Dynamics of the $O(N)$ model in a strong magnetic background field as a modified noncommutative field theory

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In the presence of a strong magnetic field, the effective action of a composite scalar field in a scalar $O(N)$ model is derived using two different methods. First, in the framework of world-line formalism, the 1PI n-point vertex function for the composites is determined in the limit of a strong magnetic field. Then, the n-point effective action of the composites is calculated in the regime of lowest Landau level dominance. It is shown that in the limit of a strong magnetic field, the results coincide and an effective field theory arises which is comparable with the conventional noncommutative field theory. In contrast to the ordinary case, however, the UV/IR mixing is absent in this modified noncommutative field theory.

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I. INTRODUCTION

In recent years, there has been remarkable interest in noncommutative geometry [1], which has made a dramatic appearance in string theory [2] and has made noncommutative field theory (NCFT) [3] an active field of study. The various aspects of noncommutative gauge theories have been extensively studied and a number of novel phenomena discovered (for a review see [4] and the references therein). The conventional noncommutative gauge theory is characterized by replacing the familiar product of functions by the Moyal $\star$ product and is therefore a nonlocal theory involving higher order derivatives between the fields. From a perturbative point of view, the theory consists therefore of planar and nonplanar diagrams. The latter are usually the source of the appearance of a certain duality between ultraviolet (UV) and infrared (IR) behavior of the theory. This UV/IR mixing phenomenon manifests itself in the singularity of the amplitudes in two limits of the small noncommutativity parameter θ and the large cutoff Λ of the theory [5].

Apart from these features noncommutative field theory exhibits certain dynamics of a quantum mechanical model in a strong magnetic field [6]. Recently the connection between the dynamics in relativistic field theories in a strong magnetic background field and that in NCFT has been studied in [7–10]. In [7], the nontrivial dynamics of the fermionic Nambu-Jona-Lasinio (NJL) model in a constant magnetic background is considered. The effective action of this theory is determined for a strong magnetic field in the regime of lowest Landau level (LLL) dominance and its dynamics is compared with a conventional noncommutative field theory. Similarly the chiral dynamics of QED and QCD is shown to be governed by a complicated nonlocal NCFT [8–10]. In all these cases, however, the emergent effective noncommutative field theory is different from the NCFT ones considered in the literature [3]. In particular, the UV/IR mixing [5], taking place in the conventional NCFT, is absent in these classes of "modified" noncommutative field theories.

In this paper, we will present another example of this phenomenon. Here, we will determine the effective action of a composite scalar field in a scalar $O(N)$ model in the presence of a strong magnetic background field using two different methods. In Sec. II, we will introduce the scalar $O(N)$ model in Euclidean spacetime and will compute the effective action for the composite scalar field. In Sec. III, in the framework of the world-line path integral method [11–14], the one-particle irreducible (1PI) n-point vertex function for the composite field will be determined in the limit of a strong magnetic background field. The emergent n-point function will be comparable with the n-point vertex function of a certain modified noncommutative field theory, where no UV/IR mixing occurs. In this case, as in the previous cases [7–10], besides the usual Moyal phase factor, an additional Gaussian-like form factor appears in the n-point vertex function of the composites. This exponentially damping form factor reflects an inner structure of the composites and is responsible for the removal of the UV/IR mixing.

In Sec. IV, the effective action of n composite fields will be calculated in the strong magnetic field limit, using the same method as in [7]. First the Green's function of the theory will be determined in the regime of LLL dominance. Then, using this effective propagator, the contribution of n composites to the full effective action of the theory will be determined. In Sec. V, the 1PI n-point vertex from Sec. III will be compared with the LLL n-point effective action from Sec. IV. We will show that here, as in the NJL model in the presence of a constant magnetic background [15], a dimensional reduction from $D = 4$ to $D = 2$ occurs in the longitudinal section of the effective theory where a free propagation of the composites is

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observed. In the transverse section, however, a modified noncommutative field theory arises where no UV/IR mixing occurs.

II. EFFECTIVE ACTION FOR THE COMPOSITE SCALAR FIELD

Let us start with the Lagrangian density of a scalar $O(N)$ model in Euclidean spacetime,

$$\mathcal{L} = -|D_\mu \Phi|^2 - m^2\Phi^*\Phi - \tfrac{1}{2}\lambda(\Phi^*\Phi)^2, \tag{2.1}$$

where $\Phi = (\phi_1, \phi_2, \ldots, \phi_N)$ and the covariant derivative D_μ is defined by

$$D_\mu \Phi = \partial_\mu \Phi + ieA_\mu \Phi. \tag{2.2}$$

To study the dynamics of the bound state formed in this theory,[1] we introduce the composite field $\sigma \equiv \lambda\Phi^*\Phi$ and a new coupling constant $g \equiv \lambda N$. The Lagrangian density (2.1) can then be given by

$$\mathcal{L} = -|D_\mu \Phi|^2 - m^2\Phi^*\Phi - \sigma\Phi^*\Phi + \frac{N}{2g}\sigma^2, \tag{2.3}$$

and the effective action for σ reads

$$\tilde{\Gamma}[\sigma] = \Gamma[\sigma] + \frac{N}{2g}\int d^4x\sigma^2, \tag{2.4}$$

where $\Gamma[\sigma]$ is found using the standard deformation

$$\begin{aligned} e^{\Gamma[\sigma]} &= \int \mathcal{D}\Phi^*\mathcal{D}\Phi \exp\Big(-\int d^4x[|D_\mu\Phi|^2 + m^2\Phi^*\Phi \\ &\quad + \sigma\Phi^*\Phi]\Big) \\ &= \int \mathcal{D}\Phi^*\mathcal{D}\Phi \exp\Big(-\int d^4x\Phi^*[-D_\mu D^\mu + m^2 \\ &\quad + \sigma]\Phi\Big) \\ &= \exp(-\mathrm{Tr}\ln[-D_\mu D^\mu + m^2 + \sigma]), \end{aligned} \tag{2.5}$$

and is therefore given by

$$\Gamma[\sigma] = -\mathrm{Tr}\ln[-D_\mu D^\mu + m^2 + \sigma]. \tag{2.6}$$

In the following two sections, we will calculate $\Gamma[\sigma]$ in the presence of a strong magnetic background field using two different methods. In Sec. III, we will calculate the 1PI n-point vertex function $\Gamma_{1\mathrm{PI}}[p_1, \ldots, p_n]$ using the world-line path integral formalism [11–14]. In Sec. IV, we will follow the method which was used in [7] to determine the effective action of a $(D + 1)$-dimensional NJL model in the regime of lowest Landau level (LLL) dominance.

[1] In Appendix A, we will show that even in a theory with one flavor a condensate is built in the presence of a strong magnetic field.

III. EFFECTIVE ACTION OF THE COMPOSITE FIELD FROM THE WORLD-LINE FORMALISM

The world-line formalism was originally introduced in [11] as a useful mathematical tool to study the effective field theory limit of the underlying string theory. In this formalism, all path integrals are manipulated into Gaussian form, and this reduces their computation to the calculation of world-line propagators and determinants. In this section, we will stay very close to the notation of [14],[2] where among many other examples the n-point amplitude of a massive φ^3 theory in the one-loop level and QED in a constant magnetic background field are calculated separately. Here, we will combine the results presented in [14], and determine the n-point vertex function of the composite field σ. At the end, we will consider the limit of a constant but strong magnetic field as background. In this way, we arrive at an effective vertex function which is comparable with the vertex function of a certain modified NCFT for the composite field σ, where, besides the Moyal $\star$ product of the conventional noncommutative field theory, a certain exponentially damping factor occurs in the interaction vertices of σ [7]. This factor is shown to play an important role in providing consistency of this class of modified NCFT [7–10].

Let us start with the world-line path integral representation for the effective action (2.6), which reads [12,14]

$$\Gamma[\sigma] = N\int_0^\infty \frac{dT}{T}e^{-m^2T} \times \int_{x(T)=x(0)} \mathcal{D}x(\tau)e^{-\int_0^T d\tau((1/4)\dot{x}^2 + ie\dot{x}\cdot A(x(\tau)) + \sigma(x(\tau)))}, \tag{3.1}$$

where coefficient N before the integral reflects the $2N$ degrees of freedom of our scalar $O(N)$ model.[3] This path integral is to be calculated using the Gauss formula

$$\int dx e^{-x\cdot\mathcal{O}\cdot x + 2b\cdot x} \sim (\det(\mathcal{O}))^{-(1/2)}e^{b\cdot\mathcal{O}^{-1}\cdot b}, \tag{3.2}$$

with $\mathcal{O}^{-1}$ the inverse of the operator $\mathcal{O}$. To build $\mathcal{O}^{-1}$, we have to be careful about the zero eigenvalues of $\mathcal{O}$, which are to be excluded from its spectrum. To deal with these zero modes,[4] contained in the coordinate path integral $\int \mathcal{D}x$ as constant loops, they are to be separated from their orthogonal nonzero modes $y(\tau)$ by evaluating $x(\tau)$ around the loop center of mass x_0. In other words, replacing $x^\mu(\tau)$ by $x^\mu(\tau) = x_0^\mu + y^\mu(\tau)$ with

[2] The reader is also referred to the excellent review of M. Strassler [12] for more useful examples and details.

[3] If we were working only with one neutral scalar field, this coefficient would be $\frac{1}{2}$.

[4] In the eigenvalue equation $\mathcal{O}\psi = \lambda\psi$ these zero modes correspond to constant eigenfunctions $\psi = \text{const}$.

$$\int_0^T d\tau y^\mu(\tau) = 0,$$

the coordinate path integral reduces to an integral over the relative coordinate $y(\tau)$,

$$\int \mathcal{D}x = \int dx_0 \int \mathcal{D}y. \tag{3.3}$$

In this way, the effective action $\Gamma[\sigma]$ is expressed by an effective Lagrangian $\mathcal{L}_{\rm eff}$, represented as an integral over the space of all loops with a fixed common center of mass x_0 [14],

$$\Gamma[\sigma] = \int dx_0 \mathcal{L}_{\rm eff}[\sigma; x_0]. \tag{3.4}$$

At this stage, we restrict the background to be constant. Using Fock-Schwinger gauge centered at x_0 we may take $A_\mu(x)$ in (3.1) to be of the form [13]

$$A_\mu(x) = \tfrac{1}{2} y^\nu F_{\nu\mu}, \tag{3.5}$$

where $F_{\mu\nu}$ is the constant field-strength tensor of A_μ. After removing the zero modes, the operator $\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau}$ becomes invertible. The world-line Green's function is then given by [14]

$$2\left\langle \tau_i \left| \left(\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau}\right)^{-1} \right| \tau_j \right\rangle \equiv \mathcal{G}_B(\tau_i, \tau_j),$$

where

$$\mathcal{G}_B(\tau_i, \tau_j) \equiv \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin \mathcal{Z}} e^{-i\mathcal{Z}\dot{G}_{Bij}} + i\mathcal{Z}\dot{G}_{Bij} - 1\right), \tag{3.6}$$

with $\mathcal{Z} \equiv eFT$ and

$$\begin{aligned} G_{Bij} &\equiv G_B(\tau_i, \tau_j) \equiv |\tau_i - \tau_j| - \frac{(\tau_i - \tau_j)^2}{T}, \\ \dot{G}_{Bij} &\equiv \dot{G}_B(\tau_i, \tau_j) \equiv \mathrm{sign}(\tau_i - \tau_j) - 2\frac{(\tau_i - \tau_j)}{T}. \end{aligned} \tag{3.7}$$

The Green's function (3.6) is used as the correlation function for the coordinate field

$$\langle y^\mu(\tau_i) y^\nu(\tau_j)\rangle = -g^{\mu\nu}\mathcal{G}_B(\tau_i, \tau_j). \tag{3.8}$$

To proceed, we also need to calculate the free world-line path integral

$$\begin{aligned} &\int \mathcal{D}y \exp\left(-\int_0^T d\tau \left(\frac{1}{4}\dot{y}^2 + ie\dot{y}\cdot A(y(\tau))\right)\right)\Big|_{A_\mu = (1/2)y^\nu F_{\mu\nu}} \\ &= \det{}'^{-(1/2)}\left[-\frac{d^2}{d\tau^2} + 2ieF\frac{d}{d\tau}\right] \\ &= (4\pi T)^{-(D/2)} \det{}'^{-(1/2)}\left[\mathbf{1} - 2ieF\left(\frac{d}{d\tau}\right)^{-1}\right] \\ &= (4\pi T)^{-(D/2)} \det{}^{-(1/2)}\left[\frac{\sin(eFT)}{eFT}\right]. \end{aligned} \tag{3.9}$$

This result is obtained using the method introduced in [14]. On the second and third lines, the primes denote the absence of the zero modes in the determinant.

Let us now turn back to the effective action $\Gamma[\sigma]$ from (3.1) containing the interaction term σ. To find the n-point vertex function, we recall from quantum field theory that the 1PI n-point function can be obtained from the one-loop action $\Gamma[\sigma]$ by an n-fold functional differentiation with respect to σ. In momentum space, this operation is implemented by replacing the background by a sum of plane waves,

$$\sigma(x) = \sum_{i=1}^n e^{ip_i\cdot x}, \tag{3.10}$$

and picking out the term containing every p_i only once. We arrive at

$$\Gamma_{\rm 1PI}[p_1, \ldots, p_n] = N(-1)^n \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_0^T \prod_{i=1}^n d\tau_i \int dx_0 \int \mathcal{D}y \exp\left(i\sum_{i=1}^n p_i \cdot x_i\right) e^{-\int_0^T d\tau ((1/4)\dot{x}^2 + ie\dot{x}\cdot A(x(\tau)))}. \tag{3.11}$$

Having $x_i \equiv x(\tau_i) = x_0 + y(\tau_i)$, the x_0 integral leads to energy-momentum conservation,

$$\int dx_0 \exp\left(ix_0 \cdot \sum_{i=1}^n p_i\right) = (2\pi)^D \delta\left(\sum_{i=1}^n p_i\right).$$

To perform the y integration, we use the result from (3.9). This leads to the following parameter integral:

$$\begin{aligned} \Gamma_{\rm 1PI}[p_1, \ldots, p_n] = {}& N(-1)^n (2\pi)^D \delta\left(\sum_{i=1}^n p_i\right) \int_0^\infty \frac{dT}{T}(4\pi T)^{-(D/2)} e^{-m^2 T} \det{}^{-(1/2)}\left[\frac{\sin(eFT)}{eFT}\right] \prod_{i=1}^n \int_0^T d\tau_i \\ &\times \exp\left(\frac{1}{2}\sum_{i,j=1}^n \mathcal{G}_B(\tau_i, \tau_j) p_i \cdot p_j\right). \end{aligned} \tag{3.12}$$

To proceed, we will replace $\mathcal{G}_B(\tau_i, \tau_j)$ by $\bar{\mathcal{G}}_B(\tau_i, \tau_j)$, which is defined by

$$\bar{\mathcal{G}}_B(\tau_i, \tau_j) \equiv \mathcal{G}_B(\tau, \tau) = \frac{T}{2Z}\left(\frac{e^{-i\dot{G}_{Bij}Z} - \cos Z}{\sin Z} + i\dot{G}_{Bij}\right), \tag{3.13}$$

with the coincidence limits

$$\mathcal{G}_B(\tau, \tau) = \frac{T}{2Z^2}(Z\cot Z - 1), \qquad \dot{\mathcal{G}}_B(\tau, \tau) = i\cot Z - \frac{i}{Z}, \tag{3.14}$$

which are found by applying the relations

$$\dot{G}_B(\tau, \tau) = 0, \qquad \dot{G}_B^2(\tau, \tau) = 1. \tag{3.15}$$

After replacing $\tau_i \to Tu_i$ in (3.7), the 1PI n-point vertex function in the presence of an electromagnetic background field is given by

$$\Gamma_{1\mathrm{PI}}[p_1, \ldots, p_n] = N(-1)^n(2\pi)^D\delta\left(\sum_{i=1}^n p_i\right)\int_0^\infty \frac{dT}{T^{1-n}}(4\pi T)^{-(D/2)}e^{-m^2T}\det{}^{-(1/2)}\left[\frac{\sin(eFT)}{eFT}\right] \times \prod_{i=1}^n\int_0^1 du_i \exp\left(\sum_{i<j=1}^n \bar{\mathcal{G}}_B(u_i, u_j)p_i\cdot p_j\right). \tag{3.16}$$

At this stage, a constant magnetic field is chosen for the background. With the B-field chosen along the z axis, we introduce matrices $g_\perp$ and $g_\parallel$ projecting on x, y and z, τ planes, respectively,

$$\hat{F} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad g_\perp \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad g_\parallel \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.17}$$

with $\hat{F} \equiv \frac{F}{B}$. The determinant factor in (3.16) becomes

$$\det{}^{-(1/2)}\left[\frac{\sin Z}{Z}\right] = \frac{z}{\sinh z}, \tag{3.18}$$

where we have introduced $z \equiv eBT$. Further, the world-line Green's function (3.6) is given by

$$\bar{\mathcal{G}}_B(u_i, u_j) = G_{Bij}g_\parallel - \frac{T}{2z}\left(\frac{\cosh(z\dot{G}_{Bij})}{\sinh z} - \coth z\right)g_\perp + \frac{T}{2z}\left(\frac{\sinh(z\dot{G}_{Bij})}{\sinh z} - \dot{G}_{Bij}\right)i\hat{F}. \tag{3.19}$$

As was stated before, it is desirable to obtain the effective vertex function in the presence of a strong magnetic field. Let us therefore look at the world-line Green's function (3.19) in the limit of a strong magnetic field. Rearranging the parameters as $u_1 \geq u_2 \geq \cdots \geq u_{n-1} \geq u_n$ is allowed. By this convention, it is seen that $z\dot{G}_{Bij} \leq z$. We obtain

$$\lim_{B\to\infty}\bar{\mathcal{G}}_B(u_i, u_j) = G_{Bij}g_\parallel + \frac{1}{2|eB|}g_\perp - \frac{i}{2eB}\dot{G}_{Bij}\hat{F}. \tag{3.20}$$

Moreover, the determinant behaves in this limit as

$$\lim_{B\to\infty}\frac{z}{\sinh z} \approx 2ze^{-z}. \tag{3.21}$$

In (3.20) and (3.21), we choose the sign of $eB > 0$ and we denote it by $|eB|$ whenever eB arises from a term that is even in eB. Replacing (3.20) and (3.21) in (3.16) for dimension $D = 4$ yields[5]

$$\Gamma_{1\mathrm{PI}}[p_1, \ldots, p_n] = N|eB|(-1)^n(2\pi^2)\delta\left(\sum_{i=1}^n p_i\right)\int_0^\infty \frac{dT}{T^{2-n}}e^{-(m^2+|eB|)T}\prod_{i=1}^n\int_0^1 du_i \times \exp\left(\sum_{i<j=1}^n\left[G_{Bij}g_\parallel + \frac{1}{2|eB|}g_\perp - \frac{i}{2eB}\dot{G}_{Bij}\hat{F}\right]p_i\cdot p_j\right). \tag{3.22}$$

[5] By $[\xi]\, p_i\cdot p_j$ for $\xi = g_\parallel, g_\perp$ or $\hat{F}$ in (3.22), we mean $\xi^{\mu\nu}p_{i\mu}p_{j\nu}$. Here, i and j denote the ith and jth incoming momentum; μ and ν denote the μth and νth component of the four-vector.

To make the meaning of this equation more transparent, each part will be considered independently. We refer to the three terms in the exponent as $g_{\|}$, $g_{\perp}$, and $\hat{F}$ terms. Among them, $g_{\perp}$ and $\hat{F}$ terms are independent of T, thus can be taken out of the T integral.

For the $g_{\perp}$ term we have

$$\exp\bigg(\sum_{i<j=1}^{n}\frac{g_{\perp}}{2|eB|}p_i\cdot p_j\bigg)=\exp\bigg(\frac{1}{2|eB|}(\mathbf{p}_{1\perp}\cdot\mathbf{p}_{2\perp}+\mathbf{p}_{1\perp}\cdot\mathbf{p}_{3\perp}+\cdots+\mathbf{p}_{n-1\perp}\cdot\mathbf{p}_{n\perp})\bigg), \tag{3.23}$$

where $\mathbf{p}_{\perp}\equiv(p_x,p_y)$. Applying the energy-momentum conservation $\sum_{i=1}^{n}\mathbf{p}_{i\perp}=0$, we get

$$\exp\bigg(\sum_{i<j=1}^{n}\frac{g_{\perp}}{2|eB|}(\mathbf{p}_i\cdot\mathbf{p}_j)_{\perp}\bigg)=\exp\bigg(-\frac{1}{4|eB|}\sum_{i=1}^{n}\mathbf{p}_{i\perp}^2\bigg). \tag{3.24}$$

This is exactly the damping factor which also appears in [7], where the effective action of the NJL model is calculated in the LLL approximation.

To calculate the $\hat{F}$ term, we use (3.7) with the replacement $\tau_i\to u_iT$, and are led to

$$\exp\bigg(-\frac{i}{2eB}\sum_{i<j=1}^{n}[\mathrm{sign}(u_i-u_j)-2(u_i-u_j)]\hat{F}p_i\cdot p_j\bigg). \tag{3.25}$$

To build some parallels to the (modified) noncommutative effective field theory which should arise in the large B limit, we introduce the (noncommutativity) parameter

$$\theta_{ab}\equiv\frac{1}{eB}\epsilon_{ab}, \tag{3.26}$$

with ϵ_{ab} the ordinary antisymmetric tensor of rank 2 and indices a and b denoting transverse directions, i.e. x and y coordinates. Using this new definition the product on the right-hand side of (3.25) becomes

$$\frac{1}{eB}\hat{F}p_i\cdot p_j=p_i{}^a\theta_{ab}p_j{}^b=:\mathbf{p}_i\times\mathbf{p}_j. \tag{3.27}$$

In the definition of the cross product, we will skip the subscript $\perp$ for the transverse coordinates. Adhering to our rearrangement for $u_1\geq u_2\geq\cdots\geq u_{n-1}\geq u_n$ and using the freedom to choose the zero somewhere in the world loop for setting $u_n=0$, we have

$$\sum_{i<j=1}^{n}[(\mathbf{p}_i\times\mathbf{p}_j)(u_i-u_j)]=(\mathbf{p}_1\times\mathbf{p}_2)(u_1-u_2)+(\mathbf{p}_1\times\mathbf{p}_3)(u_1-u_3)+\cdots+(\mathbf{p}_1\times\mathbf{p}_n)(u_1)(\mathbf{p}_2\times\mathbf{p}_3)(u_2-u_3)+\cdots$$
$$+(\mathbf{p}_2\times\mathbf{p}_n)(u_2)+\cdots+(\mathbf{p}_{n-1}\times\mathbf{p}_n)(u_{n-1}). \tag{3.28}$$

Using further the energy-momentum conservation and the antisymmetry property of the cross product, defined in (3.27), we obtain

$$\sum_{i<j=1}^{n}[(\mathbf{p}_i\times\mathbf{p}_j)(u_i-u_j)]=0. \tag{3.29}$$

The $\hat{F}$ term therefore becomes

$$\exp\bigg(-\frac{i}{2eB}\sum_{i<j=1}^{n}[\dot{G}_{Bij}\hat{F}]p_i\cdot p_j\bigg)=\exp\bigg(-\frac{i}{2}\sum_{i<j=1}^{n}\mathbf{p}_i\times\mathbf{p}_j\bigg). \tag{3.30}$$

The same phase factor also appears in the Fourier transform of the vertices in the ordinary noncommutative field theory [4] that is defined by replacing the ordinary product of functions by the Moyal $\star$ product.

Putting now (3.24) and (3.30) together in (3.22), we finally obtain the 1PI n-point function for the composite field σ in a strong magnetic field,

$$\Gamma_{1\mathrm{PI}}[p_1,\ldots,p_n]=N|eB|(-1)^n(2\pi^2)\delta\bigg(\sum_{i=1}^{n}p_i\bigg)\exp\bigg(-\frac{1}{4|eB|}\sum_{i=1}^{n}\mathbf{p}_{i\perp}^2\bigg)\exp\bigg(-\frac{i}{2}\sum_{i<j=1}^{n}\mathbf{p}_i\times\mathbf{p}_j\bigg)$$
$$\times\int_0^{\infty}\frac{dT}{T^{2-n}}e^{-(m^2+|eB|)T}\prod_{i=1}^{n}\int_0^1 du_i\exp\bigg(\sum_{i<j=1}^{n}[G_{Bij}g_{\|}]p_i\cdot p_j\bigg). \tag{3.31}$$

Comparison with the 1PI n-point vertex function

$$\Gamma_{1PI}[p_1, \ldots, p_n] = N(-1)^n (2\pi)^D \delta\left(\sum_{i=1}^n p_i\right) \int_0^\infty \frac{dT}{T^{1-n}} (4\pi T)^{-(D/2)} e^{-m^2 T} \prod_{i=1}^n \int_0^1 du_i \exp\left(\sum_{i<j=1}^n G_{Bij} p_i \cdot p_j\right), \quad (3.32)$$

which is obtained by turning off the electromagnetic field in (3.16), reveals that the parallel sector of (3.31), involving the τ and z directions, is up to some factor which is the same as the Γ_{1PI} (3.32) in $D = 2$ dimensions, provided that $m^2 \to m^2 + |eB|$. In the next section, using the method introduced in [7], we will find the n-point contribution to the effective action of the composites σ in the presence of a strong magnetic field in an appropriate LLL approximation. The result will have common features with (3.31). In particular, the phases (3.24) and (3.30) reappear in the final result.

IV. EFFECTIVE ACTION OF THE COMPOSITE FIELD IN THE LLL APPROXIMATION

In the first part of this section, starting from the full bosonic Green's function derived in the seminal work of J. Schwinger [16] in the framework of Schwinger proper-time formalism, we will determine the propagator of a multidimensional complex scalar field Φ in the LLL approximation. We then use the LLL propagator to determine the effective action of n composite fields σ in the LLL regime.

Let us start with the Schwinger propagator in Minkowski space,

$$G(x', x'') = P(x', x'') D(x' - x''), \quad (4.1a)$$

with

$$P(x', x'') \equiv \exp\left(ie \int_{x''}^{x'} d\xi^\mu A_\mu(\xi)\right), \quad (4.1b)$$

and

$$D(x' - x'') \equiv \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} e^{-ism^2} \times \exp\left(\frac{-1}{2} \mathrm{Tr} \ln\left[\frac{\sinh eFs}{eFs}\right]\right) \times \exp\left[-\frac{i}{4}(x' - x'') eF \coth(eFs)(x' - x'')\right]. \quad (4.1c)$$

In the symmetric gauge

$$A_\mu = \frac{B}{2}(0, x_2, -x_1, 0),$$

the part consisting of the Schwinger line integral is equal to

$$P(x', x'') = e^{(ieB/2)\epsilon^{ab} x'_a x''_b}, \qquad a, b = 1, 2. \quad (4.2)$$

Here, as in the previous section, B is chosen to be a constant magnetic background field in the x_3 direction. In the translationally invariant part $D(x' - x'')$ in (4.1c), it is easy to show that

$$x^\alpha [eF \coth(eFs)]_\alpha{}^\beta x_\beta = \frac{1}{s} \mathbf{x}_\parallel^2 - eB \cot(eBs) \mathbf{x}_\perp^2, \quad (4.3)$$

with $\mathbf{x}_\parallel^2 \equiv x_0^2 - x_3^2$ and $\mathbf{x}_\perp^2 \equiv x_1^2 + x_2^2$, and

$$\exp\left(-\frac{1}{2} \mathrm{Tr} \ln\left[\frac{\sinh(eFs)}{eFs}\right]\right) = \frac{eBs}{\sin(eBs)}. \quad (4.4)$$

Putting (4.3) and (4.4) in (4.1c) and after taking the Fourier transformation, we obtain

$$\tilde{D}(k) = -\int_0^\infty \frac{ds}{\cosh(eBs)} \times \exp\left(-s\left(m^2 - k_0^2 + \mathbf{k}_\perp^2 \frac{\tanh(eBs)}{eBs} + k_3^2\right)\right). \quad (4.5)$$

Here, we assume again that $eB > 0$ and replace $eBs \to s$. We arrive at[6]

$$\tilde{D}(k) = -\frac{1}{eB} \int_0^\infty \frac{ds}{\cosh s} e^{-s\rho - \alpha \tanh(s)}, \quad (4.6)$$

with $\alpha \equiv \mathbf{k}_\perp^2/|eB|$ and $\rho \equiv (m^2 - \mathbf{k}_\parallel^2)/|eB|$. Further, $\mathbf{k}_\parallel^2 \equiv k_0^2 - k_3^2$ and $\mathbf{k}_\perp^2 \equiv k_1^2 - k_2^2$ are introduced. To determine the regime of LLL dominance of $\tilde{D}(k)$, we first use (B4) from Appendix B to obtain[7]

$$\tilde{D}(k) = -\frac{e^{-\alpha}}{|eB|} \sum_{n'=0}^\infty (-1)^{n'} L_{n'}^{(-1)}(2\alpha) \int_0^\infty ds \frac{e^{-s(\rho + 2n')}}{\cosh s}, \quad (4.7)$$

where $L_{n'}^{(\beta)}$ is the generalized Laguerre polynomial. Then, using (C4) from Appendix C, we get

$$\tilde{D}(k) = \frac{e^{-\alpha}}{2|eB|} \sum_{n'=0}^\infty (-1)^{n'} L_{n'}^{(-1)}(2\alpha) \left[\psi\left(\frac{\rho + 2n' + 1}{4}\right) - \psi\left(\frac{\rho + 2n' + 3}{4}\right)\right]. \quad (4.8)$$

Next, we expand the digamma function $\psi(z)$ [18] according to

$$\psi(1 + z) = -\gamma + \sum_{m'=1}^\infty \frac{z}{m'(m' + z)}, \quad (4.9)$$

and arrive at

[6] As it was stated earlier eB is chosen to be $|eB|$ whenever it appears in even terms.

[7] A similar method was also used in [17].

$$\tilde{D}(k) = \frac{e^{-\alpha}}{2|eB|} \sum_{n'=0,m'=1}^{\infty} (-1)^{n'} L_{n'}^{(-1)}(2\alpha) \times \left[\frac{\rho + 2n' - 3}{m'(4m' + \rho + 2n' - 3)} - \frac{\rho + 2n' - 1}{m'(4m' + \rho + 2n' - 1)} \right]. \tag{4.10}$$

Before we continue and determine the LLL form of $\tilde{D}(k)$, we note that when the dynamics of a particle is stationary, as in the present case with pure constant magnetic background, the energy spectrum can be read from the poles of the propagator. In other words, the energy spectrum of a particle in a magnetic background that is obtained from the relativistic Klein-Gordon equation coincides with the poles of the propagator that entails corrections due to the background. This fact enables us to obtain the effective propagator in the LLL dominant regime.

The energy spectrum of a scalar field in the presence of a magnetic background is known to be (see e.g. [19])

$$E_{\ell'}(k) = \sqrt{m^2 + |eB|(2\ell' + 1) + k_3^2}, \quad \text{for } \ell' = 0, 1, 2, \ldots, \infty. \tag{4.11}$$

Choosing $\ell' = 0$ the energy of the LLL is given by

$$E_0{}^2 = k_3{}^2 + m^2 + |eB|. \tag{4.12}$$

To find the LLL form of $\tilde{D}(k)$, we determine the variables m and n in (4.10) such that the LLL energy arising from the poles of the propagator (4.10) coincides with (4.12). Using the notation introduced before, we get $E_0^2 - k_3^2 - m^2 = \mathbf{k}_\parallel^2 - m^2 = |eB|$, which yields $\rho = -1$. As it turns out, the only valid choice satisfying the first pole equation $m'(4m' + 2n' - 4) = 0$ from the first denominator in (4.10) is $(m' = 1, n' = 0)$. The second pole equation $m'(4m' + 2n' - 2) = 0$ from the second denominator in (4.10) does not have any valid solution and therefore does not contribute to the LLL propagator. Thus, plugging $(m' = 1, n' = 0)$ into (4.10) and using $L_0^{(-1)}(2\alpha) = 1$, we get

$$\tilde{D}_{\text{LLL}}(k) = e^{-(\mathbf{k}_\perp^2/|eB|)} \frac{2}{\mathbf{k}_\parallel^2 - (m^2 + |eB|)}. \tag{4.13}$$

In the coordinate space the LLL effective propagator (4.1a)–(4.1c) can therefore be written in the form

$$G_{\text{LLL}}(x', x'') = P(x', x'') \mathcal{F}^{-1}\{\tilde{D}_{\text{LLL}}(k)\}, \tag{4.14}$$

where $P(x', x'')$ is defined in (4.2) and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. From now on, we drop the subscript LLL, but it is always assumed. $G(x', x'')$ obviously factorizes into two independent transverse and longitudinal parts,

$$G(x', x'') = G_\perp(\mathbf{x}'_\perp, \mathbf{x}''_\perp) G_\parallel(\mathbf{x}'_\parallel = \mathbf{x}''_\parallel), \tag{4.15a}$$

where the transverse part is

$$G_\perp(\mathbf{x}'_\perp, \mathbf{x}''_\perp) = \frac{|eB|}{2\pi} e^{(ieB/2)\epsilon^{ab} x'_a x''_b} e^{-(|eB|/4)} (\mathbf{x}'_\perp - \mathbf{x}''_\perp)^2, \tag{4.15b}$$

including the Schwinger line integral (4.2) and the Fourier transform of the phase factor $e^{-\mathbf{k}_\perp^2/|eB|}$ from (4.13), and the longitudinal part is

$$G_\parallel(\mathbf{x}'_\parallel - \mathbf{x}''_\parallel) = \mathcal{F}^{-1}\left\{\frac{1}{\mathbf{k}_\parallel^2 - (m^2 + |eB|)}\right\}. \tag{4.15c}$$

At this stage, we have all the necessary tools to calculate the n-point vertex function $\Gamma_{n\sigma}$ for n-composite fields σ. It is obtained through

$$\Gamma_{n\sigma} = \int d^4x_1 \cdots d^4x_n [\sigma(x_1) G(x_1, x_2) \sigma(x_2) G(x_2, x_3) \cdots \sigma(x_n) G(x_n, x_1)]. \tag{4.16}$$

Using now (4.15a)–(4.15c) for $G(x', x'')$, inserting the Fourier transform of the composite fields σ, and carrying out the integrations over $\mathbf{x}$, the n-point contribution to the effective action reads

$$\Gamma_{n\sigma} = 2\pi N |eB| \int d^2x_{1\parallel} \cdots d^2x_{n\parallel} \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_n}{(2\pi)^4} \delta^2\left(\sum_{i=1}^n \mathbf{p}_{i\perp}\right) \exp\left(i \sum_{i=1}^n \mathbf{p}_{i\parallel} \cdot \mathbf{x}_{i\parallel}\right) \exp\left(-\frac{1}{4|eB|} \sum_{i=1}^n \mathbf{p}_{i\perp}^2\right) \times \exp\left(-\frac{i}{2} \sum_{i<j=1}^n \mathbf{p}_i \times \mathbf{p}_j\right) [\sigma(p_1) G_\parallel(\mathbf{x}_{1\parallel}, \mathbf{x}_{2\parallel}) \sigma(p_2) G_\parallel(\mathbf{x}_{2\parallel}, \mathbf{x}_{3\parallel}) \cdots \sigma(p_n) G_\parallel(\mathbf{x}_{n\parallel}, \mathbf{x}_{1\parallel})]. \tag{4.17}$$

The cross product on the second line is defined in (3.27) and includes only the transverse coordinates of p_i i.e. $\mathbf{p}_{i\perp} = (p_{i1}, p_{i2})$. In the next section we compare this result with the 1PI n-point vertex function (3.31), which we obtained in the previous section in the limit of a strong magnetic field.

V. MODIFIED NONCOMMUTATIVE FIELD THEORY

Let us consider the n-point vertex function (3.31) which was found in the framework of the world-line formalism and the n-point effective action (4.17) which was obtained in the LLL approximation. As for the parallel sector of the effective theory, similar to (3.31) and (4.17), it shows a free propagation of the composite field σ in the longitudinal coordinates, with the effective square mass $m^2 + |eB|$. The same dimensional reduction from $D = 4$ to $D = 2$ dimensions was also observed in the dynamics of fermion pairing in a constant magnetic field for an effective NJL model [15].

As for the transverse part, the general structure of both results (3.31) and (4.17) can be compared with the general structure of an n-point vertex of a conventional NCFT (see for instance [4]),

$$\int d^D x \overbrace{\phi(x) \star \cdots \star \phi(x)}^{n\text{-times}} = \int \frac{d^D p_1}{(2\pi)^D} \cdots \frac{d^D p_n}{(2\pi)^D} \delta^D\Big(\sum_{i=1}^n p_i\Big) \exp\Big(-\frac{i}{2} \sum_{i<j=1}^n \mathbf{p}_i \times \mathbf{p}_j\Big) \phi(p_1) \cdots \phi(p_n). \tag{5.1}$$

Here, the ordinary Moyal $\star$ product is defined by

$$\phi(x) \star \phi(x) = e^{(i/2)\theta^{ij}(\partial/\partial y^i)(\partial/\partial z^j)} \sigma(y)\sigma(z)|_{y=z=x}, \tag{5.2}$$

which reflects the noncommutativity in the x_i and x_j coordinates

$$[x_i, x_j] = i\theta_{ij}, \tag{5.3}$$

with the noncommutativity parameter θ_{ij}. Similarly, as was originally shown in [7] for a fermionic NJL model, the noncommutative feature of (3.31) and (4.17) manifests itself in the phase factor containing the cross product $\mathbf{p}_i \times \mathbf{p}_j = p_i^a \theta_{ab} p_j^b$ with $a, b = 1, 2$ and θ defined in (3.26). However, in contrast to the ordinary noncommutative field theory, in (3.31) as well as in (4.17) an additional phase factor,

$$\exp\Big(-\frac{1}{4|eB|} \sum_{i=1}^n \mathbf{p}_{i\perp}^2\Big), \tag{5.4}$$

appears which modifies the noncommutativity between the longitudinal coordinates (5.3) to

$$[x^a, x^b] = i\hat{\theta}^{ab}, \qquad a, b = 1, 2. \tag{5.5}$$

The modified noncommutative parameter $\hat{\theta}$ is given by [7]

$$\hat{\theta} = \frac{1}{|eB|} \begin{pmatrix} i & \text{sign}(eB) \\ -\text{sign}(eB) & i \end{pmatrix}. \tag{5.6}$$

Using this definition, the full phase factor which manifests the noncommutative properties of the effective n-point vertex function can be rewritten as

$$\exp\Big(-\frac{1}{4|eB|} \sum_{i=1}^n \mathbf{p}_{i\perp}^2 - \frac{i}{2} \sum_{i<j=1}^n \mathbf{p}_i \times \mathbf{p}_j\Big) = \exp\Big(-\frac{i}{2} \sum_{i<j=1}^n \mathbf{p}_i \hat{\times} \mathbf{p}_j\Big),$$

with $\mathbf{p}_i \hat{\times} \mathbf{p}_j = p_i^a \hat{\theta}_{ab} p_j^b$, $a, b = 1, 2$. In the special case where the composites are independent of the longitudinal coordinates, $\mathbf{x}_\parallel$, the effective action of n composites, (4.17), in coordinate space is

$$\Gamma_{n\sigma} \sim \int d^2 x_\parallel d^2 x_\perp \underbrace{\sigma(\mathbf{x}_\perp) \hat{\star} \cdots \hat{\star} \sigma(\mathbf{x}_\perp)}_{n\text{-times}}, \tag{5.7}$$

with the modified Moyal $\hat{\star}$ product defined by

$$\sigma(\mathbf{x}_\perp) \hat{\star} \sigma(\mathbf{x}_\perp) = e^{(i/2)\hat{\theta}^{ab}(\partial/\partial y^a)(\partial/\partial z^b)} \sigma(y)\sigma(z)|_{y=z=x}, \quad a, b = 1, 2. \tag{5.8}$$

Alternatively, the above phase factor (5.4) can be absorbed in the definition of the composite field σ, leading to a new *smeared* field [7]

$$\Sigma(x) \equiv e^{(\vec{\nabla}_\perp^2/4|eB|)} \sigma(x). \tag{5.9}$$

In terms of the smeared fields Σ, the effective action $\Gamma_{n\Sigma}$ of n composites in coordinate space is similar to (5.7) with σ replaced by Σ and the modified $\hat{\star}$ product replaced by the ordinary $\star$ product (5.2).

It is worth mentioning that the damping phase factor (5.4) protects the modified NCFT from the appearance of the UV/IR mixing [5] which appears in the ordinary NCFT. This can be shown by writing the one-loop correction to the tree-level propagator $G_\sigma^{\text{tree}}(p)$. The one-loop vertices can be read e.g. from (4.17) for $n = 4$. The explicit calculation, similar to what is performed in [7] for the NJL model, shows that when the loop momentum $\ell - \infty$, the loop integral is convergent for all external legs momenta p, even for $\mathbf{p}_\perp \to 0$.

VI. CONCLUSION

In this paper, using two apparently different methods, the effective action of n-composite scalar fields is derived in the presence of a strong magnetic background field. In Sec. III, the 1PI n-point amplitude of these composites is determined in the framework of world-line formalism. Following the standard manipulations, we arrived first at the path integral representation of the n-point vertex function in the presence of a constant magnetic field. As a novelty, we then took the limit of a strong magnetic field and ended up with an effective n-point vertex function which is similar to the vertex function of a modified noncommutative field theory. In comparison to the standard noncommutative field theory a phase factor occurs which protects the effective field theory from inconsistencies due to the appearance of UV/IR mixing.

In Sec. IV, we followed the method presented in [7] and calculated the contribution of n composites to the effective action in the regime of LLL dominance. First, starting from the full Schwinger propagator of the scalar field in the presence of a background field, we derived the effective propagator of the theory in an appropriate LLL approximation. Then, using this effective Green's function, we determined the n-point effective action in this approximation.

In Sec. V, we compared both results from Secs. III and IV. We showed that here, as in the NJL model in the presence of a constant magnetic background [15], a dimensional reduction from $D = 4$ to $D = 2$ occurs in the longitudinal section of the effective theory where a free propagation of the composites is observed. In the transverse section, however, a modified noncommutative field theory arises where no UV/IR mixing occurs. The emergence of a modified noncommutativity is due to a breakdown of the translational invariance which, in Sec. IV, exhibits itself in an explicit dependence of the full Green's function of the theory on a Schwinger phase line integral. Although the Schwinger line integral does not appear explicitly in the world-line formalism in Sec. III, in the heart of this formalism is, in fact, the expectation value of the Wilson loop which is the Schwinger phase line integral taken along a closed loop.

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APPENDIX A

In this appendix, we show that in four dimensions and in the presence of a constant magnetic background field, dynamical symmetry breaking occurs and a condensate always develops. This authenticates our assumption for the introduction of the composite field σ in Sec. II. The condensate is expressed through the complex boson propagator $G(x', x'') \equiv i\langle 0|\phi^*(x')\phi(x'')|0\rangle$,

$$\langle 0|\phi^*\phi|0\rangle = -i \lim_{x'\to x''} G(x', x'') = -\frac{i|eB|}{(4\pi)^2}\int_0^\infty \frac{ds}{s\sin(|eB|s)} e^{-ism^2}. \quad \text{(A1)}$$

To deal with the infinity in the integral arising from the $s \to 0$ limit, we can either restrict the lower limit of $s \to \frac{1}{\Lambda^2}$ or alternatively render it finite through ζ-function regularization. Here we opt for the latter. After rotating the contour of the s integration by $s \to -is$, we introduce μ in the exponent of (A1) to get

$$\langle 0|\phi^*\phi|0\rangle = -\frac{|eB|}{(4\pi)^2}\int_0^\infty ds \frac{s^{\mu-1}}{\sinh(s)} e^{-s(m^2/|eB|)}. \quad \text{(A2)}$$

In the spirit of dimensional regularization, we will analytically continue to $\mu \to 0$. By making use of the integral representation of the Hurwitz ζ function,

$$\zeta(r, \nu) = \frac{1}{\Gamma(t)}\int_0^\infty dt t^{r-1}\frac{e^{-t\nu}}{1-e^{-t}}, \qquad \mathrm{Re}t > 1, \qquad \mathrm{Re}\nu > 0. \quad \text{(A3)}$$

Thus, the integration in (A2) can be written as

$$\langle 0|\phi^*\phi|0\rangle = \frac{-|eB|2^{1-\mu}}{(4\pi)^2}\Gamma(\mu)\zeta\left(\mu, \frac{m^2}{2|eB|} + \frac{1}{2}\right). \quad \text{(A4)}$$

Expansion around the poles yields

$$\langle 0|\phi^*\phi|0\rangle = -\frac{|eB|}{8\pi^2}\left(\frac{1}{\mu} - \gamma + \mathcal{O}(\mu)\right)\left(\frac{m^2}{2|eB|} + \left[\ln\Gamma\left(\frac{m^2}{2|eB|} + \frac{1}{2}\right) - \frac{1}{2}\ln(2\pi)\right]\mu\right), \quad \text{(A5)}$$

where we have used the identities

$$\zeta(0, \nu) = \frac{1}{2} - \nu, \qquad \frac{\partial}{\partial r}\zeta(r, \nu)|_{r=0} = \ln\Gamma(\nu) - \frac{1}{2}\ln(2\pi). \quad \text{(A6)}$$

Keeping the finite terms in (A5) and taking the limit $m^2 \to 0$, we obtain

$$\langle 0|\phi^*\phi|0\rangle = \frac{|eB|}{(4\pi)^2}\ln(2). \quad \text{(A7)}$$

This relation shows that in the presence of a constant magnetic background field the expectation value of the condensate is nonzero and the composite field σ is always formed. As it was implied earlier the condensate is proportional to $|eB|$.

APPENDIX B

In this appendix, we derive the useful formulas which will help us to determine the LLL effective propagator in Sec. IV. We start with the identity

$$(1-z)^{(-\beta-1)}\exp\left(\frac{\lambda_z}{z-1}\right) = \sum_{n'=0}^\infty L_{n'}^{(\beta)}(\lambda)z^{n'}, \qquad \text{for } |z| < 1, \quad \text{(B1)}$$

where $L_{n'}^{(\beta)}$ are the generalized Laguerre polynomials [18]. For $\beta = -1$, (B1) can be written as

$$\exp\left[\frac{\lambda}{2}\left(\frac{z+1}{z-1}\right)\right] = e^{(-\lambda/2)}\sum_{n'=0}^\infty L_{n'}^{(-1)}(\lambda)z^{n'}, \qquad \text{for } |z| < 1, \quad \text{(B2)}$$

with $L_{n'}^{(\beta)}$ satisfying

$$L_{n'}^{(-1)}(\lambda) = L_{n'}(\lambda) - L_{n'-1}(\lambda). \tag{B3}$$

Defining $z = -e^{-2s}$ and using (B2), we obtain[8]

$$\begin{aligned} e^{-\alpha \tanh(s)} &= \exp\left[-\alpha\left(\frac{e^s - e^{-s}}{e^s + e^{-s}}\right)\right] \\ &= e^{-\alpha} \sum_{n'=0}^{\infty} (-1)^{n'} L_{n'}^{(-1)}(2\alpha) e^{-2n's}. \end{aligned} \tag{B4}$$

APPENDIX C

In order to perform the integrals of the form

$$\int_0^\infty dt \frac{e^{-zt}}{\cosh(t)}, \tag{C1}$$

we start from the definition of the digamma function [18]

$$\begin{aligned} \psi(z) + \gamma &= \int_0^\infty dt \left[\frac{e^{-t} - e^{-zt}}{1 - e^{-t}}\right] \\ &= \int_0^\infty dt \left[\frac{e^{-(t/2)} - e^{-(z-(1/2))t}}{(e^{t/4} - e^{-(t/4)})(e^{t/4} + e^{-(t/4)})}\right]. \end{aligned} \tag{C2}$$

Then it is easy to show that

$$\psi\left(z + \frac{3}{4}\right) - \psi\left(z + \frac{1}{4}\right) = \int_0^\infty dt \left[\frac{e^{-zt}}{(e^{t/4} + e^{-(t/4)})}\right]. \tag{C3}$$

Replacing $t \to 4t$, we finally arrive at

$$\int_0^\infty dt \frac{e^{-zt}}{\cosh(t)} = \frac{1}{2}\left[\psi\left(\frac{z+3}{4}\right) - \psi\left(\frac{z+1}{4}\right)\right]. \tag{C4}$$

[8] The condition $|z| < 1$ is satisfied because in (4.7) $s = 0$ is a singular point. To deal with this infinity the integral is always proper-time regularized by replacing the integration interval from $[0, +\infty)$ to $[\frac{1}{\Lambda^2}, \infty)$.

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