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Thermal operator representation of finite temperature graphs. II.

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Using the mixed space representation, we extend our earlier analysis to the case of Dirac and gauge fields and show that in the absence of a chemical potential, the finite temperature Feynman diagrams can be related to the corresponding zero temperature graphs through a thermal operator. At nonzero chemical potential we show explicitly in the case of the fermion self-energy that such a factorization is violated because of the presence of a singular contact term. Such a temperature dependent term which arises only at finite density and has a quadratic mass singularity cannot be related, through a regular thermal operator, to the fermion self-energy at zero temperature which is infrared finite. Furthermore, we show that the thermal radiative corrections at finite density have a screening effect for the chemical potential leading to a finite renormalization of the potential.

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I. INTRODUCTION

In an earlier paper [1] (referred to as Paper I), we gave a simple derivation of an interesting relation [2–4] between finite temperature Feynman graphs and the corresponding zero temperature graphs within the context of a scalar field theory in real as well as in imaginary time formalisms. We showed that the derivation is particularly simple if one uses a mixed space representation of the graphs in the $(t, \vec{p})$ space [5–7] and the proof of the correspondence is particularly simple in the closed time path formalism [5,8,9]. Explicitly, for any N-point graph (at any loop) in a scalar field theory at finite temperature, the relation can be written as

$$\begin{aligned}\Gamma_N^{(T)} &= \int \prod_{i=1}^{I} \frac{d^3k_i}{(2\pi)^3} \prod_{v=1}^{V} (2\pi)^3 \delta_v^{(3)}(k, p)\gamma_N^{(T)} \\ &= \int \prod_{i=1}^{I} \frac{d^3k_i}{(2\pi)^3} \prod_{v=1}^{V} (2\pi)^3 \delta_v^{(3)}(k, p)\mathcal{O}^{(T)}\gamma_N^{(0)}, \qquad (1)\end{aligned}$$

where

$$\mathcal{O}^{(T)} = \prod_{i=1}^{I}(1 + n_i(1 - S_i)), \qquad (2)$$

with $E_i = \sqrt{\vec{k}^2 + m^2}$, $n_i = n(E_i)$ denoting the Bose-Einstein distribution function associated with the internal propagators and $S_i = S(E_i)$ represents a reflection operator that changes $E_i \rightarrow -E_i$. In (1), I characterizes the number of internal propagators, V the total number of vertices in the graph (with the usual relation for the number of loops $L = I - V + 1$) and $\delta_v^{(3)}(k, p)$ enforces the conservation of momentum at the vertex v. We denote the internal and the external three momenta of a graph generically by $\vec{k}$, $\vec{p}$ respectively. Furthermore, $\gamma_N^{(T)}$ represents the integrand of the finite temperature graph (after the internal time coordinates have been integrated in the mixed space) so that it has the dependence

$$\gamma_N^{(T)} = \gamma_N^{(T)}(T, \vec{k}_i, t_\alpha), \qquad (3)$$

where t_α, $\alpha = 1, 2, \ldots, N$ denote the external time coordinates of the graph while $\gamma_N^{(0)}$ is the integrand of the same graph at zero temperature (with the internal time coordinates integrated). The operator (2) relating the integrands of the two graphs was termed the *thermal operator* and the most important property of this operator is that it is independent of time coordinates and carries the entire temperature dependence of the (finite temperature) graph. This interesting result is calculationally quite useful and allows us to study directly many questions of interest at finite temperature such as Ward identities and analyticity [10–12]. In Paper I, we had shown that this simple relation arises as a consequence of the factorization of the finite temperature propagator in the scalar field theory into a basic thermal operator acting on the zero temperature propagator and had studied various properties associated with this thermal operator. In particular, we had shown that the basic thermal operator for the propagator corresponds to a projection operator that projects onto the space of periodic functions. (We recall that while the finite temperature propagator for the scalar field satisfies periodic conditions following from the Kubo-Martin-Schwinger condition [13], the zero temperature propagator does not.) For a complex scalar field with a chemical potential, on the other hand, we showed that the basic thermal operator is much more complex involving time derivative terms. In this case, we could not give a general proof of a thermal operator representation such as (1) although we showed, for specific complicated graphs, that a nontrivial factorization nonetheless arises.

In this paper, we extend our analysis in Paper I to theories involving fermions as well as gauge theories. The analysis for gauge theories is particularly of interest since the interaction terms (non-Abelian three point interaction as well as the interaction of the ghost fields) involve derivative terms. We find in all cases that if there is no chemical potential present, a thermal operator representation for finite temperature graphs naturally follows. On the other hand, for a fermion with a chemical potential, as in the case of the complex scalar field discussed in Paper I, the basic thermal operator is complicated involving time derivatives and we find that a thermal operator representation for graphs fails. This failure is traced to the fact that in such theories, the self-energy develops a quadratic mass singularity because of radiative corrections at finite density. The paper is organized as follows. In Sec. II, we discuss fermion theories at finite temperature (without a chemical potential) and show that the basic factorization of the thermal propagator arises much as in the scalar field theory. The proof of the thermal operator representation for an interacting theory involving scalar and fermion fields is direct in the closed time path formalism which we discuss. In Sec. III, this analysis is extended to gauge theories where we show that the basic factorization of the thermal propagator leads to a thermal operator representation for any graph at finite temperature in spite of interaction terms involving derivatives. The thermal operator representation is explicitly worked out for the contribution of the ghost loop to the self-energy of the gauge field. In Sec. IV, we study an interacting theory of gauge fields and fermions with a chemical potential (for example, QED at finite density) and show that in this case the basic factorization of the thermal propagator for the fermion involves a dependence on time derivatives. In this case, the basic thermal operator can also be written equivalently as one without a time derivative but with a matrix structure. We work out the fermion self-energy in this theory explicitly and show that a thermal operator representation fails. We trace this failure in Sec. V to the fact that the quantum corrections in this theory lead to a quadratic mass singularity at finite density. By analyzing the pole of the fermion propagator in this theory, we show that the chemical potential has a finite renormalization due to radiative corrections and we discuss some interesting aspects of this phenomenon (see, for example, Ref. [14] for a discussion from the point of view of the renormalization group evolution). We conclude with a brief summary in Sec. VI. In Appendix A, we study the $0+1$ dimensional Chern-Simons QED to bring out some interesting features of the thermal operator representation in lower dimensions while Appendix B describes briefly the derivation of some of the formulas used in the text.

II. FERMIONS (WITHOUT CHEMICAL POTENTIAL)

In this section, we will study an interacting theory of scalar and fermion fields at finite temperature (without chemical potential). As we have shown in Paper I, the proof of the thermal operator representation for any graph is direct in the closed time path formalism [5,8,9]. Therefore, for simplicity, we will discuss the theory in this formalism although everything we say also holds in the imaginary time formalism. Indeed, in Ref. [3] the validity of the thermal operator representation has been shown to hold when there is no chemical potential. Let us consider the theory described by the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{M^2}{2}\phi^2 - g\bar{\psi}\psi\phi - \frac{\lambda}{4!}\phi^4. \tag{4}$$

The factorization of the scalar propagator has already been discussed in Paper I and we simply recapitulate here the essential results. In the closed time path formalism, the propagator has a 2×2 structure which can be written as

$$\Delta^{(T)}(t, E) = \mathcal{O}_{\mathrm{B}}^{(T)}\Delta^{(0)}(t, E), \tag{5}$$

where the 2×2 matrix structure of the propagator is labeled at any temperature as

$$\Delta^{(T)}(t, E) = \begin{pmatrix} \Delta_{++}^{(T)}(t, E) & \Delta_{+-}^{(T)}(t, E) \\ \Delta_{-+}^{(T)}(t, E) & \Delta_{--}^{(T)}(t, E) \end{pmatrix}, \tag{6}$$

and the basic thermal operator is the scalar operator

$$\mathcal{O}_{\mathrm{B}}^{(T)}(E) = 1 + n_{\mathrm{B}}(E)(1 - S(E)), \tag{7}$$

where

$$n_{\mathrm{B}}(E) = \frac{1}{e^{E/T} - 1}, \qquad E = E(M) = \sqrt{\vec{p}^2 + M^2}. \tag{8}$$

The components of the propagator at zero temperature have the following explicit forms in the mixed space:

$$\begin{aligned} \Delta_{++}^{(0)}(t, E) &= L(\epsilon)\frac{1}{2E}[\theta(t)e^{-i(E-i\epsilon)t} + \theta(-t)e^{i(E-i\epsilon)t}], \\ \Delta_{+-}^{(0)}(t, E) &= \frac{1}{2E}e^{iEt}, \qquad \Delta_{-+}^{(0)}(t, E) = \frac{1}{2E}e^{-iEt}, \\ \Delta_{--}^{(0)}(t, E) &= L(\epsilon)\frac{1}{2E}[\theta(t)e^{i(E+i\epsilon)t} + \theta(-t)e^{-i(E+i\epsilon)t}], \end{aligned} \tag{9}$$

where the operator $L(\epsilon)$ takes the limit $\epsilon \to 0$.

For fermions, on the other hand, we know that the components of the 2×2 matrix propagator (in the closed time path formalism) at finite temperature have the momentum space representation

$$S^{(T)}(p) = \begin{pmatrix} S^{(T)}_{++}(p) & S^{(T)}_{+-}(p) \\ S^{(T)}_{-+}(p) & S^{(T)}_{--}(p) \end{pmatrix}, \tag{10}$$

with [$L(\epsilon)$ is the operator taking the limit $\epsilon \longrightarrow 0$ introduced earlier]

$$\begin{aligned} S^{(T)}_{++}(p) &= (\not p + m)\bigg(L(\epsilon)\frac{i}{p^2 - m^2 + i\epsilon} \\ &\qquad - 2\pi n_{\mathrm{F}}(|p_0|)\delta(p^2 - m^2)\bigg), \\ S^{(T)}_{+-}(p) &= 2\pi(\not p + m)(\theta(-p_0) - n_{\mathrm{F}}(|p_0|))\delta(p^2 - m^2), \\ S^{(T)}_{-+}(p) &= 2\pi(\not p + m)(\theta(p_0) - n_{\mathrm{F}}(|p_0|))\delta(p^2 - m^2), \\ S^{(T)}_{--}(p) &= (\not p + m)\bigg(-L(\epsilon)\frac{i}{p^2 - m^2 - i\epsilon} \\ &\qquad - 2\pi n_{\mathrm{F}}(|p_0|)\delta(p^2 - m^2)\bigg). \end{aligned} \tag{11}$$

Here $n_{\mathrm{F}}(|p_0|)$ represents the Fermi-Dirac distribution function

$$n_{\mathrm{F}}(|p_0|) = \frac{1}{e^{|p_0|/T} + 1}, \tag{12}$$

and the temperature dependent terms reflect the antiperiodicity condition satisfied by the fermion propagator.

The components of the fermion propagator in (11) can be Fourier transformed in the energy variable to give

$$S^{(T)}(t, \vec{p}) = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} e^{-ip_0 t} S^{(T)}(p) = \mathcal{O}^{(T)}_{\mathrm{F}}(E) S^{(0)}(t, \vec{p}), \tag{13}$$

where $E = E(m) = \sqrt{\vec{p}^2 + m^2}$ and

$$\mathcal{O}^{(T)}_{\mathrm{F}}(E) = 1 - n_{\mathrm{F}}(E)(1 - S(E)). \tag{14}$$

The components of the zero temperature propagator have the explicit forms

$$\begin{aligned} S^{(0)}_{++}(t, \vec{p}) &= L(\epsilon)\frac{1}{2E}[\theta(t)A(E)e^{-i(E-i\epsilon)t} \\ &\qquad + \theta(-t)B(E)e^{i(E-i\epsilon)t}], \\ S^{(0)}_{+-}(t, \vec{p}) &= \frac{1}{2E}B(E)e^{iEt}, S^{(0)}_{-+}(t, \vec{p}) = \frac{1}{2E}A(E)e^{-iEt}, \\ S^{(0)}_{--}(t, \vec{p}) &= L(\epsilon)\frac{1}{2E}[\theta(t)B(E)e^{i(E+i\epsilon)t} \\ &\qquad + \theta(-t)A(E)e^{-i(E+i\epsilon)t}], \end{aligned} \tag{15}$$

where

$$\begin{aligned} A(E) &= \gamma^0 E - \vec{\gamma}\cdot\vec{p} + m, \\ B(E) &= -\gamma^0 E - \vec{\gamma}\cdot\vec{p} + m. \end{aligned} \tag{16}$$

It is worth remarking here that, as in the case of the scalar propagator, it is easy to verify that the basic thermal operator in (14) is a projection operator, namely,

$$(\mathcal{O}^{(T)}_{\mathrm{F}}(E))^2 = \mathcal{O}^{(T)}_{\mathrm{F}}(E), \tag{17}$$

and in the present case projects onto functions satisfying antiperiodicity properties.

Thus, we see that in spite of a matrix structure (from the Dirac gamma matrices) of the fermion propagator, the thermal propagator factorizes in terms of a basic thermal operator (14) which is a scalar quantity much like in the case of the scalar field theory. Furthermore, it is independent of the time coordinates and as a result, the thermal operator representation for any graph can be obtained as follows. First, let us suppose that we have a graph with only external vertices (that is, a one-loop graph). A typical N-vertices graph involving fermion propagators will have the general form shown in Fig. 1. [The external vertices can be of "±" type, but we choose all of them to be of "+" for illustrative purposes only. The same derivation will go through for vertices of any type since the basic thermal operators in (7) and (14) are scalar quantities and have the same form for any component of the propagator.] In this case, at finite temperature, the value of the graph can be written as (we set $g = 1 = \lambda$ and ignore all the multiplicative factors coming from the vertices for simplicity. The external momenta are all assumed to be flowing into a vertex, and the internal momentum k_i flows from vertex i to vertex $i + 1$ and we identify $t_{N+1} = t_1$, $k_{N+1} = k_1$, $p_{N+1} = p_1$.)

$$\Gamma^{(T)}_N = \int \prod_{i=1}^{N} \frac{d^3 k_i}{(2\pi)^3}(2\pi)^3 \delta^{(3)}(k_i - k_{i+1} + p_{i+1})\gamma^{(T)}_N, \tag{18}$$

with

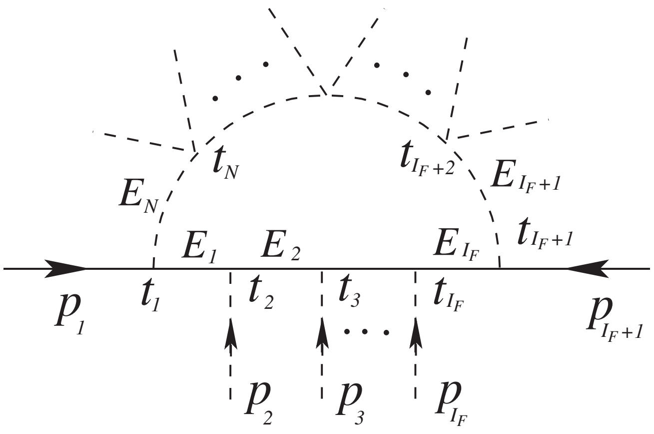

FIG. 1. A typical one-loop diagram involving fermions (solid lines) and scalar fields (dashed lines). For simplicity, the vertices are all assumed to be of "+" type.

$$\begin{aligned}\gamma_N^{(T)} &= \prod_{i=1}^{I_{\rm F}} S_{++}^{(T)}(t_i - t_{i+1}, k_i) \prod_{i=I_{\rm F}+1}^{N} \Delta_{++}^{(T)}(t_i - t_{i+1}, E_i)\\ &= \prod_{i=1}^{I_{\rm F}} \mathcal{O}_{\rm F}^{(T)}(E_i) S_{++}^{(0)}(t_i - t_{i+1}, k_i)\\ &\quad \times \prod_{i=I_{\rm F}+1}^{N} \mathcal{O}_{\rm B}^{(T)}(E_i) \Delta_{++}^{(0)}(t_i - t_{i+1}, E_i)\\ &= \mathcal{O}^{(T)} \prod_{i=1}^{I_{\rm F}} S_{++}^{(0)}(t_i - t_{i+1}, k_i) \qquad (19)\end{aligned}$$

$$\times \prod_{i=I_{\rm F}+1}^{N} \Delta_{++}^{(0)}(t_i - t_{i+1}, E_i) = \mathcal{O}^{(T)} \gamma_N^{(0)}, \qquad (20)$$

where we have identified the thermal operator for the graph as

$$\mathcal{O}^{(T)} = \prod_{i=1}^{I_{\rm F}} \mathcal{O}_{\rm F}^{(T)}(E_i) \prod_{i=I_{\rm F}+1}^{N} \mathcal{O}_{\rm B}^{(T)}(E_i). \qquad (21)$$

The basic thermal operators $\mathcal{O}_B^{(T)}(E)$, $\mathcal{O}_{\rm F}^{(T)}(E)$ are defined in Eqs. (7) and (14) respectively and we have to remember that in (21)

$$E_i = \begin{cases} \sqrt{\vec{k}_i^2 + m^2} & \text{for } i = 1, 2, \ldots, I_{\rm F},\\ \sqrt{\vec{k}_i^2 + M^2} & \text{for } i = I_{\rm F} + 1, \ldots, N. \end{cases} \qquad (22)$$

As a result, we can write

$$\Gamma_N^{(T)} = \int \prod_{i=1}^{N} \frac{d^3 k_i}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(k_i - k_{i+1} + p_{i+1}) \mathcal{O}^{(T)} \gamma_N^{(0)}, \qquad (23)$$

showing that in this case, the finite temperature graph can be given a thermal operator representation. Furthermore, since in the closed time path formalism the range of time integration at finite temperature continues to be the same as at zero temperature and since the basic thermal operators (7) and (14) are independent of time coordinates (so that they can be taken outside the integral), such a factorization of any graph with internal time coordinate (that needs to be integrated over) continues to hold and in general, for any N-point graph (at any loop) at finite temperature, we have the thermal operator representation

$$\Gamma_N^{(T)} = \int \prod_{i=1}^{I} \frac{d^3 k_i}{(2\pi)^3} \prod_{v=1}^{V} (2\pi)^3 \delta_v^{(3)}(k, p) \mathcal{O}^{(T)} \gamma_N^{(0)}, \qquad (24)$$

where the thermal operator follows from (21) to be

$$\mathcal{O}^{(T)} = \prod_{i=1}^{I_{\rm F}} \mathcal{O}_{\rm F}^{(T)}(E_i) \prod_{i=I_{\rm F}+1}^{I} \mathcal{O}_{\rm B}^{(T)}(E_i), \qquad (25)$$

with $I_{\rm F}$, I representing, respectively, the number of internal fermion propagators and the total number of internal propagators.

III. GAUGE THEORIES

We have seen thus far that the thermal operator representation for any Feynman graph at finite temperature holds for theories involving scalar and fermion fields (without a chemical potential). However, physically gauge theories are more interesting and in this section we will discuss a non-Abelian gauge theory at finite temperature. Let us consider a Yang-Mills theory [where the gauge fields belong to $SU(n)$] in the Feynman gauge described by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} - \tfrac{1}{2}(\partial \cdot A^a)^2 + \partial^\mu \bar{c}^a D_\mu c^a, \qquad (26)$$

where $a = 1, 2, \ldots, n^2 - 1$ and (we set the coupling to unity for simplicity)

$$\begin{aligned} D_\mu c^a &= \partial_\mu c^a + f^{abc} A_\mu^b c^c,\\ F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c. \end{aligned} \qquad (27)$$

In this case, in the closed time path formalism, the gauge and the ghost propagators at finite temperature have the momentum space representation

$$\begin{aligned} D_{\mu\nu;\alpha\beta}^{ab(T)}(p) &= -\eta_{\mu\nu} \delta^{ab} \Delta_{\alpha\beta}^{(T)}(p),\\ D_{\alpha\beta}^{ab(T)}(p) &= \delta^{ab} \Delta_{\alpha\beta}^{(T)}(p), \qquad \alpha, \beta = \pm, \end{aligned} \qquad (28)$$

where $\Delta_{\alpha\beta}^{(T)}(p)$ represent the components of a massless scalar propagator at finite temperature and have the explicit forms

$$\begin{aligned} \Delta_{++}^{(T)}(p) &= \left(L(\epsilon) \frac{i}{p^2 + i\epsilon} + 2\pi n_{\rm B}(|p_0|) \delta(p^2) \right),\\ \Delta_{+-}^{(T)}(p) &= 2\pi(\theta(-p_0) + n_{\rm B}(|p_0|)) \delta(p^2),\\ \Delta_{-+}^{(T)}(p) &= 2\pi(\theta(p_0) + n_{\rm B}(|p_0|)) \delta(p^2),\\ \Delta_{--}^{(T)}(p) &= \left(-L(\epsilon) \frac{i}{p^2 - i\epsilon} + 2\pi n_{\rm B}(|p_0|) \delta(p^2) \right), \end{aligned} \qquad (29)$$

where $L(\epsilon)$ is the limiting operator introduced earlier.

By taking the Fourier transform of (29) with respect to p_0, we can obtain the components of the gauge and the ghost propagators in the mixed space. We already know from (6) and (9) that at finite temperature the components of the scalar propagator factorize in the mixed space representation. It follows, therefore, that the components of the gauge and the ghost propagators also factorize in the mixed space representation as

$$\begin{aligned} D_{\mu\nu;\alpha\beta}^{ab(T)}(t, \vec{p}) &= \mathcal{O}_{\rm B}^{(T)}(E) D_{\mu\nu;\alpha\beta}^{(0)}(t, \vec{p}),\\ D_{\alpha\beta}^{ab(T)}(t, \vec{p}) &= \mathcal{O}_{\rm B}^{(T)}(E) D_{\alpha\beta}^{(0)}(t, \vec{p}), \end{aligned} \qquad (30)$$

where the same basic thermal operator $\mathcal{O}_{\rm B}^{(T)}(E)$ leading to a factorization of the gauge and the ghost propagators coincides with that for a scalar propagator defined in (7) (with $E = |\vec{p}|$ for a massless field). Furthermore, all the components of the propagator factorize in the same manner and the basic thermal operator is independent of the time coordinate. It is worth remarking at this point that we have chosen to work in the Feynman gauge for simplicity. In any other covariant gauge fixing, only the Lorentz structure of the gauge propagator generalizes, but the basic factorization of the thermal propagator continues to hold. Furthermore, we consider the case of a vanishing chemical potential here for simplicity.

Since the gauge and the ghost propagators factorize in the same way as in a scalar field theory, the thermal operator representation of any graph at finite temperature would seem obvious. However, unlike in a scalar field theory, the interactions in a non-Abelian gauge theory involve time derivatives in the mixed space (for example, the three gluon vertex or the ghost interaction vertex) and, in principle, may complicate the general proof of the thermal operator representation of an arbitrary graph. On the other hand, we note that the basic thermal operator (7) in the factorization of the propagators (30) is independent of time coordinates. As a result, it follows trivially that

$$\partial_t \mathcal{O}_{\mathrm{B}}^{(T)}(E) = \mathcal{O}_{\mathrm{B}}^{(T)}(E)\partial_t, \tag{31}$$

so that the basic thermal operators in a propagator can be trivially commuted past the derivatives in the vertices leading to a thermal operator representation of any arbitrary graph. Let us illustrate this with the example of the one-loop graph (see Fig. 2) depicting the ghost contribution to the gauge self-energy. In this case, the graph can be written as (we suppress all multiplicative factors associated with the diagram for simplicity, identify $k_3 = k_1$ and look only at the contributions coming from a time derivative in the vertex which may present a challenge to a thermal operator representation)

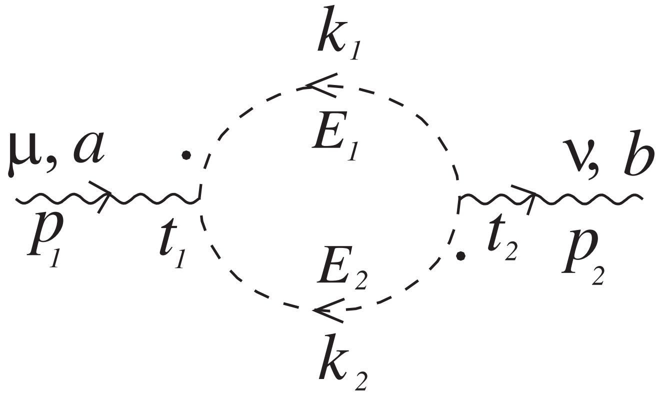

FIG. 2. The ghost loop contribution to the self-energy of the gauge boson. The "dots" represent the action of the time derivatives.

$$\begin{aligned}
\Gamma_2^{ab(T)} &= \int \prod_{i=1}^{2} \frac{d^3k_i}{(2\pi)^3}(2\pi)^3\delta^3(k_i + p_i + k_{i+1})\gamma_2^{ab(T)} \\
&= \int \prod_{i=1}^{2} \frac{d^3k_i}{(2\pi)^3}(2\pi)^3\delta^3(k_i + p_i + k_{i+1})f^{apc}f^{bqr}\partial_{t_1}D_{++}^{qc(T)}(t_2 - t_1, E_1)\partial_{t_2}D_{++}^{pr(T)}(t_1 - t_2, E_2) \\
&= \int \prod_{i=1}^{2} \frac{d^3k_i}{(2\pi)^3}(2\pi)^3\delta^3(k_i + p_i + k_{i+1})f^{apc}f^{bqr}(\partial_{t_1}\mathcal{O}_{\mathrm{B}}^{(T)}(E_1)D_{++}^{qc(0)}(t_2 - t_1, E_1))(\partial_{t_2}\mathcal{O}_{\mathrm{B}}^{(T)}(E_2)D_{++}^{pr(0)}(t_1 - t_2, E_2)) \\
&= \int \prod_{i=1}^{2} \frac{d^3k_i}{(2\pi)^3}(2\pi)^3\delta^3(k_i + p_i + k_{i+1})\mathcal{O}_{\mathrm{B}}^{(T)}(E_1)\mathcal{O}_{\mathrm{B}}^{(T)}(E_2)f^{apc}f^{bqr}\partial_{t_1}D_{++}^{qc(0)}(t_2 - t_1, E_1)\partial_{t_2}D_{++}^{pr(0)}(t_1 - t_2, E_2) \\
&= \int \prod_{i=1}^{2} \frac{d^3k_i}{(2\pi)^3}(2\pi)^3\delta^3(k_i + p_i + k_{i+1})\mathcal{O}_{\mathrm{B}}^{(T)}(E_1)\mathcal{O}_{\mathrm{B}}^{(T)}(E_2)\gamma_2^{ab(0)} \\
&= \int \prod_{i=1}^{2} \frac{d^3k_i}{(2\pi)^3}(2\pi)^3\delta^3(k_i + p_i + k_{i+1})\mathcal{O}^{(T)}\gamma_2^{ab(0)},
\end{aligned} \tag{32}$$

where in the last step we have identified the thermal operator for the graph to be

$$\mathcal{O}^{(T)} = \mathcal{O}_{\mathrm{B}}^{(T)}(E_1)\mathcal{O}_{\mathrm{B}}^{(T)}(E_2). \tag{33}$$

Thus, we see that in spite of the presence of time derivatives in the vertices in the mixed space representation, the thermal operator representation holds simply because the basic thermal operator commutes with the time derivative operator. With this observation, it is clear that for an interacting non-Abelian gauge theory, we can write the thermal operator representation for any arbitrary N-point graph at finite temperature (involving gauge and ghost vertices of ± type) at any loop as

$$\Gamma_N^{a_1\cdots a_N(T)} = \int \prod_{i=1}^{I} \frac{d^3k_i}{(2\pi)^3} \prod_{v=1}^{V} (2\pi)^3 \delta_v^{(3)}(k,p)\gamma_N^{a_1\cdots a_N(T)} = \int \prod_{i=1}^{I} \frac{d^3k_i}{(2\pi)^3} \prod_{v=1}^{V} (2\pi)^3 \delta_v^{(3)}(k,p)\mathcal{O}^{(T)}\gamma_N^{a_1\cdots a_N(0)}, \tag{34}$$

where we have suppressed the Lorentz indices associated with the graph and the thermal operator for the graph has the form

$$\mathcal{O}^{(T)} = \prod_{i=1}^{I} \mathcal{O}_{\mathrm{B}}^{(T)}(E_i). \tag{35}$$

In a similar manner, it can be shown that in the absence of chemical potentials any diagram in an interacting theory involving gauge fields, fermions and scalar fields at finite temperature will have a thermal operator representation. The interesting and challenging case, however, seems to be in the presence of a chemical potential which we discuss in the next section.

IV. FERMIONS WITH A CHEMICAL POTENTIAL

Let us next consider QED at finite temperature and density. In the Feynman gauge, the theory is described by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi - \tfrac{1}{2}(\partial\cdot A)^2 + \mu\bar{\psi}\gamma^0\psi, \tag{36}$$

where μ represents the chemical potential associated with the fermion and the covariant derivative is defined to be

$$D_\mu\psi = \partial_\mu\psi - ieA_\mu\psi. \tag{37}$$

In (36) we have neglected the free Lagrangian density for the ghosts which is not relevant for our discussions.

As we have argued before, there is no chemical potential associated with the photon. As a result, the propagator for the gauge boson in the Feynman gauge will continue to be what we have discussed in the last section (without any internal indices). On the other hand, in momentum space in the closed time path formalism, the components of the fermion propagator in the presence of a chemical potential take the forms

$$\begin{aligned}
S_{++}^{(T,\mu)}(p) &= (\not{p} + m + \mu\gamma^0)\left[L(\epsilon)\frac{i}{(p_0+\mu)^2 - E^2 + i\epsilon} - 2\pi n_{\mathrm{F}}(\mathrm{sgn}(p_0+\mu)p_0)\delta((p_0+\mu)^2 - E^2)\right],\\
S_{+-}^{(T,\mu)}(p) &= 2\pi(\not{p} + m + \mu\gamma^0)(\theta(-p_0-\mu) - n_{\mathrm{F}}(\mathrm{sgn}(p_0+\mu)p_0))\delta((p_0+\mu)^2 - E^2),\\
S_{-+}^{(T,\mu)}(p) &= 2\pi(\not{p} + m + \mu\gamma^0)(\theta(p_0+\mu) - n_{\mathrm{F}}(\mathrm{sgn}(p_0+\mu)p_0))\delta((p_0+\mu)^2 - E^2),\\
S_{--}^{(T,\mu)}(p) &= (\not{p} + m + \mu\gamma^0)\left[-L(\epsilon)\frac{i}{(p_0+\mu)^2 - E^2 - i\epsilon} - 2\pi n_{\mathrm{F}}(\mathrm{sgn}(p_0+\mu)p_0)\delta((p_0+\mu)^2 - E^2)\right],
\end{aligned} \tag{38}$$

where $E = \sqrt{\vec{p}^2 + m^2}$. Equation (38) clearly reduces to (11) when $\mu = 0$. The Fourier transform of (38) in the p_0 variable leads to the propagator in the mixed space which can be seen to have a nontrivial factorization (as is the case with the complex scalar field discussed in Ref. [1])

$$S_{\alpha\beta}^{(T,\mu)}(t,\vec{p}) = e^{i\mu t}\mathcal{O}_{\mathrm{F}}^{(T,\mu)}(E)S_{\alpha\beta}^{(0,0)}(t,\vec{p}), \tag{39}$$

where, as before, $\alpha, \beta = \pm$ and the basic thermal operator in (39) has the form

$$\mathcal{O}_{\mathrm{F}}^{(T,\mu)}(E) = 1 - \frac{n_{\mathrm{F}}^{+} + n_{\mathrm{F}}^{-}}{2}(1 - S(E)) + \frac{n_{\mathrm{F}}^{+} - n_{\mathrm{F}}^{-}}{2}(1 + S(E))\frac{i\partial_t}{E}, \tag{40}$$

where we have defined

$$n_{\mathrm{F}}^{\pm} = n_{\mathrm{F}}(E \pm \mu), \tag{41}$$

and $S_{\alpha\beta}^{(0,0)}(t,\vec{p})$ represent the components of the fermion propagator at zero temperature and zero chemical potential given in (15).

We note that the structure of the basic thermal operator in (40) is quite analogous to the case of the complex scalar field with a chemical potential discussed in Ref. [1]. This is a scalar operator which involves a time derivative operator (it does not depend on the time coordinate). Unlike in the case of the scalar field, however, in this case, we can equivalently define a basic thermal operator which is independent of the time derivative, but instead is a matrix (in the Dirac space), namely, in this case we can also write

$$S_{\alpha\beta}^{(T,\mu)}(t,\vec{p}) = e^{i\mu t}\tilde{\mathcal{O}}_{\mathrm{F}}^{(T,\mu)}S_{\alpha\beta}^{(0,0)}(t,\vec{p}), \tag{42}$$

where

$$\tilde{\mathcal{O}}_{\mathrm{F}}^{(T,\mu)} = 1 - (n_{\mathrm{F}}^{-}A(E) - n_{\mathrm{F}}^{+}B(E))\frac{\gamma^0}{2E}(1 - S(E)), \tag{43}$$

where $A(E)$, $B(E)$ are matrices defined in (16). The two forms of the basic thermal operator are related through the first order equation satisfied by the fermion propagator in the mixed space. However, the scalar form of the basic thermal operator (in spite of the time derivative operator) is easier to use than the matrix one and we will carry out our discussions in terms of the factorization (39). We note that both forms of the basic thermal operator in (40) and (43) can be checked to be projection operators, namely,

$$(\mathcal{O}_{\rm F}^{(T,\mu)})^2 = \mathcal{O}_{\rm F}^{(T,\mu)}, \qquad (\tilde{\mathcal{O}}_{\rm F}^{(T,\mu)})^2 = \tilde{\mathcal{O}}_{\rm F}^{(T,\mu)}, \tag{44}$$

and enforce the necessary antiperiodicity in the present case.

As in the case of the complex scalar field with a chemical potential (in Ref. [1]), here we note that the presence of the time derivative term in the basic thermal operator (40) makes it difficult to give a general proof of a thermal operator representation for any arbitrary graph, particularly in cases involving internal time coordinates which are integrated over. In the scalar theory, we had shown in a specific complicated graph that a (nontrivial) thermal operator representation results in spite of the presence of the time derivative operator and this led to the hope that a thermal operator representation may result in general. In the fermionic theory, we will show through an explicit calculation that the thermal operator representation fails in the presence of a chemical potential and we will trace this failure (in the next section) to a renormalization of the chemical potential because of radiative corrections. For the purpose of explicit calculations, we will use the imaginary time formalism [5,15,16] where there is no doubling of fields. In this case, the factorization of the thermal propagator (39) can be written as

$$S^{(T,\mu)}(\tau,\vec{p}) = e^{\mu\tau}\mathcal{O}_{\rm F}^{(T,\mu)}(E)S^{(0,0)}(\tau,\vec{p}), \tag{45}$$

where $\mathcal{O}_{\rm F}^{(T,\mu)}(E)$ is the rotation of the operator in (40) to imaginary time

$$\begin{aligned}\mathcal{O}_{\rm F}^{(T,\mu)}(E) = 1 &- \frac{n_{\rm F}^+ + n_{\rm F}^-}{2}(1 - S(E)) \\ &- \frac{n_{\rm F}^+ - n_{\rm F}^-}{2}(1 + S(E))\frac{\partial_\tau}{E},\end{aligned} \tag{46}$$

and the propagator at zero temperature in the absence of a chemical potential has the form

$$S^{(0,0)}(\tau,\vec{p}) = \frac{1}{2E}[\theta(\tau)A(E)e^{-E\tau} + \theta(-\tau)B(E)e^{E\tau}]. \tag{47}$$

Here $A(E)$, $B(E)$ denote the Euclidean rotation of the matrices in (16), namely,

$$\begin{aligned}A(E) &= i\gamma_0 E - \vec{\gamma}\cdot\vec{p} + m,\\ B(E) &= -i\gamma_0 E - \vec{\gamma}\cdot\vec{p} + m.\end{aligned} \tag{48}$$

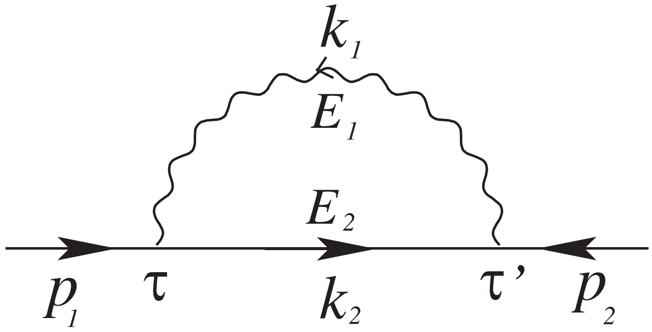

FIG. 3. The fermion self-energy at one-loop order.

Let us next analyze the one-loop fermion self-energy graph at finite temperature and density in QED [see Lagrangian density (36)] at one loop in the imaginary time formalism. The two point function in Fig. 3 can be explicitly evaluated and leads to (we note that the fermion self-energy is simply the two point function with momentum conserving delta functions factored out and we identify $k_3 = k_1$ in the derivation below)

$$\begin{aligned}\Gamma_2^{(T,\mu)} &= \int\prod_{i=1}^{2}\frac{d^3k_i}{(2\pi)^3}(2\pi)^3\delta^3(k_i + p_i - k_{i+1})\tilde{\gamma}_2^{(T,\mu)}\\ &= (2\pi)^3\delta^3(p_1 + p_2)\int\frac{d^3k}{(2\pi)^3}\tilde{\gamma}_2^{(T,\mu)},\end{aligned} \tag{49}$$

where we have identified the integrand with $\tilde{\gamma}_2^{(T,\mu)}$ (to avoid confusion with the Dirac gamma matrices) and have carried out the integration over $\vec{k}_2$ using one of the delta functions (and have identified $\vec{k}_1 = \vec{k}$). The integrand has the explicit form

$$\begin{aligned}\tilde{\gamma}_2^{(T,\mu)} &= -e^2\gamma_\mu S^{(T,\mu)}(\tau - \tau',\vec{k}_2)\gamma_\mu D^{(T)}(\tau - \tau', E_1)\\ &= -e^2\bigg[e^{\mu(\tau-\tau')}\mathcal{O}_{\rm B}^{(T)}(E_1)\bar{\mathcal{O}}_{\rm F}^{(T,\mu)}(E_2)(\gamma_\mu S^{(0,0)}(\tau-\tau',\vec{k}_2)\gamma_\mu D^{(0)}(\tau - \tau', E_1)) + \frac{2i(n_{\rm F}^+(E_2) - n_{\rm F}^-(E_2))\gamma_0}{E_1^2 - E_2^2}\delta(\tau - \tau')\bigg]\\ &= e^{\mu(\tau-\tau')}\mathcal{O}_{\rm B}^{(T)}(E_1)\bar{\mathcal{O}}_{\rm F}^{(T,\mu)}(E_1,E_2)\tilde{\gamma}_2^{(0,0)} - \frac{2ie^2(n_{\rm F}^+(E_2) - n_{\rm F}^-(E_2))\gamma_0}{E_1^2 - E_2^2}\delta(\tau - \tau'),\end{aligned} \tag{50}$$

where the modified fermion operator $\bar{\mathcal{O}}_{\rm F}^{(T,\mu)}$ is similar to the one in (46) except that it involves $\partial_\tau/(E_1 + E_2)$ and

$$\begin{aligned}E_1 = |\vec{k}|, \qquad & E_2 = \sqrt{(\vec{k}+\vec{p})^2 + m^2},\\ \vec{p}_1 = \vec{p}, \qquad & \vec{k}_2 = \vec{k} + \vec{p}.\end{aligned} \tag{51}$$

Thus, we see explicitly from (50) that, unlike the case of the complex scalar field discussed in Ref. [1], here a thermal operator representation for the fermion two point function at one loop breaks down in the presence of a chemical potential. The additional term leading to the breakdown of the thermal operator representation is a contact term which vanishes for $\mu \to 0$ since

$$\lim_{\mu\to 0}(n_{\rm F}^{+}(E_2) - n_{\rm F}^{-}(E_2)) \to 0. \tag{52}$$

V. RENORMALIZATION OF CHEMICAL POTENTIAL

Let us note that the chemical potential can be thought of as a constant background electrostatic potential. This is particularly clear if we note that the free part of the fermion Lagrangian density in (36) can be written in the Euclidean space as

$$\mathcal{L}_{f,\rm free} = \bar\psi(\gamma_0(\partial_\tau - \mu) + \vec\gamma\cdot\vec\nabla)\psi + m\bar\psi\psi. \tag{53}$$

As a result of this structure of the theory, one can derive certain identities as follows. Adding the sources for the fields, we can write the generating functional for the theory in (36) (in imaginary time) as (we set $\hbar = 1$)

$$Z[\mu, J_\mu, \eta, \bar\eta] = e^{-W[\mu,J_\mu,\eta,\bar\eta]} = \int \mathcal{D}A_\mu\mathcal{D}\bar\psi\mathcal{D}\psi e^{-\int_0^{1/T}d\tau\int d^3x(\mathcal{L}+J_\mu A_\mu + i(\bar\eta\psi - \bar\psi\eta))}. \tag{54}$$

This, in turn, leads to the identity

$$\frac{\partial W}{\partial\mu} = \int_0^{1/T} d\tau\int d^3x\bigg(i\frac{\delta W}{\delta\eta(x)}\gamma_0\frac{\delta W}{\delta\bar\eta(x)} - i\,{\rm Tr}\gamma_0\frac{\delta^2 W}{\delta\eta(x)\delta\bar\eta(x)}\bigg). \tag{55}$$

By taking the second order derivative with respect to $\eta, \bar\eta$ and setting all sources to zero, this leads to the identity (when we Fourier transform the spatial coordinates)

$$\frac{\partial S^{(T,\mu)}(\tau_1 - \tau_2, \vec p)}{\partial\mu} = i\int_0^{1/T} d\tau S^{(T,\mu)}(\tau_1 - \tau, \vec p) \times \gamma_0 S^{(T,\mu)}(\tau - \tau_2, \vec p). \tag{56}$$

Such an identity has already proved quite useful in the solution of the $0 + 1$ dimensional Chern-Simons QED [7] and with the explicit form of the fermion propagator in (45) it can be checked that this is true. It also follows from (56) that

$$\frac{1}{n!}\frac{\partial^n S^{(T,\mu)}}{\partial\mu^n} = (i)^n S^{(T,\mu)}\gamma_0\cdots S^{(T,\mu)}\gamma_0 S^{(T,\mu)}, \tag{57}$$

where there are n insertions of γ_0 on the right-hand side and we have used a compact notation suppressing the internal time integrations.

Using the above result, we can see that a correction to the chemical potential may arise from successive insertions of the operator $\delta\mu\gamma_0$ in the fermion propagator $S^{(T,\mu)}$. The resulting corrected propagator $S^{(T,\mu+\delta\mu)}$ is then obtained by summing the geometric series

$$\sum_{n=0}^{\infty}(i)^n S^{(T,\mu)}(\delta\mu\gamma_0)\cdots S^{(T,\mu)}(\delta\mu\gamma_0)S^{(T,\mu)} = \sum_{n=0}^{\infty}(i)^n\frac{(\delta\mu)^n}{n!}\frac{\partial^n S^{(T,\mu)}}{\partial\mu^n} = S^{(T,\mu+\delta\mu)}. \tag{58}$$

Thus, the effect of such insertions is to shift the chemical potential to an effective value given by $\mu + \delta\mu$. In order to evaluate this shift (finite renormalization) we have to perform a more systematic analysis of the fermion self-energy. To this end, let us calculate the complete self-energy in momentum space [5] [namely, Fourier transform (49) in the time variable and factor out overall energy-momentum conserving delta functions]

$$\Sigma^{(T,\mu)}(p) = \frac{1}{2}\int_{-1/T}^{1/T} d\tau e^{ip_0\tau}\Sigma^{(T,\mu)}(\tau, \vec p). \tag{59}$$

A direct evaluation yields

$$\begin{aligned}\Sigma^{(T,\mu)}(p) = -\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{e^2}{4E_1E_2} &\times\bigg[\bar A\bigg(\frac{(1 + n_{\rm B}(E_1) - n_{\rm F}^{-}(E_2))}{E_1 + E_2 - (ip_0 + \mu)} - \frac{(n_{\rm B}(E_1) + n_{\rm F}^{-}(E_2))}{E_1 - E_2 + (ip_0 + \mu)}\bigg)\\ &+ \bar B\bigg(\frac{(1 + n_{\rm B}(E_1) - n_{\rm F}^{+}(E_2))}{E_1 + E_2 + (ip_0 + \mu)} - \frac{(n_{\rm B}(E_1) + n_{\rm F}^{+}(E_2))}{E_1 - E_2 - (ip_0 + \mu)}\bigg)\bigg],\end{aligned} \tag{60}$$

where we have identified

$$\begin{aligned}\bar A &= 2(i\gamma_0 E_2 - \vec\gamma\cdot\vec k_2 - 2m),\\ \bar B &= 2(-i\gamma_0 E_2 - \vec\gamma\cdot\vec k_2 - 2m).\end{aligned} \tag{61}$$

It is clear from the explicit form of (60) that the self-energy (in fact, any thermodynamic function) is a function of $(ip_0 + \mu, \mu)$ where the extra μ dependence comes from the explicit dependence of the fermion distribution functions on the chemical potential. The analytic continuation of the self-energy can be carried out appropriately [17] through

$$iP_0 = ip_0 + \mu \to P_0 = (p_0 + \mu)(1 + i\epsilon), \tag{62}$$

say for the time ordered amplitude. We recognize from (60) that the real part of the coefficient of the γ^0 term in (60) coincides exactly with the coefficient of the contact term in the mixed space in (50) when $p_0 + \mu = 0$.

In fact, let us note that after analytic continuation, we can write the coefficient of the γ^0 term in (60) as

$$\begin{aligned}\frac{1}{4}\,\mathrm{Tr}\gamma^0\Sigma^{(T,\mu)}(p) = -\frac{e^2}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{E_1}\bigg[\bigg\{&\frac{1}{E_1+E_2-P_0}-\frac{1}{E_1+E_2+P_0}+n_{\mathrm{B}}(E_1)\bigg(\frac{1}{E_1+E_2-P_0}-\frac{1}{E_1+E_2+P_0}\\ &+\frac{1}{E_1-E_2-P_0}-\frac{1}{E_1-E_2+P_0}\bigg)-\frac{n_{\mathrm{F}}^{+}(E_2)+n_{\mathrm{F}}^{-}(E_2)}{2}\bigg(\frac{1}{E_1+E_2-P_0}-\frac{1}{E_1+E_2+P_0}\\ &-\frac{1}{E_1-E_2-P_0}+\frac{1}{E_1-E_2+P_0}\bigg)\bigg\}+\frac{n_{\mathrm{F}}^{+}(E_2)-n_{\mathrm{F}}^{-}(E_2)}{2}\bigg(\frac{1}{E_1+E_2-P_0}+\frac{1}{E_1+E_2+P_0}\\ &+\frac{1}{E_1-E_2-P_0}+\frac{1}{E_1-E_2+P_0}\bigg)\bigg].\end{aligned}\tag{63}$$

Let us denote the terms involving the braces as Σ_1 and the terms in the last parenthesis as Σ_2 (after integration) so that we can write

$$\tfrac{1}{4}\,\mathrm{Tr}\gamma^0\Sigma^{(T,\mu)}(p) = \Sigma_1(P_0,\vec{p},\mu)+\Sigma_2(P_0,\vec{p},\mu).\tag{64}$$

From the explicit form of the terms in (63), we note that under $P_0\rightarrow -P_0$, $\mu\rightarrow-\mu$, both Σ_1, Σ_2 change sign. However, since Σ_1 is manifestly symmetric under $\mu\rightarrow-\mu$, it is antisymmetric under $P_0\rightarrow-P_0$ and, consequently, vanishes when $P_0=0$. On the other hand, Σ_2 is antisymmetric under $\mu\rightarrow-\mu$ and, consequently, is symmetric under $P_0\rightarrow-P_0$ and does not vanish when $P_0=0$ and, in fact, yields the contact term in (50). Using the results in Appendix B, we can evaluate Σ_2 explicitly in the high T limit and when $\vec{p}=0$, it has the form (for $T\gg m$)

$$\Sigma_2(P_0=0,\vec{p}=0)\simeq -\frac{e^2\mu(\pi^2T^2+\mu^2)}{6\pi^2m^2}.\tag{65}$$

This has a quadratic mass singularity that arises only when $\mu\neq 0$ (and, therefore, at finite charge density). The self-energy at zero temperature and chemical potential on the other hand, is infrared finite [18] (logarithmic infrared divergences only appear if one expands around the singular point $p^2=m^2$). Consequently, the singular term (65) cannot be related to the zero temperature fermion self-energy through a regular thermal operator. We believe that the presence of this strongly divergent infrared behavior is responsible for the failure of the thermal operator representation in the case of a nonvanishing chemical potential.

To understand the renormalization of the chemical potential, we have to analyze the poles of the fermion propagator. We note that with the self-energy corrections, the complete two point function at one loop (in Minkowski space) can be written as

$$i(S^{(T,\mu)})^{-1}(p) = \not{p}-m+\mu\gamma^0-\Sigma(p,T,\mu),\tag{66}$$

where Σ can only be calculated in some limit such as the high T limit. The analysis of the poles of the propagator in this limit, even in the absence of a chemical potential, is highly nontrivial. It is known in the absence of a chemical potential [16] that in the leading order at high temperature ($T\gg m$), the fermion propagator has an absolute pole at $\vec{p}=0$, $p_0=\pm m_f$ where $m_f=eT/2\sqrt{2}$ represents the thermal mass of the fermion. For nonzero $\vec{p}$, the fermion propagator has only partial poles corresponding to two quasiparticle modes. Here we will follow the same analysis restricting ourselves to only the absolute pole in the leading order at high temperature. Also, we will disregard those thermal corrections which yield a finite renormalization of the vacuum fermion mass, because such nonleading terms are not relevant for the analysis of the renormalization of the chemical potential. In the leading order at high temperature ($T\gg m$), the terms in (60) which are even under $\mu\rightarrow-\mu$ lead to

$$\Sigma_{\mathrm{even}} = \frac{m_f^2\gamma^0}{2p}\ln\frac{P_0+p}{P_0-p}+\frac{m_f^2\vec{\gamma}\cdot\hat{p}}{p}\bigg(1-\frac{P_0}{2p}\ln\frac{P_0+p}{P_0-p}\bigg),\tag{67}$$

where we have defined $p=|\vec{p}|$ and $\hat{p}$ denotes the unit vector along $\vec{p}$. In the presence of a chemical potential we have

$$m_f^2=\frac{e^2}{8}\bigg(T^2+\frac{\mu^2}{\pi^2}\bigg),\tag{68}$$

which is well known (see, for example, [16]). The nonanalytic behavior of Σ_{even} at $P_0=0$, $p=0$ is obvious from (67). We also note that all the $\vec{\gamma}\cdot\hat{p}$ terms in (67) vanish in the limit $p=0$ (basically because in this case, there is no direction available to contract the gamma matrix) so that we can write

$$\Sigma_{\mathrm{even}}(\vec{p}=0)\simeq\frac{m_f^2}{P_0}\gamma^0=\gamma^0\Sigma_1(P_0,\vec{p}=0).\tag{69}$$

The absolute pole in the fermion propagator continues to be at $\vec{p}=0$ and at this point, the terms proportional to γ_0 in (60) which are odd under $\mu\rightarrow-\mu$ yield

$$\begin{aligned}\Sigma_{\mathrm{odd}} &= -\frac{e^2(P_0^2+m^2)\gamma^0}{2\pi^2}\int_0^\infty dkk^2\frac{(n_{\mathrm{F}}^{-}(k)-n_{\mathrm{F}}^{+}(k))}{4k^2P_0^2-(P_0^2-m^2)^2}\\ &=\gamma^0\Sigma_2(P_0,\vec{p}=0).\end{aligned}\tag{70}$$

Using the results in Appendix B, this can be evaluated in the high temperature limit and shows that at $|P_0| = |M| \sim eT \gg m$ it is well behaved and has the value

$$\Sigma_{\rm odd}(P_0 = M, \vec{p} = 0) = \gamma^0\Sigma_2(P_0 = M, \vec{p} = 0) \simeq \frac{e^2\mu\gamma^0}{8\pi^2}. \tag{71}$$

With these results, the analysis of the pole when $\vec{p} = 0$ becomes quite straightforward. We note from (66) that when $\vec{p} = 0$, the propagator will have a pole provided

$$\gamma^0 P_0 - m - \Sigma_{\rm even}(\vec{p} = 0) - \Sigma_{\rm odd}(\vec{p} = 0) = 0. \tag{72}$$

We note from (68) and (69) that at very high temperature $m_f \gg m$ so that the fermion mass may be neglected in the above equation. All the other terms are proportional to γ^0. If we expand $\Sigma_{\rm odd}$ around $P_0 = M$, the equation (72) takes the form

$$\gamma^0\left(P_0 - \frac{m_f^2}{P_0} - \frac{e^2\mu}{8\pi^2}\right) - (P_0 - M)\Sigma'_{\rm odd}(P_0 = M) + \cdots = 0. \tag{73}$$

Here we have used (71) and $\Sigma'_{\rm odd}$ denotes the derivative of $\Sigma_{\rm odd}$ with respect to P_0. The root of this equation and, therefore, the location of the pole is given by

$$P_0 = M = \frac{e^2\mu}{16\pi^2} \pm m_f\left(1 + \frac{e^2\mu^2}{32\pi^2(\pi^2T^2 + \mu^2)}\right)^{1/2}, \tag{74}$$

which can be equivalently written as

$$P_0 - \frac{e^2\mu}{16\pi^2} = \pm m_f\left(1 + \frac{e^2\mu^2}{32\pi^2(\pi^2T^2 + \mu^2)}\right)^{1/2}. \tag{75}$$

To the order that we are working, this can be simplified to give the location of the pole at

$$p_0 + \mu\left(1 - \frac{e^2}{16\pi^2}\right) = \pm m_f, \tag{76}$$

or

$$p_0 + \mu_{\rm R} = \pm\frac{e}{2\sqrt{2}}\left(T^2 + \frac{\mu^2}{\pi^2}\right)^{1/2}.$$

Here we have identified

$$\mu_{\rm R} = \mu\left(1 - \frac{e^2}{16\pi^2}\right), \tag{77}$$

which can be interpreted as the renormalized chemical potential due to the radiative corrections of the theory. Since it is associated with a physical pole of the propagator, we expect this result to be gauge independent which we have explicitly checked. Such a finite renormalization of the chemical potential has the effect of screening the chemical potential because of thermal interactions at a finite density. This is consistent with our earlier observation that the chemical potential can be thought of as a constant Abelian electrostatic potential and Abelian gauge fields lead to a screening effect.

VI. SUMMARY

In this paper, we have extended our analysis of the thermal operator representation for Feynman graphs at finite temperature to theories involving fermions as well as gauge fields. We have shown that as long as there is no chemical potential, the thermal operator representation holds. We have also discussed in an appendix how a thermal operator representation naturally arises in 0 + 1 dimensional Chern-Simons QED. However, in QED at finite temperature and density (nonzero chemical potential), we have shown that such a factorization is violated because of the appearance of singular contact terms. This is explicitly worked out in the case of the fermion self-energy at one loop. The reason for this failure of the thermal operator representation is traced to the presence of a quadratically divergent thermal infrared singularity in the self-energy for a nonzero chemical potential (finite density). In this case, we find that the chemical potential undergoes a finite renormalization due to radiative corrections. The renormalized chemical potential is determined from an analysis of the pole of the fermion propagator at high temperature and shows that the radiative corrections lead to a screening of the chemical potential. This is argued to be consistent with the observation that a chemical potential can be thought of as a constant electrostatic potential and screening is a phenomenon associated with Abelian gauge fields.

In conclusion, we would like to point out that the lack of a complete factorization in the presence of a chemical potential may be related to our choice of generalizing the basic thermal operator in terms of the simple reflection operator $S(E)$. Finding an alternate basic thermal operator possibly dependent on other nontrivial operators and determining its consequences on factorization is an interesting issue which is presently under study.

ACKNOWLEDGMENTS

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APPENDIX A: 0 + 1 DIMENSIONAL CHERN-SIMONS QED

As we have shown in Ref. [1] as well as in this paper, a thermal operator representation for any finite temperature graph holds in any theory in the absence of chemical potential. The reflection operator $S(E)$ in the thermal operator simply changes $E \rightarrow -E$ and in this way incorporates the negative energy contributions into the graph.

Furthermore, the thermal operator is independent of the time coordinate which plays a significant role in the general proof of the thermal operator representation. All of this is true in higher dimensional field theories. However, in a $0 + 1$ dimensional field theory (quantum mechanics), the situation is different because the energy is positive and the question arises as to whether a thermal operator representation holds for such a theory as well. Furthermore, as is well known, in Chern-Simons QED in $0 + 1$ dimensions amplitudes beyond the one point function vanish at zero temperature while all the higher point functions are nonzero at finite temperature [7]. Therefore, it is interesting to analyze how the nonzero finite temperature amplitudes in such a theory can arise from a thermal operator acting on trivial amplitudes.

Let us recall that the Lagrangian for the $0 + 1$ dimensional Chern-Simons QED in the Euclidean space is given by

$$\mathcal{L} = \bar{\psi}(\partial_\tau - iA + m)\psi - i\kappa A, \tag{A1}$$

where κ represents the Chern-Simons coefficient and we have set the coupling for the gauge field to unity for simplicity. In this case, the fermion propagator (in the imaginary time formalism) has the form

$$S^{(T,m)}(\tau) = e^{-m\tau}(\theta(\tau) - n_{\mathrm{F}}(m)) = e^{-m\tau}\mathcal{O}_{\mathrm{F}}^{(T,m)}(\tau)S^{(0,0)}(\tau), \tag{A2}$$

where we have identified the basic thermal operator of the theory with

$$\mathcal{O}_{\mathrm{F}}^{(T,m)}(\tau) = 1 - n_{\mathrm{F}}(m)(1 + S(\tau)). \tag{A3}$$

There are two things to note here. First, in the $0 + 1$ dimensional case, the mass term corresponds to a chemical potential and second, the basic thermal operator contains a reflection operator that reflects the time coordinate and, therefore, is manifestly time dependent. This is quite different from the higher dimensional cases we have studied where the basic thermal operator is independent of the time coordinate. In the $0 + 1$ dimensional theory, on the other hand, we do not have higher loop diagrams (the photon is nondynamical) and, consequently, the time dependence of the basic thermal operator does not pose a problem in deriving a thermal operator representation for any graph.

For any graph of the form in Fig. 4, we can immediately write down the thermal operator representation as (using the identification $\tau_{N+1} = \tau_1$ and the fact that the exponential term around a closed loop vanishes)

$$\begin{aligned}\Gamma_N^{(T,m)} &= -\frac{(i)^N}{N!}\prod_{i=1}^{N} S^{(T,m)}(\tau_i - \tau_{i+1})\\ &= \prod_{i=1}^{N}\mathcal{O}_{\mathrm{F}}^{(T,m)}(\tau_i - \tau_{i+1})\bigg(-\frac{(i)^N}{N!}\prod_{i=1}^{N} S^{(0,0)}(\tau_i - \tau_{i+1})\bigg)\\ &= \mathcal{O}_{\mathrm{F}}^{(T,m)}\Gamma_N^{(0,0)},\end{aligned} \tag{A4}$$

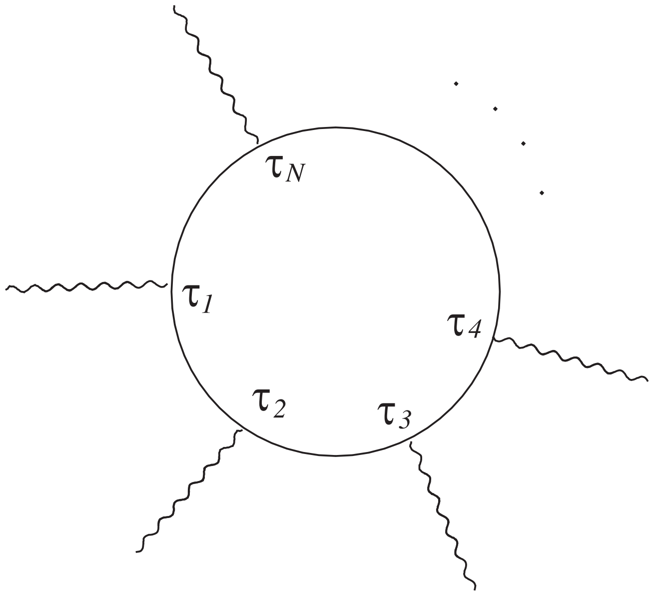

FIG. 4. A typical N-point graph in $0 + 1$ dimensional Chern-Simons QED.

where we have identified

$$\mathcal{O}_{\mathrm{F}}^{(T,m)} = \prod_{i=1}^{N}\mathcal{O}_{\mathrm{F}}^{(T,m)}(\tau_i - \tau_{i+1}). \tag{A5}$$

Indeed we see that formally there is a thermal operator representation for any graph in the $0 + 1$ dimensional Chern-Simons QED.

In practice, on the other hand, we know that in this theory

$$\Gamma_N^{(0,0)} = 0, \quad \text{for } N \geq 2. \tag{A6}$$

The way (A4) works in practice is as follows. Let us identify

$$\tau_{i,i+1} = \tau_i - \tau_{i+1}, \tag{A7}$$

and define the amplitude in (A4) in the limiting manner

$$\Gamma_N^{(T,m)} = \lim_{\tau_{N+1}\rightarrow\tau_1}\mathcal{O}_{\mathrm{F}}^{(T,m)}\Gamma_N^{(0,0)}. \tag{A8}$$

The limit $\tau_{N+1} \rightarrow \tau_1$ is assumed to be taken only at the end of the calculation (after the action of the thermal operator). With this, let us show explicitly how the correct finite temperature two point function arises from a trivial zero temperature amplitude.

We note that the two point amplitude at zero temperature is given by

$$\Gamma_2^{(0,0)} = -\frac{(i)^2}{2!}S^{(0,0)}(t_{1,2})S^{(0,0)}(t_{2,3}) = \frac{1}{2}\theta(\tau_{12})\theta(\tau_{2,3}). \tag{A9}$$

This, of course, vanishes if we identify $\tau_3 = \tau_1$ (or $\tau_{2,3} = \tau_{2,1}$). However, we are not supposed to take the limit until the thermal operator has acted on the zero temperature amplitude. Letting the thermal operator act on (A9), we obtain

$$\begin{aligned}\Gamma_2^{(T,m)} &= \lim_{\tau_3\to\tau_1} \mathcal{O}_{\rm F}^{(T,m)}(\tau_{12})\mathcal{O}_{\rm F}^{(T,m)}(\tau_{23})\tfrac{1}{2}\theta(\tau_{12})\theta(\tau_{23})\\ &= \lim_{\tau_3\to\tau_1} \tfrac{1}{2}(\theta(\tau_{12}) - n_{\rm F}(m))(\theta(\tau_{23}) - n_{\rm F}(m))\\ &= -\tfrac{1}{2}n_{\rm F}(m)(1 - n_{\rm F}(m)). \qquad \text{(A10)}\end{aligned}$$

This is indeed the correct finite temperature result (for a single fermion flavor) [7] and this shows how the thermal operator correctly reproduces the nonzero finite temperature amplitudes from trivial zero temperature ones if the operation is carried out in a limiting manner.

APPENDIX B: DERIVATION OF THE HIGH *T* LIMIT

In this Appendix, we evaluate some integrals which are used in the text. Let us first consider

$$\Omega_{\rm odd}^{(0)}(\mu, m, T) = \int_0^\infty dk(n_{\rm F}(E-\mu) - n_{\rm F}(E+\mu)), \qquad \text{(B1)}$$

which is essential for obtaining the high temperature limit in (71). Here $E = \sqrt{\vec{k}^2 + m^2}$ and in the high temperature limit $T \gg m$, it is possible to obtain a closed form expression for $\Omega_{\rm odd}^{(0)}$ for an arbitrary μ in the following way. Let us expand the integrand in a power series in μ and integrate term by term. Every term in this series is well behaved and leads to

$$\Omega_{\rm odd}^{(0)}\left(\mu, \frac{m}{T}\to 0\right) = 2\mu\sum_{\ell=0}^\infty \frac{\mu^{2\ell}}{(2\ell+1)!}\frac{\partial^{2\ell}n_{\rm F}(t)}{\partial t^{2\ell}}\bigg|_{t=0} + O\left(\frac{m}{T}\right). \qquad \text{(B2)}$$

Since the distribution function for the fermion can be expanded as [19]

$$n_{\rm F}(t) = \frac{1}{e^t+1} = \frac{1}{2}\sum_{n=0}^\infty E_n(0)\frac{t^n}{n!}, \qquad \text{(B3)}$$

where $E(x)$ represents the Euler polynomials, the expression (B2) can be evaluated in terms of the Euler functions. A further simplification results from the fact that

$$E_0(0) = 1, \qquad E_{2\ell}(0) = 0 \quad \text{for } \ell > 0. \qquad \text{(B4)}$$

Consequently, in this limit, (B2) has the form

$$\Omega_{\rm odd}^{(0)}\left(\mu, \frac{m}{T}\to 0\right) = \mu + O\left(\frac{m}{T}\right). \qquad \text{(B5)}$$

This result is true for any value of μ.

However, when $\frac{\mu}{T}$ is also small, one can determine the next order correction to $\Omega_{\rm odd}^{(0)}$ as follows. We recall an alternative expansion of the fermion distribution function as

$$\frac{1}{e^z+1} = \frac{1}{2} - 2\sum_{n=0}^\infty \frac{z}{(2n+1)^2\pi^2 + z^2}. \qquad \text{(B6)}$$

If we substitute this expansion into (B1) and regularize the integral by multiplying $p^{-\epsilon}$ with $\epsilon \to 0$ taken at the end [20,21], the integrand can be expanded in a power series in $\frac{m}{T}, \frac{\mu}{T}$. In this case, each term in the series can be integrated and the series can be summed to give Riemann's zeta function $\zeta(2n+1, \frac{1}{2})$ [19]. A straightforward calculation leads to

$$\Omega_{\rm odd}^{(0)}\left(\frac{\mu}{T} \ll 1, \frac{m}{T} \ll 1\right) = \mu + \mu\sum_{n=1}^\infty (-1)^n \zeta\left(2n+1, \frac{1}{2}\right) \times H_{n-1}(r^2)\left(\frac{m}{2\pi T}\right)^{2n}, \qquad \text{(B7)}$$

where $r^2 = \mu^2/m^2$ and $H_n(r^2)$ are polynomials of order n in r^2. For example, for the first few we have

$$H_0(r^2) = 1, \qquad H_1(r^2) = \tfrac{1}{2}(3 + 4r^2), \ldots. \qquad \text{(B8)}$$

In the limit, $\frac{m}{T} \to 0$, we recover (B5).

Using similar techniques, we can furthermore show that

$$\begin{aligned}\Omega_{\rm odd}^{(2)}(\mu, m, T) &= \int_0^\infty dk k^2(n_{\rm F}(E-\mu) - n_{\rm F}(E+\mu))\\ &= \frac{\mu}{3}(\pi^2T^2 + \mu^2) + O\left(\frac{m}{T}\right), \qquad \text{(B9)}\end{aligned}$$

$$\begin{aligned}\Omega_{\rm even}^{(1)}(\mu, m, T) &= \int_0^\infty dk k(n_{\rm F}(E-\mu) + n_{\rm F}(E+\mu))\\ &= \frac{1}{2}\left(\frac{\pi^2T^2}{3} + \mu^2\right) + O\left(\frac{m}{T}\right). \qquad \text{(B10)}\end{aligned}$$

One may extend the set of formulas (B5), (B9), and (B10) with the help of the basic integral:

$$I(p, u) = \int_0^\infty \frac{x^p}{e^{x-u}+1}dx = -\Gamma(p+1)Li_{p+1}(-e^u), \qquad \text{(B11)}$$

which can be obtained by expanding the integrand in powers of e^u and then integrating term by term. $Li_n(z)$ is the polylogarithm function, which is the analytic continuation to the whole complex z plane of the series (valid for $n \geq 1$ and $|z| < 1$)

$$Li_n(z) = \sum_{k=1}^\infty \frac{z^k}{k^n}.$$

Using a generalization of the method employed by Haber and Weldon in Appendix A of Ref. [21], one can find a power series expansion of the function $Li_{p+1}(-e^u)$, leading to

$$I(p,u) = \Gamma(p+1)\sum_{n=0}^{\infty}\frac{(1-2^{n-p})\zeta(p+1-n)}{n!}u^n, \tag{B12}$$

where $\zeta(z)$ is the Riemann zeta function, and where the singular numerator corresponding to $n = p$ must be interpreted as its limiting value, ln2. The formula is valid for all real values of $p > -1$. Notice that for integer p the series in (B12), starting from the power u^{p+2}, contains powers of the same parity only, because the zeta function vanishes at all negative even integers.

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