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Casimir piston and cylinder, perturbatively

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According to perturbation theory, a thin piston in a thin-walled semi-infinite cylinder closed at its head, both weakly reflecting dielectrics, experiences a force which is directed towards the cylinder head at small separations a, but which changes sign as a rises. By contrast, for thick enough material the force remains attractive for all a (as it does for perfect reflection, according to the recent paper by Hertzberg *et al.* [M. P. Hertzberg, R. L. Jaffe, M. Kardar, and A. Scardicchio, Phys. Rev. Lett. **95**, 250402 (2005)]).

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I. INTRODUCTION

Casimir energies and forces are those arising from the interaction of the quantized Maxwell field with macroscopic bodies, when the bodies are described purely by their geometry and by their electromagnetic response functions, plus, if necessary, by cutoffs that impose either maximum wave numbers on the fields or minimum distances between parts of the bodies. The cutoffs make a minimal allowance for the granular structure of true macroscopic media; and after 40 years or so of confusion it has become clear recently that one cannot understand the Casimir energy or the internal stresses of any single connected body, solid or hollow, if one simply drops all cutoff-dependent terms, and ignores energy localized on the charge carriers inside the material. For instance, it is next to impossible to elucidate arguments confined to perfect reflectors from the start. In the language of conventional but nonrenormalizable nonrelativistic QED, the basic physics for dielectric media is spelled out in [1–4], and by Marachevsky [5,6]; and for plasmas in [7–10]. For renormalizable models of scalar fields subject to potentials due to otherwise inert material, the theory, initiated by Graham *et al.* [11], can be traced very accessibly from, say, Graham *et al.* [12] and from Jaffe [13,14].

In particular, for the paradigmatic case of hollow shells it is now understood that the Casimir stress has just the sign one would naturally expect from the interactions between their constituent parts: inward if these interactions are attractive (as in dielectrics), and outward if they are repulsive (as for jellium models of electron plasmas, which pretend that the overall-neutralizing ion cores have been smeared out into an inert continuum). This negates the historic but mistaken belief that the stress on, say, an infinitesimally thin but perfectly reflecting spherical or approximately cubic shell is necessarily directed outwards. On the other hand, the interaction between two mutually disjoint bodies made of normal material is always attractive; moreover, at large enough separations it becomes independent of cutoffs, and can make sense even in the limit of perfect reflection.

The next important question concerns the net Casimir force F experienced by a freely movable piston, in contact but unattached, inside a semi-infinite hollow cylinder with one closed end. This geometry features a wholly enclosed finite volume between piston and cylinder head, and a laterally bounded but axially semi-infinite volume above the piston. The theory in its present state does not immediately reveal whether (and if yes then just how reliably) one can determine F without ascribing at least modestly realistic electromagnetic properties to piston and cylinder wall. The problem is brought into focus by Cavalcanti [15], who finds that F is always inwards (towards the closed end) in a 2D model with scalar fields, if perfect reflection is enforced by Dirichlet boundary conditions on all material surfaces. His reasoning requires no cutoff; and is extended to perfectly reflected Maxwell fields in 3D in a remarkable paper by Hertzberg, Jaffe, Kardar, and Scardicchio [16], of which more in Section II B below.

Hence it becomes interesting to study the same geometry in the opposite extreme of weak instead of perfect reflection. For insulators this is a task accessible perturbatively and therefore relatively straightforward. Section II sets up a rough but reasonable material model already used to obtain exact solutions for simply shaped single bodies [7–10], and resumes the appropriate perturbation formalism [1–4]. Section III determines the short- and the long-distance asymptotics of F, by exploiting related but much simpler geometries dealt with in the appendix. This suffices to show that F changes sign [17]. Section IV spells out the complete perturbative expression for right circular cylinders; as an illustration, Section V then evaluates it in closed form, and plots it, in the so-called Casimir-Polder regime where all distances are much larger than the absorption wavelengths characterizing the material. Finally, Section VI summarizes our results, and discusses how sensitive they are likely to be to departures from our assumptions and approximations.

II. GENERALITIES

A. Model and perturbation theory

The material is modeled by infinitesimally thin nonconducting surfaces, with optical properties aping those

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of n atoms per unit area. The atoms are taken to be isotropic simple-harmonic oscillators, with atomic polarizability

$$\Pi(\omega) = \alpha\Omega^2/[\Omega^2 - (\omega + i0)^2]. \quad (1)$$

The entire interaction between the material and the Coulomb-gauge-quantized Maxwell field is treated as a perturbation. This is reasonable if, loosely speaking, reflection is weak, i.e. if $n\alpha$ is small enough. More precisely, since $n\alpha$ has the dimensions of length, it must be much smaller than the smallest pertinent dimension of the system (the smaller of the parameters a and b to be introduced presently), and much smaller also than c/Ω. (Further, one requires $\alpha n^{3/2} \ll 1$ to avoid local-field effects like those normally embodied in the Lorenz-Lorentz formula.) The analogous problem with three-dimensional distributions of materials is discussed in [2]. Crucially, the second-order shift is identically the same as the interatomic potential V summed over all pairs of atoms; the first-order shift is irrelevant for our purposes, being merely a feeble attempt to mimic the self-energies of the individual atoms. These connections are spelled out in [1,4] and in Appendix E of [2].

The potential at separation ρ, and a useful set of associated moments, are written as [18]

$$V(\rho) = -\alpha^2 f(\rho), \qquad \mathcal{J}_n(x) \equiv \int_x^\infty d\rho f(\rho)\rho^n. \quad (2)$$

At zero temperature (reviewed say in [2]) , the asymptotic Van der Waals (VdW) and the Casimir-Polder (CP) regimes have, respectively,

$$f(\Omega\rho \ll 1) \simeq 3\Omega/4\rho^6, \qquad f(\Omega\rho \gg 1) \simeq C/\rho^7, \quad C = 23/4\pi; \quad (3)$$

at high temperatures [19], albeit still with $k_BT \ll \Omega$, and long distance (reviewed say in [3]), the pseudo-VdW regime has

$$f \simeq 3k_BT/\rho^6, \qquad (1/2\pi\rho \ll k_BT \ll \Omega). \quad (4)$$

The total second-order shift reads

$$B = -(n\alpha)^2\tfrac{1}{2}\iint dSdS'f(\rho), \qquad \rho \equiv |\mathbf{r} - \mathbf{r}'|, \quad (5)$$

where each integral runs over all the material surfaces, with elements of area dS, dS' at $\mathbf{r}$, $\mathbf{r}'$. The force F between two disjoint surfaces S_i and S_j, separation parameter a, is

$$F(a) = -\partial U/\partial a, \quad U \equiv B(a) - B(\infty) = -(n\alpha)^2\int_{S_i} dS\int_{S_j} dS'f(\rho), \quad (6)$$

with the first integral over one surface and the second over the other. Equivalently, one can get U from B by keeping only terms that depend on a.

B. Cylinder and piston

We study a semi-infinite cylinder parallel to the z axis, extending over all $z > 0$, with constant cross-sectional area $A \sim b^2$ and perimeter of length [20] $P \sim b$, and closed by a fixed flat baffle in the xy plane. We shall need to consider also the corresponding infinite cylinder, extending over all z, with a similar fixed baffle at $z = 0$. The piston too is flat, is parallel to the baffle, and situated at $z = a > 0$. Eventually we shall specialize to a circular cylinder, with radius b, $A = \pi b^2$, $P = 2\pi b$.

Hertzberg *et al.* [16] consider the same geometry but with perfectly reflecting surfaces; determine the a-dependent part of the energy of the quantized Maxwell field (i.e. of the energy localized in the vacuum); find that the attendant force F is attractive for cross sections of any shape, and for all values of a/b; and give an expression for F (their Eq. (6)) when the cylinder cross section is a square with edge length b. Here we shall show that, by contrast, for the weakly reflecting material modeled above, there is attraction for small a/b but repulsion for large a/b. The force vanishes (the piston is in unstable equilibrium) at a value of a/b dependent on the shape of the cross section and on $f(\rho)$. For instance, in the CP regime and for circular cross sections, F will turn out to vanish at $a/b \simeq 1.83$.

In an infinite cylinder, it is obvious from translational invariance that the net force $F^{(1)}(a) = -\partial U^{(1)}/\partial a$ experienced by the piston comes wholly from the baffle. Hence $U^{(1)}(a)$ is the interaction between two parallel flats each of area A, a distance a apart. In the semi-infinite cylinder, the piston experiences also an opposite force $F^{(2)} = -(-\partial U^{(2)}/\partial a)$, where $U^{(2)}$ is (or rather would be) the interaction between the piston and the (in fact missing) semi-infinite cylinder with $z < 0$. The minimum separation between these is also a. Thus

$$U = U^{(1)} + U^{(2)}, \quad (7)$$

$$U^{(1)} = -(n\alpha)^2\int_{\text{piston}} dS\int_{\text{baffle}} dS'f(\rho), \quad U^{(2)} = (n\alpha)^2\int_{\text{piston}} dS\int_{\text{cylinder},z<0} dS'f(\rho). \quad (8)$$

Correspondingly, $F = F^{(1)} + F^{(2)}$, where $F^{(1)}$ is negative, $F^{(2)}$ is positive, and the total force on the piston towards the baffle reads

$$F = |F^{(1)}| - |F^{(2)}|. \quad (9)$$

III. ASYMPTOTICS AND SIGN REVERSAL

Elementary reasoning about its asymptotics shows that F changes sign. One requires only the forces in three simple geometries, namely, (i) the force $-u'(a)$ *per unit area* between infinitely extended parallel surfaces; (ii) the force $-v'(a)$ *per unit edge length* between two half-planes at right angles with their edges parallel and a distance a apart (e.g. the half-planes ($z = 0$, $y > 0$) and ($y = 0$, $z > a$)); and (iii) the force $-w'(a)$ between a number nA of atoms concentrated at the center of the piston (at 0, 0, a), and a number nP of atoms per unit length concentrated on the axis of the missing half-cylinder (on $z < 0$). These are derived in the appendix, and read

$$\begin{aligned} -u'(a) &= -(n\alpha)^2 2\pi a f(a), \\ -v'(a) &= -(n\alpha)^2 \pi \mathcal{J}_1(a), \\ -w'(a) &= -(n\alpha)^2 APf(a). \end{aligned} \tag{10}$$

As $b \to \infty$ at fixed a (i.e. as $b/a \to \infty$) we can evidently approximate [21] $F^{(1)} \simeq -Au'(a)$ and $F^{(2)} \simeq -Pv'(a)$. On the other hand, when $a \to \infty$ at fixed b (i.e. as $a/b \to \infty$) piston and baffle can be viewed as points a distance a apart, whence $F^{(1)} \simeq -(n\alpha)^2 A^2 f'(a)$; likewise we can approximate the absent cylinder as if it were collapsed onto its axis $z < 0$, whence $F^{(2)} \simeq (n\alpha)^2 w'(a)$. But from these expressions it is easy to verify that any practicable combination of asymptotics from (3) and (4) entails

$$\begin{aligned} |F^{(1)}(a/b \to 0)| &> |F^{(2)}(a/b \to 0)|, \\ |F^{(1)}(a/b \to \infty)| &< |F^{(2)}(a/b \to \infty)|. \end{aligned} \tag{11}$$

To illustrate this we specialize to a circular cross section of radius b, and to the CP regime $f \simeq C/\rho^7$. Then

$$\begin{aligned} |F^{(1)}(a \ll b)| &\simeq (n\alpha)^2 C[2\pi^2] b^2/a^6 \gg |F^{(2)}(a \ll b)| \\ &\simeq (n\alpha)^2 C[2\pi^2/5] b/a^5, \end{aligned} \tag{12}$$

$$\begin{aligned} |F^{(1)}(a \gg b)| &\simeq (n\alpha)^2 C[7\pi^2] b^4/a^8 \ll |F^{(2)}(a \gg b)| \\ &\simeq (n\alpha)^2 C[2\pi^2] b^3/a^7. \end{aligned} \tag{13}$$

Changing the shape of the cross section merely changes the numerical coefficients inside the square brackets. To get the corresponding estimates for the pseudo-VdW regime (4) one replaces C with $ak_B T$, and changes the numerical coefficients.

The asymptotic repulsion $F \sim F^{(2)} \sim (n\alpha)^2 Cb^3/a^7$ may be contrasted with the perfect-reflector attraction for a square cross section of edge length b, which reads [22] $F \sim -(ab^3)^{-1/2} \exp(-2\pi a/b)$.

IV. CALCULATION

We find U via B, adapting to surface-distributed the formalism developed in [2] for bulk-distributed atoms:

$$\begin{aligned} B &= -(n\alpha)^2 \tfrac{1}{2} \int_{\rho(\min)}^{\infty} d\rho \rho^2 f(\rho) g(\rho), \\ g(\rho) &\equiv \iint dS dS' \delta(|\mathbf{r} - \mathbf{r}'| - \rho), \end{aligned} \tag{14}$$

where $\rho(\min)$ is a minimum interatomic distance. Recall from Section II A that to obtain U one simply drops from B, in this instance from g, all the terms independent of a: in the following this will be done without further comment. Thereafter the correlation function $g(\rho)$ features only $\rho \geq a$, with possibly small but macroscopic $a \gg \rho(\min)$, making the precise value of $\rho(\min)$ irrelevant.

For the very simple geometries considered in the appendix one can determine g directly from its definition (14), but generally it is best found through the form-factors $\mathcal{F}(\mathbf{k})$ of the surfaces in question:

$$\begin{aligned} g(\rho) &= \int d\Omega_\rho \int \frac{d^3k}{(2\pi)^3} \exp(-i\mathbf{k} \cdot \rho) |\mathcal{F}(\mathbf{k})|^2, \\ \mathcal{F}(\mathbf{k}) &\equiv \int dS \exp(i\mathbf{k} \cdot \mathbf{r}), \end{aligned} \tag{15}$$

where $d\Omega_\rho$ stands for the element of the solid angle of ρ. The present section will simply quote the results of this method (which avoids much tedious algebra at the cost of an analytic detour through Bessel functions).

A. Piston and baffle

Here, S consists of two circular disks both of radius b, both parallel to the xy plane, one centered on (0, 0, 0) and the other on (0, 0, a). Eventually one finds

$$\begin{aligned} g^{(1)}(\rho) = \Theta(\rho - a)\Theta(2b - \sqrt{\rho^2 - a^2}) \frac{8\pi b^2}{\rho} \\ \times \{\cos^{-1}(\sigma) - \sigma\sqrt{1 - \sigma^2}\}, \end{aligned} \tag{16}$$

$$\sigma^2 \equiv (\rho^2 - a^2)/4b^2, \qquad 0 \leq \sigma \leq 1, \tag{17}$$

where $\Theta(x)$ is the Heaviside step function defined by $\Theta(x < 0) = 0$, $\Theta(x > 1) = 1$, $d\Theta(x)/dx = \delta(x)$. Substituting (16) into U via (14) as explained above, we obtain

$$\begin{aligned} U^{(1)} &= -(n\alpha)^2 4\pi b^2 \int_a^{\sqrt{a^2+4b^2}} d\rho \rho f(\rho) \{\cos^{-1}(\sigma) \\ &\quad - \sigma\sqrt{1 - \sigma^2}\} \\ &= -(n\alpha)^2 16\pi b^4 \int_0^1 d\sigma \sigma f(\rho) \{\cos^{-1}(\sigma) \\ &\quad - \sigma\sqrt{1 - \sigma^2}\}, \\ \rho &= \sqrt{4b^2\sigma^2 + a^2}. \end{aligned} \tag{18}$$

As $b \to \infty$ at fixed a, we check the first line of (18) against the parallel-planes result from the appendix, already used in Section III. The upper limit of the integral recedes to ∞, while $\sigma \to 0$, entailing $\{\ldots\} \to \pi/2$. Accordingly,

$$\begin{aligned} U^{(1)}(b/a \to \infty)/\pi b^2 &= -(n\alpha)^2 2\pi \int_a^{\infty} d\rho f(\rho) \rho \\ &= -(n\alpha)^2 2\mathcal{J}_1(a), \end{aligned} \tag{19}$$

consistently with (A1). As $a \to \infty$ at fixed b, we use the second line of (18), observe that now $\rho \to a$, replace $f(\rho) \to f(a)$, and have

$$U^{(1)}(a/b \to \infty) = -(n\alpha)^2 16\pi b^4 f(a) \int_0^1 d\sigma \sigma \{\cos^{-1}(\sigma) - \sigma\sqrt{1-\sigma^2}\} = -(n\alpha)^2 f(a)(\pi b^2)^2, \tag{20}$$

as anticipated in Section III.

The most convenient expression for $F^{(1)} = -\partial U^{(1)}/\partial a$ turns out to be

$$F^{(1)}(a) = (n\alpha)^2 16\pi b^4 \int_0^1 d\sigma \frac{\partial f(\sqrt{4b^2\sigma^2 + a^2})}{\partial a} \times \sigma\{\cos^{-1}(\sigma) - \sigma\sqrt{1-\sigma^2}\}. \tag{21}$$

B. Piston and cylinder

Again S consists of two surfaces: a right circular cylinder, plus a circular disk centered on the cylinder axis a distance a outside its end, both of radius b. Eventually one finds

$$g^{(2)}(\rho) = \Theta(\rho - a) 8\pi b \int_{a/\rho}^1 d\cos\vartheta\, \Theta(2b/\rho - \sin\vartheta) \times \cos^{-1}[\rho\sin(\vartheta)/2b]. \tag{22}$$

From the right-angled triangle with sides a, $2b$, $\rho = \sqrt{a^2 + 4b^2}$ one can see that for $\rho \lesseqgtr \sqrt{a^2+4b^2}$ the effective lower limits on $\cos\vartheta$ are, respectively, a/ρ or $\sqrt{1 - 4b^2/\rho^2}$:

$$g^{(2)}(\rho) = \Theta(\rho - a) 8\pi b \left\{\Theta(\sqrt{a^2+4b^2} - \rho)\int_{a/\rho}^1 + \Theta(\rho - \sqrt{a^2+4b^2})\int_{\sqrt{1-4b^2/\rho^2}}^1\right\} \times d(\cos\vartheta)\cos^{-1}[\rho\sin(\vartheta)/2b]. \tag{23}$$

This appears to be as simple an expression as one can get, reducible though it is to elliptic integrals.

As $b \to \infty$, only the first term of (23) survives, with $\cos^{-1}[\ldots] \to \cos^{-1}[0] = \pi/2$. Thus $g^{(2)}(\rho) \to \Theta(\rho - a)4\pi^2 b(1 - a/\rho)$, which substituted into (14) indeed reproduces the expected result $U^{(2)} \to 2\pi b w(a)$. The limit $a \to \infty$ is less obvious, and easier to check from (22). Start by observing that the inner step function makes $\sin\vartheta \ll 1$, whence $\sin\vartheta \simeq \vartheta < 2b/a \ll 1$; then $d\cos\vartheta \to d\vartheta\vartheta$, and $\cos^{-1}[\ldots] \to \cos^{-1}x$ with $x \equiv \rho\vartheta/2b$. Changing the integration variable to x, and noting that $0 \le x \le 1$, we obtain

$$g^{(2)} \to 8\pi b(2b/\rho)^2 \int_0^1 dx x\cos^{-1}x = 4\pi^2 b^3/\rho^2. \tag{24}$$

Substituted into (14) this reproduces the expected result $(n\alpha)^2 2\pi^2 b^3 \mathcal{J}_0(a)$ for the interaction between the piston shrunk to a point and the half-cylinder shrunk to a half-line.

It pays to give some thought to simplifying the force

$$F^{(2)} = -(n\alpha)^2(\partial/\partial a)\tfrac{1}{2}\int_a^\infty d\rho f(\rho)\rho^2 g^{(2)}(\rho), \tag{25}$$

with $g^{(2)}$ from (23). There is no contribution from $\partial/\partial a$ acting on the lower limit of $\int_a^\infty d\rho \ldots$. Differentiating the step functions under $\int_a^\infty d\rho \ldots$ produces delta functions with equal and opposite coefficients, so that these contributions cancel too. Thus the only survivor stems from differentiating the first inner integral with respect to its lower limit:

$$F^{(2)} = -(n\alpha)^2 4\pi b \int_a^{\sqrt{a^2+4b^2}} d\rho f(\rho)\rho\cos^{-1}[\rho\sin(\vartheta)/2b], \quad \cos\vartheta = a/\rho. \tag{26}$$

On changing the integration variable to σ as in (17), this reduces to [23]

$$F^{(2)}(a) = -(n\alpha)^2 16\pi b^3 \int_0^1 d\sigma f(\sqrt{4b^2\sigma^2 + a^2})\sigma\cos^{-1}\sigma, \tag{27}$$

to be compared with (21).

V. THE CASIMIR-POLDER REGIME

When f varies as an inverse power of ρ, the force F emerges in closed form. As an illustration we determine and plot it for the CP regime where $f = C/\rho^7$, an approximation adequate if $\min(a, b) \gg 1/\Omega$. It proves convenient to introduce parameters y and s, and (following MAPLE's notation) short symbols for two complete elliptic integrals:

$$y \equiv a/2b, \qquad s \equiv 1/\sqrt{1+y^2}, \tag{28}$$

$$E(s) \equiv \int_0^{\pi/2} d\theta\sqrt{1 - s^2\sin^2\theta}, \qquad K(s) \equiv \int_0^{\pi/2} d\theta/\sqrt{1 - s^2\sin^2\theta}. \tag{29}$$

We construct $F = F^{(1)} + F^{(2)}$ from (21) and (27), set $f \to C(4b^2\sigma^2 + a^2)^{-7/2}$, evaluate the integrals, and scale the

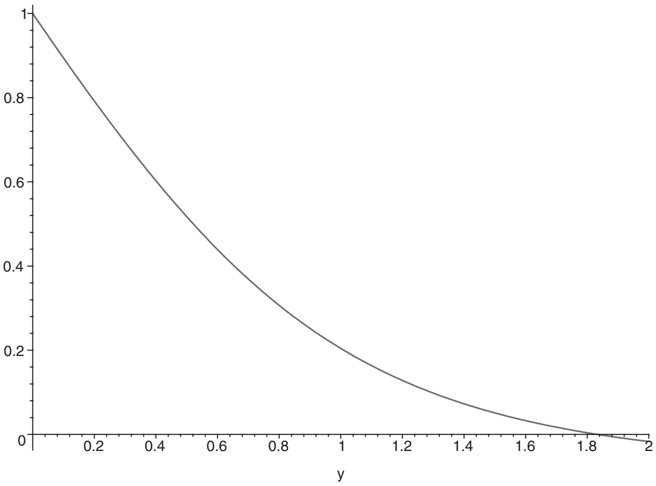

FIG. 1. $T(y)$ from Eq. (30) as a function of $y = a/2b$. This is the force F in the CP regime, normed to its asymptotic form for $y \to 0$.

result to its short-distance asymptotic form:

$$T(y) \equiv \frac{F}{(n\alpha)^2 C(2\pi^2 b^2/a^6)} = \frac{\pi(5-2y)}{20y^6} + \frac{s^3}{15y^5}(-3y^4 + 4y^3 - 13y^2 + 2y - 8)E(s) + \frac{s^3}{15y^3}(3y^2 - y + 4)K(s). \quad (30)$$

The asymptotics read

$$T(y \ll 1) = 1 - \left(\frac{2}{5} + \frac{16}{15\pi}\right)y + \frac{8}{15\pi}y^2 + \ldots,$$
$$T(y \gg 1) = -\frac{1}{2y} + \frac{7}{8y^2} + \frac{21}{32y^3} + \ldots, \quad (31)$$

and T is plotted in Fig. 1. The zero occurs at $y \simeq 1.834$. The approach to the long-distance asymptotic form is very slow, as shown by Fig. 2, which plots $-2yT$.

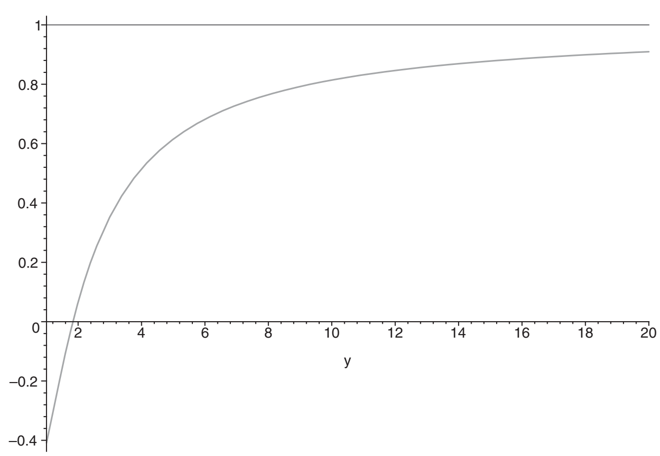

FIG. 2. $-2yT(y)$ as a function of $y = a/2b$. This is the force F in the CP regime, normed to its asymptotic form for $y \to \infty$.

VI. DISCUSSION

We have used leading-order perturbation theory to derive the force F on the piston in a cylinder, both of them infinitesimally thin and weakly (but finitely) reflecting. It changes from attractive to repulsive (relative to the cylinder head) as the ratio height/diameter rises through a value close to 2. By contrast, in the same geometry with perfect reflection F is always attractive [16]. Evidently one must consider (i) how sensitive our conclusions are to the details of our model, and (ii) possible inadequacies of perturbation theory.

(i) Under this heading it should be stressed that a zero of F is not a universal consequence of weak reflection *per se*. Consider for instance the corresponding cylindrical cavity ($z > 0$, $\sqrt{x^2 + y^2} < b$) hollowed out of an optically dilute medium occupying all the rest of space, with ν atoms per unit volume (so that the dielectric function is $\varepsilon \simeq 1 + 4\pi\nu\Pi$). Now insert our infinitesimally thin piston into this cavity at $z = a$, and note that if the cavity were filled, then the total force on the piston would, of course, vanish. Hence, by the same arguments as in Sections II and III, the net force on the piston in the otherwise empty cavity points towards negative z, and is equal in magnitude to the attraction between the piston and a solid cylinder ($z > 2a$, $\sqrt{x^2 + y^2} < b$): in other words it is given by $(\nu/n)\int_a^\infty dz F^{(1)}(z) = (\nu/n)U^{(1)}(a)$.

It seems plausible to infer that, if the cylinder and the baffle of our originally thin-walled system were gradually thickened, the zero of F predicted by perturbation theory would shift to higher values of a/b, with $a/b \to \infty$ at some critical finite thickness, and that for still thicker material F would remain attractive for all a/b.

(ii) Since second-order perturbation theory proves equivalent to summing two-body forces, it ceases to be reliable, even for weak reflectors, whenever three-body or other many-body forces contribute appreciably. As discussed in [2], the latter become increasingly important at long distances, simply because, with rising separations, the number of multiplets rises faster than the number of pairs [24]. But our thin-wall-perturbative F reverses with increasing a precisely in virtue of a change in the balance between the attraction of the piston to the increasingly distant baffle and its attraction to comparably distant parts of the cylinder wall. Accordingly, if the reversal were predicted to happen at large values of a/b, it would certainly become vulnerable to the quite general doubts that afflict perturbation theory at long range. On the other hand, the actually predicted aspect ratio $a/b \simeq 2$ is of order unity rather than very large, and the present writer is inclined to trust perturbation theory, pending explicit calculations to the contrary.

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APPENDIX: SIMPLE GEOMETRIES

We derive the simple expressions already used in the estimates of Section III, and in Section II (with $A = \pi b^2$, $P = 2\pi b$) as checks on the far more complicated calculations for circles and circular cylinders.

1. Parallel planes

The interaction $u(a)$ per unit area between two parallel planes a distance a apart is $n\alpha$ times the interaction between a single atom say at $(0, 0, -a)$ and the plane $z = 0$. Starting with cylindrical-polar coordinates, and then changing the integration variable from s to ρ, one sees that

$$u(a) = -(n\alpha)^2 \int_0^\infty 2\pi ds s f(\rho), \qquad \rho^2 = a^2 + s^2,$$
$$u(a) = -(n\alpha)^2 2\pi \int_a^\infty d\rho \rho f(\rho) = -(n\alpha)^2 2\pi \mathcal{J}_1(a),$$
$$-\partial u/\partial a = -(n\alpha)^2 2\pi a f(a). \tag{A1}$$

2. Orthogonal half-planes

Consider the two half-planes ($z = 0$, $y > 0$) and ($y = 0$, $z > a$), at right angles, with their edges parallel and a distance a apart. Let $w(a)$ be the interaction per unit length (measured in the x direction). Changing from Cartesian to spherical polars (with $z = \rho\cos\vartheta$) one has

$$v(a) = -(n\alpha)^2 \int_{-\infty}^\infty dx \int_0^\infty dy \int_a^\infty dz f(\rho),$$
$$\rho^2 = x^2 + y^2 + z^2,$$
$$v(a) = -(n\alpha)^2 \int_a^\infty d\rho \rho^2 f(\rho) \int_{a/\rho}^1 d\cos\vartheta \int_0^\pi d\phi$$
$$= -(n\alpha)^2 \pi \int_a^\infty d\rho \rho^2 f(\rho)(1 - a/\rho),$$
$$v(a) = -(n\alpha)^2 \pi \{\mathcal{J}_2(a) - a\mathcal{J}_1(a)\},$$
$$-\partial v/\partial a = -(n\alpha)^2 \pi \mathcal{J}_1(a). \tag{A2}$$

3. Point and line

Consider a number nA of atoms at the point $(0, 0, z = a)$, and the line $(0, 0, z < 0)$ with nP atoms per unit length. Their interaction energy is

$$w(a) = -(n\alpha)^2 AP \int_a^\infty dz f(z) = -(n\alpha)^2 AP \mathcal{J}_0(a),$$
$$-\partial w/\partial a = -(n\alpha)^2 AP f(a). \tag{A3}$$

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[17] With *perfect reflectors*, older views would probably have predicted a change from attraction to repulsion as the totally enclosed volume elongates. It might be thought entertaining that, while the proper calculation [16] then yields only attraction, it is for *weak reflectors* that the sign changes.
[18] We set $\hbar = 1 = c$ and use unrationalized Gaussian units for electrodynamics.
[19] At nonzero temperature B in (5) stands for the Helmholtz free energy.
[20] We exclude pathologically shaped cross sections.
[21] The approximations of this section are verified in the next.
[22] M. P. Hertzberg (private communication).
[23] $F^{(2)}(a)$ features $f(\rho)$ only for $a < \rho < \sqrt{4b^2 + a^2}$, which might suggest, incorrectly, that there are no contributions from the cylinder walls at axial distances greater than a from the piston. In fact all of the cylinder contributes: this becomes obvious if one starts with $U^{(2)}$ for a cylinder of finite length L, and only then takes the limit $L \to \infty$.
[24] This has the *prima facie* paradoxical consequence that perturbation theory is a safer guide at small distances, where interactions are strong, than it is at long distances, where interactions are weak.