Bootstrapping multiparton loop amplitudes in QCD

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We present a new method for computing complete one-loop amplitudes, including their rational parts, in nonsupersymmetric gauge theory. This method merges the unitarity method with on-shell recursion relations. It systematizes a unitarity-factorization bootstrap approach previously applied by the authors to the one-loop amplitudes required for next-to-leading order QCD corrections to the processes $e^+e^- \rightarrow Z, \gamma^* \rightarrow 4$ jets and $pp \rightarrow W + 2$ jets. We illustrate the method by reproducing the one-loop color-ordered five-gluon helicity amplitudes in QCD that interfere with the tree amplitude, namely $A_{5;1}(1^-, 2^-, 3^+, 4^+, 5^+)$ and $A_{5;1}(1^-, 2^+, 3^-, 4^+, 5^+)$. Then we describe the construction of the six- and seven-gluon amplitudes with two adjacent negative-helicity gluons, $A_{6;1}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$ and $A_{7;1}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+, 7^+)$, which uses the previously-computed logarithmic parts of the amplitudes as input. We present a compact expression for the six-gluon amplitude. No loop integrals are required to obtain the rational parts.

I. INTRODUCTION

The approaching dawn of the experimental program at CERN's Large Hadron Collider calls for theoretical support in a number of areas. A key ingredient in the quest to find and understand the new physics at the TeV scale will be our ability to deliver precise predictions for a variety of observable processes. Fulfilling this demand will depend in turn on having versatile tools for calculating multiparticle, loop-level scattering amplitudes in the component gauge theories of the standard model. Tree-level amplitudes provide a first but insufficient step. The size and scale-variation of the strong coupling constant imply that even for a basic quantitative understanding, one must also include the one-loop amplitudes which enter into next-to-leading order corrections to cross sections [1]. An important class of computations are of perturbative QCD and QCD-associated processes. Extending the set of available processes to $W +$ multi-jet production, and beyond, will demand computations of new one-loop amplitudes in perturbative QCD.

In this paper we will describe a new approach to computing complete one-loop scattering amplitudes in nonsupersymmetric theories such as QCD. This approach systematizes a unitarity-factorization bootstrap approach applied by the authors to the computation of the one-loop scattering amplitudes needed for $Z \rightarrow 4$ jets and $pp \rightarrow W + 2$ jets at next-to-leading order in the QCD coupling [2]. As in that paper, the cut-containing logarithmic and polylogarithmic terms are computed using the unitarity method [3–8] and four-dimensional tree-level amplitudes as input. The remaining rational-function pieces are com-

puted via a factorization bootstrap, in the form of an on-shell recurrence relation [9–12]. (In Ref. [2] the rational functions were constructed as ansätze with the assistance of the factorization limits, and verified by numerical comparison to a direct Feynman diagram computation.)

The unitarity method has proven to be an effective means of computing the logarithmic and polylogarithmic terms in gauge theory amplitudes at one and two loops. In massless supersymmetric theories the complete one-loop amplitudes may be determined from the four-dimensional cuts [4]. This method has been applied in a variety of amplitude calculations in QCD [2,13–17] and in supersymmetric gauge theories [3,4,18–20]. A recent improvement to the unitarity method [7] uses *complex* momenta within generalized unitarity [2,16,19], and allows a simple determination of box integral coefficients. (The name "generalized unitarity," as applied to amplitudes for massive particles, can be traced back to Ref. [21].) The unitarity method has spawned a number of related techniques, including the very beautiful application of maximally-helicity-violating (MHV) vertices to loop calculations [8,22] and the use [23,24] of the holomorphic anomaly [25] to evaluate the cuts. The unitarity method can also be used to determine complete amplitudes, including all rational pieces [5,13,14,17] by applying full D-dimensional unitarity, where $D = 4 - 2\epsilon$ is the parameter of dimensional regularization [26]. This approach requires the computation of tree amplitudes where at least two of the momenta are in D dimensions. For one-loop amplitudes containing only external gluons, these tree amplitudes can be interpreted as four-dimensional amplitudes but with massive scalars. Recent work has used on-shell recursive techniques [9,10] to extend the number of known massive-scalar amplitudes [27]. At present, the D-dimensional unitarity approach has been applied to all n-gluon ampli-

tudes with $n = 4$ [17] and to special helicity configurations with n up to 6 [13,17].

The somewhat greater complexity of the D-dimensional cuts suggests that it is worthwhile to explore other methods of obtaining the rational terms. We have additional information about these terms, after all, beyond the knowledge that their D-dimensional cuts are D-dimensional tree amplitudes. Because we know *a priori* the factorization properties of the complete one-loop amplitude [3,28], we also know the factorization properties of the pure rational terms. It would be good to bring this information to bear on the problem. This idea was behind the "bootstrap" approach used in Ref. [2]. The idea was used to produce compact expressions for the $Z \rightarrow q\bar{q}gg$ amplitudes. However, it was not presented in a systematic form, and indeed, for sufficiently complicated amplitudes it can be difficult to find ansätze with the proper factorization properties. This shortcoming has prevented wider application of these ideas.

Recent progress in calculations of gauge-theory amplitudes has led us to reexamine the bootstrap approach. This progress has been stimulated by Witten's proposal of a weak-weak duality between $\mathcal{N} = 4$ supersymmetric gauge theory and the topological open-string B model in twistor space [29]. (The roots of the duality lie in Nair's description [30] of the simplest gauge-theory amplitudes.) Witten also made the beautiful conjecture that the amplitudes are supported on a set of algebraic curves in twistor space. The underlying twistor structure of gauge theories, as revealed by further investigation [23,31–36], has turned out to be even simpler than originally conjectured. (For a recent review, see Ref. [37].) The underlying twistor structure was made manifest by Cachazo, Svrček and Witten [38], in a new set of diagrammatic rules for computing all tree-level amplitudes, which use MHV amplitudes as vertices. These MHV rules led to further progress in the computation of tree-level [9,10,27,38–42] amplitudes. Brandhuber, Spence, and Travaglini [22] provided the link between loop computations using MHV vertices and those done in the unitarity-based method. This development in turn opened the way for further computations and insight at one loop [7,8,19,20,24,43]. The remarkable conclusion of all these studies is that gauge-theory amplitudes, especially in supersymmetric theories, are *much* simpler than had been anticipated, even in light of known, simple, results. Several groups have also studied multiloop amplitudes, and have found evidence for remarkable simplicity, at least for maximal supersymmetry [18].

Recently, Britto, Cachazo and Feng wrote down [9] a new set of tree-level recursion relations. Recursion relations have long been used in QCD [44,45], and are an elegant and efficient means for computing tree-level amplitudes. The new recursion relations differ in that they employ only *on-shell* amplitudes (at *complex* values of the external momenta). These relations were stimulated by the

compact forms of seven- and higher-point tree amplitudes [19,20,41] that emerged from studying infrared consistency equations [46] for one-loop amplitudes. A simple and elegant proof of the relation using special complex continuations of the external momenta has been given by Britto, Cachazo, Feng and Witten [10]. Its application yields compact expressions for tree amplitudes in gravity as well as gauge theory [42], and extends to massive theories as well [27].

In principle, recursion relations of this type could provide a systematic way to carry out the factorization bootstrap at one loop. One must however confront a number of subtleties in attempting to extend them from tree to loop level. The most obvious problem is that the proof of the tree-level recursion relations relies on the amplitudes having only simple poles; loop amplitudes in general have branch cuts. Moreover, the factorization properties of loop amplitudes evaluated at complex momenta are not fully understood; unlike the case of real momenta, there are no theorems specifying these properties. Indeed, there are double pole and "unreal" pole contributions that must be taken into account [11,12].

In a pair of previous papers [11,12] we have applied on-shell recursion relations to the study of finite one-loop amplitudes in QCD. These helicity amplitudes vanish at tree level. Accordingly, the one-loop amplitudes are finite, and possessing no four-dimensional cuts, are purely rational functions. Through careful choices of shift variables and studies of known amplitudes, we found appropriate double and unreal pole contributions for the recursion relations, and used them to recompute known gluon amplitudes, and to compute fermionic ones for the first time.

While we will not give a derivation of complex factorization in the present paper, it is heartening that no new subtleties of this sort arise in the amplitudes studied here, beyond those studied in Refs. [11,12]. The systematization we shall present suggests that a proper and general derivation of the complex factorization behavior should indeed be possible.

In this paper, we focus on the issue of setting up on-shell recursion relations in the presence of branch cuts. We describe a new method for merging the unitarity technique with the on-shell recursion procedure. As mentioned above, we follow the procedure introduced in Ref. [2], determining the cut-containing logarithms and polylogarithms via the unitarity method, and then determining the rational functions via a factorization bootstrap. We derive on-shell recursion relations for accomplishing the bootstrap. In general, both the rational functions and cut pieces have spurious singularities which cancel against each other. These spurious singularities would interfere with the recursion because their factorization properties are not universal. We solve this problem by using functions which are manifestly free of the spurious singularities, at the price of adding some rational functions to the cut parts.

These added rational functions have an overlap with the on-shell recursion. To handle this situation, we derive a recursion relation which accounts for these overlap terms.

To illustrate our bootstrap method we recompute the rational-function parts of the known [47] five-gluon amplitudes. We present all the intermediate steps determining the rational functions of one of the five-gluon amplitudes, in order to underline the algebraic simplicity of the procedure. As a demonstration of its utility, we also compute two new results, the six- and seven-gluon amplitudes with two color-adjacent negative helicities. We present the complete six-gluon amplitude in a compact form. These results have all the required factorization properties in real momenta, a highly nontrivial consistency check. A computation based purely on the unitarity method, that is to say based on full D-dimensional unitarity, would provide a further check.

This paper is organized as follows. In the next section, we review our notation and the elements entering into a decomposition of QCD amplitudes at tree level and one loop. In Sec. III, we derive a new on-shell recursion-based formula for general one-loop amplitudes. In Sec. IV, we review the relevant known amplitudes, and pieces thereof, and lay out the vertices that will be used for the recomputation of the five-point amplitude and the computation of the six- and seven-point amplitudes. In Sec. V, we display the recomputation of the five-point amplitude in great detail. In Sec. VI, we compute and quote the six-point amplitude, and present the diagrams for the seven-point amplitude. We then give our conclusions.

II. NOTATION

In this section we summarize the notation used in the remainder of the paper, following the notation of our previous papers [11,12]. We use the spinor helicity formalism [48,49], in which the amplitudes are expressed in terms of spinor inner-products,

$$\langle jl \rangle = \langle j^- | l^+ \rangle = \bar{u}_-(k_j)u_+(k_l),$$
$$[jl] = \langle j^+ | l^- \rangle = \bar{u}_+(k_j)u_-(k_l),$$

(2.1)

where $u_\pm(k)$ is a massless Weyl spinor with momentum k and positive or negative chirality. We follow the convention that all legs are outgoing. The notation used here follows the standard QCD literature, with $[ij] = \text{sign}(k_i^0 k_j^0)\langle ji \rangle^*$ so that,

$$\langle ij \rangle [ji] = 2k_i \cdot k_j = s_{ij}.$$

(2.2)

These spinors are connected to Penrose's twistors [50] via a Fourier transform of half the variables, e.g. the u_- spinors [29,50]. (Note that the QCD-literature square bracket $[ij]$ employed here differs by an overall sign compared to the notation commonly found in twistor-space studies [29].) We also define, as in the twistor-string literature,

$$\lambda_i \equiv u_+(k_i), \qquad \tilde{\lambda}_i \equiv u_-(k_i).$$

(2.3)

We denote the sums of cyclicly-consecutive external momenta by

$$K_{i\cdots j}^\mu \equiv k_i^\mu + k_{i+1}^\mu + \cdots + k_{j-1}^\mu + k_j^\mu,$$

(2.4)

where all indices are mod n for a n-gluon amplitude. The invariant mass of this vector is $s_{i\cdots j} = K_{i\cdots j}^2$. Special cases include the two- and three-particle invariant masses, which are denoted by

$$s_{ij} \equiv K_{i,j}^2 \equiv (k_i + k_j)^2 = 2k_i \cdot k_j,$$
$$s_{ijk} \equiv (k_i + k_j + k_k)^2.$$

(2.5)

We also define spinor strings,

$$\langle i^- | (a + b) | j^- \rangle = \langle ia \rangle [aj] + \langle ib \rangle [bj],$$
$$\langle i^+ | (a + b)(c + d) | j^- \rangle = [ia]\langle a^- | (c + d) | j^- \rangle$$
$$+ [ib]\langle b^- | (c + d) | j^- \rangle,$$

(2.6)

and gamma matrix traces,

$$\text{tr}_+[abcd] = [ab]\langle bc \rangle [cd]\langle da \rangle,$$

(2.7)

$$\text{tr}_+[abc(d + e)] = [ab]\langle bc \rangle [cd]\langle da \rangle$$
$$+ [ab]\langle bc \rangle [ce]\langle ea \rangle.$$

(2.8)

We use the trace-based color decomposition of amplitudes [49,51–53]. For tree-level amplitudes with n external gluons, this decomposition is,

$$\mathcal{A}_n^{\text{tree}}(\{k_i, h_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}})$$
$$\times A_n^{\text{tree}}(\sigma(1^{h_1}, \ldots, n^{h_n})),$$

(2.9)

where g is the QCD coupling, S_n/Z_n is the group of non-cyclic permutations on n symbols, and j^{h_j} denotes the jth gluon, with momentum k_j, helicity h_j, and adjoint color index a_j. The T^a are SU(N_c) color matrices in the fundamental representation, normalized so that $\text{Tr}(T^a T^b) = \delta^{ab}$. The color-ordered amplitude A_n^{tree} is invariant under a cyclic permutation of its arguments.

When all internal particles transform in the adjoint representation of SU(N_c), the color decomposition for one-loop n-gluon amplitudes is given by [54],

$$\mathcal{A}_n^{\text{adjoint}}(\{k_i, h_i, a_i\})$$
$$= g^n \sum_J n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \text{Gr}_{n;c}(\sigma) A_{n;c}^{[J]}(\sigma), \quad (2.10)$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. The sum $J \in \{0, 1/2, 1\}$ runs over all spins of particles and n_J is the multiplicity of each spin. We assume all particles are massless. The leading-color structure,

$$\text{Gr}_{n;1}(1) = N_c \text{Tr}(T^{a_1} \cdots T^{a_n}),$$

(2.11)

is N_c times the tree color structure. The subleading-color

structures are given by

$$\text{Gr}_{n;c}(1) = \text{Tr}(T^{a_1} \cdots T^{a_{c-1}})\text{Tr}(T^{a_c} \cdots T^{a_n}). \quad (2.12)$$

In Eq. (2.10), S_n is the set of all permutations of n objects, and $S_{n;c}$ is the subset leaving $\text{Gr}_{n;c}$ invariant. For adjoint particles circulating in the loop, the subleading-color partial amplitudes, $A_{n;c}$ for $c > 1$, are given by a sum over permutations of the leading-color ones [3]. Therefore we need to compute directly only the leading-color, single-trace, partial amplitudes $A_{n;1}(1^{h_1}, \ldots, n^{h_n})$. The leading-color amplitude is again invariant under cyclic permutation of its arguments.

In QCD, of course, there can be fundamental representation quarks present in the loop. In this case only the single-trace color structure contributes, but it is smaller by a factor N_c. In general, scalars or fermions in the $N_c + \overline{N}_c$ representation give a contribution,

$$\mathcal{A}_n^{\text{fund}}(\{k_i, h_i, a_i\}) = g^n \sum_{J=0,1/2} \frac{n_J}{N_c} \sum_{\sigma \in S_n/Z_n} \text{Gr}_{n;1}(\sigma) A_{n;1}^{[J]}(\sigma),$$

$$(2.13)$$

to the one-loop amplitude. We use a supersymmetric convention in which the number of states for a single complex scalar (squark) is $4N_c$, in order to match the number of states of a Dirac fermion (quark).

Helicity amplitudes that do not vanish at tree level develop infrared and ultraviolet divergences at one loop. We regulate these dimensionally. Following Ref. [47], for the divergent one-loop amplitudes we write,

$$A_{n;1}^{[0]} = c_\Gamma(V_n^s A_n^{\text{tree}} + iF_n^s), \quad (2.14)$$

$$A_{n;1}^{[1/2]} = -c_\Gamma((V_n^f + V_n^s)A_n^{\text{tree}} + i(F_n^f + F_n^s)), \quad (2.15)$$

$$A_{n;1}^{[1]} = c_\Gamma((V_n^g + 4V_n^f + V_n^s)A_n^{\text{tree}} + i(4F_n^f + F_n^s)), \quad (2.16)$$

where

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}. \quad (2.17)$$

The V_n^x parts contain the divergences, while the F_n^x are finite. (Of course, there is some ambiguity in the separation between divergent and finite terms.) These pieces have a natural interpretation in terms of supersymmetric and non-supersymmetric parts [47],

$$A_{n;1}^{\mathcal{N}=4} = c_\Gamma A_n^{\text{tree}} V_n^g, \quad (2.18)$$

$$A_{n;1}^{\mathcal{N}=1} = -c_\Gamma(A_n^{\text{tree}} V_n^f + iF_n^f), \quad (2.19)$$

$$A_{n;1}^{\mathcal{N}=0} = c_\Gamma(A_n^{\text{tree}} V_n^s + iF_n^s). \quad (2.20)$$

Here $A_{n;1}^{\mathcal{N}=4}$ sums over the contributions of a $\mathcal{N} = 4$ multiplet consisting of one gluon, four Majorana fermions,

and three complex scalars, all in the adjoint representation. The $\mathcal{N} = 1$ amplitude contains the contributions of an adjoint chiral multiplet, consisting of one complex scalar and one Weyl fermion. The nonsupersymmetric amplitudes, denoted by $\mathcal{N} = 0$, are just the contributions of a complex scalar in the loop, $A_{n;1}^{\mathcal{N}=0} = A_{n;1}^{[0]}$.

The utility of separating QCD amplitudes into supersymmetric and nonsupersymmetric pieces follows from their differing analytic properties. The supersymmetric pieces can be constructed completely from four-dimensional unitarity cuts [3,4] and have no remaining rational contributions (in the limit $\epsilon \to 0$). The examples discussed in this paper, n-gluon amplitudes with two negative-helicity gluons, have been known for quite some time [3,4]. (Such amplitudes are often referred to as "MHV" in the supersymmetric cases, because amplitudes with fewer (zero or one) negative-helicity gluons vanish.) The logarithmic and polylogarithmic terms in the nonsupersymmetric (scalar) pieces can also be obtained from four-dimensional cuts, or from MHV vertices. These cut-containing terms are also known for all n-gluon amplitudes with two negative-helicity gluons. They were computed first for the case where the two negative-helicity gluons are color-adjacent [4], and more recently for the general case [8].

Here we focus on the unsolved problem of computing the rational-function terms in the $\mathcal{N} = 0$ contributions in six and higher-point amplitudes, given the knowledge of the logarithmic and polylogarithmic terms. Finding an effective computational approach to the rational terms in $A_{n;1}^{\mathcal{N}=0}$ is tantamount to solving the problem in QCD.

The leading-color QCD amplitudes are expressible in terms of the different components in Eqs. (2.18), (2.19), and (2.20) via

$$A_{n;1}^{\text{QCD}} = c_\Gamma\Bigg[(V_n^g + 4V_n^f + V_n^s)A_n^{\text{tree}} + i(4F_n^f + F_n^s)$$
$$- \frac{n_f}{N_c}(A_n^{\text{tree}}(V_n^s + V_n^f) + i(F_n^s + F_n^f))\Bigg], \quad (2.21)$$

where n_f is the number of active quark flavors in QCD. We will present the formulas for unrenormalized amplitudes. To carry out a $\overline{\text{MS}}$ subtraction, one should subtract from the leading-color partial amplitudes $A_{n;1}$ the quantity

$$c_\Gamma\Bigg[\frac{(n-2)}{2}\frac{1}{\epsilon}\Big(\frac{11}{3} - \frac{2}{3}\frac{n_f}{N_c} - \frac{1}{3}\frac{n_s}{N_c}\Big)\Bigg]A_n^{\text{tree}}, \quad (2.22)$$

where we also included a term proportional to the number of active fundamental representation scalars n_s, which vanishes in QCD.

We will need to consider additional objects (parts of amplitudes), beyond the V^x and F^x defined here, in order to construct and apply appropriate on-shell recursion relations for one-loop amplitudes. The definition of these

objects and derivation of the relations is the subject of the next section.

III. ON-SHELL RECURSION RELATIONS FOR LOOP AMPLITUDES

On-shell recursion relations provide an effective means for obtaining remarkably compact forms for tree-level amplitudes [9,10,41], and have led to a variety of new results [27,42]. In previous work [11,12], we have shown how to use on-shell recursion relations to compute all the finite loop amplitudes of QCD. These special helicity amplitudes vanish at tree level. Hence the one-loop amplitudes are free of infrared and ultraviolet divergences, and they are "treelike" in that they contain no cuts (absorptive parts) in four dimensions. The derivation of these loop recursion relations is similar in spirit to the tree-level case, but it does require the treatment of factorizations which differ from the "ordinary" factorization in *real* momentum.

In this paper, we will extend the analysis of Refs. [11,12] to cut-containing one-loop amplitudes (for which the corresponding tree-level amplitudes do not vanish), deriving new recursion relations for the rational functions appearing in such amplitudes. The new recursion relations allow us to systematize the factorization bootstrap approach of Ref. [2]. We assume that the cut-containing terms have already been determined via the unitarity method or some other means.

A. Analytic behavior of shifted loop amplitudes

The starting point for our analysis, as for the finite loop amplitudes, is to consider [10] a complex-valued shift of the momentum of a pair of external particles in a n-point amplitude, $k_j \rightarrow \hat{k}_j(z)$, $k_l \rightarrow \hat{k}_l(z)$. This shift is best described in terms of the spinor variables λ and $\tilde{\lambda}$ defined in Eq. (2.3),

$$\tilde{\lambda}_j \rightarrow \tilde{\lambda}_j - z\tilde{\lambda}_l, \qquad \lambda_l \rightarrow \lambda_l + z\lambda_j. \quad (3.1)$$

This (j, l) shift maintains overall momentum conservation, because

$$\begin{aligned} k\!\!\!/_j + k\!\!\!/_l &= \lambda_j \tilde{\lambda}_j + \lambda_l \tilde{\lambda}_l \rightarrow \hat{k}\!\!\!/_j + \hat{k}\!\!\!/_l \\ &= \lambda_j(\tilde{\lambda}_j - z\tilde{\lambda}_l) + (\lambda_l + z\lambda_j)\tilde{\lambda}_l = k\!\!\!/_j + k\!\!\!/_l, \quad (3.2) \end{aligned}$$

as well as the masslessness of the external momenta, $\hat{k}_j^2 = \hat{k}_l^2 = 0$. Denote the original n-point amplitude by $A_n \equiv A_n(0)$, and the shifted one by $A_n(z)$. We wish to determine $A_n(0)$ by making use of the analytic properties of $A_n(z)$.

In the case of tree-level or finite one-loop amplitudes, $A_n(z)$ is a meromorphic function of z. Here we also encounter branch cuts, which may terminate at poles, as depicted in Fig. 1. Branch cuts arise from logarithms or polylogarithms in the amplitudes. Consider, for example, the scalar contributions to the five-gluon amplitude with
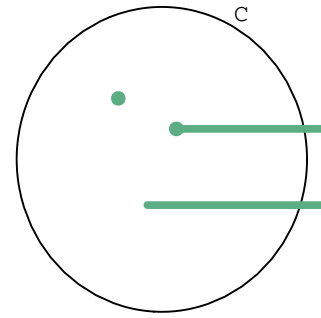

FIG. 1 (color online). A configuration of poles and branch cuts for a term in a one-loop amplitude. The contour C is a circle at ∞.

color-ordered helicity assignment $(- - + + +)$, recalled in Eq. (4.20). It contains a logarithm, $\ln((-s_{23})/(-s_{51}))$, multiplied by a rational coefficient. If we perform a shift (3.1) with $(j, l) = (1, 2)$, the logarithm becomes

$$\ln\left(\frac{-s_{23} - z\langle 1^-|3|2^-\rangle}{-s_{51} + z\langle 1^-|5|2^-\rangle}\right) = \ln\left(\frac{[23](\langle 23\rangle + z\langle 13\rangle)}{\langle 15\rangle([15] - z[25])}\right). \quad (3.3)$$

This function has two branch cuts in z, one starting at

$$z = \frac{[15]}{[25]}, \quad (3.4)$$

the other starting at

$$z = -\frac{\langle 23\rangle}{\langle 13\rangle}. \quad (3.5)$$

Because of the form of the rational coefficient of $\ln((-s_{23})/(-s_{51}))$ in this case, neither branch cut starts at a pole.

We assume that j and l can be chosen so that $A_n(z) \rightarrow 0$ as $z \rightarrow \infty$. We consider the following quantity,

$$\frac{1}{2\pi i} \oint_C \frac{dz}{z} A_n(z), \quad (3.6)$$

where the contour integral is taken around the circle at ∞. A typical configuration for a term in a one-loop amplitude is shown in Fig. 1. Even though the contour crosses branch cuts, the integral still vanishes, because $A_n(z)$ vanishes at infinity. Unlike the rational cases studied previously, however, this does not mean that it is given simply by a sum of residues at its poles. We need to include those contributions, of course; but we also need to integrate around the branch cuts, with special handling for poles at the end of branch cuts.

We start with the ordinary branch cuts, with no pole touching the branch cut. We can imagine a related contour, going along the circle at infinity, but avoiding the branch cuts by integrating inwards along one side, and then outwards along the other, as shown in Fig. 2. (We will route the branch cuts so that no two overlap.) The integral along *this*
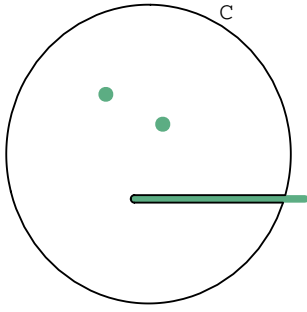
FIG. 2 (color online). A configuration of poles and branch cuts for a term in a one-loop amplitude, with a branch-cut-hugging contour.

contour *is* given by the sum of residues. The difference between the two integrals is given by the branch-cut-hugging integral,

$$\frac{1}{2\pi i}\int_{B^\uparrow + i\epsilon}\frac{dz}{z}A_n(z) + \frac{1}{2\pi i}\int_{B^\downarrow - i\epsilon}\frac{dz}{z}A_n(z), \qquad (3.7)$$

where $B^\uparrow$ is directed from an endpoint B_0 to infinity, and $B^\downarrow$ is directed in the opposite way. Now, $A_n(z)$ has a branch cut along B, which means that it has a nonvanishing discontinuity,

$$2\pi i\mathrm{Disc}_B A_n(z) = A_n(z + i\epsilon) - A_n(z - i\epsilon), \quad z \text{ on } B. \tag{3.8}$$

Thus our original vanishing integral can be written as follows,

$$0 = A_n(0) + \sum_{\text{poles }\alpha}\mathrm{Res}_{z=z_\alpha}\frac{A_n(z)}{z} + \int_{B_0}^\infty\frac{dz}{z}\mathrm{Disc}_B A_n(z). \tag{3.9}$$

The case with a pole at the end of the branch cut—arising, for example, from terms containing $\ln(-s_{ab})/\langle ab\rangle$—can be handled similarly, but care must be taken with the evaluation of the integral along the branch cut. From a conceptual point of view, we can also compute this case by moving the pole away from the end of the branch cut by an amount δ, computing the branch-cut-hugging and residue terms separately, and taking the limit $\delta \to 0$ at the end.

B. Cut-containing terms, and their "completion"

To proceed further, let us assume that we have already computed all terms having branch cuts, plus certain closely related terms that can generally be obtained from the same computation. That is, we have computed all polylog terms, all log terms, and all π^2 terms. There are also certain classes of rational terms that are natural to include with the cut-containing terms.

In particular, there are rational terms whose presence is required to cancel spurious singularities in the logarithmic terms. Spurious singularities arise in the course of integral reductions. They cannot be singularities of the final ampli-

tude, because they are unphysical, and not singularities of any Feynman diagram. A simple example comes from a "two-mass" triangle integral for which two of the three external legs are off-shell (massive), with momentum invariants s_1 and s_2, say. When there are sufficiently many loop momenta inserted in the numerator of this integral, it gives rise to functions such as,

$$\frac{\ln(r)}{(1 - r)^2}, \tag{3.10}$$

where r is a ratio of momentum invariants (here $r = s_1/s_2$). The limit $r \to 1$ (that is, $s_1 \to s_2$) is a spurious singularity; it does not correspond to any physical factorization. Indeed, this function always shows up in the amplitude together with appropriate rational pieces,

$$\frac{\ln(r) + 1 - r}{(1 - r)^2}, \tag{3.11}$$

in a combination which is finite as $r \to 1$. From a practical point of view, it is most convenient to "complete" the unitarity-derived answer for the cuts by replacing functions like Eq. (3.10) with nonsingular combinations like Eq. (3.11). Such completions are of course not unique; one could add additional rational terms free of spurious singularities.

There are other kinds of spurious singularities connected with polylogarithms. For example, in the scalar contributions to the five-gluon $(- + - + +)$ amplitude, there are factors of $\langle 24\rangle$ and $\langle 25\rangle$ appearing in the denominators of certain coefficients. These might appear to give rise to nonadjacent collinear singularities in complex momenta; but by expanding the polylogarithms and logarithms in that limit, one can show that these singularities are in fact absent.

Let us accordingly define two decompositions of the amplitude. The first is into "pure-cut" and "rational" pieces. The rational parts are defined by setting all logarithms, polylogarithms, and π^2 terms to zero,

$$R_n(z) \equiv \frac{1}{c_\Gamma}A_n\bigg|_{\text{rat}} = \frac{1}{c_\Gamma}A_n\bigg|_{\ln,\mathrm{Li},\pi^2\to 0}. \tag{3.12}$$

(Note that the normalization constant c_Γ, defined in Eq. (2.17), plays no essential role in the following arguments, and is just carried along for completeness.) The "pure-cut" terms are the remaining terms, all of which must contain logarithms, polylogarithms, or π^2 terms,

$$C_n(z) \equiv \frac{1}{c_\Gamma}A_n\bigg|_{\text{pure-cut}} = \frac{1}{c_\Gamma}A_n\bigg|_{\ln,\mathrm{Li},\pi^2}. \tag{3.13}$$

In other words,

$$A_n(z) = c_\Gamma[C_n(z) + R_n(z)], \tag{3.14}$$

where we have explicitly taken the ubiquitous one-loop factor c_Γ outside of $C_n(z)$ and $R_n(z)$.

The second decomposition uses the "completed-cut" terms, obtained from $C_n(z)$ by replacing logarithms and polylogarithms by corresponding functions free of spurious singularities. We shall call this completion $\hat{C}_n$. The decomposition defines the remaining rational pieces $\hat{R}_n$,

$$A_n(z) = c_\Gamma[\hat{C}_n(z) + \hat{R}_n(z)]. \tag{3.15}$$

We also need to define the rational part of the completed-cut terms, $\widehat{CR}_n(z)$. We write,

$$\hat{C}_n(z) = C_n(z) + \widehat{CR}_n(z), \tag{3.16}$$

where

$$\widehat{CR}_n(z) \equiv \hat{C}_n(z)|_{\text{rat}}. \tag{3.17}$$

Combining Eqs. (3.14), (3.15), and (3.16), we see that the full rational part is the sum of the rational part of the completed-cut terms, and the remaining rational pieces,

$$R_n(z) = \widehat{CR}_n(z) + \hat{R}_n(z). \tag{3.18}$$

Now, because we know all the terms containing branch cuts, we could compute the branch-cut-hugging integral,

$$\int_{B_0}^\infty \frac{dz}{z} \text{Disc}_B \hat{C}_n(z). \tag{3.19}$$

However, there is no need to do the integral explicitly, because we already know the answer for the integral, plus the corresponding residues. It is just $\hat{C}_n(0)$, part of the final answer. That is, applying the same logic to $\hat{C}_n(z)$ as was applied to $A_n(z)$ in Eq. (3.9), we have,

$$\hat{C}_n(0) = -\sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{C}_n(z)}{z} - \int_{B_0}^\infty \frac{dz}{z} \text{Disc}_B \hat{C}_n(z). \tag{3.20}$$

In arriving at this result, we need to assume that $\hat{C}_n(z)$, in addition to the full amplitude, also vanishes as $z \to \infty$. (This constraint may place some restrictions on the allowable rational completions of the cut terms.)

Using Eq. (3.9), the split-up (3.15) and (3.20) to evaluate the terms involving $\hat{C}_n(z)$, we can write our desired answer as follows,

$$A_n(0) = -c_\Gamma\Bigg[\int_{B_0}^\infty \frac{dz}{z} \text{Disc}_B \hat{C}_n(z) + \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{C}_n(z)}{z}$$

$$+ \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{R}_n(z)}{z} \Bigg]$$

$$= c_\Gamma\Bigg[\hat{C}_n(0) - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{R}_n(z)}{z} \Bigg]. \tag{3.21}$$

Because we have completed the cut terms so that $\hat{C}_n(z)$ contains no spurious singularities, the sums over the poles in Eq. (3.21) are only over the genuine, "physical" poles in the amplitude. (As explained elsewhere, these are the poles

that arise for *complex* momenta, and not merely those that arise for *real* momenta.)

C. Separate factorization of pure-cut and rational terms

The residues of the completed-cut terms $\hat{C}_n(z)$ at the genuine poles contain both rational and cut-containing functions. The residues of cut-containing functions necessarily have cut-containing functions (and the residues of π^2-containing terms will necessarily have factors of π^2, etc.), so that they will arise from cut-containing parts of the factorized amplitudes. The intuition from collinear factorization of one-loop amplitudes suggests, however, that the pure-cut terms arise from pure-cut terms, and the rational terms arise from rational terms. The purpose of this subsection is to flesh out this intuition of separate factorization for pure-cut and rational terms.

More concretely, the arguments of the logarithms or polylogarithms are ratios of invariants $s_{i\cdots j}$. (In a limited number of logarithms, the arguments are ratios of invariants to the renormalization scale squared.) When we shift these ratios by shifting momenta according to Eq. (3.1), one of three things can happen to any specific argument:

 (1) The ratio may be invariant under the shift, so that the residue simply has the original cut-containing function in it;
 (2) the ratio may acquire a dependence on z, but neither vanish nor diverge at any of the poles in z of the given term;
 (3) or the ratio may acquire a dependence on z, and either vanish or diverge at one of the poles in the given term.

Because the branch cuts in a massless theory start at either a vanishing ratio or at a vanishing of its inverse, the last situation corresponds to a pole touching the end of a branch cut.

In the first of these three cases, the cut-containing function clearly arises from a cut-containing function in the residue of the pole; and any associated rational terms arise from rational terms in the residue. In the second case, behavior under factorization in *real* momenta suggests that we need consider only single poles. In particular, for multiparticle invariants, even for complex momenta we have only single poles to consider. For two-particle channels, double poles can arise (for complex momenta) in the kinematic invariant of the momenta of two nearest-neighboring legs in the color ordering, but only at one loop and only for certain helicity configurations.

In two-particle factorizations, one-loop splitting amplitudes appear in addition to tree-level ones. The remaining amplitude left behind in the former case, however, is a tree-level amplitude, which is of course purely rational. In the latter case, we need consider only single poles.

We are left to consider the separation inside the cut-containing one-loop splitting amplitudes themselves.

Because we are only interested in the scalar-loop contributions, the relevant splitting amplitudes we need are scalar-loop ones [3,4,15,28,55]. For the helicity configurations for which the tree-level splitting amplitude $\text{Split}_{-\lambda}^{\text{tree}}(x; a^{\lambda_a}, b^{\lambda_b})$ is nonvanishing (where λ is the helicity of the outgoing off-shell leg, and x is the longitudinal-momentum fraction carried by leg a), we write the loop splitting amplitude as,

$$\text{Split}_{-}^{1-\text{loop},[0]}(x; a^{\lambda_a}, b^{\lambda_b}) = r_S^{1-\text{loop}\lambda_a\lambda_b,[0]}(x, s_{ab})$$
$$\times \text{Split}_{-}^{\text{tree}}(x; a^{\lambda_a}, b^{\lambda_b}), \quad (3.22)$$

where

$$r_S^{1-\text{loop}-+,[0]}(x, s) = 0, \quad (3.23)$$

$$r_S^{1-\text{loop}++,[0]}(x, s) = c_\Gamma\left(\frac{\mu^2}{-s}\right)^\epsilon \frac{2x(1-x)}{(1-2\epsilon)(2-2\epsilon)(3-2\epsilon)}. \quad (3.24)$$

The vertex corresponding to the first splitting amplitude will indeed vanish, and so it will not affect the separation between cut-containing and rational pieces. The situation with the second helicity configuration remains to be studied. Indeed, it appears in this case that for complex momenta, there are genuinely nonfactorizing contributions in addition to pole contributions. The shift choices we will make in this paper avoid the appearance of this vertex, and so we postpone the analysis of this case to future work.

Of course, outside of cut-containing terms, double poles do arise in general, as indicated by the finite one-loop splitting amplitude (the helicity configuration that vanishes at tree level),

$$\text{Split}_{+}^{1-\text{loop},[0]}(x; a^+, b^+) = -\frac{1}{48\pi^2}\sqrt{x(1-x)}\frac{[ab]}{\langle ab\rangle^2}. \quad (3.25)$$

They would be handled as in Refs. [11,12]; given the shift choices we will make in the present paper, they will in any event not arise.

So long as we are indeed considering only single poles, extracting the residue will not force us to expand $\hat{C}_n(z)$ in a series around the pole. In this case, the two types of contributions, pure-cut and rational, will remain separate in this class of contributions. As explained above, the third and last case, that of the pole hitting the branch cut, can effectively be reduced to the second case by artificially separating the pole from the end of the branch cut, and removing the separation at the end of the calculation.

As an example of the different kinds of behavior, consider the following expression,

$$\frac{\langle 23\rangle[34]^2\langle 41\rangle^2\langle 12\rangle[15]}{\langle 34\rangle\langle 45\rangle} \frac{L_1\left(\frac{-s_{23}}{s_{51}}\right)}{s_{51}^2} \quad (3.26)$$

where $L_1(r) = (\ln(r) + 1 - r)/(1 - r)^2$, under a $(1, 5)$ and $(3, 4)$ shift, respectively. Under the first shift, we obtain a simple pole at $z = -\langle 45\rangle/\langle 41\rangle$, but the argument of the L_1 function is unchanged, and accordingly, the logarithm and rational parts are left unaltered. In the context of a factorization, we would interpret the logarithm as arising from a logarithm in the factorized amplitude, and the rational part inside the L_1 from a rational term.

Under the second shift, we obtain a simple pole at $z = -\langle 45\rangle/\langle 35\rangle$. The argument of L_1 at the pole is now,

$$\frac{-\langle 23\rangle([32] - z_{\text{pole}}[42])}{-s_{51}} = \frac{\langle 23\rangle(\langle 53\rangle[32] + \langle 54\rangle[42])}{-\langle 35\rangle s_{51}}$$
$$= \frac{\langle 23\rangle\langle 15\rangle[12]}{-\langle 35\rangle s_{51}}, \quad (3.27)$$

but because the pole is a simple one, the separation between logarithm and rational part is again left undisturbed. If instead the original expression had contained not a $\langle 45\rangle$ in the denominator but rather a $\langle 45\rangle^2$, we would have had to expand the logarithm inside L_1 in order to extract the residue, and this expansion would have led to rational terms whose origin was in a "pure-cut" term, rather than a rational one, mixing up the two types of contributions. Fortunately, such combinations never arise in the analysis of the amplitudes discussed in this paper.

D. Residues of the remaining rational pieces $\hat{R}_n(z)$

Thanks to the analysis of the previous subsection, we can simplify our calculation by separating the two classes of terms—pure-cut and rational—in the factorized amplitudes. Because we already know the cut-containing pieces, we need to analyze only the rational terms. (This separation also avoids the need to treat poles hitting the ends of branch cuts explicitly.)

We now examine more carefully the residues of the poles of the purely rational terms, $R_n(z)$, and related to that, of the shifted remaining rational terms, $\hat{R}_n(z)$, appearing in Eq. (3.21). Given the (j, l) shift (3.1), we define a partition P to be a set of two or more cyclically-consecutive momentum labels containing j, such that the complementary set $\overline{P}$ consists of two or more cyclically-consecutive labels containing l:

$$P \equiv \{P_1, P_2, \ldots, j, \ldots, P_{-1}\},$$
$$\overline{P} \equiv \{\overline{P}_1, \overline{P}_2, \ldots, l, \ldots, \overline{P}_{-1}\}, \quad (3.28)$$
$$P \cup \overline{P} = \{1, 2, \ldots, n\}.$$

This definition ensures that the sum of momenta in each partition is z-dependent, so that it can go on shell for a suitable value of z. At tree level, the sum over residues becomes,

$$-\sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{A_n^{\text{tree}}(z)}{z} = A_n^{\text{tree}}(k_1, \ldots, k_n)$$

$$= \sum_{\text{partitions } P} \sum_{h=\pm} A_L^{\text{tree}}(k_{P_1}, \ldots, \hat{k}_j, \ldots, k_{P_{-1}}, -\hat{P}^h) \frac{i}{P^2 - m_P^2} A_R^{\text{tree}}(k_{\overline{P}_1}, \ldots, \hat{k}_l, \ldots, k_{\overline{P}_{-1}}, \hat{P}^{-h}). \quad (3.29)$$

The complex on-shell momenta $\hat{k}_j$, $\hat{k}_l$ and $\hat{P}$ are determined by solving the on-shell condition, $\hat{P}^2 = m_P^2$, for z. Although the examples we discuss in this paper are for massless particles, we allow for a mass m_P^2 to indicate that there is no special restriction to massless amplitudes, as has already been noted at tree level [27].

At one loop, the sum analogous to Eq. (3.29) will have an additional two-fold sum. In each term in this sum, either A_L or A_R will be a tree amplitude, and the other one will be a loop amplitude; in general both terms will appear. Taking the rational parts of the one-loop amplitudes appearing in this expression, the one-loop physical-pole recursion for the rational terms is,

$$-\sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{R_n(z)}{z} \equiv R_n^D(k_1, \ldots, k_n)$$

$$= \sum_{\text{partitions } P} \sum_{h=\pm} \left\{ R(k_{P_1}, \ldots, \hat{k}_j, \ldots, k_{P_{-1}}, -\hat{P}^h) \frac{i}{P^2} A^{\text{tree}}(k_{\overline{P}_1}, \ldots, \hat{k}_l, \ldots, k_{\overline{P}_{-1}}, \hat{P}^{-h}) \right.$$

$$\left. + A^{\text{tree}}(k_{P_1}, \ldots, \hat{k}_j, \ldots, k_{P_{-1}}, -\hat{P}^h) \frac{i}{P^2} R(k_{\overline{P}_1}, \ldots, \hat{k}_l, \ldots, k_{\overline{P}_{-1}}, \hat{P}^{-h}) \right\}, \quad (3.30)$$

where we now assume that the intermediate states are massless. This result follows directly from the general factorization behavior of one-loop amplitudes, plus the separate factorization of pure-cut and rational terms that was established in the previous subsection. Just as in the case of the tree-level recursion (3.29), it exhibits the required factorization properties in each channel P (dropping the terms with logarithms, polylogarithms, and π^2). Although the R functions are not complete amplitudes, they can be thought of as vertices from a diagrammatic perspective, and this equation lends itself to the same kind of diagrammatic interpretation available for Eq. (3.29).

However, the factorization cannot distinguish between the rational terms we have already included in the completed-cut terms and the remaining ones. That is, there would be an overlap or double count if we were simply to combine the recursive diagrams with the completed-cut terms. To remove this overlap, we separate the physical-pole contributions into those already included in the completed-cut terms and those in the remaining rational terms. Using Eq. (3.18), we know that

$$-\sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{R_n(z)}{z} = R_n^D$$

$$= -\sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\widehat{CR}_n(z)}{z}$$

$$- \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\hat{R}_n(z)}{z}. \quad (3.31)$$

Because we know the completed-cut terms $\hat{C}_n(z)$ and their

rational parts $\widehat{CR}_n(z)$ explicitly, we can compute the first term on the right-hand side, and solve for the remaining terms using the determination of R_n^D via Eq. (3.30). Inserting the result into Eq. (3.21) then gives us the basic on-shell recursion relation for complete one-loop amplitudes,

$$A_n(0) = c_\Gamma \left[\hat{C}_n(0) + R_n^D + \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\widehat{CR}_n(z)}{z} \right]. \quad (3.32)$$

To compute with this equation, we construct R_n^D via "direct recursion" diagrams; that is, via Eq. (3.30). We call the elements of the last term "overlap" terms. Because each pole is associated with a specific diagram, these can also be given a diagrammatic interpretation. Although the definition of the completed-cut terms $\hat{C}_n$ is not unique, the ambiguity cancels between $\hat{C}_n(0)$ and the sum over $\widehat{CR}_n$ residues. In the calculations in this paper, an astute choice of completion terms can simplify the calculation, by simplifying the extraction of the residues in the last term.

The reader may wonder how the calculation would have proceeded if we had started with "pure" cut terms, not including any of the rational pieces needed to eliminate the spurious singularities. In this case, the intermediate stages, and, in particular, Eq. (3.21), would have included a sum over the spurious singularities as well. Since these singularities include double and triple poles, we *would* have needed to expand the logarithms in extracting the residues for the overlap terms, and this expansion would have produced rational terms. In contrast, with our approach, we never evaluate residues at values of z corresponding to

unphysical spurious singularities, and we never have to expand logarithmic functions.

In Sec. II, we separated the one-loop amplitudes into divergent and finite parts. The amplitude as a whole satisfies the bootstrap relation Eq. (3.32); but it turns out that for the amplitudes we consider in the present paper, it can be applied separately to the V and F terms. As the recursion relation for the pure-scalar parts of the former are basically the same as at tree level, we will focus on the computation of the F terms. For this purpose, we shall use quantities analogous to C_n, R_n, $\hat{C}_n$, $\hat{R}_n$, and $\widehat{CR}_n$, defined as in Eqs. (3.13), (3.14), (3.15), (3.16), and (3.17), but with respect to F_n^s of Eq. (2.14) instead of A_n. Note that this shift of convention generates a relative factor of i in the quantities we use below, due to the relative i in Eq. (2.14).

IV. REVIEW OF KNOWN RESULTS

In this section we summarize the previously-computed amplitudes, and pieces thereof, that feed into our recursive construction. In this paper, we consider n-gluon amplitudes with two color-adjacent negative helicities, $A_{n;1}(1^-, 2^-, 3^+, 4^+, \ldots, (n-1)^+, n^+)$. As mentioned in Sec. II, the $\mathcal{N} = 4$ and $\mathcal{N} = 1$ components of these amplitudes have been known for a while, so the only issue is the computation of the $\mathcal{N} = 0$ or scalar contribution,

$$A_{n;1}^{[0]}(1^-, 2^-, 3^+, 4^+, \ldots, (n-1)^+, n^+). \quad (4.1)$$

Assigning intermediate helicities to all possible factorizations of this amplitude, as encountered in Eq. (3.30), allows us to determine which lower-point amplitudes are required as input. Besides the Parke-Taylor tree amplitudes with two (adjacent) negative helicities [52,53,56], we shall need the one-loop scalar contributions with one negative helicity [57], for which we recently found a compact form [12]. The one-loop amplitudes with two adjacent negative helicities and smaller values of n are also needed, and are obtained recursively, given a suitable starting point.

By choosing to shift the two negative-helicity legs we can avoid some factorizations. For example, for a generic choice of j and l in the (j, l) shift (3.1), the amplitude could factorize onto products containing a one-loop amplitude with all positive external helicities and an internal helicity of either sign. We avoid these factorizations by choosing to shift the two negative-helicity legs, $(j, l) = (1, 2)$. One external negative helicity then appears in each partition. In addition to lower-point amplitudes, we also need the logarithmic parts of the n-point amplitude (4.1), obtained in Ref. [4]. (The logarithmic parts of the more general set of amplitudes for two nonadjacent negative helicities were recently obtained in Ref. [8].)

In Sec. V we recompute the five-gluon helicity amplitude $A_{5;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+)$. For that computation, we take the four-point amplitude $A_{4;1}^{[0]}(1^-, 2^-, 3^+, 4^+)$ as an input, but here we quote the previous five-gluon result

too, in order to demonstrate that the method works. (We also outline the recursive construction of $A_{5;1}^{[0]}(1^-, 2^+, 3^-, 4^+, 5^+)$, which uses $A_{4;1}^{[0]}(1^-, 2^+, 3^-, 4^+)$ as an input, and agrees with the known result.) In Sec. VI we feed the five-point result back into the recursion to construct the six-point result, and following that, outline the construction of the seven-point result.

Finally, we also need the three-point amplitudes, which vanish for real momenta, but are nonvanishing for generic complex momenta. The one-loop three-vertex that we need may be deduced from a one-loop splitting amplitude [3].

Let us start with the tree amplitudes. The tree amplitudes that enter into our calculation are just the Parke-Taylor amplitudes [52,53,56],

$$A_n^{\text{tree}}(1^\pm, 2^+, 3^+, \ldots, n^+) = 0, \quad (4.2)$$

$$A_n^{\text{tree}}(1^-, 2^-, 3^+, \ldots, n^+) = i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \cdots \langle n1 \rangle}. \quad (4.3)$$

For some complex momenta, the three-point amplitudes

$$A_3^{\text{tree}}(1^-, 2^-, 3^+) = i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \quad (4.4)$$

$$A_3^{\text{tree}}(1^+, 2^+, 3^-) = -i \frac{[12]^4}{[12][23][31]}, \quad (4.5)$$

are nonvanishing. (They are vanishing for real momenta.)

The finite one-loop amplitudes that feed into our recursion are also relatively simple. The four-point finite amplitude with a single negative-helicity leg is [58–60],

$$A_{4;1}^{[0]}(1^-, 2^+, 3^+, 4^+) = \frac{i c_\Gamma}{3} \frac{\langle 24 \rangle [24]^3}{[12] \langle 23 \rangle \langle 34 \rangle [41]}. \quad (4.6)$$

Since the amplitude is entirely composed of rational functions, the vertex is proportional to the amplitude

$$R_4(1^-, 2^+, 3^+, 4^+) = \frac{1}{i c_\Gamma} A_{4;1}^{[0]}(1^-, 2^+, 3^+, 4^+). \quad (4.7)$$

We have removed an extra i from this vertex (and all others in the section) compared to the vertices R_n appearing in Sec. III. As mentioned at the end of that section, we wish to perform the recursion directly on the finite parts F_n^s defined in Eq. (2.14). For this reason, a factor of i from the vertex is removed from the vertex, compared with the one that would be used for constructing the amplitudes A_n.

The five-point finite amplitudes are also rather simple. We will need the finite amplitude

$$A_{5;1}^{[0]}(1^-, 2^+, 3^+, 4^+, 5^+) = i\frac{c_\Gamma}{3}\frac{1}{\langle 34\rangle^2}\left[-\frac{[25]^3}{[12][51]}\right.$$

$$+\frac{\langle 14\rangle^3[45]\langle 35\rangle}{\langle 12\rangle\langle 23\rangle\langle 45\rangle^2}$$

$$\left.-\frac{\langle 13\rangle^3[32]\langle 42\rangle}{\langle 15\rangle\langle 54\rangle\langle 32\rangle^2}\right].$$

This amplitude, along with all the other one-loop five-gluon helicity amplitudes, was first calculated using string-based methods [47]. Again because the amplitude is purely rational, the vertex is proportional to the amplitude,

$$R_5(1^-, 2^+, 3^+, 4^+, 5^+) = \frac{1}{ic_\Gamma}A_{5;1}^{[0]}(1^-, 2^+, 3^+, 4^+, 5^+).$$

$$(4.8)$$

It is worth noting that compact expressions now exist for the n-point generalization of this amplitude [11,12], $A_{n;1}^{[0]}(1^-, 2^+, 3^+, \ldots, n^+)$, which agree numerically with Mahlon's [57] original determination.

Using the decompositions (2.14)–(2.21), we express the divergent four-point amplitudes $A_{4;1}^{[J]}(1^-, 2^-, 3^+, 4^+)$ in terms of the functions [58–60],

$$V_4^g = -\frac{2}{\epsilon^2}\left[\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{23}}\right)^\epsilon\right] + \ln^2\left(\frac{-s_{12}}{-s_{23}}\right)$$

$$+ \pi^2 - \frac{\delta_R}{3}, \qquad (4.9)$$

$$V_4^f = -\frac{1}{\epsilon}\left(\frac{\mu^2}{-s_{23}}\right)^\epsilon - 2, \qquad (4.10)$$

$$V_4^s = -\frac{V_4^f}{3} + \frac{2}{9}. \qquad (4.11)$$

In this case the finite parts are trivial,

$$F_4^f = 0, \qquad F_4^s = 0. \qquad (4.12)$$

The version of dimensional regularization under consideration is determined by the δ_R parameter; for the 't Hooft-Veltman scheme [26] we take $\delta_R = 1$ while for the four-dimensional helicity scheme [58,61] we take $\delta_R = 0$.

Equation (4.12) shows that all the rational terms for the $(- - ++)$ case are constant multiples of the tree amplitude, so they can easily be absorbed into the V^x terms. Because the tree amplitude obeys its own on-shell recursion relation [9], it does not really matter whether we put constant terms like the $2/9$ term in V_4^s into the V or F category, but we will be able to drop one recursive diagram by assigning it to V. In general, then, we define V_n^s by,

$$V_n^s = -\frac{V_n^f}{3} + \frac{2}{9}. \qquad (4.13)$$

Because F_4^s vanishes, we take the loop vertex to also

vanish,

$$R_4(1^-, 2^-, 3^+, 4^+) = 0. \qquad (4.14)$$

For the divergent amplitudes with five or more legs, it is useful to introduce a set of auxiliary functions [47],

$$L_0(r) = \frac{\ln(r)}{1-r}, \qquad L_1(r) = \frac{\ln(r) + 1 - r}{(1-r)^2},$$

$$L_2(r) = \frac{\ln(r) - (r - 1/r)/2}{(1-r)^3}, \qquad (4.15)$$

in which the pole at $r = 1$ is removable. As discussed in Sec. III, we can therefore use the functions to construct the completed-cut terms, out of logarithms deduced from four-dimensional cuts.

We shall be quoting the functional form of the cut-containing pieces in the Euclidean region; a discussion of analytic continuations to the physical region may be found in, for example, Ref. [2].

We will also need the five-gluon amplitude $A_{5;1}^{[J]}(1^-, 2^-, 3^+, 4^+, 5^+)$. Reference [47] gives us,

$$V_5^g = -\frac{1}{\epsilon^2}\sum_{j=1}^{5}\left(\frac{\mu^2}{-s_{j,j+1}}\right)^\epsilon$$

$$+ \sum_{j=1}^{5}\ln\left(\frac{-s_{j,j+1}}{-s_{j+1,j+2}}\right)\ln\left(\frac{-s_{j+2,j-2}}{-s_{j-2,j-1}}\right) + \frac{5}{6}\pi^2 - \frac{\delta_R}{3}, \qquad (4.16)$$

$$V_5^f = -\frac{1}{2\epsilon}\left[\left(\frac{\mu^2}{-s_{23}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{51}}\right)^\epsilon\right] - 2, \qquad (4.17)$$

$$V_5^s = -\frac{1}{3}V_5^f + \frac{2}{9}, \qquad (4.18)$$

for the functions appearing in the decompositions (2.14)–(2.21). [The $1/\epsilon$ singularities differ from those in Ref. [47] because the amplitudes there were renormalized, whereas here we are using unrenormalized amplitudes; the difference is given simply by the renormalization subtraction (2.22).] The finite parts of this amplitude are,

$$F_5^f = -\frac{1}{2}\frac{\langle 12\rangle^2(\langle 23\rangle[34]\langle 41\rangle + \langle 24\rangle[45]\langle 51\rangle)}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\frac{L_0(\frac{-s_{23}}{-s_{51}})}{s_{51}}, \qquad (4.19)$$

$$F_5^s = -\frac{1}{3}F_5^f$$

$$-\frac{1}{3}\frac{[34]\langle 41\rangle\langle 24\rangle[45](\langle 23\rangle[34]\langle 41\rangle + \langle 24\rangle[45]\langle 51\rangle)}{\langle 34\rangle\langle 45\rangle}$$

$$\times\frac{L_2(\frac{-s_{23}}{-s_{51}})}{s_{51}^3} + \hat{R}_5, \qquad (4.20)$$

where

$$\hat{R}_5 = -\frac{1}{3}\frac{\langle 35\rangle[35]^3}{[12][23]\langle 34\rangle\langle 45\rangle[51]} + \frac{1}{3}\frac{\langle 12\rangle[35]^2}{[23]\langle 34\rangle\langle 45\rangle[51]}$$
$$+ \frac{1}{6}\frac{\langle 12\rangle[34]\langle 41\rangle\langle 24\rangle[45]}{s_{23}\langle 34\rangle\langle 45\rangle s_{51}}. \tag{4.21}$$

In Sec. V, we shall recompute the explicit rational terms $\hat{R}_5$. The other pieces are all either trivial or obtainable from four-dimensional unitarity.

Following the discussion of the previous section, we define a recursion vertex (3.12), composed of all the rational terms in F_5^s,

$$R_5(1^-, 2^-, 3^+, 4^+, 5^+) = F_5^s|_{\text{rat}}, \tag{4.22}$$

including those contained in L_2. As in the four-point case the effect of the $2/9$ term in V^s is trivial, so we do not need to make it part of the recursion vertex.

Now consider the known six-point results for $A_{6;1}^{[J]}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$. Except for rational terms associated with the $J = 0$ scalar loop, these amplitudes were determined from the unitarity method in Refs. [3,4], where the calculations were performed for the more general n-point amplitudes with two adjacent negative helicities. These results have recently been confirmed in Refs. [8,22].

From the results of Ref. [3], after setting $n = 6$, we have the $\mathcal{N} = 4$ contribution,

$$V_6^g = \sum_{i=1}^6 \left[-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^\epsilon \right.$$
$$\left. - \ln\left(\frac{-s_{i,i+1}}{-s_{i,i+1,i+2}}\right)\ln\left(\frac{-s_{i+1,i+2}}{-s_{i,i+1,i+2}}\right) \right]$$
$$+ D_6 + L_6 + \pi^2, \tag{4.23}$$

where all indices are to be taken mod 6 and

$$D_6 = -\sum_{i=1}^3 \text{Li}_2\left(1 - \frac{s_{i,i+1}s_{i+3,i+4}}{s_{i,i+1,i+2}s_{i-1,i,i+1}}\right),$$
$$L_6 = -\frac{1}{4}\sum_{i=1}^6 \ln\left(\frac{-s_{i,i+1,i+2}}{-s_{i+1,i+2,i+3}}\right)\ln\left(\frac{-s_{i+1,i+2,i+3}}{-s_{i,i+1,i+2}}\right). \tag{4.24}$$

From Ref. [4], the $\mathcal{N} = 1$ components are,

$$V_6^f = -\frac{1}{2\epsilon}\left(\left(\frac{\mu^2}{-s_{23}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{61}}\right)^\epsilon\right) - 2, \tag{4.25}$$

$$F_6^f = \frac{1}{2s_{12}}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\left[\frac{1}{s_{16}}\text{L}_0\left(\frac{s_{234}}{s_{16}}\right)(\text{tr}_+[1256]\right.$$
$$- \text{tr}_+[12(1 + 6)5]) + \frac{1}{s_{234}}\text{L}_0\left(\frac{s_{23}}{s_{234}}\right)(\text{tr}_+[1234]$$
$$\left. - \text{tr}_+[124(2 + 3)])\right], \tag{4.26}$$

while the scalar-loop contributions are,

$$V_6^s = -\frac{1}{3}V_6^f + \frac{2}{9}, \tag{4.27}$$

$$F_6^s = -\frac{1}{3s_{12}^3}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$
$$\times \left[\frac{1}{s_{16}^3}\text{L}_2\left(\frac{s_{234}}{s_{16}}\right)\left((\text{tr}_+[1256])^2\text{tr}_+[12(1 + 6)5]\right.\right.$$
$$\left. - \text{tr}_+[1256](\text{tr}_+[12(1 + 6)5])^2\right)$$
$$+ \frac{1}{s_{234}^3}\text{L}_2\left(\frac{s_{23}}{s_{234}}\right)\left(\text{tr}_+[124(2 + 3)](\text{tr}_+[1234])^2\right.$$
$$\left.\left. - (\text{tr}_+[124(2 + 3)])^2\text{tr}_+[1234]\right)\right] - \frac{1}{3}F_6^f + \hat{R}_6,$$
$$\tag{4.28}$$

where $\hat{R}_6$ are the rational terms not contained in L_2. A key task of this paper will be to obtain an explicit formula for the unknown rational function $\hat{R}_6$. We shall do so in Sec. VI.

Similarly, for the seven-point amplitude $A_{7;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+, 7^+)$, it is not difficult to extract the functions from the n-point forms given in Refs. [3,4]. Although we shall not discuss the seven-point case in any detail, these functions enter into the computation of the rational terms of this amplitude, as outlined in Sec. VI.

We still need a one-loop three-vertex for the recursion. We determine this vertex by inspecting the one-loop splitting amplitudes. Because the loop splitting amplitude with opposite on-shell helicities and a scalar circulating in the loop vanishes [see Eq. (3.23)],

$$\text{Split}_{-\lambda}^{[0]}(1^\pm, 2^\mp) = 0, \tag{4.29}$$

the corresponding three-vertex should also be taken to vanish,

$$R_3(\hat{1}^-, 2^+, -\hat{K}_{12}^\pm) = 0. \tag{4.30}$$

For the cases where the two external lines have the same helicity, the situation is much more subtle, with the appearance of ''unreal poles'' [11,12] and nonfactorizing contributions. However, because we choose to shift the two negative-helicity legs, we do not encounter such vertices.

Finally, we need the rational functions $\widehat{CR}_n$ contained in the completed-cut part defined in Eqs. (3.15) and (3.17), for $n = 5$, 6, and 7. These functions will be used to obtain the overlap contributions in Secs. V and VI. We easily obtain these from the L_2 terms by replacing the L_2 function with its rational part, using Eq. (4.15). For the five-point amplitude $A_{n;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+)$, the explicit value is,

$$\widehat{CR}_5 = -\frac{1}{6}\frac{s_{15}+s_{23}}{s_{23}s_{15}(s_{15}-s_{23})^2}\frac{[34]\langle41\rangle\langle24\rangle[45](\langle23\rangle[34]\langle41\rangle+\langle24\rangle[45]\langle51\rangle)}{\langle34\rangle\langle45\rangle}. \tag{4.31}$$

For the six-point amplitude $A_{6;1}^{[0]}(1^-,2^-,3^+,4^+,5^+,6^+)$, the rational function appearing in the completed-cut part is,

$$\widehat{CR}_6 = -\frac{1}{6}\frac{1}{\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle\langle61\rangle}\Bigg[\frac{1}{s_{16}s_{234}}\frac{s_{16}+s_{234}}{(s_{16}-s_{234})^2}\langle25\rangle[56]\langle61\rangle\langle51\rangle\langle2^-|(3+4)|5^-\rangle\Big(\langle25\rangle[56]\langle61\rangle+\langle2^-|(3+4)|5^-\rangle\langle51\rangle\Big)$$

$$+\frac{1}{s_{23}s_{234}}\frac{s_{234}+s_{23}}{(s_{234}-s_{23})^2}\langle24\rangle\langle4^+|(5+6)|1^+\rangle\langle23\rangle[34]\langle41\rangle\Big(\langle23\rangle[34]\langle41\rangle+\langle24\rangle\langle4^+|(5+6)|1^+\rangle\Big)\Bigg]. \tag{4.32}$$

Similarly, for the seven-point amplitude $A_{7;1}^{[0]}(1^-,2^-,3^+,4^+,5^+,6^+,7^+)$ which we briefly discuss in Sec. VI, it is not difficult to extract the explicit value of $\widehat{CR}_7$ from eq. (7.1) of Ref. [4], although we shall not quote the result here.

V. RECOMPUTATION OF FIVE-GLUON QCD AMPLITUDES

In this section we illustrate our method for determining loop amplitudes, by recomputing the known five-gluon QCD amplitudes, given the four-dimensional cut-constructible parts of the amplitudes. There are two independent helicity amplitudes, $A_{5;1}^{[0]}(1^-,2^-,3^+,4^+,5^+)$ and $A_{5;1}^{[0]}(1^-,2^+,3^-,4^+,5^+)$. We will discuss the first of these in some detail, and merely summarize the calculation of the latter. In both cases, we correctly reproduce the results of ref. [47].

Begin with $A_{5;1}^{[0]}(1^-,2^-,3^+,4^+,5^+)$. For this amplitude, use a (1, 2) shift,

$$\lambda_1\rightarrow\lambda_1,\qquad\tilde\lambda_1\rightarrow\tilde\lambda_1-z\tilde\lambda_2,$$
$$\lambda_2\rightarrow\lambda_2+z\lambda_1,\qquad\tilde\lambda_2\rightarrow\tilde\lambda_2. \tag{5.1}$$

It is not difficult to verify that this shift has the required property that the rational part of the cut term $\widehat{CR}_5$, given in Eq. (4.31), vanishes at large z, as required.

This shift yields a version of the rational-recursion (3.30), where each term is represented by one of the recursive diagrams depicted in Fig. 3. We have dropped diagrams with a trivially vanishing tree amplitude. Consider the first diagram in Fig. 3,

$$D_5^{(a)} = A_3^{\text{tree}}(\hat2^-,3^+,-\hat K_{23}^-)\times\frac{i}{s_{23}}\times R_4(\hat1^-,\hat K_{23}^+,4^+,5^+). \tag{5.2}$$

It vanishes,

$$D_5^{(a)} = 0, \tag{5.3}$$

because [9]

$$A_3^{\text{tree}}(\hat2^-,3^+,-\hat K_{23}^-)\propto\langle\hat2\hat K_{23}\rangle^3$$

$$\propto\Bigg(\langle2^-|(2+3)|2^-\rangle$$

$$-\frac{s_{23}}{\langle1^-|(2+3)|2^-\rangle}\langle1^-|(2+3)|2^-\rangle\Bigg)^3$$

$$= 0. \tag{5.4}$$

Diagram 3(b) also vanishes,

$$D_5^{(b)} = 0, \tag{5.5}$$

because the loop three-vertex (4.30) vanishes. Similarly, it is not difficult to show that diagrams 3(d) and 3(e) vanish,

$$D_5^{(d)} = D_5^{(e)} = 0. \tag{5.6}$$

We are left with just two direct-recursion diagrams. Diagram 3(c) is given by

$$D_5^{(c)} = A_3^{\text{tree}}(\hat2^-,3^+,-\hat K_{23}^+)\times\frac{i}{s_{23}}\times R_4(\hat1^-,\hat K_{23}^-,4^+,5^+). \tag{5.7}$$

As we saw in Eq. (4.14), the loop vertex vanishes, and so

$$D_5^{(c)} = 0. \tag{5.8}$$

The last diagram is,

$$D_5^{(f)} = A_3^{\text{tree}}(5^+,\hat1^-,-\hat K_{51}^-)\frac{i}{s_{51}}R_4(\hat2^-,3^+,4^+,\hat K_{51}^+)$$

$$= -\frac{1}{3}\frac{\langle\hat1(-\hat K_{51})\rangle^3}{\langle5\hat1\rangle\langle(-\hat K_{51})5\rangle}\frac{1}{s_{51}}\frac{\langle3\hat K_{51}\rangle[3\hat K_{51}]^3}{[\hat23]\langle34\rangle\langle4\hat K_{15}\rangle[\hat K_{51}\hat2]}$$

$$= \frac{1}{3}\frac{\langle1^-|5|2^-\rangle^3}{\langle51\rangle\langle5^-|1|2^-\rangle}\frac{1}{\langle51\rangle[15]\langle1^-|5|2^-\rangle^2}$$

$$\times\frac{\langle3^-|4|2^-\rangle\langle1^-|5|3^-\rangle^3}{[23]\langle34\rangle\langle4^-|3|2^-\rangle\langle1^-|5|2^-\rangle}$$

$$= -\frac{1}{3}\frac{[24][35]^3}{\langle34\rangle[12][15][23]^2}. \tag{5.9}$$

This direct-recursion diagram is the only one that does not vanish.

Next we must evaluate the overlap contributions from Eq. (3.32), depicted in Fig. 4. We start from the rational
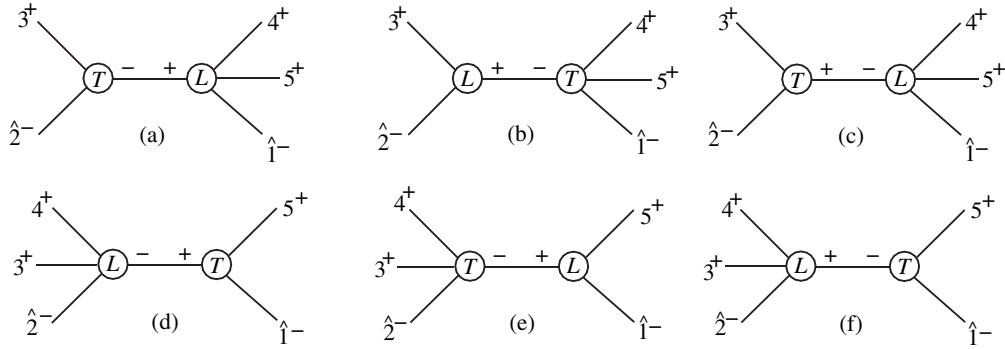
FIG. 3.   The recursive diagrams for computing the rational parts of $A_{5;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+)$ with the shift of legs 1,2 given in Eq. (5.1). T signifies a tree vertex and L a loop vertex.

parts of the cut contributions, $\widehat{CR}_5$ as given in Eq. (4.31). Applying the shift (5.1) to these contributions, we have,

$$\widehat{CR}_5(z) = -\frac{1}{6}\frac{[34]\langle 41\rangle(\langle 24\rangle + z\langle 14\rangle)[45]}{\langle 34\rangle\langle 45\rangle}\left((\langle 23\rangle + z\langle 13\rangle)[34]\langle 41\rangle + (\langle 24\rangle + z\langle 14\rangle)[45]\langle 51\rangle\right)$$

$$\times \frac{s_{51} + s_{23} - z\langle 1^-|5|2^-\rangle + z\langle 1^-|3|2^-\rangle}{(\langle 23\rangle + z\langle 13\rangle)[32]\langle 15\rangle([51] - z[52])(s_{15} - s_{23} - z\langle 1^-|(5+3)|2^-\rangle)^2}. \tag{5.10}$$

The residues of $\widehat{CR}_5(z)/z$ that we need to evaluate are located at the values of z,

$$z^{(a)} = -\frac{\langle 23\rangle}{\langle 13\rangle}, \qquad z^{(b)} = \frac{[15]}{[25]}, \tag{5.11}$$

corresponding to the two overlap diagrams in Fig. 4. Evaluating the residue corresponding to the first of these overlap diagrams is very simple and gives,

$$O_5^{(a)} = -\frac{1}{6}\frac{\langle 12\rangle^2\langle 14\rangle[34]}{\langle 15\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle[23]}. \tag{5.12}$$

Similarly, the overlap diagram (b) gives

$$O_5^{(b)} = \frac{1}{6}\frac{\langle 14\rangle[34][35](\langle 14\rangle[34] - \langle 15\rangle[35])}{\langle 15\rangle\langle 34\rangle\langle 45\rangle[15][23]^2}. \tag{5.13}$$

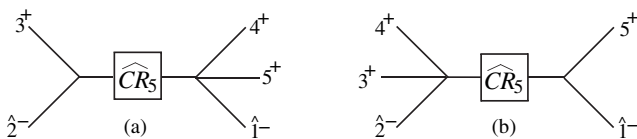Summing over the nonvanishing diagrammatic contributions, we get the simple result,

$$\hat{R}_5 = D_5^{(f)} + O_5^{(a)} + O_5^{(b)}$$

$$= \frac{1}{6}\left(-\frac{2[24][35]^3}{\langle 34\rangle[12][15][23]^2} - \frac{\langle 12\rangle^2\langle 14\rangle[34]}{\langle 15\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle[23]}\right.$$

$$\left.+\frac{\langle 14\rangle^2[34]^2[35]}{\langle 15\rangle\langle 34\rangle\langle 45\rangle[15][23]^2} - \frac{\langle 14\rangle[34][35]^2}{\langle 34\rangle\langle 45\rangle[15][23]^2}\right). \tag{5.14}$$

With a few spinor manipulations this result can be brought into manifest agreement with the known result (4.21). Thus, we have correctly reproduced the rational parts of $A_{5;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+)$, without performing any loop integrals.

We have also verified that our method properly reproduces $A_{5;1}^{[0]}(1^-, 2^+, 3^-, 4^+, 5^+)$, computed in Ref. [47]. As for the previous case, we choose the negative-helicity legs as the shifted ones, i.e. we use a (1, 3) shift,

$$\lambda_1 \to \lambda_1, \qquad \tilde{\lambda}_1 \to \tilde{\lambda}_1 - z\tilde{\lambda}_3,$$
$$\lambda_3 \to \lambda_3 + z\lambda_1, \qquad \tilde{\lambda}_3 \to \tilde{\lambda}_3. \tag{5.15}$$

For this computation, we completed the cut terms with the Ls_1 function of Ref. [47], along with the L_1 and L_2 functions also used there,

FIG. 4.   The five-point overlap diagrams using the (1,2) shift given in Eq. (5.1).

$$\hat{C}_5 = -\frac{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle^2[24]^2}{\langle 45\rangle\langle 51\rangle\langle 24\rangle^2} \frac{2\mathrm{Ls}_1(\frac{-s_{23}}{-s_{51}}, \frac{-s_{34}}{-s_{51}}) + \mathrm{L}_1(\frac{-s_{23}}{-s_{51}}) + \mathrm{L}_1(\frac{-s_{34}}{-s_{51}})}{s_{51}^2}$$

$$+ \frac{\langle 32\rangle\langle 21\rangle\langle 15\rangle\langle 53\rangle^2[25]^2}{\langle 54\rangle\langle 43\rangle\langle 25\rangle^2} \frac{2\mathrm{Ls}_1(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}}) + \mathrm{L}_1(\frac{-s_{12}}{-s_{34}}) + \mathrm{L}_1(\frac{-s_{51}}{-s_{34}})}{s_{34}^2} + \frac{2}{3}\frac{\langle 23\rangle^2\langle 41\rangle^3[24]^3}{\langle 45\rangle\langle 51\rangle\langle 24\rangle} \frac{\mathrm{L}_2(\frac{-s_{23}}{-s_{51}})}{s_{51}^3}$$

$$- \frac{2}{3}\frac{\langle 21\rangle^2\langle 53\rangle^3[25]^3}{\langle 54\rangle\langle 43\rangle\langle 25\rangle} \frac{\mathrm{L}_2(\frac{-s_{12}}{-s_{34}})}{s_{34}^3} + \frac{\mathrm{L}_2(\frac{-s_{34}}{-s_{51}})}{s_{51}^3}\left(\frac{1}{3}\frac{\langle 13\rangle[24][25](\langle 15\rangle[52]\langle 23\rangle - \langle 34\rangle[42]\langle 21\rangle)}{\langle 45\rangle} + \frac{2}{3}\frac{\langle 12\rangle^2\langle 34\rangle^2\langle 41\rangle[24]^3}{\langle 45\rangle\langle 51\rangle\langle 24\rangle}\right.$$

$$\left. - \frac{2}{3}\frac{\langle 32\rangle^2\langle 15\rangle^2\langle 53\rangle[25]^3}{\langle 54\rangle\langle 43\rangle\langle 25\rangle}\right) + \frac{1}{6}\frac{\langle 13\rangle^3(\langle 15\rangle[52]\langle 23\rangle - \langle 34\rangle[42]\langle 21\rangle)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \frac{\mathrm{L}_0(\frac{-s_{34}}{-s_{51}})}{s_{51}}. \tag{5.16}$$

This function satisfies $\hat{C}_5(z) \to 0$ as $z \to \infty$, thanks to cancellations between the polylogarithmic or logarithmic functions and the rational terms. The attentive reader will note that this function has spurious double poles involving nonadjacent legs in the color ordering, e.g. $1/\langle 24\rangle^2$. These poles do not invalidate the calculation with a $(1, 3)$ shift, because they acquire no z dependence, and hence produce no poles in z at spurious locations. The spurious singularities cancel in the complete answer for F^s, not only at order $1/\langle 24\rangle^2$, but also at order $1/\langle 24\rangle$, even for complex momenta. (The completed-cut term $\hat{C}_5$ given here does contain a $1/\langle 24\rangle$ pole for complex momenta, but it is canceled in the full answer by the additional diagrammatic terms in our construction.)

VI. SIX- AND SEVEN-POINT QCD AMPLITUDES

In this section we describe the computations of the unknown rational functions for the six- and seven-point helicity amplitudes $A_{6;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$ and $A_{7;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+, 7^+)$. Were one to attempt the calculation by traditional means, one would encounter large numbers of Feynman diagrams. The total number of one-loop diagrams for the six-gluon process in QCD is 10 860 (including gluon, ghost and fermion loops, but dropping those which vanish trivially in dimensional regularization). For seven and eight external gluons the numbers grow to 168 925 and 3 017 490, respectively. But the number of diagrams only hints at the full complexity of the calculation, because it does not take into account the explosion of terms resulting from tensor integral reductions, which is what renders a brute-force Feynman diagram computation impractical even on a modern computer. The approach we take in the present paper avoids this explosion by focusing on analytic properties that all amplitudes must satisfy. We computed the logarithmic terms in the amplitudes long ago [3,4], using the unitarity-based method. Here we complete the QCD calculation by computing the rational terms in the scalar contributions, namely, Eq. (4.1) for $n = 6, 7$. For the six-point case we present a compact analytical expression.

First consider the six-point amplitude $A_{6;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$. To obtain the rational terms of the six-point amplitude, $\hat{R}_6$ in Eq. (4.28), we first evaluate the recursive diagrams shown in Fig. 5, corresponding to the terms in the recursion (3.30). We again shift the two negative-helicity legs, using Eq. (5.1). Following a similar discussion as for the five-point amplitude, it is not difficult to show that diagrams 5(a), 5(b), 5(d), and 5(e) vanish,

$$D_6^{(a)} = D_6^{(b)} = D_6^{(d)} = D_6^{(e)} = 0. \tag{6.1}$$

Four diagrams remain to be evaluated,

$$D_6 = D_6^{(c)} + D_6^{(f)} + D_6^{(g)} + D_6^{(h)}$$

$$= A_3^{\mathrm{tree}}(\hat{2}^-, 3^+, -\hat{K}_{23}^+) \times \frac{i}{s_{23}} \times R_5(\hat{1}^-, \hat{K}_{23}^-, 4^+, 5^+, 6^+) + R_5(\hat{2}^-, 3^+, 4^+, 5^+, \hat{K}_{61}^+) \times \frac{i}{s_{61}} \times A_3^{\mathrm{tree}}(6^+, \hat{1}^-, -\hat{K}_{61}^-)$$

$$+ R_4(\hat{2}^-, 3^+, 4^+, -\hat{K}_{234}^+) \times \frac{i}{s_{234}} \times A_4^{\mathrm{tree}}(\hat{1}^-, \hat{K}_{234}^-, 5^+, 6^+)$$

$$+ A_4^{\mathrm{tree}}(-\hat{K}_{234}^-, \hat{2}^-, 3^+, 4^+) \times \frac{i}{s_{234}} \times R_4(\hat{1}^-, \hat{K}_{234}^+, 5^+, 6^+). \tag{6.2}$$

We do not present a detailed evaluation of these diagrams, because it is similar to the five-point evaluation discussed in the previous section.

Next consider the overlap contributions of Eq. (3.32) displayed in Fig. 6. These are determined by evaluating the residues of the rational part of the cut contributions given in Eq. (4.32) after performing the shift (5.1). The residues of $\widehat{CR}_6(z)/z$ that need to be computed are at the values of z located at,
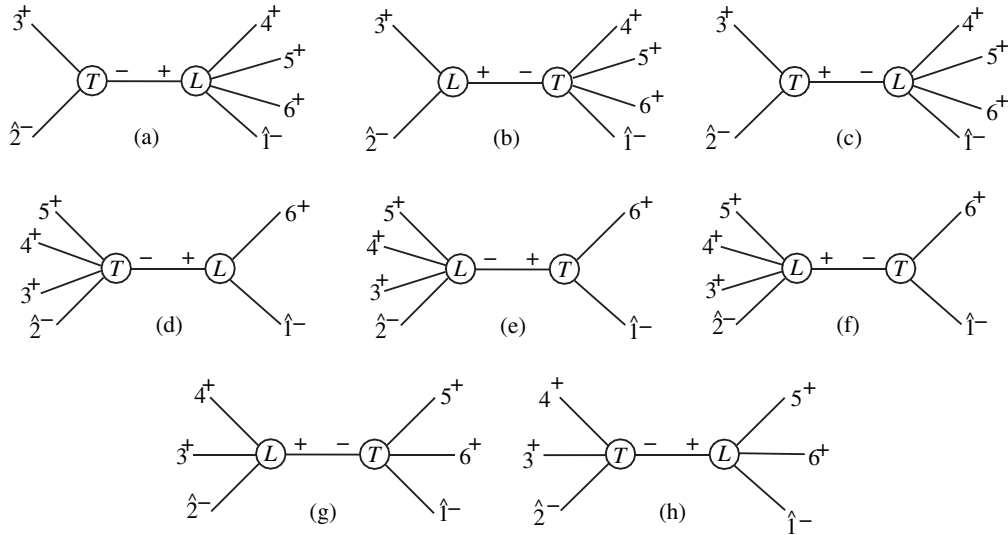
FIG. 5.   The recursive diagrams for $A_{6;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, using the (1, 2) shift. Diagrams (a), (b), (d) and (e) vanish.

$$z^{(a)} = \frac{[16]}{[26]}, \qquad z^{(b)} = -\frac{\langle 23\rangle}{\langle 13\rangle}, \qquad z^{(c)} = -\frac{s_{234}}{\langle 1^-|(3+4)|2^-\rangle}, \tag{6.3}$$

corresponding to the three overlap diagrams in Fig. 6. It is straightforward to evaluate the residues since we encounter only simple poles. For diagram (a) of Fig. 6, for example, we have,

$$O_6^{(a)} = -\frac{1}{6}\frac{\langle 15\rangle[56]s_{612}\langle 5^-|(1+2)|6^-\rangle(2\langle 15\rangle s_{612} - \langle 1^-|2(1+6)|5^+\rangle)}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle[61]\langle 3^-|(1+2)|6^-\rangle\langle 5^-|(3+4)|2^-\rangle^2}, \tag{6.4}$$

which has been simplified using spinor-product identities. Similarly, it is not difficult to evaluate the other two residues, corresponding to diagrams (b) and (c) in Fig. 6.

The sum of the nonvanishing recursive and overlap diagrams for $\hat{R}_6$ in Eq. (4.28) is given by

$$\hat{R}_6 = D_6^{(c)} + D_6^{(f)} + D_6^{(g)} + D_6^{(h)} + O_6^{(a)} + O_6^{(b)} + O_6^{(c)}. \tag{6.5}$$

One of the terms in the expression for the overlap diagram in the $\langle 23\rangle$ channel, $O^{(b)}$, contains an unusual factor in its denominator, namely, the square of the quantity

$$\langle 1^-|5(1+2)|3^+\rangle + \langle 13\rangle s_{56} = \langle 1^-|(2+3)4|3^+\rangle$$
$$- \langle 1^-|6(1+2)|3^+\rangle. \tag{6.6}$$

However, the recursive diagram in the $\langle 23\rangle$ channel, $D^{(c)}$, shown in Fig. 5(c), also contains a term with this behavior, and the two terms cancel against each other. After simplifying the sum over diagrams we obtain,
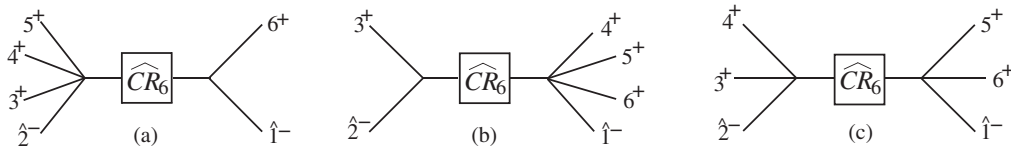

FIG. 6.   The overlap diagrams for $A_{6;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, using the (1,2) shift.

$$\hat{R}_6 = \frac{1}{6}\Bigg\{ -2\frac{\langle 35\rangle[35]\langle 4^-|(1+2)|3^-\rangle\langle 4^-|(1+2)|6^-\rangle\langle 5^-|(1+2)|6^-\rangle}{[12]\langle 34\rangle^2\langle 45\rangle^2[61]\langle 5^-|(3+4)|2^-\rangle\langle 6^-|(1+2)|3^-\rangle} - 2\frac{\langle 35\rangle[36]\langle 4^-|(1+2)|6^-\rangle^2}{[12]\langle 34\rangle^2\langle 45\rangle^2[61]\langle 5^-|(3+4)|2^-\rangle}$$

$$+ 2\frac{\langle 12\rangle\langle 24\rangle\langle 35\rangle[35]^2[56]\langle 5^-|(1+2)|6^-\rangle}{\langle 34\rangle^2\langle 45\rangle[61]\langle 2^-|(1+6)|5^-\rangle\langle 5^-|(3+4)|2^-\rangle\langle 6^-|(1+2)|3^-\rangle}$$

$$+ 2\frac{\langle 12\rangle^2[35]^2(\langle 5^-|(3+4)2|1^+\rangle + \langle 5^-|35|1^+\rangle)}{\langle 34\rangle\langle 45\rangle\langle 61\rangle\langle 2^-|(1+6)|5^-\rangle\langle 5^-|(3+4)|2^-\rangle\langle 6^-|(1+2)|3^-\rangle}$$

$$- \frac{\langle 12\rangle^3\langle 35\rangle[46][56]}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 1^-|(2+3)|4^-\rangle\langle 3^-|(1+2)|6^-\rangle} + 2\frac{[36]^3}{[12][23]\langle 45\rangle^2[61]}$$

$$- \frac{[56]\langle 5^-|(1+2)|6^-\rangle^2(2\langle 4^-|(3+5)(1+2)|5^+\rangle + \langle 12\rangle[12]\langle 45\rangle)}{[12]\langle 34\rangle\langle 45\rangle^2\langle 56\rangle[61]\langle 3^-|(1+2)|6^-\rangle\langle 5^-|(3+4)|2^-\rangle}$$

$$+ 2\frac{\langle 15\rangle^2[34]^2[56](\langle 16\rangle[34]\langle 45\rangle - \langle 1^-|(2+4)|3^-\rangle\langle 56\rangle)}{[23]\langle 45\rangle\langle 56\rangle^2 s_{234}\langle 1^-|(2+3)|4^-\rangle\langle 5^-|(3+4)|2^-\rangle} - \frac{\langle 12\rangle\langle 15\rangle[34][56]\langle 1^-|(5+6)(3+4)|5^+\rangle}{\langle 34\rangle\langle 45\rangle\langle 56\rangle s_{234}\langle 1^-|(2+3)|4^-\rangle\langle 5^-|(3+4)|2^-\rangle}$$

$$+ 2\frac{\langle 35\rangle\langle 1^-|(2+4)|3^-\rangle^3}{[23]\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle s_{234}\langle 5^-|(3+4)|2^-\rangle} - \frac{\langle 12\rangle\langle 1^-|(2+4)|3^-\rangle(2\langle 1^-|(2+4)|3^-\rangle + \langle 1^-|4|3^-\rangle)}{[23]\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle s_{234}}$$

$$+ 2\frac{\langle 12\rangle^3[46]^2\langle 5^-|(4+6)|5^-\rangle}{\langle 23\rangle\langle 45\rangle\langle 56\rangle s_{123}\langle 1^-|(2+3)|4^-\rangle\langle 3^-|(1+2)|6^-\rangle} + 2\frac{\langle 12\rangle^3[35]^2\langle 4^-|(3+5)|4^-\rangle}{\langle 34\rangle\langle 45\rangle\langle 61\rangle s_{612}\langle 2^-|(1+6)|5^-\rangle\langle 6^-|(1+2)|3^-\rangle}$$

$$- \frac{\langle 12\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}\left[\frac{\langle 1^-|4|3^-\rangle}{[23]} + \frac{\langle 2^-|5|6^-\rangle}{[61]}\right]\Bigg\}, \tag{6.7}$$

determining the previously unknown rational function in Eq. (4.28).

As in the past, for both theoretical and practical reasons, it probably will be important to have the simplest representations of amplitudes. It is likely that the result in Eq. (6.7) can be simplified even further. For example, individual terms contain spurious singularities due to the following denominator factors: $\langle 1^-|(2+3)|4^-\rangle$, $\langle 2^-|(1+6)|5^-\rangle$, $\langle 3^-|(1+2)|6^-\rangle$, $\langle 5^-|(3+4)|2^-\rangle$, and $\langle 6^-|(1+2)|3^-\rangle$. We have checked numerically that these singularities cancel between different terms, so that the full expression is nonsingular. Nevertheless, one might expect a simpler form with fewer such cancellations. Some such cancellations have already been carried out to arrive at Eq. (6.7); besides the cancellation involving the unusual factor (6.6), individual recursive and overlap diagrams also contained factors of $1/\langle 5^-|(3+4)|2^-\rangle^2$, as in Eq. (6.4). These squared factors were eliminated using spinor-product identities to combine and rearrange terms.

Removing *all* of the factors of the form $\langle a^-|(b+c)|d^-\rangle$ is unlikely to be desirable; after all, the simple forms of tree amplitudes found in Refs. [9,19,20,41,42] are simpler than previous forms precisely *because* of the presence of some such denominators. However, the denominators present in Eq. (6.7) occur in a rather asymmetric fashion, preventing the flip symmetry of the expression—symmetry under

$$1 \leftrightarrow 2, \quad 3 \leftrightarrow 6, \quad 4 \leftrightarrow 5 \tag{6.8}$$

—from being manifest. Indeed, the factor $\langle 5^-|(3+4)|2^-\rangle$ occurs in the denominator many times, yet its image under the flip (6.8), $\langle 4^-|(5+6)|1^-\rangle$ never appears.

We have carried out several numerical checks of Eq. (4.28), after inserting into it the value of $\hat{R}_6$ from Eq. (6.7). Besides verifying the absence of any spurious singularities, we checked that the amplitude satisfies the nonmanifest flip symmetry (6.8). We checked the multiparticle factorization in the s_{123} and s_{234} channels. (The s_{612} channel is related trivially to the s_{123}channel by the flip symmetry.) We also confirmed the proper collinear behavior, for real momenta, in all the independent channels s_{12}, s_{23}, s_{34} and s_{45}. These checks leave little doubt that Eq. (6.7) is the correct expression.

Using our expression for the six-point amplitude as well as the seven-point cut terms from ref. [4] as input, it is then straightforward to evaluate the complete seven-point scalar-loop amplitude, $A_{7;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+, 7^+)$. The needed recursive diagrams are displayed in Fig. 7, while the overlap diagrams are given in Fig. 8. We have evaluated these diagrams and have confirmed that the resulting expression for the amplitude has all the proper factorization properties in real momenta, and that all spurious singularities cancel. Combining this result with the known $\mathcal{N} = 4$ [3] and $\mathcal{N} = 1$ [4] supersymmetric amplitudes, yields a complete solution for the seven-gluon QCD amplitude with the same helicity configuration. Although its construction is entirely straightforward, and parallels the six-point case, the seven-point result is rather lengthy, so we refrain from presenting it here. Since the original version of this paper appeared, the result has been presented [62], as a member of the infinite series of n-point amplitudes $A_{n;1}^{\mathrm{QCD}}(1^-, 2^-, 3^+, 4^+, \ldots, n^+)$, constructed using the methods of the present paper.
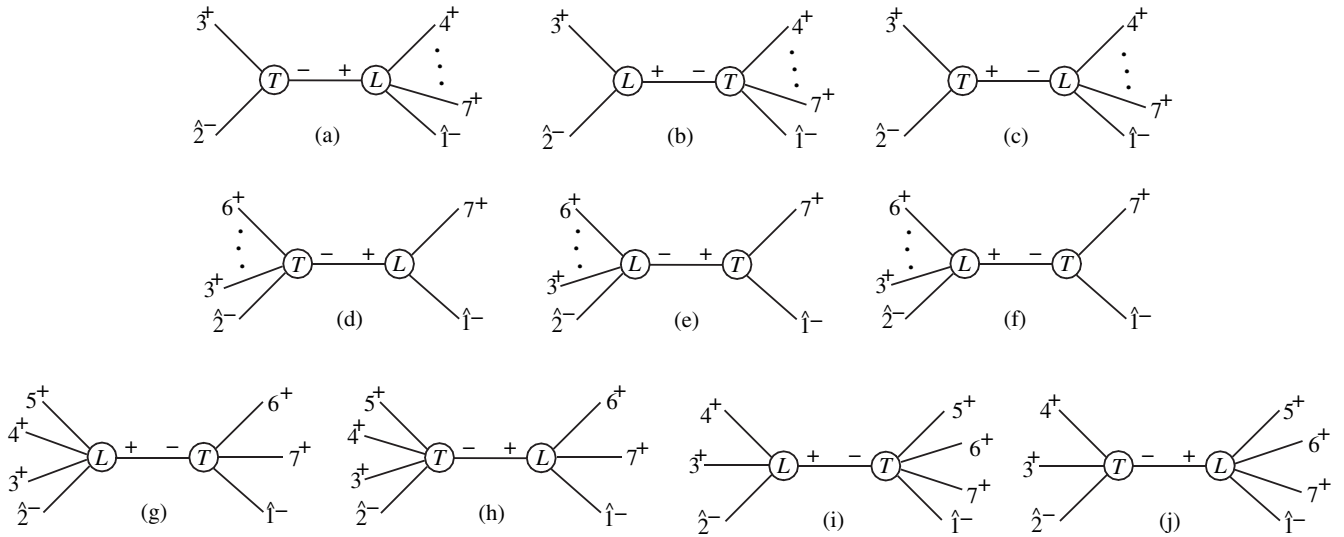
FIG. 7.   The recursive diagrams for the amplitude $A_{7;1}^{[0]}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+, 7^+)$, using the (1,2) shift. Diagrams (a), (b), (d) and (e) vanish.
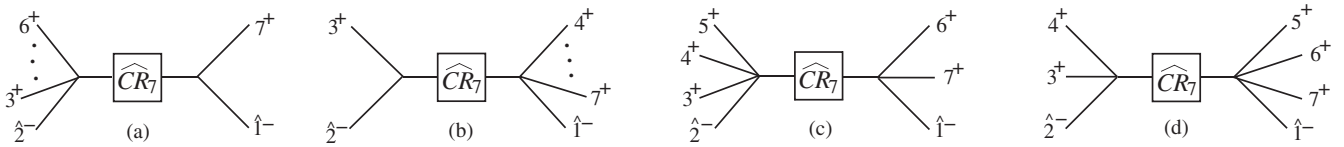

FIG. 8.   The overlap diagrams for the seven-point amplitude.

These examples demonstrate the power of the factorization-bootstrap approach, systematized here, as a complement to the unitarity-based method, for evaluating complete QCD amplitudes, including purely rational parts. The required diagrams are surprisingly simple to evaluate, not really more involved than tree-level diagrams. It is striking that what had previously been the most difficult part of a one-loop QCD calculation has been reduced to a simple computation.

VII. CONCLUSIONS

In this paper we presented a new method for computing the rational functions in nonsupersymmetric gauge theory loop amplitudes. The unitarity method [3–7] has already proven itself to be an effective means for obtaining the cut-containing terms in amplitudes, so we may rely on this approach for obtaining such terms. To obtain the rational terms we took a recursive approach, systematizing an earlier unitarity-factorization bootstrap [2].

Our systematic loop-level recursion uses the proof of tree-level on-shell recursion relations by Britto, Cachazo, Feng and Witten [10] as a starting point. There are, however, a number of issues and subtleties that arise, which are not present at tree level. The most obvious issue is that the

tree-level proof relies on the amplitudes having only simple poles and no branch cuts; loop amplitudes in general contain branch cuts. Furthermore, as we have already discussed in Refs. [11,12], there are subtleties resulting from the differences of one-loop factorizations in complex momenta as compared to those in real momenta. These differences have important effects, unlike the tree-level case. At loop-level there are also spurious poles present, which would interfere with a naive recursion on the rational terms. In this paper we showed how to overcome these potential difficulties.

As an illustrative example of our approach, we described in some detail a computation of the rational terms appearing in the five-gluon QCD amplitudes with nearest-neighboring negative helicities in the color ordering, reproducing the results [47] of the string-based calculation of the same amplitudes. Although we did not describe it in any detail, we also confirmed that our new approach properly reproduces the other independent color-ordered five-gluon helicity amplitude.

Next we computed the six- and seven-point QCD amplitudes $A_{6;1}^{\mathrm{QCD}}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$ and $A_{7;1}^{\mathrm{QCD}}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+, 7^+)$. The rational terms of these amplitudes had not been computed previously. Our computations of these terms use as input lower-point am-

plitudes [11,47,57–59], and the cut-containing terms of the amplitudes under consideration, obtained previously via the unitarity method [4]. For the six-point case we presented a compact expression for the complete amplitude.

Another possible approach to obtaining complete loop amplitudes is via D-dimensional unitarity [5,13,14,17]. It would be worthwhile to corroborate the results of this paper starting from the known D-dimensional tree amplitudes [5,13,27]. It would be also be desirable to develop a first-principles understanding of loop-level factorization with complex momenta, instead of the heuristic one of Refs. [11,12].

The computation of rational function terms has been a bottleneck for calculating one-loop amplitudes in nonsupersymmetric gauge theories with six or more external particles. We expect the technique discussed in this paper to apply to all one-loop multiparton amplitudes in QCD with massless quarks. It should also work, without modification, for amplitudes that contain external massive vec-

tor bosons, or Higgs bosons (in the limit of a large top-quark mass), in addition to massless partons. Finally, we expect suitable modifications of the method to be applicable to processes with massive particles propagating in the loop.

[1] E. W. N. Glover, Nucl. Phys. B, Proc. Suppl. **116**, 3 (2003).
[2] Z. Bern, L. J. Dixon, and D. A. Kosower, Nucl. Phys. **B513**, 3 (1998).
[3] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B425**, 217 (1994).
[4] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B435**, 59 (1995).
[5] Z. Bern and A. G. Morgan, Nucl. Phys. **B467**, 479 (1996).
[6] Z. Bern, L. J. Dixon, and D. A. Kosower, Annu. Rev. Nucl. Part. Sci. **46**, 109 (1996); Z. Bern, L. J. Dixon, and D. A. Kosower, Nucl. Phys. B, Proc. Suppl. **51**, 243 (1996).
[7] R. Britto, F. Cachazo, and B. Feng, Nucl. Phys. **B725**, 275 (2005).
[8] J. Bedford, A. Brandhuber, B. Spence, and G. Travaglini, Nucl. Phys. **B712**, 59 (2005).
[9] R. Britto, F. Cachazo, and B. Feng, Nucl. Phys. **B715**, 499 (2005).
[10] R. Britto, F. Cachazo, B. Feng, and E. Witten, Phys. Rev. Lett. **94**, 181602 (2005).
[11] Z. Bern, L. J. Dixon, and D. A. Kosower, Phys. Rev. D **71**, 105013 (2005).
[12] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **72**, 125003 (2005).
[13] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Phys. Lett. B **394**, 105 (1997).
[14] Z. Bern, L. J. Dixon, and D. A. Kosower, J. High Energy Phys. 01 (2000) 027; Z. Bern, A. De Freitas, and L. J. Dixon, J. High Energy Phys. 09 (2001) 037; Z. Bern, A. De Freitas, and L. J. Dixon, J. High Energy Phys. 03 (2002) 018.
[15] D. A. Kosower, Nucl. Phys. **B552**, 319 (1999); D. A. Kosower and P. Uwer, Nucl. Phys. **B563**, 477 (1999).
[16] Z. Bern, L. J. Dixon, and D. A. Kosower, J. High Energy Phys. 08 (2004) 012.

[17] A. Brandhuber, S. McNamara, B. Spence, and G. Travaglini, J. High Energy Phys. 10 (2005) 011.
[18] Z. Bern, J. S. Rozowsky, and B. Yan, Phys. Lett. B **401**, 273 (1997); Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein, and J. S. Rozowsky, Nucl. Phys. **B530**, 401 (1998); C. Anastasiou, Z. Bern, L. J. Dixon, and D. A. Kosower, Phys. Rev. Lett. **91**, 251602 (2003); Z. Bern, L. J. Dixon, and V. A. Smirnov, Phys. Rev. D **72**, 085001 (2005); E. I. Buchbinder and F. Cachazo, J. High Energy Phys. 11 (2005) 036.
[19] Z. Bern, V. Del Duca, L. J. Dixon, and D. A. Kosower, Phys. Rev. D **71**, 045006 (2005).
[20] Z. Bern, L. J. Dixon, and D. A. Kosower, Phys. Rev. D **72**, 045014 (2005).
[21] R. E. Cutkosky, J. Math. Phys. (N.Y.) **1**, 429 (1960); R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S Matrix*, (Cambridge University Press, Cambridge, 1966).
[22] A. Brandhuber, B. Spence, and G. Travaglini, Nucl. Phys. **B706**, 150 (2005); J. Bedford, A. Brandhuber, B. Spence, and G. Travaglini, Nucl. Phys. **B706**, 100 (2005).
[23] F. Cachazo, hep-th/0410077.
[24] R. Britto, F. Cachazo, and B. Feng, Phys. Rev. D **71**, 025012 (2005).
[25] F. Cachazo, P. Svrček, and E. Witten, J. High Energy Phys. 10 (2004) 077.
[26] G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972).
[27] S. D. Badger, E. W. N. Glover, V. V. Khoze, and P. Svrček, J. High Energy Phys. 07 (2005) 025; D. Forde and D. A. Kosower, Phys. Rev. D **73**, 065007 (2006).
[28] Z. Bern and G. Chalmers, Nucl. Phys. **B447**, 465 (1995).
[29] E. Witten, Commun. Math. Phys. **252**, 189 (2004).
[30] V. P. Nair, Phys. Lett. B **214**, 215 (1988).
[31] R. Roiban, M. Spradlin, and A. Volovich, J. High Energy

Phys. 04 (2004) 012; R. Roiban and A. Volovich, Phys. Rev. Lett. **93**, 131602 (2004); R. Roiban, M. Spradlin, and A. Volovich, Phys. Rev. D **70**, 026009 (2004); E. Witten, Adv. Theor. Math. Phys. **8**, 779 (2004).

[32] S. Gukov, L. Motl, and A. Neitzke, hep-th/0404085.

[33] I. Bena, Z. Bern, and D. A. Kosower, Phys. Rev. D **71**, 045008 (2005).

[34] F. Cachazo, P. Svrček, and E. Witten, J. High Energy Phys. 10 (2004) 074.

[35] F. Cachazo, P. Svrček, and E. Witten, J. High Energy Phys. 10 (2004) 077.

[36] I. Bena, Z. Bern, D. A. Kosower, and R. Roiban, Phys. Rev. D **71**, 106010 (2005).

[37] F. Cachazo and P. Svrček, in *Proceedings of the RTN Winter School on Strings, Supergravity and Gauge Theories*, edited by M. Bertolini *et al.* (Proceedings of Science, 2005).

[38] F. Cachazo, P. Svrček, and E. Witten, J. High Energy Phys. 09 (2004) 006; C. J. Zhu, J. High Energy Phys. 04 (2004) 032; G. Georgiou and V. V. Khoze, J. High Energy Phys. 05 (2004) 070; J. B. Wu and C. J. Zhu, J. High Energy Phys. 07 (2004) 032; 09 (2004) 063; D. A. Kosower, Phys. Rev. D **71**, 045007 (2005); G. Georgiou, E. W. N. Glover, and V. V. Khoze, J. High Energy Phys. 07 (2004) 048; Y. Abe, V. P. Nair, and M. I. Park, Phys. Rev. D **71**, 025002 (2005).

[39] L. J. Dixon, E. W. N. Glover, and V. V. Khoze, J. High Energy Phys. 12 (2004) 015; S. D. Badger, E. W. N. Glover, and V. V. Khoze, J. High Energy Phys. 03 (2005) 023.

[40] Z. Bern, D. Forde, D. A. Kosower, and P. Mastrolia, Phys. Rev. D **72**, 025006 (2005).

[41] R. Roiban, M. Spradlin, and A. Volovich, Phys. Rev. Lett. **94**, 102002 (2005).

[42] M. Luo and C. Wen, J. High Energy Phys. 03 (2005) 004; Phys. Rev. D **71**, 091501 (2005); J. Bedford, A. Brandhuber, B. Spence, and G. Travaglini, Nucl. Phys. **B721**, 98 (2005); F. Cachazo and P. Svrček, hep-th/0502160; R. Britto, B. Feng, R. Roiban, M. Spradlin, and A. Volovich, Phys. Rev. D **71**, 105017 (2005).

[43] C. Quigley and M. Rozali, J. High Energy Phys. 01 (2005) 053; S. J. Bidder, N. E. J. Bjerrum-Bohr, L. J. Dixon, and D. C. Dunbar, Phys. Lett. B **606**, 189 (2005); S. J. Bidder, N. E. J. Bjerrum-Bohr, D. C. Dunbar, and W. B. Perkins, Phys. Lett. B **608**, 151 (2005); **612**, 75 (2005); R. Britto, E. Buchbinder, F. Cachazo, and B. Feng, Phys. Rev. D **72**, 065012 (2005); S. J. Bidder, D. C. Dunbar, and W. B.

Perkins, J. High Energy Phys. 08 (2005) 055.

[44] F. A. Berends and W. T. Giele, Nucl. Phys. **B306**, 759 (1988).

[45] D. A. Kosower, Nucl. Phys. **B335**, 23 (1990).

[46] W. T. Giele and E. W. N. Glover, Phys. Rev. D **46**, 1980 (1992); Z. Kunszt, A. Signer, and Z. Trócsányi, Nucl. Phys. **B420**, 550 (1994); S. Catani, Phys. Lett. B **427**, 161 (1998).

[47] Z. Bern, L. J. Dixon, and D. A. Kosower, Phys. Rev. Lett. **70**, 2677 (1993).

[48] F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, and T. T. Wu, Phys. Lett. B **103B**, 124 (1981); P. De Causmaecker, R. Gastmans, W. Troost, and T. T. Wu, Nucl. Phys. **B206**, 53 (1982); Z. Xu, D. H. Zhang, and L. Chang, Report No. TUTP-84/3-TSINGHUA; R. Kleiss and W. J. Stirling, Nucl. Phys. **B262**, 235 (1985); J. F. Gunion and Z. Kunszt, Phys. Lett. **161B**, 333 (1985); Z. Xu, D. H. Zhang, and L. Chang, Nucl. Phys. **B291**, 392 (1987).

[49] M. L. Mangano and S. J. Parke, Phys. Rep. **200**, 301 (1991); L. J. Dixon, in *QCD & Beyond: Proceedings of TASI '95*, edited by D. E. Soper (World Scientific, Singapore, 1996).

[50] R. Penrose, J. Math. Phys. (N.Y.) **8**, 345 (1967).

[51] J. E. Paton and H. M. Chan, Nucl. Phys. **B10**, 516 (1969); P. Cvitanović, P. G. Lauwers, and P. N. Scharbach, Nucl. Phys. **B186**, 165 (1981); D. Kosower, B. H. Lee, and V. P. Nair, Phys. Lett. B **201**, 85 (1988).

[52] F. A. Berends and W. Giele, Nucl. Phys. **B294**, 700 (1987).

[53] M. L. Mangano, S. J. Parke, and Z. Xu, Nucl. Phys. **B298**, 653 (1988).

[54] Z. Bern and D. A. Kosower, Nucl. Phys. **B362**, 389 (1991).

[55] Z. Bern, V. Del Duca, and C. R. Schmidt, Phys. Lett. B **445**, 168 (1998); Z. Bern, V. Del Duca, W. B. Kilgore, and C. R. Schmidt, Phys. Rev. D **60**, 116001 (1999).

[56] S. J. Parke and T. R. Taylor, Phys. Rev. Lett. **56**, 2459 (1986).

[57] G. Mahlon, Phys. Rev. D **49**, 4438 (1994).

[58] Z. Bern and D. A. Kosower, Nucl. Phys. **B379**, 451 (1992).

[59] Z. Kunszt, A. Signer, and Z. Trócsányi, Nucl. Phys. **B411**, 397 (1994).

[60] Z. Bern, L. J. Dixon, and D. A. Kosower, Nucl. Phys. **B437**, 259 (1995).

[61] Z. Bern, A. De Freitas, L. J. Dixon, and H. L. Wong, Phys. Rev. D **66**, 085002 (2002).

[62] D. Forde and D. A. Kosower, hep-ph/0509358.