat{\omega})\psi_\nu \equiv \left(\partial_\mu + \frac{1}{2}\sigma_{ab}\hat{\omega}^{ab}_\mu - \frac{3i}{4}\gamma_5 B_\mu\right)\psi_\nu + \Gamma_{\mu\nu}{}^\lambda\psi_\lambda.$$

The action is also invariant under conformal transformations:

$$e'^a{}_\mu = \Omega e^a_\mu, \qquad \psi'_\mu = \Omega^{1/2}\psi_\mu, \qquad B'_\mu = B_\mu,$$
$$A'^i{}_\mu = A^i_\mu, \qquad \lambda' = \Omega^{-(3/2)}\lambda^i, \qquad \mathcal{D}'^i = \Omega^{-2}\mathcal{D}^i.$$

There is an additional invariance under so-called R symmetry, a local axial $U(1)$ (the associated gauge field is B_μ):

$$\delta A^i_\mu = \delta\mathcal{D}^i = \delta e^a_\mu = 0, \qquad \delta\lambda^i = \frac{3i}{4}\alpha\gamma_5\lambda^i,$$
$$\delta\psi_\mu = \frac{3i}{4}\alpha\gamma_5\psi_\mu, \qquad \delta B_\mu = \partial_\mu\alpha.$$

Finally, the action is also invariant under local superconformal transformations:

$$\delta e^a_\mu = 0, \qquad \delta\psi_\mu = -\gamma_\mu\eta, \qquad \delta B_\mu = i\bar{\eta}\gamma_5\psi_\mu,$$
$$\delta A^i_\mu = 0, \qquad \delta\lambda^i = 0, \qquad \delta\mathcal{D}^i = 0.$$

As in Sec. II, we have a complex action. There are no kinetic terms for the tetrad field e^a_μ, its superpartner Majorana ψ_μ, and the auxiliary axial gauge field B_μ.

A. Equations of motion

Variation of the action with respect to various fields (A^i_μ, λ^i, B_μ, e^a_μ, and ψ_μ) leads to the following equations of motion:

$$\delta A^i_\mu\!: \ \mathcal{D}_\mu(F^{i\mu\nu} + \bar{\lambda}^i\gamma^\alpha\sigma^{\mu\nu}\psi_\alpha) = \tfrac{1}{2}\epsilon^{ijk}\bar{\lambda}^j\gamma^\nu\lambda^k,$$
$$\delta\lambda^i\!: \ \hat{\not{D}}(\hat{\omega})\lambda^i = 0,$$
$$\delta B_\mu\!: \ \bar{\lambda}^i\gamma_5\gamma_\mu\lambda^i = 0 \Rightarrow \gamma_5\lambda^i = \pm\lambda^i,$$
$$\delta\psi_\mu\!: \ \sigma^{\alpha\beta}\hat{F}^i{}_{\alpha\beta}\gamma^\mu\lambda^i = 0,$$
$$\delta e^a_\mu\!: \ T_{\mu\nu} \equiv [F^i_{\mu\alpha} + \bar{\lambda}^i\gamma^\beta\sigma_{\mu\alpha}\psi_\beta]F^i{}_\nu{}^\alpha - \tfrac{1}{4}g_{\mu\nu}[F^{i\alpha\beta} + \bar{\lambda}^i\gamma^\rho\sigma^{\alpha\beta}\psi_\rho]F^i_{\alpha\beta} = 0, \tag{44}$$

where we have used the earlier equations in simplifying the last two equations, and the derivative $\mathcal{D}_\mu$ in the first equation is covariant with respect to both the complex $SU(2)$ gauge transformations and general coordinate transformations. While variations with respect to the gauge field A^i_μ and the fermions λ^i yield genuine equations of motion, those with respect to the fields B_μ, ψ_μ, and the tetrad e^a_μ give only constraints.

We now try to solve these equations. It is straightforward to check that the last three equations in (44) are solved by

$$\gamma_5\lambda^i = \mp\lambda^i, \qquad F^i_{\mu\nu} + \bar{\lambda}^i\gamma^\alpha\sigma_{\mu\nu}\psi_\alpha = \pm{}^*F^i_{\mu\nu}.$$

These, in turn, imply a generalized self-duality or anti-self-duality constraint equation for the supercovariant field strength of (43):

$$\gamma_5\lambda^i = \mp\lambda^i, \qquad \hat{F}^i_{\mu\nu} = \pm{}^*\hat{F}^i_{\mu\nu}. \tag{45}$$

These constraints make the δA^i_μ equation of motion (the first equation in (44) above) hold identically. As in the nonsupersymmetric case of Sec. II, for configurations satisfying these constraints, the $N = 1$ super Yang-Mills Lagrangian density (41) is a total divergence: $\mathcal{L} = \mp(e/4)F^i_{\mu\nu}{}^*F^{i\mu\nu} = \mp\partial_\mu J^\mu$.

There is a supersymmetric generalization of Urbantke type formulas (7) as

$$g^{-(1/4)}g_{\mu\nu} = (\det\hat{\phi}_{ij})^{-(1/2)}\hat{X}_{\mu\nu}, \qquad g^{1/4}g^{\mu\nu} = (\det\hat{\phi}_{ij})^{-(1/2)}\hat{Y}^{\mu\nu}, \tag{46}$$

where quantities $\hat{\phi}_{ij}$, $\hat{X}_{\mu\nu}$, and $\hat{Y}^{\mu\nu}$ are given in terms of the self-dual or anti-self-dual supercovariant field strength as

$$\hat{\phi}_{ij} = \pm\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}\hat{F}^i_{\mu\nu}\hat{F}^j_{\alpha\beta}$$

and

$$\hat{X}_{\mu\nu} = \pm\tfrac{2}{3}\epsilon^{\alpha\beta\sigma\delta}\epsilon^{ijk}\hat{F}^i_{\mu\alpha}\hat{F}^j_{\beta\sigma}\hat{F}^k_{\delta\nu},$$
$$\hat{Y}^{\mu\nu} = \tfrac{1}{3}\epsilon^{\mu\alpha\beta\gamma}\epsilon^{\lambda\nu\rho\sigma}\epsilon^{ijk}\hat{F}^i_{\beta\gamma}\hat{F}^j_{\alpha\lambda}\hat{F}^k_{\rho\sigma}.$$

To develop solutions of the constraint equations, we next write

$$\hat{F}^{\,i}_{\mu\nu} = \hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta}\Sigma^i_{\alpha\beta}. \tag{47}$$

We trade three complex fermions λ^i with six Majorana fermions $\lambda^{\alpha\beta}$ through the relation $\lambda^i = \lambda^{\alpha\beta}\Sigma^i_{\alpha\beta}$. Then the supercovariant $\hat{\mathcal{R}}$ tensor introduced above can be written as

$$\hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta} = \mathcal{R}_{\mu\nu}{}^{\alpha\beta} - \tfrac{1}{2}\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^{\alpha\beta}, \tag{48}$$

where $\mathcal{R}_{\mu\nu}{}^{\alpha\beta}$ is the same tensor as introduced in Sec. II: $F^i_{\mu\nu} = \mathcal{R}_{\mu\nu}{}^{\alpha\beta}\Sigma^i_{\alpha\beta}$.

Self-duality or anti-self-duality of the supercovariant field strength (45) implies

$${}^*\hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta} = \pm\tilde{\hat{\mathcal{R}}}_{\mu\nu}{}^{\alpha\beta}, \tag{49}$$

where as earlier * represents duality with respect to the first pair of indices $[\mu\nu]$ and tilde ˜ is duality with respect to the second pair $[\alpha\beta]$. As in Sec. II, this equation in turn leads to the supersymmetric generalization of Eq. (11):

$$\pm\hat{\mathcal{R}}_{\mu\nu}{}^{ab} - \hat{\mathcal{R}}^{ab}{}_{\mu\nu} = \Sigma_{\mu\nu}{}^{ab}\hat{\mathcal{R}} - e^{[a}_{[\mu}\hat{\mathcal{R}}^{b]}{}_{\nu]}. \tag{50}$$

We shall consider here only the self-dual case. In this case the constraint equation above is equivalent to the following two independent equations:

$$\hat{\mathcal{R}}_{\mu\nu} + \hat{\mathcal{R}}_{\nu\mu} = \tfrac{1}{2}g_{\mu\nu}\hat{\mathcal{R}}, \tag{51a}$$
$$\hat{\mathcal{R}}_{\mu\nu}{}^{ab} - \hat{\mathcal{R}}^{ab}{}_{\mu\nu} = \tfrac{1}{2}e^{[a}_{[\mu}(\hat{\mathcal{R}}_{\nu]}{}^{b]} - \hat{\mathcal{R}}^{b]}{}_{\nu]}). \tag{51b}$$

These fix 18 of the 36 independent components of $\hat{\mathcal{R}}_{\mu\nu}{}^{ab}$.

B. A self-dual solution of equations of motion: $N = 1$ supergravity

To solve the self-duality constraint and also the fermion equation of motion above, we parametrize

$$A^i_\mu = \hat{\omega}_\mu{}^{\alpha\beta}\Sigma^i_{\alpha\beta} \equiv [\omega_\mu{}^{\alpha\beta}(e) + \kappa_\mu{}^{\alpha\beta}]\Sigma^i_{\alpha\beta} \tag{52}$$

with contortion tensor $\kappa_{\mu\alpha\beta}$ as given in Eq. (42). Then

$$\mathcal{R}_{\mu\nu}{}^{\alpha\beta} \equiv R_{\mu\nu}{}^{\alpha\beta}(\hat{\omega}) = R_{\mu\nu}{}^{\alpha\beta}(\omega(e)) + s_{\mu\nu}{}^{\alpha\beta},$$

where

$$s_{\mu\nu}{}^{\alpha\beta} = \nabla_{[\mu}\kappa_{\nu]}{}^{\alpha\beta} + \kappa_{[\mu}{}^{\alpha\lambda}\kappa_{\nu]\lambda}{}^{\beta}.$$

For $s_{\mu\nu} = g^{\alpha\beta}s_{\mu\alpha\nu\beta}$ and $s = g^{\mu\nu}s_{\mu\nu}$ we have

$$s_{\mu\nu} = \nabla_\mu\kappa_\nu - (\nabla_\alpha - \kappa_\alpha)\kappa_{\mu\nu}{}^\alpha - \kappa_{\mu\alpha\beta}\kappa^{\alpha\beta}{}_\nu,$$
$$s = 2\nabla\cdot\kappa - \kappa^\mu\kappa_\mu - \kappa_{\mu\alpha\beta}\kappa^{\alpha\beta\mu},$$

where $\kappa_\mu = \kappa_{\alpha\mu}{}^\alpha$. Straightforward calculation yields

$$\begin{aligned}\hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta} - \hat{\mathcal{R}}^{\alpha\beta}{}_{\mu\nu} = {}& \tfrac{1}{2}\bar{\psi}^{[\alpha}\gamma^{\beta]}(\psi_{\mu\nu} + \lambda_{\mu\nu}) \\ & - \tfrac{1}{2}\bar{\psi}_{[\mu}\gamma_{\nu]}(\psi^{\alpha\beta} + \lambda^{\alpha\beta}) \\ & + \tfrac{1}{8}g^{\sigma[\alpha}\bar{\psi}^{\beta]}\gamma_{[\sigma}\psi_{\mu\nu]} \\ & - \tfrac{1}{8}g_{\sigma[\mu}\bar{\psi}_{\nu]}\gamma^{[\sigma}\psi^{\alpha\beta]},\end{aligned} \tag{53}$$

where $\psi_{\mu\nu} = D_{[\mu}(\hat{\omega})\psi_{\nu]} \equiv \partial_{[\mu}\psi_{\nu]} + \frac{1}{2}\sigma_{ab}\hat{\omega}_{[\mu}{}^{ab}\psi_{\nu]} - \frac{3i}{4}\gamma_5 B_{[\mu}\psi_{\nu]}$. From this we have

$$\hat{\mathcal{R}}_{\mu\nu} - \hat{\mathcal{R}}_{\nu\mu} = -\tfrac{1}{4}\bar{\psi}_{[\mu}\gamma^\sigma(\psi_{\nu\sigma]} + \lambda_{\nu\sigma]}) + \tfrac{1}{4}\bar{\psi}^\sigma\gamma_{[\mu}\lambda_{\nu\sigma]}.$$

Next, like in the last section, we use the generalized master equation (51a) to construct an expression for $[R_{\mu\nu}(\omega(e)) - \frac{1}{2}g_{\mu\nu}R(\omega(e))] = -t_{\mu\nu}$. This can easily be seen to be

$$t_{\mu\nu} = \tfrac{1}{2}[s_{\mu\nu} + s_{\nu\mu} - \tfrac{1}{2}g_{\mu\nu}s] + \tfrac{1}{4}g_{\mu\nu}R(\omega(e)) - \tfrac{1}{4}[\bar{\psi}_{[\mu}\gamma_{\alpha]}\lambda_\nu{}^\alpha + \bar{\psi}_{[\nu}\gamma_{\alpha]}\lambda_\mu{}^\alpha - \tfrac{1}{2}g_{\mu\nu}\bar{\psi}_{[\alpha}\gamma_{\beta]}\lambda^{\alpha\beta}]. \quad (54)$$

We seek solutions of $\nabla^\mu t_{\mu\nu} = 0$. Along with this the fermion equation $\hat{\not{\mathcal{D}}}(\hat{\omega})\lambda^i = 0$ with constraint $\gamma_5\lambda^i = -\lambda^i$ is also to be solved. Finally, the solution is given by $\lambda_{\mu\nu} = -\psi_{\mu\nu}$ and the following set of equations:

$$B_\mu = 0, \qquad \gamma_5\gamma^{\nu *}\psi_{\alpha\nu} = 0, \qquad R_{\mu\nu}(\hat{\omega}) = \tfrac{1}{2}\bar{\psi}^\alpha\gamma_5\gamma_\mu{}^*\psi_{\nu\alpha}. \quad (55)$$

Notice the second equation implies $\lambda_{\mu\nu} = -\psi_{\mu\nu} = \gamma_5{}^*\psi_{\mu\nu}$ and also $\gamma_{[\mu}\psi_{\nu\alpha]} = 0$. These in turn make the right-hand side of (53) identically zero:

$$\hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta} = R_{\mu\nu}{}^{\alpha\beta} + \tfrac{1}{2}\psi_{[\mu}\gamma_{\nu]}\psi^{\alpha\beta} = \hat{\mathcal{R}}^{\alpha\beta}{}_{\mu\nu} = {}^*\hat{\hat{\mathcal{R}}}_{\mu\nu}{}^{\alpha\beta}. \quad (56)$$

The last equation in (55) can be rewritten as

$$\hat{\mathcal{R}}_{\mu\nu} \equiv R_{\mu\nu}(\hat{\omega}) - \tfrac{1}{2}\bar{\psi}_{[\mu}\gamma_{\alpha]}\lambda_\nu{}^\alpha = 0.$$

Both the constraints (51) are satisfied. Thus (55) then provide a solution to the generalized self-duality constraint. Also the fermion equation of motion of super Yang-Mills theory is satisfied. To verify that this is so, using the equations (55) and (56) above and the implied equation $\gamma^\alpha\sigma_{\lambda\rho}\hat{\mathcal{R}}_{\alpha\mu}{}^{\lambda\rho} = 0$, it can be checked that the gravitino field strength $\psi_{ab} \equiv e_a^\mu e_b^\nu\psi_{\mu\nu}$ satisfies the following equation:

$$\not{\mathcal{D}}(\hat{\omega})\psi_{ab} - \tfrac{1}{2}\gamma^\mu\sigma^{cd}\hat{\mathcal{R}}_{cdab}\psi_\mu = \gamma_c(\kappa_{[a}{}^{cd} + \kappa^{cd}{}_{[a})\psi_{b]d} - \tfrac{1}{4}\gamma^e\sigma_{cd}\psi_{[a}\bar{\psi}_b\gamma_{e]}\psi^{cd} = 0, \quad (57)$$

where the second step follows by Fierz rearrangement. This equation is equivalent to the λ^i equation of motion in (44).

Equations (55) can readily be recognized as the equations of motion of $N = 1$ Poincaré supergravity [13,15]. These describe the dynamics of the Poincaré supermultiplet of physical fields e_μ^a and ψ_μ and the auxiliary axial vector field B_μ, and are governed by an effective linear R Lagrangian density:

$$e^{-1}\mathcal{L}_{\text{eff}} = \frac{1}{2}R(e,\hat{\omega}) - \frac{1}{2e}\epsilon^{\mu\nu\alpha\beta}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\alpha(\hat{\omega})\psi_\beta + \frac{3}{4}B^\mu B_\mu. \quad (58)$$

Thus, starting from the $N = 1$ supersymmetric complex $SU(2)$ gauge theory we have obtained a solution for its equations of motion which is described by $N = 1$ Poincaré supergravity equations of motion. Clearly this is not the most general solution of the self-dual case. A more general solution would include supermultiplets of a dilaton and an axion coupled to the gravity supermultiplet along with a cosmological constant in a (super)conformally invariant manner as a complete supersymmetric generalization of the gravity solution of the self-duality constraint obtained in Sec. II. Such a solution, though more involved, can be developed by the same method as described above. It would exhibit all the symmetries, including conformal and superconformal symmetries, of the starting action (41).

IV. $N = 2$ SUPERSYMMETRIC COMPLEX $SU(2)$ GAUGE THEORY

The next level of supersymmetric generalization of Einstein gravity is $N = 2$ supergravity [15–18]. As earlier, this is to be obtained from the self-dual sector of the conformally invariant $N = 2$ supersymmetric complex $SU(2)$ Yang-Mills theory. The $N = 2$ Yang-Mills multiplet consists of two complex $SU(2)$ triplet fermion fields, each made up of two Majorana fermions, $\Psi^{iI} = \Psi^{(1)iI} + i\Psi^{(2)iI}$ ($I = 1, 2$), containing eight off-shell complex triplet fermionic degrees of freedom. We shall split their left and right handed chiral components: $\lambda_I^i = (1/2)(1 + \gamma_5)\Psi^{iI}$ and $\lambda^{iI} = (1/2)(1 - \gamma_5)\Psi^{iI}$ so that $\gamma_5\lambda_I^i = \lambda_I^i$ and $\gamma_5\lambda^{iI} = -\lambda^{iI}$. There is additional compact $SU(2)$ symmetry [which, along with a $U(1)$ axial symmetry, forms the R symmetry group] which acts on the upper and lower chiral indices I so that chirality and transformation properties under this real $SU(2)$ are in direct correspondence. An equal number of off-shell bosonic degrees of freedom consist of (1) a complex gauge field A_μ^i, (2) two scalar fields, $X^i = X^{(1)i} + iX^{(2)i}$ and its charge conjugate $\bar{X}^i = X^{(1)i} - iX^{(2)i}$ (both $X^{(1)i}$ and $X^{(2)i}$ are complex), and (3) a symmetric auxiliary field $Y_{IJ}^i = Y_{IJ}^{(1)i} + iY_{IJ}^{(1)i} = Y_{JI}^i$ and its conjugate $\bar{Y}^{iIJ} = Y^{(1)iIJ} - iY^{(2)iIJ} = \epsilon^{IK}\epsilon^{JL}(Y_{KL}^{(1)i} - iY_{KL}^{(2)i})$. We need a conformally invariant supersymmetric action coupling these fields to the background $N = 2$ off-shell superconformal gravity multiplet. This background supermultiplet contains 24 off-shell fermionic degrees of freedom consisting of two Majorana gravitinos with chiral components ψ_μ^I and $\psi_{I\mu}$ ($\gamma_5\psi_\mu^I = \psi_\mu^I$, $\gamma_5\psi_{I\mu} = -\psi_{I\mu}$) and additional Majorana fermion fields with chiral components ϕ^I and ϕ_I ($\gamma_5\phi_I = \phi_I$, $\gamma_5\phi^I = -\phi^I$). An equal number of bosonic degrees of freedom are contained in the tetrad e_μ^a, antisymmetric $T_{\mu\nu}^{-IJ}$ (anti-self-dual in μ, ν and antisymmetric in I, J) and its charge conjugate self-dual $T_{IJ}^{+\mu\nu}$, a scalar field f, and an anti-Hermitian gauge field $V_\mu{}^I{}_J = (V_{\mu I}{}^J)^* = -V_{\mu J}{}^I$ ($V_\mu{}^I{}_I = 0$) and an axial gauge field B_μ of the associated real $SU(2)$ and $U(1)$ of the R symmetry group.

Action for general $N = 2$ super Yang-Mills theory in $N = 2$ superconformal gravity background has been worked out in Refs. [16,19]. Complex $SU(2)$ super Yang-Mills Lagrangian density $\mathcal{L}$ is given by

$$e^{-1}\mathcal{L} = -D_\mu \bar{X}^i D^\mu X^i + 2f\bar{X}^i X^i + \frac{1}{8}Y^i_{IJ}\bar{Y}^{iIJ} + (\epsilon^{ijk}X^j\bar{X}^k)^2 - \frac{1}{4}(\hat{F}^{i+}_{\mu\nu})^2 + X^i\hat{F}^{i\mu\nu}T^+_{\mu\nu IJ}\epsilon^{IJ} - \frac{1}{2}X^iX^i(T^+_{\mu\nu IJ}\epsilon^{IJ})^2$$
$$+ 2\bar{\phi}_I\lambda^{iI}X^i - \frac{1}{2}\bar{\lambda}^{iI}\not{D}\lambda^i_I - \epsilon^{IJ}\epsilon^{ijk}\bar{\lambda}^i_I\bar{X}^j\lambda^k_J - 2\bar{\lambda}^{iI}\gamma_\mu\psi^J_\nu T^{+\mu\nu}_{IJ}X^i + \bar{\psi}^I_\mu\not{D}\bar{X}^i\gamma^\mu\lambda^i_I - \epsilon^{ijk}\bar{\lambda}^{iI}\gamma^\mu\psi^J_\mu\epsilon_{IJ}X^j\bar{X}^k$$
$$+ \bar{\psi}^\mu_I\psi^\nu_J T^{+IJ}_{\mu\nu}X^i\bar{X}^i + \frac{1}{2}(\bar{\lambda}^{iI}\gamma_\mu\psi^J_\nu\epsilon_{IJ} + \bar{\psi}_{I\mu}\psi_{J\nu}\epsilon^{IJ}X^i)\,{}^*\hat{F}^{i\mu\nu} - \frac{1}{2e}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{I\mu}\gamma_\nu\psi^I_\rho D_\sigma X^i\bar{X}^i - \frac{1}{4}\bar{\lambda}^{iI}\gamma^\mu\gamma^\nu\psi_{I\mu}\bar{\psi}^J_\nu\lambda^i_J$$
$$- \frac{1}{4}\epsilon^{IK}\epsilon^{JL}(\bar{\psi}_{I\mu}\sigma^{\mu\nu}\psi_{J\nu}\bar{\lambda}^i_K\lambda^i_L - \bar{\psi}_{I\mu}\psi_{J\nu}\bar{\lambda}^i_K\sigma^{\mu\nu}\lambda^i_L) - \frac{1}{2e}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{I\mu}\gamma_\nu\psi^J_\rho(\bar{\psi}^I_\sigma\lambda^i_J - \delta^I_J\bar{\psi}^K_\sigma\lambda^i_K)\bar{X}^i$$
$$+ \frac{1}{8e}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{I\mu}\psi_{J\nu}\epsilon^{IJ}\epsilon^{KL}(2\bar{\psi}_{K\rho}\gamma_\sigma\lambda^i_L + \bar{\psi}_{K\rho}\psi_{L\sigma}X^i)X^i + \text{c.c.} \quad (59)$$

where the supercovariant complex $SU(2)$ gauge field strength is

$$\hat{F}^{\,i}_{\mu\nu} = F^i_{\mu\nu} - (\tfrac{1}{2}\bar{\psi}_{I[\mu}\gamma_{\nu]}\lambda^i_J\epsilon^{IJ} + \bar{\psi}_{I\mu}\psi_{J\nu}\epsilon^{IJ}X^i + \text{c.c.}) \quad (60)$$

and $F^{i\pm}_{\mu\nu} \equiv 1/2(F^i_{\mu\nu} \pm {}^*F^i_{\mu\nu})$ are self- and anti-self-dual combinations of the field strength. The covariant derivatives of scalar fields are

$$D_\mu X^i = \left(\partial_\mu - \frac{i}{2}B_\mu\right)X^i - \epsilon^{ijk}A^j_\mu X^k,$$
$$D_\mu \bar{X}^i = \left(\partial_\mu + \frac{i}{2}B_\mu\right)\bar{X}^i - \epsilon^{ijk}A^j_\mu \bar{X}^k \quad (61)$$

and those for fermions are

$$D_\mu\lambda^i_I = \left(\partial_\mu + \frac{1}{2}\sigma^{ab}\hat{\omega}_{\mu ab} - \frac{i}{4}B_\mu\right)\lambda^i_I - \epsilon^{ijk}A^j_\mu\lambda^k_I + V_{\mu I}{}^J\lambda^i_J,$$
$$D_\mu\lambda^{iI} = \left(\partial_\mu + \frac{1}{2}\sigma^{ab}\hat{\omega}_{\mu ab} + \frac{i}{4}B_\mu\right)\lambda^{iI} - \epsilon^{ijk}A^j_\mu\lambda^{kI} + V_\mu{}^I{}_J\lambda^{iJ}. \quad (62)$$

The supercovariant spin connection contains the ψ^I_μ torsion and is

$$\hat{\omega}_{\mu ab} = \omega_{\mu ab}(e) + \kappa_{\mu ab},$$
$$\kappa_{\mu ab} = \tfrac{1}{4}(\bar{\psi}^I_\mu\gamma_a\psi_{Ib} - \bar{\psi}^I_\mu\gamma_b\psi_{Ia} + \bar{\psi}^I_a\gamma_\mu\psi_{Ib} + \text{c.c.}). \quad (63)$$

Here c.c. stands for charge conjugation which for various fields acts as $X^i \leftrightarrow \bar{X}^i$, $Y^i_{IJ} \leftrightarrow \bar{Y}^{iIJ}$, $\hat{F}^{i+}_{\mu\nu} \leftrightarrow \hat{F}^{i-}_{\mu\nu}$, $T^+_{\mu\nu IJ} \leftrightarrow T^{-IJ}_{\mu\nu}$, $\lambda^{iI} \leftrightarrow \lambda^i_I$, $V_\mu{}^I{}_J \leftrightarrow (V_\mu{}^I{}_J)^* = V_{\mu I}{}^J$, $\psi_{I\mu} \leftrightarrow \psi^I_\mu$, $\phi^I \leftrightarrow \phi_I$, and also $e \rightarrow e^* = -e$. Further, it is useful to introduce a generalized supercovariant complex $SU(2)$ gauge field strength:

$$\mathcal{F}^{\,i}_{\mu\nu} = \hat{F}^i_{\mu\nu} - X^iT^+_{\mu\nu IJ}\epsilon^{IJ} - \bar{X}^iT^{-IJ}_{\mu\nu}\epsilon_{IJ}. \quad (64)$$

Like in earlier sections, we introduce the tensors $\mathcal{R}_{\mu\nu}{}^{\alpha\beta}$, $\hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta}$ and in addition the fermionic tensor fields $\lambda^I_{\mu\nu}$, $\lambda_{I\mu\nu}$ and bosonic tensor fields $\phi_{\mu\nu}$, $\bar{\phi}_{\mu\nu}$ through

$$F^i_{\mu\nu} = \mathcal{R}_{\mu\nu}{}^{\alpha\beta}\Sigma^i_{\alpha\beta}, \qquad \mathcal{F}^i_{\mu\nu} = \hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta}\Sigma^i_{\alpha\beta},$$
$$\lambda^i_I = \lambda^{\mu\nu}_I\Sigma^i_{\mu\nu}, \qquad \lambda^{iI} = \lambda^{I\mu\nu}\Sigma^i_{\mu\nu}, \quad (65)$$
$$X^i = \phi^{\mu\nu}\Sigma^i_{\mu\nu}, \qquad \bar{X}^i = \bar{\phi}^{\mu\nu}\Sigma^i_{\mu\nu}.$$

Then from the generalized supercovariant field strength (64), we have the generalized covariant curvature tensor as

$$\hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta} = \mathcal{R}_{\mu\nu}{}^{\alpha\beta} - [\tfrac{1}{2}\bar{\psi}^I_{[\mu}\gamma_{\nu]}\lambda^{J\alpha\beta}\epsilon_{IJ} + \bar{\psi}^I_\mu\psi^J_\nu\epsilon_{IJ}\bar{\phi}^{\alpha\beta} + T^{-IJ}_{\mu\nu}\epsilon_{IJ}\bar{\phi}^{\alpha\beta} + \text{c.c.}], \quad (66)$$

where $(\phi_{\mu\nu})^{\text{c.c.}} = \bar{\phi}_{\mu\nu}$ and $(\lambda_{I\mu\nu})^{\text{c.c.}} = \lambda^I_{\mu\nu}$.

A. Equations of motion

Variations of the action with respect to fields f, ϕ_I, ϕ^I, $T^+_{\mu\nu IJ}$, and $T^{-IJ}_{\mu\nu}$ yield the following equations, respectively:

$$X^i\bar{X}^i = 0, \quad \lambda^i_I\bar{X}^i = 0, \quad \lambda^{iI}X^i = 0, \quad X^i[\hat{F}^{i+}_{\mu\nu} - X^iT^+_{\mu\nu IJ}\epsilon^{IJ} + \tfrac{1}{2}(\bar{\psi}_{I\mu}\psi_{J\nu}\epsilon^{IJ})^+\bar{X}^i - \tfrac{1}{2}(\bar{\lambda}^{iI}\gamma_{[\mu}\psi^J_{\nu]}\epsilon_{IJ})^+] = 0,$$
$$\bar{X}^i[\hat{F}^{i-}_{\mu\nu} - \bar{X}^iT^{-IJ}_{\mu\nu}\epsilon_{IJ} + \tfrac{1}{2}(\bar{\psi}^I_\mu\psi^J_\nu\epsilon_{IJ})^-X^i - \tfrac{1}{2}(\bar{\lambda}^i_I\gamma_{[\mu}\psi_{\nu]J}\epsilon^{IJ})^-] = 0.$$

These equations have two sets of solutions:

$$\text{(i)} \quad \lambda^i_I = 0, \quad X^i = 0, \quad \hat{F}^{i-}_{\mu\nu} = \bar{X}^iT^{-IJ}_{\mu\nu}\epsilon_{IJ}, \quad (67)$$

$$\text{(ii)} \quad \lambda^{iI} = 0, \quad \bar{X}^i = 0, \quad \hat{F}^{i+}_{\mu\nu} = X^iT^+_{\mu\nu IJ}\epsilon^{IJ}. \quad (68)$$

The generalized supercovariant complex $SU(2)$ gauge field strength $\mathcal{F}^i_{\mu\nu}$ introduced in (64) is self-dual and anti-self-dual, respectively, for these two solutions.

Variations of the action with respect to fields B_μ, $V_\mu{}^I{}_J$, gauge field A^i_μ, tetrad e^a_μ, and gravitinos ψ^I_μ, $\psi_{I\mu}$ are identically zero when solution (67) or (68) is used. In particular, the stress-energy tensor $T_{\mu\nu}$ obtained by variation with respect to e^a_μ is zero for the two solutions (67) and (68).

For the case of self-dual solution (67), variations of the action with respect to λ^{iI} and $\bar{X}^i$ are also identically zero. But variations with respect to λ^i_I and X^i yield additional equations of motion for the fermion field λ^{iI} and the scalar field $\bar{X}^i$. The fermion equation of motion is

$$\hat{\not{D}}\lambda^{iI} - 2\bar{X}^i\phi^I = 0, \tag{69}$$

where the supercovariant derivative of the fermion is

$$\hat{D}_\mu\lambda^{iI} \equiv D_\mu\lambda^{iI} - \tfrac{1}{2}\sigma^{ab}\hat{F}^{i+}_{ab}\psi_{J\mu}\epsilon^{IJ} - \hat{\not{D}}\bar{X}^i\psi^I_\mu \tag{70}$$

and the supercovariant derivative of the scalar field is

$$\hat{D}_\mu\bar{X}^i \equiv D_\mu\bar{X}^i - \tfrac{1}{2}\bar{\lambda}^{iJ}\psi_{J\mu}. \tag{71}$$

The scalar field equation of motion is

$$D_a\hat{D}^a\bar{X}^i - \tfrac{1}{2}\bar{\psi}_{Ia}\hat{D}^a\lambda^{iI} + \tfrac{1}{2}\epsilon^{ijk}\bar{\psi}^I_a\gamma^a\lambda^{jJ}\epsilon_{IJ}\bar{X}^k - \tfrac{1}{2}\bar{\lambda}^{iI}\gamma_a\psi^J_bT^{+ab}{}_{IJ} + \tfrac{1}{2}\epsilon^{ijk}\bar{\lambda}^{jI}\lambda^{kJ}\epsilon_{IJ} + \tfrac{1}{2}\hat{F}^i_{ab}T^{+ab}{}_{IJ}\epsilon^{IJ} + \tfrac{1}{2}\bar{\lambda}^{iI}(\phi_I + \sigma^{ab}\psi_{Iab}) + [2f + \bar{\psi}^a_I\gamma_a\phi^I - \tfrac{1}{4}\bar{\psi}^I_a\gamma^*_b\psi^{ab}_I - \tfrac{1}{4}\bar{\psi}_{Ia}\gamma^*_b\psi^{Iab}]\bar{X}^i = 0, \tag{72}$$

where the derivative with the Lorentz index $D_a = e^\mu_a D_\mu$ and the supercovariant gravitino field strengths are

$$\psi_{I\mu\nu} \equiv \hat{D}_{[\mu}\psi_{\nu]I} \equiv D_{[\mu}\psi_{\nu]I} - \gamma^\sigma T^+_{IJ\sigma[\mu}\psi^J_{\nu]}, \qquad \psi^I_{\mu\nu} \equiv \hat{D}_{[\mu}\psi^I_{\nu]} \equiv D_{[\mu}\psi^I_{\nu]} - \gamma^\sigma T^{-IJ}_{\sigma[\mu}\psi_{\nu]J}. \tag{73}$$

On the other hand, for the anti-self-dual case (68), variations of the action with respect to λ^i_I and X^i are identically zero and those with respect to λ^{iI} and $\bar{X}^i$ yield equations of motion for the fermion λ^i_I and the scalar field X^i which are the conjugate versions of Eqs. (69) and (72) above:

$$\hat{\not{D}}\lambda^i_I - 2X^i\phi_I = 0,$$

$$D_a\hat{D}^aX^i - \tfrac{1}{2}\bar{\psi}^I_a\hat{D}^a\lambda^i_I + \tfrac{1}{2}\epsilon^{ijk}\bar{\psi}_{Ia}\gamma^a\lambda^j_J\epsilon^{IJ}X^k - \tfrac{1}{2}\bar{\lambda}^i_I\gamma^a\psi^b_JT^{-IJ}_{ab} + \tfrac{1}{2}\epsilon^{ijk}\bar{\lambda}^j_I\lambda^k_J\epsilon^{IJ} + \tfrac{1}{2}\hat{F}^{iab}T^{-IJ}_{ab}\epsilon_{IJ} + \tfrac{1}{2}\bar{\lambda}^i_I(\phi^I + \sigma^{ab}\psi^I_{ab}) + [2f + \bar{\psi}^I_a\gamma^a\phi_I - \tfrac{1}{4}\bar{\psi}^I_a\gamma_b{}^*\psi^{ab}_I - \tfrac{1}{4}\bar{\psi}_{Ia}\gamma_b{}^*\psi^{Iab}]X^i = 0, \tag{74}$$

where the supercovariant derivatives are

$$\hat{D}_\mu\lambda^i_I \equiv D_\mu\lambda^i_I - \tfrac{1}{2}\sigma^{ab}\hat{F}^{i-}_{ab}\psi^J_\mu\epsilon_{IJ} - \hat{\not{D}}X^i\psi_{I\mu}, \qquad \hat{D}_\mu X^i \equiv D_\mu X^i - \tfrac{1}{2}\bar{\lambda}^i_I\psi^I_\mu.$$

As in earlier sections, for configurations obeying self-dual (67) or anti-self-dual (68) solutions, the Lagrangian density (59) is a total divergence: $\mathcal{L} = \pm(e/4)\,{}^*F^{i\mu\nu}F^i_{\mu\nu} = \pm\partial_\mu J^\mu$.

B. $N = 2$ supergravity as a self-dual solution of the equations of motion

We wish to solve the self-duality constraints (67) and the associated fermion equation (69) and scalar field equation (72). We parametrize the complex gauge field A^i_μ as in (52): $A^i_\mu = \hat{\omega}_\mu{}^{\alpha\beta}\Sigma^i_{\alpha\beta}$, where now $\hat{\omega}_\mu{}^{\alpha\beta}$ is the $N = 2$ supercovariantized spin connection given by (63). Self-duality implies the same two independent constraints (51) as in the $N = 1$ case but with the tensor $\hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta}$ now given by the generalized $N = 2$ expression (66). Notice for this solution that $\lambda^I_{\mu\nu}$ and $\bar{\phi}_{\mu\nu}$ are self-dual and $\lambda_{I\mu\nu}$ and $\phi_{\mu\nu}$ are anti-self-dual.

An analogous calculation to that for Eq. (53) of the $N = 1$ theory here yields the relation

$$\begin{aligned}\hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta} - \hat{\mathcal{R}}^{\alpha\beta}{}_{\mu\nu} = {}& \tfrac{1}{2}[\bar{\psi}^{[\alpha}_I\gamma^{\beta]}(\psi^I_{\mu\nu} + \epsilon^{IJ}\lambda_{J\mu\nu}) - \bar{\psi}_{I[\mu}\gamma_{\nu]}(\psi^{I\alpha\beta} + \epsilon^{IJ}\lambda^{\alpha\beta}_J)] + \tfrac{1}{8}[g^{\sigma[\alpha}\bar{\psi}^{\beta]}_I\gamma_{[\sigma}\psi^I_{\mu\nu]} - g_{\sigma[\mu}\bar{\psi}_{\nu]I}\gamma^{[\sigma}\psi^{\alpha\beta]I}] \\ & + [\bar{\psi}_{I\mu}\psi_{J\nu}(T^{-\alpha\beta IJ} - \epsilon^{IJ}\phi^{\alpha\beta}) - \bar{\psi}^\alpha_I\psi^\beta_J(T^{-IJ}{}_{\mu\nu} - \epsilon^{IJ}\phi_{\mu\nu})] \\ & - [T^+_{\mu\nu IJ}\epsilon^{IJ}\phi^{\alpha\beta} - T^{+\alpha\beta}{}_{IJ}\epsilon^{IJ}\phi_{\mu\nu}] + \text{c.c.}\end{aligned} \tag{75}$$

$N = 2$ generalization of the equation (54) of the $N = 1$ case is

$$\hat{\mathcal{R}}_{\mu\nu} - \hat{\mathcal{R}}_{\nu\mu} = -\tfrac{1}{4}[\bar{\psi}_{I[\mu}\gamma^\sigma(\psi^I_{\nu\sigma]} + \epsilon^{IJ}\lambda_{\nu\sigma]J}) - \bar{\psi}^{I\sigma}\gamma_{[\mu}\lambda^I_{\nu\sigma]}\epsilon_{IJ}] + [\bar{\psi}_{I[\mu}\psi^\sigma_J(T^{-IJ}{}_{\nu]\sigma} - \epsilon^{IJ}\phi_{\nu]\sigma}) - T^{-\sigma IJ}_{[\mu}\bar{\phi}_{\nu]\sigma}\epsilon_{IJ}] + \text{c.c.} \tag{76}$$

Self-duality constraints (67) are solved if the following hold:

$$B_\mu = 0, \qquad V_\mu{}^I{}_J = 0, \qquad \phi^I = 0, \qquad \phi_I = 0, \qquad f = 0, \qquad \lambda^I_{\mu\nu} = \lambda^{I+}_{\mu\nu} = \epsilon^{IJ}\psi_{J\mu\nu},$$
$$\lambda_{I\mu\nu} = \lambda^-_{I\mu\nu} = \epsilon_{IJ}\psi^J_{\mu\nu}, \qquad T^{-IJ}_{\mu\nu} = \epsilon^{IJ}\phi_{\mu\nu} = \frac{1}{\sqrt{2}}\hat{F}^-_{\mu\nu}\epsilon^{IJ}, \qquad T^+_{\mu\nu IJ} = \epsilon_{IJ}\bar{\phi}_{\mu\nu} = \frac{1}{\sqrt{2}}\hat{F}^+_{\mu\nu}\epsilon_{IJ},$$

where the supercovariant field strength for the Abelian gauge field A_μ is

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{1}{\sqrt{2}}(\bar{\psi}^I_\mu\psi^J_\nu\epsilon_{IJ} + \bar{\psi}_{I\mu}\psi_{J\nu}\epsilon^{IJ}), \qquad F_{\mu\nu} = \partial_{[\mu}A_{\nu]}.$$

Then self-duality equations are satisfied if

$$\gamma^\mu\psi^I_{\mu\nu} = 0, \qquad \gamma^\mu\psi_{I\mu\nu} = 0,$$
$$\hat{\mathcal{R}}_{\mu\nu} = R_{\mu\nu}(\hat{\omega}, e) - \frac{1}{2}(\bar{\psi}^\alpha_I\gamma_\mu\psi^I_{\nu\alpha} + \bar{\psi}^{I\alpha}\gamma_\mu\psi_{I\nu\alpha}) - \frac{1}{\sqrt{2}}(\bar{\psi}^I_\mu\psi^J_\alpha\epsilon_{IJ}\hat{F}^+{}_\nu{}^\alpha + \bar{\psi}_{I\mu}\psi_{J\alpha}\epsilon^{IJ}\hat{F}^-{}_\nu{}^\alpha) - 2\hat{F}^+_{\mu\alpha}\hat{F}^-_\nu{}^\alpha = 0,$$
$$\hat{D}_a\hat{F}^{+ab} \equiv D_a(\hat{\omega})\hat{F}^{+ab} - \frac{1}{\sqrt{2}}\bar{\psi}_{Ia}\psi^{ab}_J\epsilon^{IJ} = 0, \qquad \hat{D}_a\hat{F}^{-ab} \equiv D_a(\hat{\omega})\hat{F}^{-ab} - \frac{1}{\sqrt{2}}\bar{\psi}^I_a\psi^{Jab}\epsilon_{IJ} = 0. \tag{77}$$

These make the right-hand sides of Eqs. (75) and (76) identically zero. Other equations of motion of the Yang-Mills theory are also satisfied. It can be checked that Eqs. (77) imply the following equations for the gravitino field strengths:

$$\begin{aligned} \not{D}(\hat{\omega})\psi^I_{ab} - \frac{1}{2}\gamma^\mu\sigma_{cd}\hat{\mathcal{R}}^{cd}{}_{ab}\psi^I_\mu + \frac{1}{\sqrt{2}}\gamma^\mu\hat{\not{D}}\hat{F}^-_{ab}\psi_{J\mu}\epsilon^{IJ} = 0, \\ \not{D}(\hat{\omega})\psi_{Iab} - \frac{1}{2}\gamma^\mu\sigma_{cd}\hat{\mathcal{R}}^{cd}{}_{ab}\psi_{I\mu} + \frac{1}{\sqrt{2}}\gamma^\mu\hat{\not{D}}\hat{F}^+_{ab}\psi^J_\mu\epsilon_{IJ} = 0, \end{aligned} \tag{78}$$

where

$$\hat{\mathcal{R}}_{ab}{}^{cd} = R_{ab}{}^{cd}(\hat{\omega}, e) + \left[\frac{1}{2}\bar{\psi}^I_{[a}\gamma_{b]}\psi^{cd}_I - \frac{1}{\sqrt{2}}\bar{\psi}^I_a\psi^J_b\epsilon_{IJ}\hat{F}^{+cd} - \hat{F}^-_{ab}\hat{F}^{+cd} + \text{c.c.}\right].$$

Contracting (78) with Σ^i_{ab}, the left-hand side of the first equation is identically zero (ψ^I_{ab} is anti-self-dual) and the second equation is the fermion equation (69) for $\phi^I = 0$. It can also be checked that the field equation (72) for scalar field $\bar{X}^i$ is satisfied by the above solution.

The self-dual solutions (77) are equations of motion of the $N = 2$ supergravity action [17,18]:

$$e^{-1}\mathcal{L}_{\text{eff}} = \frac{1}{2}R(\hat{\omega}, e) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2e}\epsilon^{\mu\nu\alpha\beta}[\bar{\psi}^I_\mu\gamma_\nu D_\alpha(\hat{\omega})\psi_{I\beta} - \bar{\psi}_{I\mu}\gamma_\nu D_\alpha(\hat{\omega})\psi^I_\beta] + \frac{1}{2\sqrt{2}}[\bar{\psi}^I_\mu\psi^J_\nu\epsilon_{IJ}(F^{+\mu\nu} + \hat{F}^{+\mu\nu}) + \bar{\psi}_{I\mu}\psi_{J\nu}\epsilon^{IJ}(F^{-\mu\nu} + \hat{F}^{-\mu\nu})].$$

Clearly a more general solution of the self-dual constraint (67) and the associated fermion equation (69) and scalar equation (72) would involve $N = 2$ supermultiplets of a dilaton and an axion coupled to a gravity supermultiplet in a (super)conformally invariant manner as an $N = 2$ generalization of the gravity solution of the self-dual constraint of Sec. II.

V. CONCLUDING REMARKS

We have presented a gauge theory formulation of gravity based on the complex $SU(2)$ group. The action functional is quadratic in field strength. Here both the complex gauge field A^i_μ and the metric $g_{\mu\nu}$ are varied. There is no dynamics for the metric to start with. Varying the action with respect to the metric gives a constraint equation, which is solved by self-dual or anti-self-dual field strengths. This then relates the metric to the gauge field. Einstein gravity equations of motion follow from the self-dual constraint. Though the starting action has only a complex dimensionless coupling, dimensionful constants, in particular, Newton's gravitational constant, appear as parameters in the space of solutions.

This theory has some similarities with the Ashtekar formulation of gravity. But there are some characteristic differences: (i) The action in the Ashtekar approach is linear in field strength, whereas it is quadratic here. (ii) The equations of motion ultimately obtained here are not pure gravity but gravity coupled to a dilaton and an axion in a conformally invariant manner. (iii) Symplectic structure is distinctly different. Canonical momentum conjugate to the gauge field A^i_I is not a densitized spatial triad as in the Ashtekar theory ($\kappa^{-2}e\Sigma^{itI}$), but like in ordinary gauge theories, it is given by $\Pi^{iI} = \tau e F^{itI}$. However, unlike other ordinary gauge theories, there is an additional constraint given by the self-duality or anti-self-duality condition of the field strength. Thus the symplectic structure is also different from other gauge theories. In fact, this makes the constrained Poisson bracket (Dirac bracket) of two gauge fields $A_I^i(t, \mathbf{x})$ and $A_J^j(t, \mathbf{y})$ nonzero.

The analysis has been extended to the $N = 1$ complex $SU(2)$ super Yang-Mills theory. This results in a generalized self-duality/anti-self-duality condition not for ordinary gauge field strength, but for supercovariantized field strength. Finally, for the self-dual case a solution of equations of motion is given by the equations of the $N = 1$ supergravity theory.

The discussion has also been extended to $N = 2$ complex $SU(2)$ super Yang-Mills theories. Results are similar to those for the $N = 1$ case. The self-duality/anti-self-duality holds for a generalized field strength which not only contains the usual gauge field strength and terms involving fermions but also other fields of the supersymmetric Yang-Mills and gravity multiplets. For the self-dual case, the analysis leads to $N = 2$ supergravity equations of motion.

This analysis can also be extended to the $N = 4$ complex $SU(2)$ supersymmetric gauge theory. Here self-duality of a more complicated generalized supercovariant $SU(2)$ field strength leads to the equations of motion of $N = 4$ supergravity.

A detailed discussion of $N = 4$ supergravity obtained from the self-duality constraint in the $N = 4$ complex $SU(2)$ super Yang-Mills theory and one-loop quantum corrections in such a theory will be presented elsewhere.

ACKNOWLEDGMENTS

Discussions with G. Date, S. Kalyana Rama, and J. Maharana are gratefully acknowledged.

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