Self-forces on extended bodies in electrodynamics

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(Received 30 August 2005; published 6 March 2006)

In this paper, we study the bulk motion of a classical extended charge in flat spacetime. A formalism developed by W. G. Dixon is used to determine how the details of such a particle's internal structure influence its equations of motion. We place essentially no restrictions (other than boundedness) on the shape of the charge, and allow for inhomogeneity, internal currents, elasticity, and spin. Even if the angular momentum remains small, many such systems are found to be affected by large self-interaction effects beyond the standard Lorentz-Dirac force. These are particularly significant if the particle's charge density fails to be much greater than its 3-current density (or vice versa) in the center-of-mass frame. Additional terms also arise in the equations of motion if the dipole moment is too large, and when the ''center-of-electromagnetic mass'' is far from the ''center-of-bare mass'' (roughly speaking). These conditions are often quite restrictive. General equations of motion were also derived under the assumption that the particle can only interact with the radiative component of its self-field. These are much simpler than the equations derived using the full retarded self-field; as are the conditions required to recover the Lorentz-Dirac equation.

DOI: 10.1103/PhysRevD.73.065006 PACS numbers: 03.50.De, 04.25.−g

I. INTRODUCTION

Originally motivated by the discovery of the electron, the behavior of small (classical) electric charges has been studied in various contexts for over a century. The first results were obtained by Abraham [1] for a nonrelativistic nonspinning rigid sphere. This calculation was later repeated by Schott [2] and Lorentz [3] within special relativity, and by Crowley and Nodvik [4] in general (background) spacetimes. Similar results have since been obtained by a number of authors for more general charge distributions [5–11] in flat spacetime. These results have very recently been extended to curved backgrounds as well [12]. Still, these derivations did not allow for significant elasticity, charge-current coupling, and/or rotation.

Detailed reviews of different aspects of the self-force problem (in electromagnetism as well as scalar field theory and general relativity) have been given by Poisson [13], Havas [14], and Spohn [15]. To summarize, though, a common theme throughout all of these works has been that the equations of motion describing sufficiently small particles were found to be independent of the details of their internal structure to a considerable degree of precision. Only a few parameters such as the rest mass and total charge entered into these equations. Ignoring the effects of spin and internal currents, the apparently universal correction to the Lorentz force law is given by the well-known Lorentz-Dirac self-force. If a particle with charge q has a center-of-mass position $z^a(\bar{s})$ (with $\bar{s}$ being a proper time), then this self-force is equal to (using a metric with signature -2, and units in which $c = 1$)

$$\frac{2}{3}q^2(\dddot{z}^a(\bar{s}) + \dot{z}^a(\bar{s})\ddot{z}^b(\bar{s})\ddot{z}_b(\bar{s})). \tag{1.1}$$

Although this is only an approximation to the full self-force for any realistic (extended) charge, it is natural to introduce a new concept into the theory for which it is exact—that of a point particle. The immediate problem with this is of course that the self-field diverges at the location of any pointlike source, which would appear to imply that its self-force and self-energy are not well-defined. Dirac [16] removed this problem by noting that self-forces acting on a structureless particle should only arise as a reaction to emitted radiation. Letting $F^{ab}_{\text{self}(+)}$ denote the advanced self-field and $F^{ab}_{\text{self}(-)}$ the retarded one, we can define ''radiative'' and ''singular'' portions of $F^{ab}_{\text{self}(-)} = F^{ab}_{\text{self}(S)} + F^{ab}_{\text{self}(R)}$:

$$F^{ab}_{\text{self}(S)} = \frac{1}{2}(F^{ab}_{\text{self}(-)} + F^{ab}_{\text{self}(+)}), \tag{1.2}$$

$$F^{ab}_{\text{self}(R)} = \frac{1}{2}(F^{ab}_{\text{self}(-)} - F^{ab}_{\text{self}(+)}). \tag{1.3}$$

As the names imply, the singular field contains the entire divergent part of the retarded field. It is also derived from a time-symmetric Green function, so it would not be expected to contain any radiation. The self-force on a point particle should therefore be determined entirely by the radiative self-field. Combining it with the Lorentz force law immediately recovers (1.1) [16].

This prescription is much simpler than any direct derivation of the equations of motion for a finite charge, and for this reason, it has been generalized to work in curved spacetime, as well as for scalar and gravitational self-forces [13,17] (although this is not the only way of ''renormalizing'' point particle self-fields [18,19]). As before, the relevance of these extensions to realistic extended bodies has also been established in certain special cases. In linearized gravity, for example, a small nearly-

Schwarzchild black hole has been found to obey the same equations of motion as a point particle [13,20]. More generally, it has been shown that a nonspinning body's internal structure is irrelevant to very high order within Post-Newtonian theory [21]. Still, there remain questions of exactly how universal these results are. Will a spherical neutron star in near-equilibrium fall into a supermassive black hole in the same way as another one that is spinning rapidly and experiencing internal oscillations [22]; or one having a mountainous (solid) surface [23]? Such systems could be important sources for the upcoming Laser Interferometer Space Antenna (LISA).

Rather than addressing such questions directly, we have chosen to study the motions of charged bodies in flat spacetime (in part) as a model problem. The methods used here were specifically chosen so that very few conceptual changes would be required to consider particles (charged or not) moving in a fully dynamic spacetime. This has led to some additional complexity not strictly necessary to solve the problem at hand, although the majority of the complicating issues have been placed in the appendix.

Our derivation is based on W. G. Dixon's multipole formalism [24–28]. This gives a relatively simple, unified, and rigorous way of understanding the motions of arbitrarily-structured bodies in both electrodynamics and general relativity. Despite the fact that Dixon's theory decomposes the source functions into multipole moments, we never ignore any of them. By using generating functions, the entire infinite set is retained throughout all of our calculations. This only appears to be possible in this formalism. The laws of motion that are used are therefore exact (despite being ordinary differential equations). All of the approximations that we make are only used to compute the self-field.

Taking a hybrid point of view where an extended body can only interact with its radiative self-field, it is found that the Lorentz-Dirac equation follows for a wide range of nonspinning charges with small dipole moments. If the body instead interacts with its full retarded self-field, this result no longer holds. In this case, the equations of motion are drastically different if the charge and (3-) current densities have remotely similar magnitudes in the center-of-mass frame. Although it is impractical to define exactly what this means at this point, it will be shown that one of these quantities must be at least "second-order" compared to the other in order for these extra terms to vanish. This is because these cases usually allow the self-fields to do significant amounts of internal work (e.g. Ohmic heating). Even when the charge-current coupling in negligible, there are still extra complications in the retarded case. The conditions required to generically exclude these and other complicating effects are derived, and turn out to be surprisingly restrictive.

Sec. II reviews the various steps involved in calculating the motion of matter interacting with an electromagnetic field. Sec. III then summarizes the appropriate definitions of the center-of-mass and its laws of motion as obtained from Dixon's formalism. It also decomposes the 4-current in a particular way that happens to be convenient in this framework. Although this reduction is not strictly required for the current problem, it is adopted throughout on the grounds that it would be essential in curved spacetime. It is derived in detail in the appendix, which also contains an in-depth review of of Dixon's ideas.

With these basic ideas in place, Sec. IV goes on to derive expansions for the advanced and retarded self-fields of an arbitrarily-structured charge using a slow-motion approximation. Sec. V then combines these results with those from Sec. III to find general expressions for the self-force and self-torque. Finally, Sec. VI examines the equations of motion for certain simpler classes of charges, and derives some conditions under which the Lorentz-Dirac equation is applicable.

We use units in which $c = 1$ throughout. In order to facilitate a more direct comparison to Dixon's papers, the metric is chosen to have signature -2 (although the rest of the notation used here frequently differs from Dixon's). We also assume that the spacetime is flat, and adopt Minkowski coordinates for simplicity. Latin indices refer to these coordinates, while Greek ones are triad labels running from 1 to 3.

II. THE PROBLEM OF MOTION

In studying the dynamics of any system in a classical field theory, one has to specify "laws of motion" for both the field and matter variables. In our case, the only field is of course the electromagnetic one, $F^{ab} = F^{[ab]}$. As usual, this is governed by Maxwell's equations

$$\partial_b F^{ab} = J^a, \tag{2.1}$$

$$\partial^{[a} F^{bc]} = 0. \tag{2.2}$$

We assume that the matter in our problem is completely described by its stress-energy tensor T^{ab} and 4-current vector J^a. Taking the divergence of (2.1) immediately gives our first constraint on these quantities:

$$\partial_a J^a = 0. \tag{2.3}$$

This equation acts as one of the laws of motion for the matter fields. The other is derived from the requirement that $\partial_a(T^{ab} + T^{ab}_{\mathrm{em}}) = 0$, where T^{ab}_{em} is the stress-energy tensor of the electromagnetic field. Combining the standard form of T^{ab}_{em} [8] with Maxwell's equations then shows that the matter moves according to

$$\partial_b T^{ab} = -F^{ab} J_b. \tag{2.4}$$

(2.1), (2.2), (2.3), and (2.4) are essentially the entire content of classical continuum mechanics in flat spacetime. Of course, different types of matter do not all move in the same way, so these equations by themselves are not sufficient to determine T^{ab} and J^a for all time (even if the F^{ab} were given). One also needs to specify something analogous to a (generalized) equation of state, which can take on a rather unwieldy form.

This sort of procedure is the standard one in continuum mechanics. Unfortunately, the resulting nonlinear partial differential equations are notoriously difficult to solve. Such a detailed description of the system should not really be required, however, for problems where we are only interested in the body's bulk motion. In these cases, a representative world line could be defined inside the (convex hull of the) spacelike-compact support of T^{ab}. Given that this world line can always be parametrized by a single quantity, its tangent vector might be expected to satisfy an ordinary differential equation—at least when using certain approximations. Solving such an equation would clearly be much more straightforward than the partial differential equations that we started with.

This sort of simplification is one of the main motivations behind the many (source) multipole formalisms in the literature [8,29]. In these, one first fixes some particular reference frame which has, among other properties, a preferred time parameter s. The quantity being expanded—say $J^a(x)$—can then be written in terms of an infinite set of tensors depending only on s: $Q^a(s)$, $Q^{ab}(s), \ldots$ The reverse is also true. Given J^a, there is a well-defined way to compute any moment. The set $\{Q^{\cdots}\}$ is therefore completely equivalent to J^a. This implies that the conservation equation (2.3) may be used to find restrictions on the individual moments.

Such restrictions depend on the precise definitions that are being adopted, but can usually be divided into two general classes. The first of these consists of purely algebraic equations imposed at a fixed value of s. We call these the constraints. There are also a number of evolution equations which usually take the form of ordinary differential equations. Multipole expansions can therefore be used to convert (2.3) and (2.4) into a number of algebraic and ordinary differential equations (without any approximation).

This does not actually simplify things as much as it at first might appear. The reason is that almost all definitions for the source multipoles will lead to an infinite number of (coupled!) evolution equations. This is often dealt with in practice by assuming that all moments above a particular order are irrelevant, which leaves one with only a finite number of evolution equations. There are, however, interesting questions that require knowing the higher moments. J^a (or T^{ab}), for example, cannot be reconstructed without them. This means that the self-field cannot be calculated in the near zone from only the first few moments. Although the self-force and self-torque can be found in certain cases by examining energy and momentum fluxes in the far zone [13,16,20,30] (which can be adequately approximated using only a finite number of moments), this is considerably less accurate than integrating the force density throughout the charge's interior.

For these reasons and others, it is desirable to define a set of multipoles that do not require any cutoff. Remarkably, such a set exists [24], and is essentially unique [27]. Without any approximation, moments for both J^a and T^{ab} can be defined which satisfy a *finite* number of evolution equations. There remain (uncoupled) constraint equations for each moment, although these are easily solved. We adopt this formalism due to Dixon for the remainder of this paper. Its net effect is to allow us to relate the motion to the fields in a more rigorous way than is usually done (short of directly solving (2.3) and (2.4)). It does not, however, have anything to say about the fields themselves. We therefore obtain the self-field in a standard way, and then use Dixon's equations to find how the matter moves in response to it.

III. LAWS OF MOTION

As noted, we use Dixon's method [24–28] to decompose J^a and T^{ab} into multipole moments. Each of these sets is designed to describe as simply as possible all possible forms of J^a and T^{ab} satisfying their respective conservation equations. We first assume that these matter fields are at least piecewise continuous, and have (identical) supports W. Any intersection of a spacelike hypersurface with this worldtube is assumed to be compact.

Now choose a timelike world line $Z \subset W$, and a timelike unit vector field $n^a(s)$ defined on Z. It is assumed that this is always possible in any physically interesting system. Z is then parameterized by the coordinate function $z^a(s)$, and the tangent vector to it is denoted by $v^a(s) := \mathrm{d}z^a/\mathrm{d}s =: \dot{z}^a(s)$. v^a need not be equal to n^a, although it will be convenient to normalize s such that $n^a v_a = 1$ (so $v^a v_a \neq 1$ in general). v^a is called the kinematical velocity, while n^a is the dynamical velocity. The set $\{Z, n^a\}$ then defines a reference frame for the definition of the multipole moments. At this point, it should be thought of as arbitrary, although physical conditions will later be given that pick out a unique 'center-of-mass' frame.

A collection of spacelike hyperplanes $\{\Sigma\}$ can easily be constructed from Z and n^a. Each $\Sigma(s)$ is to pass through $z^a(s)$, and be (everywhere) orthogonal to $n^a(s)$. Assume that any point in W is contained in exactly one of these planes. Unless n^a is a constant, it is clear that this property cannot be true throughout the entire spacetime. For $x \in \Sigma(s) \cap W$, we must therefore have that $\max|\dot{n}^a(s) \times (x - z(s))_a| < 1$ (among other conditions), which gives a weak restriction on the body's maximum size. It is not really important, though, as any reasonable type of matter would be ripped apart long before this condition was violated.

The main results that we need from Dixon's theory at this point are his definitions of the linear and angular momenta. These disagree with the usual ones when either

J^a or F^{ab} are nonzero, although it is still convenient to label them by $p^a(s)$ and $S^{ab}(s)$ respectively. Unless otherwise noted, the words ''linear and angular momenta'' will always refer to the quantities [24,25]

$$p^a(s) = \int_{\Sigma(s)} \mathrm{d}\Sigma_b \left[T^{ab} + J^b r_c \int_0^1 \mathrm{d}u\, F^{ac}(z(s) + ur) \right], \tag{3.1}$$

$$S^{ab}(s) = 2 \int_{\Sigma(s)} \mathrm{d}\Sigma_c \left[r^{[a} T^{b]c} + J^c r_d \int_0^1 \mathrm{d}u\, u r^{[a} F^{b]d}(z(s) + ur) \right], \tag{3.2}$$

where $r^a := x^a - z^a(s)$, and u is just a dummy parameter used to integrate along the line segment connecting $z^a(s)$ to $x^a = z^a(s) + r^a$.

Detailed motivations for these definitions can be found in [24,25,27], as well as the appendix. In short, though, it can be shown that the given quantities are uniquely determined by demanding that stress-energy conservation directly affect only the first two moments of T^{ab} (once the concept of a moment has been defined in a reasonable way). If p^a and S^{ab} are known in some time interval and satisfy the appropriate evolution equations, the class of all stress-energy tensors with these moments can be constructed without having to solve any differential equations. Each of these will exactly satisfy (2.4).

This property implies that evolution equations for the quadrupole and higher moments of the stress-energy tensor are nearly unconstrained. They can be thought of as the ''equation of state'' of the material under consideration. This type of independence of the higher moments from the conservation laws also occurs in Newtonian theory [28], and there are considerable advantages in preserving as much of that structure as possible in the relativistic regime. In particular, there is no need to discard multipole moments above a certain order. The choices (3.1) and (3.2) allow us to retain many of the conveniences of a multipole formalism without its classic limitations.

The same definitions for the momenta can also be motivated by considering charged particles in curved spacetime [25,28]. There, one can study the conserved quantities associated with Killing vectors in appropriate spacetimes (where both the metric and electromagnetic field are assumed to share the same symmetries). Fixing Z and $n^a(s)$ allows each such quantity to be written as a linear combination of vector and antisymmetric rank 2 tensor fields on Z. Crucially, the definitions of these quantities do not depend on the Killing vector under consideration, so we can suppose that they are meaningful even in the absence of any symmetries. If the metric is taken to be flat, these objects reduce to (3.1) and (3.2) [25]. They are therefore the natural limits of what would generally be referred to as the linear and angular momenta of symmetric spacetimes.

It is now convenient to define a coordinate system on W that is more closely adapted to the system than the Minkowski coordinates used so far. First choose an orthonormal tetrad $\{n^a(s), e^a_\alpha(s)\}$ $(\alpha = 1, 2, 3)$ along Z. Requiring that each of these vectors remain orthonormal to the others implies that

$$\dot{e}^{\,a}_{\alpha} = -n^a \dot{n}_b e^b_\alpha, \tag{3.3}$$

which is essentially Fermi-Walker transport. A spatial (rotation) term may also be added to this, although our calculations would then become considerably more tedious. On its own, the choice of tetrad has no physical significance, so we choose the simplest case.

Since it was assumed that $\{\Sigma\}$ foliates W, any point $x \in \Sigma(s) \cap W$ may now be uniquely written in terms of s and a ''triad radius'' r^α

$$x^a = e^a_\alpha(s) r^\alpha + z^a(s). \tag{3.4}$$

Varying r^α with a fixed value of s clearly generates $\Sigma(s)$. Also, the Jacobian of the coordinate transformation $(x^a) \rightarrow (r^\alpha, s)$ is equal to the lapse N of the foliation,

$$\begin{aligned} N(x) &= 1 - \dot{n}_a(s) r^a, \\ &= 1 - \dot{n}_a(s) e^a_\alpha(s) r^\alpha. \end{aligned} \tag{3.5}$$

We will be extensively transforming back and forth between these two coordinate systems, so it is convenient to abuse the notation somewhat by writing $f(r^\alpha, s) = f(x^a)$ for any function f. The intended dependencies should always be clear from the context. It is also useful to denote quantities such as $\dot{n}_a e^a_\alpha$ by $\dot{n}_\alpha$. Note that in this notation, $\ddot{n}_\alpha = \ddot{n}_a e^a_\alpha \neq \mathrm{d}\dot{n}_\alpha / \mathrm{d}s$.

Using these conventions, it is natural to split J^a into the charge and 3-current densities seen by an observer at $r = 0$ (the ''center-of-mass observer'')

$$J^a = \rho n^a + e^a_\alpha j^\alpha. \tag{3.6}$$

It is then shown in the appendix that there exist ''potentials'' φ and H^α which generate J^a through the equations

$$\rho = \partial_\alpha (r^\alpha \varphi), \tag{3.7}$$

$$j^\alpha = N^{-1} [H^\alpha + v^\beta \partial_\beta (r^\alpha \varphi) - r^\alpha \dot{\varphi}]. \tag{3.8}$$

Denoting the total charge by q and letting $|r|^2 := -r^\alpha r_\alpha \geq 0$, $\varphi(r^\alpha, s)$ was found to be continuous, and equal to $q/4\pi |r|^3$ outside W. Similarly, $H^\alpha(r^\beta, s)$ is given by (A40) outside W, and is piecewise continuous in r^β. H^α also satisfies $\partial_\alpha H^\alpha = 0$. These properties guarantee that J^a has support W, and is everywhere continuous. A direct calculation also shows that

$$\partial_a J^a = N^{-1} n_b \left(\frac{\partial}{\partial s} - v^\beta \partial_\beta \right) J^b + \partial_\beta j^\beta, \tag{3.9}$$

$$= 0, \tag{3.10}$$

as required by (2.3).

Any physically reasonable current vector can now be constructed by choosing potentials satisfying these rules. A physical interpretation of one's choice is then given by substitution into (3.7) and (3.8). For example, a uniform spherical charge distribution with time-varying radius $D(s)$ is described by (assuming $n^a = v^a$)

$$\varphi(r, s) = \frac{q}{4\pi D^3(s)}\left[\Theta(D(s) - |r|) + \left(\frac{D(s)}{|r|}\right)^3 \Theta(|r| - D(s))\right], \tag{3.11}$$

$$H^\alpha(r, s) = 0, \tag{3.12}$$

where $\Theta(\cdot)$ is the Heaviside step function. (3.7) and (3.8) then show that the tetrad components of J^a are

$$\rho(r, s) = \frac{3q}{4\pi D^3}\Theta(D - |r|), \tag{3.13}$$

$$j^\alpha(r, s) = \frac{3qr^\alpha}{4\pi N(r, s) D^3(s)}\left(\frac{\dot{D}(s)}{D(s)}\right)\Theta(D(s) - |r|). \tag{3.14}$$

It is also clearly possible to calculate φ and H^α from any given J^a. This requires inverting (3.7), which acts as a partial differential equation for φ on each time slice. The solution to this equation would usually have to be obtained numerically, which is clearly inconvenient. Largely for this reason, we shall consider φ and H^α to be the given quantities for the remainder of this paper. J^a can be derived from them using operations no more complicated than differentiation.

Another reason for this unconventional choice is that φ and H^α contain all of the multipole moments of J^a in a natural way. As shown in the appendix, they are closely related to the Fourier transform of a generating function for these moments. An arbitrary current moment can be obtained essentially by differentiating the inverse Fourier transforms of the potentials a suitable number of times. For example, (A1), (A24), (A30), (A31), and (A38) can be used to show that the dipole moment has the general form

$$Q^{ab} = \int d^3r\left[2n^{[a}e^{b]}_\beta r^\beta N\left(\varphi(r, s) - \frac{q}{4\pi|r|^3}\right) + e^a_\alpha e^b_\beta \bar{H}^{\alpha\beta}(r, s)\right], \tag{3.15}$$

where $\bar{H}^{\alpha\beta} = \bar{H}^{[\alpha\beta]}$ is defined by

$$H^\alpha = \partial_\beta \bar{H}^{\alpha\beta}. \tag{3.16}$$

Given (A40), we let

$$\bar{H}^{\alpha\beta} = \frac{qv^{[\alpha}r^{\beta]}}{2\pi|r|^3} \tag{3.17}$$

outside W.

For the example given in (3.11) and (3.12), the dipole moment is equal to

$$Q^{ab} = \tfrac{1}{5}qD^2(s)n^{[a}\dot{n}^{b]}. \tag{3.18}$$

This might have been expected to vanish in spherical symmetry, although it should be noted that it is only nonzero when sphericity is being defined in an accelerated reference frame.

These constructions can now be used to simplify the definitions of p^a and S^{ab}. Letting $\bar{r}^\alpha := ur^\alpha$, the electromagnetic term in (3.1) is equivalent to

$$\int d^3\bar{r}\ e^\beta_b \bar{r}_\beta F^{ab}(\bar{r}, s)\int_0^1 du\ u^{-4}\rho(\bar{r}/u, s). \tag{3.19}$$

But (3.7) shows that

$$\int_0^1 du u^{-4}\rho(\bar{r}/u, s) = -\int_0^1 du\ \frac{\partial}{\partial u}\left[u^{-3}\left(\varphi(\bar{r}/u, s) - \frac{q}{4\pi|\bar{r}/u|^3}\right)\right], \tag{3.20}$$

$$= -\left(\varphi(\bar{r}, s) - \frac{q}{4\pi|\bar{r}|^3}\right), \tag{3.21}$$

so (3.1) and (3.2) can be rewritten as

$$p^a = \int d^3r\left[T^{ab}n_b - \left(\varphi - \frac{q}{4\pi|r|^3}\right)r_\gamma e^\gamma_c F^{ac}\right], \tag{3.22}$$

$$S^{ab} = 2\int d^3r\left[r^\alpha e^{[a}_\alpha T^{b]c}n_c - \left(\varphi - \frac{q}{4\pi|r|^3}\right)r_\gamma e^\gamma_c r^\alpha e^{[a}_\alpha F^{b]c}\right]. \tag{3.23}$$

We now need evolution equations for these quantities, which are most easily obtained by direct differentiation. For any function $I^b(x)$, relating the x-coordinates of $(r^\alpha, s + ds)$ to those of (r^α, s) shows that

$$\frac{d}{ds}\int_{\Sigma(s)} d\Sigma_b I^b(x) = \int d^3r[\dot{n}_b I^b + (v^c - n^c\dot{n}_\alpha r^\alpha)\partial_c(n_b I^b)]. \tag{3.24}$$

If I^b vanishes outside of some finite radius, this expression simplifies to

$$\frac{d}{ds}\int_{\Sigma(s)} d\Sigma_b I^b(x) = \int d^3r\ N\partial_b I^b. \tag{3.25}$$

(2.4), (3.22), and (3.23) can now be used to show that

$$\dot{p}^a = -\int d^3r\left\{NF^{ab}J_b + \frac{\partial}{\partial s}\left[\left(\varphi - \frac{q}{4\pi|r|^3}\right)r_\gamma e^\gamma_c F^{ac}\right]\right\}, \tag{3.26}$$

and

$$\dot{S}^{ab} = -2\int \mathrm{d}^3r\bigg\{Nr^{\alpha}e^{[a}_{\alpha}F^{b]c}J_c + v^{[a}T^{b]c}n_c + \frac{\partial}{\partial s}\bigg[\bigg(\varphi - \frac{q}{4\pi|r|^3}\bigg)r_{\gamma}e^{\gamma}_{c}r^{\alpha}e^{[a}_{\alpha}F^{b]c}\bigg]\bigg\},$$
$$= 2p^{[a}v^{b]} - 2\int \mathrm{d}^3r\bigg\{Nr^{\alpha}e^{[a}_{\alpha}F^{b]c}J_c + \frac{\partial}{\partial s}\bigg[\bigg(\varphi - \frac{q}{4\pi|r|^3}\bigg)r_{\gamma}e^{\gamma}_{c}r^{\alpha}e^{[a}_{\alpha}F^{b]c}\bigg] + \bigg(\varphi - \frac{q}{4\pi|r|^3}\bigg)r_{\gamma}e^{\gamma}_{c}v^{[a}F^{b]c}\bigg\}. \tag{3.27}$$

If the momenta had been defined in the usual way, the $(\varphi - q/4\pi|r|^3)$ terms would be absent from these expressions. The extra complication in the present case derives from the electromagnetic couplings in (3.1) and (3.2).

Unsurprisingly, (3.26) and (3.27) can be directly related to the monopole and dipole moments of the force density $F^{ab}J_b$. Denoting such moments by $\Psi^a(s)$ and $\Psi^{ab}(s)$ respectively, it is possible to prove (A56) and (A57). It is then natural to refer to $-\Psi^a$ as the net force, and $-2\Psi^{[ab]}$ as the net torque acting on the body. Viewing these quantities as force moments allows one to derive (A61) and (A62). But these results are no different than (3.26) and (3.27). If desired, the reader may therefore view Ψ^a and $\Psi^{[ab]}$ to be *defined* by (A56) and (A57).

In cases where F^{ab} varies slowly in the center-of-mass frame (both spatially and temporally), the expressions for the force and torque can be expanded in Taylor series involving successively higher multipole moments of the current density (denoted by $Q^{\cdots}$). Deriving such equations would be awkward using the methods introduced in this section, so we simply refer to (A63) and (A64). Despite the peculiar definitions of the current moments being used, these expansions are exactly what one would expect out of a multipole formalism, and can therefore be considered a check that Dixon's definitions are reasonable.

In most cases where (A63) and (A64) are any simpler than the exact expressions for the force and torque, only the monopole and dipole moments will be significant. These moments can be computed from (A13) and (3.15), respectively. In the limit that the particle is vastly smaller than any of the field's length scales, only the monopole moment will enter the equations of motion. In this case, the force reduces to the standard Lorentz expression and the torque vanishes, as expected.

The final ingredients required to complete this formalism are unique prescriptions for Z and n^a. Simply knowing the linear and angular momenta at any point in time does not necessarily determine the body's location in any useful way. The problem is compounded by the fact that these quantities are strongly dependent on the choice of reference frame itself. Indeed, without knowing where W is, it is essentially impossible to specify J^a in any meaningful way.

These problems can be removed by choosing Z and n^a appropriately, and then assuming that the resulting world line provides a reasonable representation of the body's "average" position. Following [25,27,28,31], we first assume that for any point $z \in W$, there exists a unique future-directed timelike unit vector $n^a(z)$ such that

$$p^a(z;n) = M(z;n)n^a(z), \tag{3.28}$$

for some positive scalar M. Here, we have temporarily changed the dependencies of p^a and n^a for clarity. It is seen from (3.1) that p^a depends nontrivially on both the base point z, and on n^a, which defines the surface of integration. (3.28) is therefore a highly implicit definition of n^a.

In any case, another condition must be also be given to fix z. We want this to lie on a 'center-of-mass line' in some sense, so it would be reasonable to expect the "mass dipole moment" defined with respect to it to vanish:

$$n_b n_c t^{abc} = 0, \tag{3.29}$$

where t^{abc} is the full dipole moment of the stress-energy tensor. This is defined by (A47), so (3.29) is equivalent to

$$n_a(z)S^{ab}(z;n) = 0. \tag{3.30}$$

The integral form of (3.30) reduces to the standard center-of-mass condition when $F^{ab} = 0$ and $n^a = v^a$. It also allows us to write the angular momentum in terms of a single 3-vector S^a

$$S^{ab} = \epsilon^{abcd}n_c S_d. \tag{3.31}$$

Of course, both of these properties would also have been satisfied by instead requiring $v_a S^{ab} = 0$. We reject this choice due to the fact that it leads to nonzero accelerations even when $F^{ab} = 0$ [32]. Replacing v^a by n^a avoids this peculiar behavior, which we consider to be an important requirement for anything deserving to be called a center-of-mass line. It also seems more natural to define the mass dipole moment in terms of n^a rather than v^a.

For the remainder of this paper, we assume that (3.28) and (3.30) are always satisfied, and call the resulting Z and $n^a(s)$ the center-of-mass frame [25,28,31]. It is not obvious that solutions to these highly implicit equations exist, although existence and uniqueness have been proven in the gravitational case [33]. There, it is also true that (given certain reasonable conditions) Z is necessarily a timelike world line inside the convex hull of W. We assume that the same results extend to electrodynamics.

The uniqueness of these definitions is actually not very critical for our purposes. The important point is that *a* solution with the given properties can presumably be chosen for a sufficiently large class of systems. While it is clearly very difficult to find the center-of-mass directly from T^{ab}, J^a, and F^{ab}, it is relatively straightforward to simply construct sets of moments which automatically incorporate (3.28) and (3.30). (3.7) and (3.8) show, for example, that these definitions do not have any effect on our ability to construct arbitrary current vectors adapted to them. There is nothing preventing the moment potentials φ and H^α from being appropriately centered around $r^\alpha = 0$. Although it is not obvious that this can also be done for the stress-energy tensor, we conjecture that it can.

Now that (3.28) and (3.30) have been assumed to hold, we need evolution equations for M, n^a, and z^a. These are easily found from (A56) and (A57):

$$\dot{M} = -n^a \Psi_a, \tag{3.32}$$

$$M\dot{n}^a = -h^a_b \Psi^b, \tag{3.33}$$

$$M(v^a - n^a) = S^{ab}\dot{n}_b - 2\Psi^{[ab]}n_b, \tag{3.34}$$

where h^a_b is the projection operator $\delta^a_b - n^a n_b$. Note that the last of these equations shows that $n^a = v^a$ if the spin and torque both vanish.

In general, (A57), (3.32), (3.33), and (3.34) may be used together to find the motion of the body's center-of-mass in terms of Ψ^a and $\Psi^{[ab]}$. Once the field is known, these quantities follow from (A61) and (A62). Some recipe for evolving the dipole and higher current moments in time – most conveniently expressed in terms of $\dot{\varphi}$ and $\dot{H}^\alpha$—is also required. Combining all of these elements together leaves us with a well-defined initial value problem that will determine z^a, n^a, M, S^{ab}, and J^a. If the stress-energy tensor is also desired, possible forms of it could in principle be constructed from (A51) and (A54) in the same way that J^a was derived from φ and H^α. Combining all of these steps would completely characterize the system, although we shall omit the last one. The result is still sufficient to answer most questions that one would be interested in asking of a nearly isolated particle.

IV. THE FIELD

It is clear from the previous section that the equations of motion will easily follow once Ψ^a and $\Psi^{[ab]}$ are known. These depend on the field, which we now calculate. For a reasonably isolated body, it is first convenient to split F^{ab} into two parts

$$F^{ab} = F^{ab}_{\rm ext} + F^{ab}_{\rm self}. \tag{4.1}$$

The external field is assumed to be generated by outside sources, while the self-field is entirely due to the charge itself. Since (A61) and (A62) are linear in F^{ab}, the force and torque can also be split up into "self" and "external" portions.

In Lorenz gauge, the vector potential sourced by the particle is given by [8]

$$A^a_{\rm self}(x) = -\int {\rm d}^4x' J^a(x')\delta(\sigma(x,x')), \tag{4.2}$$

where

$$\sigma(x,x') = \tfrac{1}{2}(x-x')^a(x-x')_a, \tag{4.3}$$

is Synge's world function [13,34]. In realistic systems, this potential will couple to the external one via the outside matter fields. These may act as reflectors or dielectrics, or there may be an n-body interaction where the self-fields influence the motion of some external charged particles (obviously affecting the fields in W). Although it would be reasonable to group together all portions of F^{ab} causally related to our particle in some way as the "self-field," this would be impossible to do with any generality. Instead, we simply define the $F^{ab}_{\rm self}$ to be the field derived from (4.2) in the usual way (this differs from the point of view taken in e.g. [35]). The interactions of the self-field with the outside world will all be categorized as "external" effects that we presume can be accounted for by separate methods.

If all of the external matter is sufficiently far away from W (and slowly varying), $F^{ab}_{\rm ext}$ will be approximately constant within each $\Sigma(s)\cap W$ slice (and from one slice to the "next"). $\Psi^a_{\rm ext}$ and $\Psi^{[ab]}_{\rm ext}$ can therefore be approximated by (A63) and (A64) in these cases. Finding the self-force and self-torque is more complicated. For this, $F^{ab}_{\rm self}$ has to be combined with the exact integral expressions for the force and torque—(A61) and (A62). The detailed structure of the self-field must therefore be known throughout W.

This easily follows from $A^a_{\rm self}$:

$$\begin{aligned} F^{ab}_{\rm self}(x) &= 2\partial^{[a}A^{b]}_{\rm self}, \\ &= -2\int {\rm d}^4x'\delta'(\sigma)(x-x')^{[a}J^{b]}. \end{aligned} \tag{4.4}$$

Writing x in terms of (r^α, s), and x' in terms of (r'^α, τ), and defining $\dot{\sigma}(x,x') := \partial\sigma/\partial\tau$, $F^{ab}_{\rm self}$ becomes

$$F^{ab}_{{\rm self}(\pm)}(x) = 2\int {\rm d}^3r'\left\{\frac{1}{|\dot{\sigma}|}\frac{\rm d}{{\rm d}\tau}\left[\frac{N}{\dot{\sigma}}(x-x')^{[a}J^{b]}\right]\right\}_{\tau=\tau_\pm}. \tag{4.5}$$

τ_+ $(>s)$ represents the advanced time, and τ_- the retarded one. These are found by solving $\sigma(x,x') = 0$ with x and r' held fixed.

Although we only consider the retarded field to be real, the advanced solution is also retained for now. This allows us to later construct the radiative self-field, which is considerably simpler than the full retarded field. It would be quite convenient if the self-forces generated by the two fields were identical (as Dirac assumed for point particles [16]), although we will show that this is not true in general.

Returning to the explicit form for $F^{ab}_{\rm self}$, splitting up J^a according to (3.6) shows that

$$F^{ab}_{\rm self}(x) = 2\int d^3r' \frac{N}{\dot\sigma|\dot\sigma|}\bigg\{\rho\bigg[(x-x')^{[a}\dot n^{b]} - \bigg(\frac{\ddot\sigma}{\dot\sigma} + N^{-1}\ddot n^\alpha r'_\alpha\bigg)(x-x')^{[a}n^{b]} - v^{[a}n^{b]}\bigg] + \dot\rho(x-x')^{[a}n^{b]} - j^\alpha\bigg[(x-x')^{[a}e^{b]}_\alpha\bigg(\frac{\ddot\sigma}{\dot\sigma} + N^{-1}\ddot n^\beta r'_\beta\bigg) + \dot n_\alpha(x-x')^{[a}n^{b]} + (Nn^{[a} + (h\cdot v)^{[a})e^{b]}_\alpha\bigg] + \dot j^\alpha(x-x')^{[a}e^{b]}_\alpha\bigg\}. \quad (4.6)$$

Here, $\dot j^\alpha(r',\tau) := \partial j^\alpha(r',\tau)/\partial\tau$, which differs from our usual convention (e.g. $\dot n^\alpha(\tau) := e^\alpha_a(\tau)\mathrm{d}n^a(\tau)/\mathrm{d}\tau$).

Moving on, (4.3) implies that

$$\dot\sigma(x,x') = -(N(r',\tau)n^a(\tau) + (h\cdot v)^a(\tau))(x-x')_a, \quad (4.7)$$

and

$$\ddot\sigma(x,x') = N^2 + v^\alpha v_\alpha - \bigg(N\dot n^a - n^a\ddot n^\beta r'_\beta + \frac{\mathrm{d}(h\cdot v)^a}{\mathrm{d}\tau}\bigg) \times (x-x')_a. \quad (4.8)$$

If we now specify how φ and H^α vary in time (which determines $\dot\rho$ and $\dot j^\alpha$), we would have all of the ingredients necessary to find the body's motion without any approximation. Unfortunately, inserting (4.6) into (A57), (A61), (A62), (3.32), (3.33), and (3.34) leads to a set of delay integro-differential equations for the object's motion. Such a system would be extremely difficult to solve (or even interpret) in general, although it could be a useful starting point for numerical simulations. It might also be interesting analytically if one were looking for the forces required to make a body move in some predetermined way (e.g. a circular orbit).

Such questions will not be discussed here. We instead consider the simplest possible approximations that allow us to gain insight into a generic class of systems. In particular, it is assumed that all of the quantities in (4.6) which depend on $\tau_\pm$ may be written in terms of quantities at s (via Taylor expansion). This requires that nothing very drastic happen on time scales less than the body's light-crossing time. This is not as trivial a condition as it might seem to be (see e.g. [36]), although we will not attempt to justify it.

Expressing everything in terms of s rather than $\tau_\pm$ first requires calculating $\Delta_\pm := s - \tau_\pm$. $\tau_\pm$ was defined by $\sigma = 0$, so $\Delta_\pm$ can be found by Taylor expanding this equation in $\Delta_\pm$. Assuming that both n^a and v^a are at least C^3 in time for $s \in [\tau_\pm, s]$,

$$e^a_\alpha(\tau_\pm) = e^a_\alpha(s) + \Delta_\pm n^a(s)\dot n_\alpha(s) - \tfrac{1}{2}\Delta^2_\pm(\dot n^a(s)\dot n_\alpha(s) + n^a(s)\ddot n_\alpha(s)) - \tfrac{1}{6}\Delta^3_\pm \dddot e^a_\alpha(\xi^{(1)}_\pm), \quad (4.9)$$

$$z^a(\tau_\pm) = z^a(s) - \Delta_\pm v^a(s) + \tfrac{1}{2}\Delta^2_\pm \dot v^a(s) - \tfrac{1}{6}\Delta^3_\pm \ddot v^a(s) + \tfrac{1}{24}\Delta^4_\pm \dddot v^a(\xi^{(2)}_\pm), \quad (4.10)$$

where $\xi^{(1)}_\pm$ and $\xi^{(2)}_\pm$ are some numbers between $\tau_\pm$ and s.

Then (3.4) shows that

$$(x-x')^a \simeq e^a_\alpha(r-r')^\alpha + \Delta_\pm(Nn^a + (h\cdot v)^a) + \tfrac{1}{2}\Delta^2_\pm(n^a\ddot n^\alpha r'_\alpha - N\dot n^a) + \tfrac{1}{6}\Delta^3_\pm \ddot n^a. \quad (4.11)$$

Everything here has been written in terms of s, and all terms involving quantities such as $|\dot n|^3$, $|\ddot n|^2$, $|\dddot n|$, $|h\cdot v|^2$, and $|\dot v - \dot n|$ have been removed. Since $|\Delta_\pm| \sim |r - r'|$, these terms can be reasonably neglected if $|\dddot n|\mathcal{R}^3 \ll |\ddot n|\mathcal{R}^2 \ll |\dot n|\mathcal{R} \ll 1$, and $|\dot v - \dot n|\mathcal{R} \ll |h\cdot v| \ll |\dot n|\mathcal{R}$, where $\mathcal{R} := \max(|r|,|r'|)$. In a sense, we are expanding up to second-order in powers of $|\dot n|\mathcal{R}$, and up to first order in $|v|$.

The requirement $|\dot n|\mathcal{R} \ll 1$ must hold for all (r,r') pairs, so it is useful to define the largest possible value of $\mathcal{R}$. We call this the body's "radius"

$$D(s) := \max_{\Sigma(s)\cap W}|r|. \quad (4.12)$$

Using it, $|\dot n|\mathcal{R} \ll 1$ implies $|\dot n|D \ll 1$. This technically restricts the allowable size of the charge, although very few reasonable systems would actually be excluded.

Interpreting the relation satisfied by $(h\cdot v)^a$ $(= v^a - n^a)$ is not quite as simple. Given (3.34), it is roughly equivalent to assuming that spin effects are present only to lowest nontrivial order. This is not completely accurate, though, and more precise statements will be given in the following section.

In any case, the inequalities following (4.11) will be assumed to hold from now on. Using them to expand $\sigma = 0$, we find that

$$R^2 := |r-r'|^2 \simeq \Delta_\pm[2v^\alpha(r-r')_\alpha + \Delta_\pm N(r,s)N(r',s) + \tfrac{1}{3}\Delta^2_\pm \ddot n^\alpha(r+2r')_\alpha + \tfrac{1}{12}\Delta^3_\pm|\dot n|^2]. \quad (4.13)$$

All but the second term here is already 'small,' but not quite negligible in our approximation. The zeroth order expression for $\Delta_\pm$ $(= \mp R)$ may therefore be substituted into each of these terms without any overall loss of accuracy. The resulting equation is easily solved:

$$\Delta_\pm \simeq \mp R[1 + \tfrac{1}{2}\dot n_\alpha(r+r')^\alpha - \tfrac{1}{2}\dot n^\alpha\dot n^\beta r_\alpha r'_\beta \pm \tfrac{1}{6}R\ddot n^\alpha(r+2r')_\alpha + \tfrac{1}{24}R^2\dot n^\alpha\dot n_\alpha + \tfrac{3}{8}(\dot n^\alpha(r+r')_\alpha)^2 \pm R^{-1}v^\alpha(r-r')_\alpha], \quad (4.14)$$

where everything is evaluated at s.

Although it would still be straightforward at this stage to compute the exact error in (4.14), it is not necessary. Dimensional analysis shows that the neglected terms have magnitudes $|\dddot n|\mathcal{R}^4$, $|\ddot n|^2\mathcal{R}^5$, $|d(h\cdot v)/ds|\mathcal{R}^2$, $|d^2(h\cdot v)/ds^2|\mathcal{R}^3$, $|h\cdot v|^2\mathcal{R}$, and so on (where each of these can be evaluated anywhere in the interval $[\tau_\pm, s]$).

Continuing to expand quantities appearing in (4.6), a long but straightforward calculation shows that (4.7) and (4.8) can be approximated by

$$\dot{\sigma} \simeq \pm R[1 - \tfrac{1}{2}\dot{n}^\alpha (r + r')_\alpha - \tfrac{1}{8}(\dot{n}^\alpha (r - r')_\alpha)^2 \mp \tfrac{1}{3} R \ddot{n}^\alpha (r + 2r')_\alpha + \tfrac{1}{8} R^2 |\dot{n}|^2], \tag{4.15}$$

and

$$\ddot{\sigma} \simeq 1 - \dot{n}^\alpha (r + r')_\alpha \mp R\ddot{n}^\alpha (r + 2r')_\alpha + \dot{n}^\alpha \dot{n}^\beta r_\alpha r'_\beta + \tfrac{1}{2} R^2 |\dot{n}|^2. \tag{4.16}$$

Looking at (4.6), the charge and current densities are the only quantities that have not yet been expanded away from $\tau_\pm$. It is instructive to temporarily leave them like this while still expressing all of the other quantities needed to compute the field in terms of objects at s. The resulting expression is quite long, so we break it up into several smaller pieces by writing $F^{ab}_{\mathrm{self}(\pm)}$ in the form

$$F^{ab}_{\mathrm{self}(\pm)} = 2\int \mathrm{d}^3 r' \frac{1}{R^3}(\rho(r', \tau_\pm) f^{ab}_{(1)} + \dot{\rho}(r', \tau_\pm) f^{ab}_{(2)} - j^\beta(r', \tau_\pm) f^{ab}_{\beta(3)} + \dot{j}^\beta (r', \tau_\pm) f^{ab}_{\beta(4)}). \tag{4.17}$$

The definitions of each of these coefficients is obvious from comparison with (4.6).

Before computing them, we first simplify the notation by defining a quantity $T(s) > 0$ such that T^{-1} remains (marginally) less than $|\dot{n}|$, $|\ddot{n}|^{1/2}$, $|h \cdot v|^{1/2}/D$, etc. This is useful because a number of different objects were assumed to be negligible in this section. Some of these may be much larger than the others, so using them to write down error estimates would become rather awkward. Introducing T removes this difficulty.

Along the same lines, we also define $\epsilon := D/T \ll 1$. A simple but exceedingly tedious calculation then shows that the coefficients in (4.17) are equal to

$$\begin{aligned} f^{ab}_{(1)} \simeq& -(r - r')^\alpha e^{[a}_\alpha n^{b]}[1 + \tfrac{1}{2}\dot{n}^\beta (r - r')_\beta - \tfrac{1}{8}(\dot{n}^\beta (r - r')_\beta)^2 \\ &+ \tfrac{1}{2}\dot{n}^\beta \dot{n}^\gamma r_\beta (r - r')_\gamma + \tfrac{1}{8}R^2|\dot{n}|^2] + \tfrac{1}{2}R^2 (r - r')^\alpha e^{[a}_\alpha \ddot{n}^{b]} \\ &- \tfrac{1}{2}R^2 n^{[a}\dot{n}^{b]}(1 + \tfrac{1}{2}\dot{n}^\beta (3r - r')_\beta) \pm \tfrac{2}{3}R^3 \ddot{n}^{[a} n^{b]} \\ &+ o([\epsilon(\mathcal{R}/D)]^3 \mathcal{R}), \end{aligned} \tag{4.18}$$

$$\begin{aligned} f^{ab}_{(2)} \simeq& \pm R\{(r - r')^\alpha e^{[a}_\alpha n^{b]}[1 + \dot{n}^\beta r_\beta + (\dot{n}^\beta r_\beta)^2 \\ &\pm \tfrac{1}{3}R\ddot{n}^\beta (2r + r')_\beta - \tfrac{1}{4}R^2 |\dot{n}|^2] \pm \tfrac{1}{2}R^2 (r - r')^\alpha e^{[a}_\alpha \ddot{n}^{b]} \\ &+ (r - r')^\alpha e^{[a}_\alpha \dot{n}^{b]} R(1 + \tfrac{1}{2}\dot{n}^\beta (3r + r')_\beta) \\ &\mp \tfrac{1}{2}R^2 n^{[a}\dot{n}^{b]}(1 + 2\dot{n}^\beta r_\beta) - \tfrac{1}{3} n^{[a}\ddot{n}^{b]} R^3 \\ &- (h \cdot v)^{[a} n^{b]} R\} + o([\epsilon(\mathcal{R}/D)]^3 \mathcal{R}^2), \end{aligned} \tag{4.19}$$

$$\begin{aligned} f^{ab}_{\beta(3)} \simeq& (r - r')^\alpha e^{[a}_\alpha e^{b]}_\beta [1 + \tfrac{1}{2}\dot{n}^\gamma (r - r')_\gamma - \tfrac{1}{8}(\dot{n}^\gamma (r - r')_\gamma)^2 \\ &+ \tfrac{1}{2}\dot{n}^\gamma \dot{n}^\lambda r_\gamma (r - r')_\lambda + \tfrac{1}{8}R^2 |\dot{n}|^2] \\ &+ \tfrac{1}{2}R^2 \ddot{n}_\beta (r - r')^\alpha e^{[a}_\alpha n^{b]} + \tfrac{1}{2}R^2 \dot{n}_\beta (r - r')^\alpha e^{[a}_\alpha \dot{n}^{b]} \\ &+ n^{[a} e^{b]}_\beta R^2 (\tfrac{1}{2}\ddot{n}^\gamma (r - r')_\gamma \mp \tfrac{1}{3}R|\dot{n}|^2 - R^{-2} v^\gamma (r - r')_\gamma) \\ &+ \tfrac{1}{2}R^2 \dot{n}^{[a} e^{b]}_\beta (1 + \tfrac{1}{2}\dot{n}^\gamma (3r - r')_\gamma) \pm \tfrac{1}{3}R^3 \ddot{n}^{[a} e^{b]}_\beta \\ &+ o([\epsilon(\mathcal{R}/D)]^3 \mathcal{R}), \end{aligned} \tag{4.20}$$

$$\begin{aligned} f^{ab}_{\beta(4)} \simeq& \pm R\{(r - r')^\alpha e^{[a}_\alpha e^{b]}_\beta [1 + \dot{n}^\beta r_\beta + (\dot{n}^\beta r_\beta)^2 \pm \tfrac{1}{3}R\ddot{n}^\beta (2r + r')_\beta - \tfrac{1}{4}R^2 |\dot{n}|^2] \mp \tfrac{1}{2}R^2 (r - r')^\alpha e^{[a}_\alpha (\dot{n}^{b]}\dot{n}_\beta - n^{b]}\ddot{n}_\beta) \\ &- (r - r')^\alpha e^{[a}_\alpha n^{b]}\dot{n}_\beta R(1 + \tfrac{1}{2}\dot{n}^\gamma (3r + r')_\gamma) - n^{[a} e^{b]}_\beta R[1 + \tfrac{1}{2}\dot{n}^\gamma (3r - r')_\gamma - \tfrac{1}{8}(\dot{n}^\gamma (r + r')^\gamma)^2 + \dot{n}^\gamma \dot{n}^\lambda r_\gamma (2r_\lambda + r'_\lambda) \\ &\pm \tfrac{1}{6}R\ddot{n}^\gamma (5r + r')_\gamma - \tfrac{7}{24}R^2 |\dot{n}|^2 \pm R^{-1} v^\gamma (r - r')_\gamma] - (h \cdot v)^{[a} e^{b]}_\beta R \mp \tfrac{1}{2}\dot{n}^{[a} e^{b]}_\beta R^2 (1 + 2\dot{n}^\gamma r_\gamma) - \tfrac{1}{6}R^3 \ddot{n}^{[a} e^{b]}_\beta\} \\ &+ o([\epsilon(\mathcal{R}/D)]^3 \mathcal{R}^2). \end{aligned} \tag{4.21}$$

A. Approximations

At this point, it is useful to consider the physical meaning behind our approximations more carefully. Although error estimates have been given for $f^{ab}_{(1)}$-$f^{ab}_{\beta(4)}$, the important errors are those in F^{ab}_{self}. Unfortunately, the unbounded integrand in (4.17) makes it difficult to compute these rigorously. We shall simply assume that J^a is sufficiently homogeneous that dimensional analysis can be used to say, for example, that the absolute value of the error in $\int \mathrm{d}^3 r' \rho(r', \tau_\pm) f^{ab}_{(1)}/R^3$ is less than about

$$\epsilon^3 \max|\rho| D. \tag{4.22}$$

With this established, we can also use the assumptions outlined in the previous section to provide bounds on the force and torque. Combining $|\dot{n}| < T^{-1}$ with (3.33),

$$|\Psi^\alpha| < M T^{-1}. \tag{4.23}$$

Since $T \gg D$, this means that the particle cannot be accelerated up to an appreciable fraction of the speed of light within a light-crossing time—a very reasonable restriction.

Differentiating (3.33),

$$M\ddot{n}^{\alpha} = -\dot{\Psi}^{\alpha} - 2\dot{M}\dot{n}^{\alpha}. \tag{4.24}$$

Generically, this implies that $|\ddot{n}| < T^{-2}$ can be ensured by assuming

$$|\dot{M}| < MT^{-1}, \tag{4.25}$$

$$|\dot{\Psi}^{\alpha}| < MT^{-2}. \tag{4.26}$$

Differentiating (3.33) a second time and using (3.32), we also have that

$$|\ddot{M}| < MT^{-2}, \tag{4.27}$$

$$|\ddot{\Psi}^{\alpha}| < MT^{-3}, \tag{4.28}$$

$$|n_a\dot{\Psi}^{a}| < MT^{-2}. \tag{4.29}$$

Similar restrictions on the spin and torque can be found by using (3.34). In particular, $|v|$ will remain less than ϵ^2 if

$$|S^{\alpha\beta}| := |S^{\alpha}| < \epsilon MD, \tag{4.30}$$

$$|\Psi^{[ab]}e^{\alpha}_{a}n_b| < \epsilon^2 M. \tag{4.31}$$

Differentiating (3.34), $|\dot{n} - \dot{v}|D \ll \epsilon^2$ is implied by

$$|\Psi^{[\alpha\beta]}| \ll \epsilon M, \tag{4.32}$$

$$|\dot{\Psi}^{[ab]}e^{\alpha}_{a}n_b| \ll \epsilon MD. \tag{4.33}$$

Although weaker restrictions than these can be adopted in special cases, we assume for simplicity that they always hold. More concisely, we are restricting ourselves to systems where the magnitude of each tetrad component of the (full self + external) force is bounded by about MT^{-1}, and the magnitude of each torque component is no larger than $\epsilon^2 M$. Bounds on the s-derivatives of these quantities are suppressed by appropriate factors of T. These conditions basically mean that the body can only change significantly over time scales larger than T. (4.30), for example, implies that this time scale sets a lower bound on the charge's rotational period.

These interpretations should not be surprising. Unfortunately, though, they are quite not as simple as they appear. Combining them with (A61) and (A62) can lead to less obvious bounds on the structure of J^a itself. One of these affects the size of the particle's dipole moment. If the self-torque is small compared to the external one, then the magnitudes of

$$Q^{[\alpha}{}_{\gamma}F^{\beta]\gamma}_{\mathrm{ext}},\, n_a e^{c}_{\gamma}Q^{a}{}_{c}F^{\beta\gamma}_{\mathrm{ext}},\, n_a e^{\gamma}_{c}Q^{\beta}{}_{\gamma}F^{ac}_{\mathrm{ext}},\, e^{[\alpha}_{a}e^{\beta]}_{b}n_c n_d Q^{ac}F^{bd}_{\mathrm{ext}} \tag{4.34}$$

should all be less than about $\epsilon^2 M$. If q is negligible, these conditions ensure that both the force and torque are sufficiently small.

In most cases of interest, though, $q \neq 0$. Bounding the quantities in (4.34) then guarantees only that the torque is acceptable. The (full) force will often be dominated by the monopole component of Ψ^{a}_{ext}, in which case (applying (4.23) and (4.25))

$$MT^{-1} > |qe^{\alpha}_{a}n_b F^{ab}_{\mathrm{ext}}|, \tag{4.35}$$

$$> |qv_{\beta}F^{\alpha\beta}_{\mathrm{ext}}|. \tag{4.36}$$

Since v_β is small, this second bound allows the magnetic field to be quite large. Despite this, we choose to assume (purely for simplicity) that both the electric and magnetic fields are bounded by $M(|q|T)^{-1}$. If the fields are in fact as large as these equations allow, then the previous bounds on the dipole moment imply that

$$|Q^{\alpha\beta}| \lesssim \epsilon|q|D, \tag{4.37}$$

along with a similar restriction on $|Q^{ab}e^{\alpha}_{a}n_b|$. This implies that the 'center-of-charge' must be very close to the center-of-mass, which considerably reduces the number of current distributions that we can allow. Note, however, that these assumptions could be relaxed somewhat if extra conditions are placed on the sizes of the field components.

To find some other implications of our approximations, we now study a particular example in detail before moving on to computing general self-forces.

V. SELF-FORCES

A. Stationary case

It was shown in the previous section that (4.17), (4.18), (4.19), (4.20), and (4.21) approximate the self-field due to a charge with an (almost) arbitrarily accelerating center-of-mass line. These equations become exact when $\dot{n}^a = 0$ and $n^a = v^a$. Temporarily assuming that these conditions are true,

$$\begin{aligned} F^{ab}_{\mathrm{self}(\pm)}(x) = -2\int \mathrm{d}^3r'\,\frac{1}{R^3}[(r-r')^{\alpha}e^{[a}_{\alpha}n^{b]}(\rho(r',\tau_\pm) \\ \mp R\dot{\rho}(r',\tau_\pm)) - (r-r')^{\alpha}e^{[a}_{\alpha}e^{b]}_{\beta}(j^{\beta}(r',\tau_\pm) \\ \pm R\dot{j}^{\beta}(r',\tau_\pm)) + n^{[a}e^{b]}_{\beta}R^2\ddot{j}^{\beta}(r',\tau_\pm)], \end{aligned} \tag{5.1}$$

where $\tau_\pm = s \pm R$.

One might at first think that the self-forces generated by this field would be identically zero, but this is not generally correct. Anything emitting a focused beam of radiation, for example, will experience some recoil. It will also lose a bit of mass over time, and can even start rotating if the the beam is offset from the source's center-of-mass. These can all be thought of as self-force and self-torque effects, and clearly do not require $\dot{v}^a \neq 0$.

Since (5.1) is exact, one could compute self-forces and self-torques for a variety of systems without any approximation. If these were nonzero, the assumptions leading to (5.1) would not be maintained unless there were also external forces and torques present that exactly balanced them. Such a situation would be rather artificial, so we instead apply the approximations of Sec. IV, and use the resulting self-force and self-torque to gain some intuition into more general cases.

Assuming that $|\varphi| \gg |\dot{\varphi}|D \gg |\ddot{\varphi}|D^2$, except perhaps at isolated points, one can show that

$$\begin{aligned}\Psi^\alpha_{\mathrm{self}(\pm)} \simeq \int \mathrm{d}^3 r \mathrm{d}^3 r' \frac{1}{R^3}\bigg\{& R^2 \rho(r,s) \dot{H}^\alpha(r',s) \\ &- 2(r-r)^\alpha H_\beta(r,s)\bigg(\frac{1}{2} r'^\beta \dot{\varphi}(r',s) \mp R\dot{H}^\beta(r',s)) \\ &+ \bigg(\varphi(r,s) - \frac{q}{4\pi|r|^3}\bigg)[(r-r')^\alpha r_\beta \\ &- \delta^\alpha_\beta (r-r')^\gamma r_\gamma]\dot{H}^\beta(r',s)\bigg\}, \end{aligned} \tag{5.2}$$

to first order in $\dot{\varphi}$ and $\dot{H}^\alpha$. This illustrates that Ψ^α is bounded by $\dot{\varphi}HD^5$ (meaning $\max|\dot{\varphi}|\max|H|D^5$), $\varphi\dot{H}D^5$, or $H\dot{H}D^5$ (whichever is largest). If H^α is very small compared to φ, then the lowest order contributions to the self-force can be shown to be of order $\varphi\ddot{\varphi}D^6$.

It is then clear that unless the system is particularly symmetric, self-forces with these magnitudes are unavoidable. They would considerably complicate the equations of motion when considering nonzero center-of-mass accelerations, so it would be convenient to ignore them. This is self-consistent with the approximations already in place if the charge's "elasticity" is bounded by

$$\dot{\varphi}D < \epsilon^2 \varphi, \tag{5.3}$$

$$\ddot{\varphi}D^2 \ll \epsilon^2 \varphi, \tag{5.4}$$

$$\dot{H}D \ll \epsilon^2 H. \tag{5.5}$$

If these relations hold, the self-force becomes negligible whenever $\dot{n}^a = 0$ and $n^a = v^a$; i.e. a charge at rest will remain at rest unless acted on by an external field. For this reason, all of the calculations that follow will assume (5.3) and (5.4) to be true. (5.5) is extremely simple to relax, so it will be eventually be replaced with $\dot{H}D < \epsilon^2 H$. The interested reader could easily weaken these conditions even further, although we regard the extra complication to be unnecessary for the present purposes.

In writing down (5.1), both $\dot{n}^a = 0$ and $n^a = v^a$ were required. This second condition was adopted here for simplicity, although it is by no means "natural." Given (3.34), it can only hold if $\Psi^{[ab]}n_b = 0$. But combining (5.3), (5.4), and (5.5) with (A62) and (5.1) shows that this portion of the torque is usually nonzero even if $\dot{\varphi} = \dot{H}^\alpha = 0$. Generically, one finds that $\Psi^{[ab]}n_b$ will only be small if either $\varphi \ll H$ or $H \ll \varphi$. $n^a \simeq v^a$ is not a particularly important requirement, so we choose not to impose these restrictions on the current structure. In general, then, $n^a \neq v^a$ even when $\dot{n}^a = 0$ (and φ can be of order H).

Of course, $|n - v| = |v|$ still cannot exceed ϵ^2. As was already shown, this remains true if each tetrad component of $\Psi^{[ab]}$ is bounded by $\epsilon^2 M$ (and $|S| < \epsilon MD$). We now strengthen this assumption slightly, and require the self and external components of the torque to be *individually* less than $\epsilon^2 M$. There are then terms in the self-torque which violate this unless more restrictions are placed on the particle's current structure.

Using (5.1) and (A62) again, one can see that the tetrad components of $\Psi^{[ab]}_{\mathrm{self}}$ will be of order φHD^5, $\varphi\dot{\varphi}D^6$, etc. These types of quantities can be conveniently simplified by defining an "electromagnetic radius" $D_{\mathrm{em}} \sim (\varphi^2 + H^2)D^6/M \lesssim D$. D_{em}/D then estimates the fraction of the particle's mass that is of (macroscopic) electromagnetic origin. This ratio will come up often, so let

$$\mathcal{E} := \left(\frac{D_{\mathrm{em}}}{D}\right) \lesssim 1. \tag{5.6}$$

In this notation, $\varphi\dot{\varphi}D^6 \to \epsilon^2 \mathcal{E} M$, and $\varphi HD^5 \to \mathcal{E}M$. If φ and H are comparable, the second of these expressions is clearly the one that estimates the self-torque. Enforcing (4.31) in this case therefore requires

$$\mathcal{E} < \epsilon^2. \tag{5.7}$$

Conditions weaker than this can be adopted if the particle is dominated by either φ or H^α, although it will still be true that $\mathcal{E} \ll 1$ (except in special cases). Surprisingly, our approximations place a severe restriction on the charge's self-energy. This makes it impossible to ever take the point particle limit in a strict sense (if such a thing is even meaningful). Such a procedure is not, however, necessary to answer whether a particle's size is important when it is much smaller than the characteristic scales of the surrounding system (the "physical point particle limit").

It is interesting to note that this condition is really independent of any of Dixon's special constructions. In almost any slow-motion approximation, one would expect that $|\dot{S}| \lesssim |S|T^{-1}$ and $|S| \lesssim MD^2T^{-1}$. So $|\Psi^{[ab]}| \sim |\dot{S}| \lesssim \epsilon^2 M$ is generic. And this is exactly the condition that led to (5.7). The specific form of the torque needed to show this did use Dixon's definition, but the more common one would not have changed anything. This can be verified by substituting (5.1) into the first term in (A62).

Also note that when $\mathcal{E} \sim 1$, it is not clear that any slow-motion assumption is even physically reasonable. It seems difficult, for example, to avoid the extremely high frequency self-sustaining oscillations discussed by Bohm and Weinstein [36], among others. When $\mathcal{E}$ is small, it is easy to imagine that oscillations like these would have extremely small amplitudes, or at least be highly damped in causal systems. But the degree of rigidity required to hold together a charge with very large self-energy might prevent this.

B. Arbitrary motion

We now turn back to analyzing the self-forces and self-torques when $\dot{n}^a \neq 0$ and $n^a \neq v^a$. For simplicity, it is assumed that the internal dynamics are always "slow" in the sense of (5.3) and (5.4). It is simple enough to relax (5.5), so we also allow the first (and only the first) derivative of H^α to appear in the self-force. Higher derivatives of both φ and H^α could be included with relatively little extra effort, although there seems to be little reason for doing so.

1. Radiative self-forces

At this point, it is useful to separately write out the retarded and radiative fields. Using (1.3) and (4.17), (4.18), (4.19), (4.20), and (4.21),

$$F^{ab}_{\mathrm{self}(R)}(x) \simeq -\tfrac{4}{3} q \ddot{n}^{[a}(s) n^{b]}(s), \tag{5.8}$$

If $n^a = v^a$, then this is the same expression found by Dirac from the Liénard-Wiechert potential [16] (we have used the opposite sign convention for the field).

The forces and torques exerted by this field are now very simple to calculate. Inserting (5.8) in (A61),

$$\Psi^\alpha_{\mathrm{self}(R)}(s) \simeq -\tfrac{2}{3} q^2 \ddot{n}^\alpha + o(\epsilon^2 \mathcal{E} M T^{-1}). \tag{5.9}$$

This is just the Lorentz-Dirac force when $n^a = v^a$ (see (1.1)).

Substituting (5.8) into (A62), the space-space components of the self-torque are

$$\Psi^{[\alpha\beta]}_{\mathrm{self}(R)}(s) \simeq \frac{1}{q} \int \mathrm{d}^3 r\, \rho(r,s) r^{[\alpha} \Psi^{\beta]}_{\mathrm{self}(R)} + o(\epsilon^3 \mathcal{E} M). \tag{5.10}$$

$\int \mathrm{d}^3 r \rho r^\alpha / q$ can be thought of as the separation vector between the "charge centroid" and the center-of-mass ($r^\alpha = 0$). Inverting (3.31), we can clearly convert this component of the self-torque into a 3-vector describing the rate of change of S^a. In this form, the (vector) self-torque is just the cross product of the self-force with this separation vector. Writing it in this way suggests that it arises due to the self-force not acting through the center-of-mass.

The time-space components of the self-torque are similar:

$$\Psi^{[ab]}_{\mathrm{self}(R)}(s) n_a e^\beta_b \simeq \frac{1}{2q} \int \mathrm{d}^3 r\, H_\alpha(r,s) \Psi^\alpha_{(R)\mathrm{self}} r^\beta + o(\epsilon^3 \mathcal{E} M). \tag{5.11}$$

This clearly vanishes if the "current centroid" coincides with the center-of-mass.

Combining all of these results with (A57), (3.32), (3.33), and (3.34), one can find that the particle's motion by simultaneously solving

$$\dot{M} \simeq -n_a \Psi^a_{\mathrm{ext}} + o(\epsilon^2 \mathcal{E} M T^{-1}), \tag{5.12}$$

$$\dot{S}^{\alpha\beta} \simeq -2\Psi^{[\alpha\beta]}_{\mathrm{ext}} + \frac{4}{3} q \int \mathrm{d}^3 r \rho(r,s) r^{[\alpha} \ddot{n}^{\beta]} + o(\epsilon^3 \mathcal{E} M), \tag{5.13}$$

$$M\dot{n}^\alpha \simeq -\Psi^\alpha_{\mathrm{ext}} + \tfrac{2}{3} q^2 \ddot{n}^\alpha + o(\epsilon^2 \mathcal{E} M T^{-1}), \tag{5.14}$$

$$M\dot{v}^\alpha \simeq M\dot{n}^\alpha + S^{\alpha\beta}\ddot{n}_\beta - \dot{M} v^\alpha - 4\Psi^{[\alpha\beta]}_{\mathrm{ext}} \dot{n}_\beta - 2\dot{\Psi}^{[ab]}_{\mathrm{ext}} e^\alpha_a n_b + o(\epsilon^2 \mathcal{E} M T^{-1}). \tag{5.15}$$

If the external field varies slowly over $\Sigma(s) \cap W$, (A63) and (A64) can be used to approximate Ψ^a_{ext} and $\Psi^{[ab]}_{\mathrm{ext}}$. Let the minimum characteristic length scale of the external field be denoted by $\lambda \gg D$, so that $|F^{ab}_{\mathrm{ext}}| \gtrsim \lambda |\partial F^{ab}_{\mathrm{ext}}|$ (where the absolute value signs are meant to act on each tetrad component of the quantity inside them). It is then convenient to assume that

$$\lambda \gtrsim T/\mathcal{E}. \tag{5.16}$$

This ensures such that the dipole and higher contributions to the external force are negligible compared to the Lorentz-Dirac self-force.

Without any loss of accuracy, (5.16) allows (5.11), (5.12), (5.13), (5.14), and (5.15) to be written as (making some weak assumptions on the magnitudes of the quadrupole and higher moments)

$$\dot{M} \simeq -q n_a e^\beta_b v_\beta F^{ab}_{\mathrm{ext}}, \tag{5.17}$$

$$\dot{S}^{\alpha\beta} \simeq -2 e^\alpha_a e^\beta_b (Q^{[a}{}_c F^{b]c}_{\mathrm{ext}} + Q^{d[a}{}_c \partial_d F^{b]c}_{\mathrm{ext}}) + \frac{4}{3} q \int \mathrm{d}^3 r \rho(r,s) r^{[\alpha} \ddot{n}^{\beta]}, \tag{5.18}$$

$$M\dot{n}^\alpha \simeq -q e^\alpha_a v_b F^{ab}_{\mathrm{ext}} + \tfrac{2}{3} q^2 \ddot{n}^\alpha, \tag{5.19}$$

$$M\dot{v}^\alpha \simeq M\dot{n}^\alpha + S^{\alpha\beta}\ddot{n}_\beta - \dot{M} v^\alpha - 4 e^\alpha_a e^\beta_b \dot{n}_\beta Q^{[a}{}_c F^{b]c}_{\mathrm{ext}} - 2 e^\alpha_a n_b \dot{Q}^{[a}{}_c F^{b]c}_{\mathrm{ext}}, \tag{5.20}$$

where the error terms have not changed. Similarly, v^α can be recovered from (3.34):

$$M v^\alpha \simeq S^{\alpha\beta}\dot{n}_\beta - 2 e^\alpha_a n_b (Q^{[a}{}_c F^{b]c}_{\mathrm{ext}} + Q^{d[a}{}_c \partial_d F^{b]c}_{\mathrm{ext}}) - \frac{2}{3} q \int \mathrm{d}^3 r\, H_\beta(r,s) \ddot{n}^\beta r^\alpha + o(\epsilon^3 \mathcal{E} M). \tag{5.21}$$

The assumptions we have made so far—although fairly restrictive—are clearly not sufficient to recover the standard point particle result. Still, there must exist some class of charges which do behave in this way, and it is interesting to characterize it. Before doing so, it must first be mentioned that the $\bar{s}$ in (1.1) is a proper time, while s is not. It is close, though. Letting $s = s(\bar{s})$ and $s'(\bar{s}) := \mathrm{d}s/\mathrm{d}\bar{s}$,

$$s' = \frac{1}{\sqrt{v^a v_a}}, \tag{5.22}$$

$$\sim 1 + o(\epsilon^4). \tag{5.23}$$

The fractional difference between the center-of-mass 4-velocity and v^a is therefore of order ϵ^4. In particular, the triad components of this 4-velocity will differ from v^α by terms no larger than ϵ^6. Given the error estimate in (5.21), these differences are negligible whenever $\mathcal{E} \gtrsim \epsilon^3$.

The proper acceleration can now be written in terms of v^a and $\dot{v}^a$:

$$z''^a = s'' v^a + (s')^2 \dot{v}^a. \tag{5.24}$$

It follows from (5.20) and (5.22) that $s'' \sim \epsilon^4/T$, so z''^a differs from $\dot{v}^a$ by terms of this same order. (5.15) then implies that these two quantities are interchangeable whenever $\mathcal{E} \gtrsim \epsilon^2$.

Applying similar arguments, our results can be easily compared to the Lorentz-Dirac equation by writing it in the form

$$(M\dot{v}^\alpha)_{\rm LD} \simeq -q e^\alpha_a v_b F^{ab}_{\rm ext} + \tfrac{2}{3} q^2 \ddot{n}^\alpha + o(\epsilon^3 M T^{-1} \max(\epsilon, \mathcal{E})), \tag{5.25}$$

$$(M n_a \dot{v}^a)_{\rm LD} \simeq -q n_a e^\beta_b v_\beta F^{ab}_{\rm ext} + o(\epsilon^3 M T^{-1} \max(\epsilon, \mathcal{E})). \tag{5.26}$$

In order to avoid overcomplicating the discussion, we shall say that our equations of motion reduce to the Lorentz-Dirac result if $M\dot{v}^\alpha$ and $M n_a \dot{v}^a$ match (5.25) and (5.26) up to terms of order $\epsilon^2 M T^{-1} \max(\epsilon^2, \mathcal{E})$. Noting that $n_a \dot{v}^a = -\dot{n}^\alpha v_\alpha$, (5.19) shows that the "temporal component" of the body's acceleration always matches (5.26) to the required accuracy.

The same is not true for the spatial acceleration. This should not be too surprising, though, as the Lorentz-Dirac equation was never intended to describe spinning particles. S^{ab} should therefore be set to zero (at least instantaneously) before any reasonable comparison can be made. Even this is not quite sufficient, though. The situation can be remedied by assuming that

$$|S| < \epsilon M D \max(\epsilon^2, \mathcal{E}), \tag{5.27}$$

$$|Q^{[a}{}_c F^{b]c}_{\rm ext} e^\alpha_a e^\beta_b| < \epsilon^2 M \max(\epsilon^2, \mathcal{E}), \tag{5.28}$$

along with a similar restriction on $|Q^{[a}{}_c F^{b]c}_{\rm ext} e^\alpha_a n_b|$. Then

$$|v| < \epsilon^2 \max(\epsilon^2, \mathcal{E}), \tag{5.29}$$

$$|\dot{M}| < \epsilon^2 M T^{-1} \max(\epsilon^2, \mathcal{E}), \tag{5.30}$$

$$|\dot{S}| < \epsilon^2 M \max(\epsilon^2, \mathcal{E}). \tag{5.31}$$

If $\mathcal{E} \gtrsim \epsilon^2$, $\dot{M}$ vanishes up to the maximum order that we can calculate it. The same is not necessarily true for $\dot{S}^{\alpha\beta}$ and v^α, although they remain sufficiently small that (5.25) can now be recovered to the desired accuracy. There therefore exists a regime in which the equations of motion derived here reduce to the Lorentz-Dirac equation. Considering z^a to be the only observable (as would be reasonable for an extremely small particle), this completely recovers the usual point particle result.

While this conclusion is not particularly surprising, it is interesting to note how restrictive the required assumptions are. (5.28) is particularly difficult to satisfy. For example, when the force is approximately given by the Lorentz (monopole) expression, external field magnitudes up to $\sim M(|q|T)^{-1}$ are allowed. In fields this large, (5.28) implies that the magnitudes of the dipole moment must be less than $\sim \epsilon \mathcal{E} q D$ (when $\mathcal{E} \gtrsim \epsilon^2$). This is an extremely limiting condition in cases where the self-energy is small. Given (3.15), there do not appear to be any rigid ($\dot{\varphi} = \dot{H}^\alpha = 0$) or nearly rigid charge distributions that could satisfy it. One either needs to choose a very special class of charges, or considerably restrict the maximum allowable field strength.

Before continuing, it should first be mentioned that our definitions of the charge and three-current densities are slightly unusual. These quantities would usually be defined with respect to an orthonormal tetrad adapted to $z'^a(s(\bar{s}))$. We have instead defined these quantities in terms of a tetrad with temporal component n^a. Translating from one of these definitions to the other would only introduce fractional changes of order ϵ^2, which are usually irrelevant at our level of approximation.

2. *Retarded self-forces*

It is generally accepted that detectors placed outside of W will measure $F^{ab}_{\rm ext} + F^{ab}_{{\rm self}(-)}$ rather than $F^{ab}_{\rm ext} + F^{ab}_{{\rm self}(R)}$. But infinitesimal elements of an extended body cannot "know" that they are part of a larger whole, so they must couple to this same field. It is therefore reasonable to consider only the retarded self-fields to be "physical." Recalling that this is equal to the sum of the radiative and singular self-fields, any relevance of $F^{ab}_{{\rm self}(R)}$ itself should be derived by showing the effects of the singular self-field are irrelevant in certain cases.

To this end, we now examine the (presumably) realistic case in which the particle interacts with its full retarded self-field. Essentially all of the steps in the previous section carry over identically in this case, although most involve considerably more calculation. Starting with (3.7), (3.8), (4.17), (4.18), (4.19), (4.20), and (4.21), the retarded field can be shown to be (dropping the '−' subscript)

$$F^{ab}_{\rm self}(x) \simeq 2\int \mathrm{d}^3 r' \frac{1}{R^3}\{\rho[-(r-r')^\alpha e^{[a}_\alpha n^{b]}(1+\tfrac{1}{2}\dot n^\beta(r-r')_\beta - \tfrac{1}{8}(\dot n^\beta(r-r')_\beta)^2 + \tfrac{1}{8}R^2|\dot n|^2 + \tfrac{1}{2}\dot n^\beta \dot n^\gamma r_\beta (r-r')_\gamma) + \tfrac{1}{2}R^2(r-r')^\alpha e^{[a}_\alpha \ddot n^{b]} - \tfrac{1}{2}R^2 n^{[a}\dot n^{b]}(1+\tfrac{1}{2}\dot n^\beta(3r-r')_\beta) - \tfrac{2}{3}R^3 \ddot n^{[a} n^{b]}] - (H^\beta + v^\sigma \partial'_\sigma(r'^\beta \varphi)) \times[(r-r')^\alpha e^{[a}_\alpha e^{b]}_\beta(1+\tfrac{1}{2}\dot n^\gamma(r+r')_\gamma + \tfrac{3}{8}(\dot n^\gamma(r+r')_\gamma)^2 + \tfrac{1}{8}R^2|\dot n|^2 - \tfrac{1}{2}\dot n^\gamma \dot n^\lambda r_\gamma r'_\lambda) + n^{[a}e^{b]}_\beta R^2(\tfrac{1}{2}\ddot n^\gamma (r+r')_\gamma + \tfrac{1}{3}R|\dot n|^2 - R^{-2}v^\gamma(r-r')_\gamma) + \tfrac{1}{2}R^2 \dot n_\beta (r-r')^\alpha e^{[a}_\alpha \dot n^{b]} + \tfrac{1}{2}R^2 \ddot n_\beta (r-r')^\alpha e^{[a}_\alpha n^{b]} + \tfrac{1}{2}R^2 \dot n^{[a} e^{b]}_\beta(1+\tfrac{1}{2}\dot n^\gamma(3r+r')_\gamma) - \tfrac{1}{3}R^3 \ddot n^{[a}e^{b]}_\beta] + \dot\varphi (r-r')^\alpha e^{[a}_\alpha e^{b]}_\beta r'^\beta - \dot H^\beta n^{[a}e^{b]}_\beta R^2\}. \tag{5.32}$$

The self-force is now obtained by inserting this into (A61), which results in an expression of the form

$$\Psi^a_{\rm self}(s) \simeq -\int \mathrm{d}^3 r \mathrm{d}^3 r' \mathcal{F}^a(r,r',s), \tag{5.33}$$

$$\simeq -\frac{1}{2}\int \mathrm{d}^3 r\mathrm{d}^3 r'(\mathcal{F}^a(r,r',s) + \mathcal{F}^a(r',r,s)), \tag{5.34}$$

where $\mathcal{F}^a(r,r',s)$ represents the force density exerted by a charge element at r on a charge element at r'. If Newton's third law were correct, $\mathcal{F}^a(r,r',s) = -\mathcal{F}^a(r',r,s)$ (so $\Psi^a_{\rm self}$ would vanish). Of course, this is does not quite hold for the electromagnetic field (or any other fully observable field), so there is a nonzero self-force. (5.34) therefore gives a precise form to the intuitive idea that self-forces measure the degree of failure of Newton's third law.

After removing the components of $\mathcal{F}^a$ which reverse sign under interchange of r and r', it can be shown that

$$M\dot n^\alpha \simeq -\Psi^\alpha_{\rm ext} + \frac{2}{3}q^2\ddot n^\alpha - M_{\rm em}\dot n^\alpha - \int \mathrm{d}^3 r\mathrm{d}^3 r' \frac{1}{R^3}\bigg\{\rho(r,s)H^\beta(r',s)R^2\bigg[\delta^\alpha_\beta\bigg(\frac{1}{2}\ddot n^\gamma(r+r')_\gamma - R^{-2}v^\gamma(r-r')_\gamma\bigg) - (r-r')^\alpha \ddot n_\beta\bigg] + \rho(r,s)\dot H^\beta(r',s)R^2 + v^\gamma\partial_\gamma(r_\beta\varphi(r,s))H^\alpha(r',s)(r-r')^\beta - \dot\varphi(r,s)H^\gamma(r',s)(r-r')^\alpha r_\gamma - \bigg(\varphi(r,s) - \frac{q}{4\pi|r|^3}\bigg)\bigg[\frac{1}{2}H^\gamma(r',s)(R^2\ddot n^\alpha r_\gamma - R^2\delta^\alpha_\gamma \ddot n^\beta r_\beta + ((r-r')^\alpha r_\gamma - \delta^\alpha_\gamma(r-r')^\beta r_\beta)\ddot n^\sigma(r-r')_\sigma) + \dot H^\gamma(r',s)((r-r')^\alpha r_\gamma - \delta^\alpha_\gamma(r-r')^\beta r_\beta)\bigg]\bigg\} + o(\epsilon^2\mathcal{E}MT^{-1}), \tag{5.35}$$

where it was useful to define an "electromagnetic mass"

$$M_{\rm em} := \int \mathrm{d}^3 r\mathrm{d}^3 r' \frac{1}{R^3}\bigg\{\frac{1}{2}R^2[\rho(r,s)\rho(r',s) - H_\beta(r,s)H^\beta(r',s)(1+\dot n^\gamma(r+r')_\gamma)] + \bigg(\varphi(r,s) - \frac{q}{4\pi|r|^3}\bigg)\rho(r',s)\bigg[(r-r')^\beta r_\beta\bigg(1+\frac{1}{2}\dot n^\gamma(r-r')_\gamma\bigg) - \frac{1}{2}R^2\dot n^\beta r_\beta\bigg]\bigg\}. \tag{5.36}$$

The form of (5.35) suggests an effective inertial mass $m := M + M_{\rm em}$. Although M will very rarely remain constant, m is often conserved, or at least varies slowly. To see this, substitute (5.32) into (A61) and use (3.32) to show that

$$\dot M \simeq -n_a\Psi^a_{\rm ext} - \int \mathrm{d}^3 r\mathrm{d}^3 r' \frac{1}{R^3}\bigg\{\bigg(\varphi(r,s) - \frac{q}{4\pi|r|^3}\bigg)\bigg[\dot\rho(r',s)(r-r')^\beta r_\beta - \frac{1}{2}\rho(r',s)(R^2\ddot n^\beta r_\beta - \ddot n^\alpha(r-r')_\alpha(r-r')^\beta r_\beta) - H^\gamma(r',s)\bigg(\frac{1}{2}R^2(\dot n_\gamma \dot n^\beta r_\beta + r_\gamma|\dot n|^2) - (r-r')^\alpha(\dot n_\alpha r_\gamma - r_\alpha \dot n_\gamma)\bigg(1+\frac{1}{2}\dot n^\sigma(r+r')_\sigma\bigg)\bigg)\bigg] - H_\beta(r)\dot H^\beta(r')R^2 - \frac{1}{2}H_\beta(r)H^\beta(r')R^2\ddot n^\gamma(r+r')_\gamma - \rho(r',s)\bigg[R^2 H_\beta(r,s)\dot n^\beta\bigg(1+\frac{1}{2}\dot n^\alpha(3r-r')_\alpha\bigg) - v^\gamma\partial_\gamma(r_\beta\varphi(r,s))(r-r')^\beta\bigg]\bigg\} + o(\epsilon^2\mathcal{E}MT^{-1}). \tag{5.37}$$

Combining this with (5.36) and the definition of m,

$$\dot{m} \simeq -n_a \Psi^a_{\text{ext}} + \int d^3r d^3r' \frac{1}{R^3}\bigg\{\bigg(\varphi(r,s) - \frac{q}{4\pi|r|^3}\bigg) H^\gamma(r',s)\bigg[(\dot{n}_\gamma (r-r')^\alpha r_\alpha - r_\gamma \dot{n}^\alpha (r-r')_\alpha)\bigg(1 + \frac{1}{2}\dot{n}^\lambda (r+r')_\lambda\bigg) + \frac{1}{2}R^2(|\dot{n}|^2 r_\gamma + \dot{n}^\beta r_\beta \dot{n}^\gamma)\bigg] + \rho(r',s)[R^2 H_\beta(r,s)\dot{n}^\beta\bigg(1 + \frac{1}{2}\dot{n}^\alpha(3r - r')_\alpha\bigg) - v^\gamma \partial_\gamma (r_\beta \varphi(r,s))(r-r')^\beta\bigg]\bigg\} + o(\epsilon^2 \mathcal{E} M/T), \quad (5.38)$$

which is considerably simpler than (5.37).

Although we now have all of the results necessary to compute each tetrad component of $\Psi^{[ab]}_{\text{self}}$ to second-order, the resulting expressions are extremely lengthy (and correspondingly difficult to interpret). It was also seen when examining the radiative self-fields that the second-order terms in the self-torque were not necessary to reach the point particle limit. For both of these reasons, we only compute it here to first order. Combining (A62) and (5.32),

$$\Psi^{[\alpha\beta]}_{\text{self}} \simeq \int d^3r d^3r' \frac{\dot{n}^{[\alpha} r^{\beta]}}{R}\bigg[\frac{1}{2}(\rho(r,s)\rho(r',s) + H_\gamma(r,s)H^\gamma(r',s)) + \bigg(\varphi(r,s) - \frac{q}{4\pi|r|^3}\bigg)\rho(r',s)\bigg] + o(\epsilon^2 \mathcal{E} M), \quad (5.39)$$

$$n_a e^\beta_b \Psi^{[ab]}_{\text{self}} \simeq \frac{1}{2}\int d^3r d^3r' \frac{1}{R^3}\bigg\{\rho(r,s)H_\alpha(r',s)r'^\beta\bigg[(r-r')^\alpha\bigg(1 - \frac{1}{2}\dot{n}^\gamma (r-r')_\gamma\bigg) + \frac{1}{2}R^2\dot{n}^\alpha\bigg] - \bigg(\varphi(r,s) - \frac{q}{4\pi|r|^3}\bigg) \times \bigg[R^2 r_\gamma \dot{n}^{[\beta}H^{\gamma]}(r',s) + 2r_\gamma (r-r')^{[\beta}H^{\gamma]}(r',s)\bigg(1 + \frac{1}{2}\dot{n}^\gamma (r+r')_\gamma\bigg) - r_\alpha H^\alpha(r',s)(r^\beta \dot{n}^\gamma r'_\gamma - r'^\beta \dot{n}^\gamma r_\gamma) + (r-r')^\alpha r_\alpha (H^\beta(r',s)\dot{n}^\gamma r_\gamma - r^\beta \dot{n}_\gamma H^\gamma(r',s))\bigg]\bigg\} + o(\epsilon^2 \mathcal{E} M). \quad (5.40)$$

VI. SPECIAL CASES

A. Current-dominated particles

The full equations of motion derived above are obviously quite complicated, so it is useful to specialize the discussion somewhat. First assume that the particle's charge density is sufficiently small that it can be entirely dropped without losing any accuracy. This requires that $\varphi < \epsilon^2 H$. Then (5.39) becomes equivalent to

$$\dot{S}^{\alpha\beta} \simeq -2\Psi^{[\alpha\beta]}_{\text{ext}} + \int d^3r d^3r' \frac{r^{[\alpha}\dot{n}^{\beta]}}{R} H_\gamma(r,s)H^\gamma(r',s) + o(\epsilon^2 \mathcal{E} M). \quad (6.1)$$

This can be quite large. In order to make sure that it satisfies (4.32), let

$$\mathcal{E} < \epsilon. \quad (6.2)$$

Now, $M_{\text{em}} \lesssim \epsilon M$, so the remaining equations of motion can be derived from (5.35), (5.38), and (5.40):

$$\dot{m} \simeq -n_a \Psi^a_{\text{ext}} + o(\epsilon^3 m T^{-1}), \quad (6.3)$$

$$m\dot{n}^\alpha \simeq -\Psi^\alpha_{\text{ext}} + o(\epsilon^3 m T^{-1}), \quad (6.4)$$

$$m v^\alpha \simeq S^{\alpha\beta}\dot{n}_\beta - 2\Psi^{[ab]}_{\text{ext}} e^\alpha_a n_b + o(\epsilon^3 m). \quad (6.5)$$

An expression for $\dot{v}^\alpha$ is obtained by differentiating (3.34), which gives

$$m\dot{v}^\alpha \simeq -\Psi^\alpha_{\text{ext}} + n_a \Psi^a_{\text{ext}} v^\alpha + S^{\alpha\beta}\ddot{n}_\beta - 4\Psi^{[\alpha\beta]}_{\text{ext}}\dot{n}_\beta - 2\dot{\Psi}^{[ab]}_{\text{ext}} e^\alpha_a n_b + 2\int d^3r d^3r' \frac{r^{[\alpha}\dot{n}^{\beta]}\dot{n}_\beta}{R} \times H_\gamma(r,s)H^\gamma(r',s) + o(\epsilon^3 m T^{-1}). \quad (6.6)$$

The presence of a (potentially) significant self-torque here is completely different than the situation that arose when considering only the radiative component of the self-field. Even adopting conditions (5.16), (5.27), and (5.28) would not generically recover the Lorentz-Dirac equation.

Instead, we find (assuming $\mathcal{E} \gtrsim \epsilon^2$) that (5.29) and (5.30) (with $\dot{M} \to \dot{m}$) would remain correct, although (5.31) is weakened to $|\dot{S}| < \epsilon^2 m$. In this case, the center-of-mass line is governed by

$$m\dot{v}^\alpha \simeq -q e^\alpha_a v_b F^{ab}_{\text{ext}} + \tfrac{2}{3}q^2 \ddot{n}^\alpha + 2M_{\text{em}} D^{[\alpha}_{\text{H}} \dot{v}^{\beta]}\dot{v}_\beta + o(\epsilon^3 m T^{-1}), \quad (6.7)$$

where

$$D^\alpha_{\text{H}}(s) := \frac{1}{M_{\text{em}}}\int d^3r d^3r' \bigg(\frac{r^\alpha}{R}\bigg) H_\gamma(r,s)H^\gamma(r',s). \quad (6.8)$$

Given (5.36), D^α_{H} appears to be related to the shift in the position of the effective center-of-mass due to the electromagnetic self-energy.

The force that it generates clearly becomes irrelevant when $M_{\text{em}} D^{[\alpha}_{\text{H}} \dot{v}^{\beta]}\dot{v}_\beta \lesssim \epsilon^3 m/T$, which occurs when D^α_{H} (nearly) coincides with $\dot{v}^\alpha$, or more generically if

$$|D_{\rm H}| \lesssim \epsilon D. \quad (6.9)$$

This happens, for example, in cases where $H^\gamma(r, s) \simeq \pm H^\gamma(-r, s)$. (6.9) can therefore be thought of as a restriction on the allowed asymmetry in the particle's current structure about the center-of-mass. Given (3.15), a similar intuitive interpretation can also be applied to (5.28), so there is some overlap between these two conditions. It does not appear that either one strictly implies the other, however.

Also note that if (6.9) holds, (6.2) is no longer required. The case $\mathcal{E} \sim 1$ is then quite interesting if the spin and/or dipole moment are non-negligible. When this happens, the left-hand side of (6.5) needs to be replaced by Mv^α, which means that M and m must both be kept track of. The expression for $\dot{M}$ is not simple, so this is a considerable complication.

B. Charge-dominated particles

Treating the opposite case, we now consider particles with very small internal currents. In particular, let $H < \epsilon^2\varphi$. Given (A40), this is about as small as H could possibly be without fine-tuning. (3.8) and (5.3) show that the internal currents which arise due to charges rearranging themselves (via elasticity, rotation, etc.) are bounded by this same amount. So we actually have that $|j| \lesssim \epsilon^2\varphi$.

(5.39) now reduces to

$$\dot{S}^{\alpha\beta} \simeq -2\Psi^{[\alpha\beta]}_{\rm ext} + \int {\rm d}^3r{\rm d}^3r' \frac{r^{[\alpha}\dot{n}^{\beta]}}{R}\rho(r', s)\bigg[\rho(r, s) + 2\bigg(\varphi(r, s) - \frac{q}{4\pi|r|^3}\bigg)\bigg] + o(\epsilon^2\mathcal{E}m). \quad (6.10)$$

Once again, this expression can easily violate (4.32) unless $\mathcal{E} < \epsilon$. Assuming this,

$$\dot{m} \simeq -n_a\Psi^a_{\rm ext} + \int {\rm d}^3r{\rm d}^3r' \frac{1}{R} v^\alpha\partial_\alpha\rho(r, s)\rho(r', s) + o(\epsilon^3 mT^{-1}), \quad (6.11)$$

$$m\dot{n}^\alpha \simeq -\Psi^\alpha_{\rm ext} + \tfrac{2}{3}q^2\ddot{n}^\alpha + o(\epsilon^3 mT^{-1}), \quad (6.12)$$

$$mv^\alpha \simeq S^{\alpha\beta}\dot{n}_\beta - 2\Psi^{[ab]}_{\rm ext}e^\alpha_a n_b + o(\epsilon^3 m). \quad (6.13)$$

By analogy to the current-dominated case, we can now define

$$D^\alpha_\varphi := \frac{1}{M_{\rm em}} \int {\rm d}^3r{\rm d}^3r' \bigg(\frac{r^\alpha}{R}\bigg)\rho(r', s) \times \bigg[\rho(r, s) + 2\bigg(\varphi(r, s) - \frac{q}{4\pi|r|^3}\bigg)\bigg]. \quad (6.14)$$

Again, this looks like the center-of-electromagnetic mass, although the interpretation is not quite so direct as it was for $D^\alpha_{\rm H}$.

Now let $\mathcal{E} \gtrsim \epsilon^2$, and adopt (5.16), (5.27), and (5.28). As before, these conditions imply that $|v| < \epsilon^3$, so $\dot{m} < \epsilon^3 mT^{-1}$. The Lorentz-Dirac equation is recovered when $M_{\rm em}D^{[\alpha}_\varphi\dot{v}^{\beta]}\dot{v}_\beta \lesssim \epsilon^3 mT^{-1}$. If the acceleration is not restricted to be in a specific direction, then it is convenient to satisfy this condition by requiring

$$|D_\varphi| \lesssim \epsilon D. \quad (6.15)$$

This clearly holds when (for example) $\varphi(r, s) \simeq \pm\varphi(-r, s)$, so $|D_\varphi|/D$ can be considered a measure of the charge's internal symmetry.

The two examples discussed here are by the far the simplest, although other ones may also be of interest. For example, the condition $|j| \lesssim \epsilon\varphi$ could have replaced $|j| \lesssim \epsilon^2\varphi$ in this section. This would allow for some charge-current coupling without introducing undue complexity. Similarly, one might be interested in the case where $\varphi \lesssim \epsilon H$, or the general scenario where φ and H^α are symmetric (or antisymmetric) about $r = 0$.

VII. CONCLUSIONS

A. Results

Starting from the fundamental Eqs. (2.1), (2.2), (2.3), and (2.4), we have now derived equations of motion describing a very wide variety of classical extended charges in flat spacetime. One of our main motivations has been to investigate the validity of the commonly-held notion that "small" charges can often be treated as though they were perfectly pointlike. Because of this, the precise implications of each approximation have been emphasized, and all of our results have been kept as general as (reasonably) possible.

To review, charge distributions were considered where all significant length scales remained of order $D(s)$ within each time slice $\Sigma(s)$. An acceleration time scale $T \gg D$ was then defined to place lower bounds on $|\dot{n}|^{-1}$ and $|\ddot{n}|^{-1/2}$. This was assumed to relate to the time scales of the body's internal motions as well. Specifically, the s-derivatives of the "charge potential" φ were required to satisfy (5.3) and (5.4). Similar restrictions were placed on the derivatives of H^α as well. The last major restriction required that the spin and torque play a relatively small but non-negligible role in the system's behavior. This was formalized by assuming that $|n - v| = |v|$ remained less than about $\epsilon^2 = (D/T)^2 \ll 1$. All of these assumptions are kinematic, and cannot automatically be assumed to hold without any regard for the sizes of the external fields, spin, and so on. Self-consistency is preserved by requiring the angular momentum, force, and torque obey the bounds discussed in Sec. IVA.

The restriction given on S^{ab} essentially states that the body's rotational period cannot be less than T. The relations satisfied by the force and torque are not quite as simple, though. Both of these quantities depend on J^a, so any bounds placed on them must affect the class of allowable current distributions. While this can be taken into account in different ways, we have chosen to impose restrictions on the dipole moment and the 'relative self-energy' $D_{\rm em}/D = \mathcal{E}$. If the dominant force is just $-qF^{ab}_{\rm ext}v_b$, Q^{ab} must satisfy (4.37).

The bound on $\mathcal{E}$ is more complicated to summarize, as it is sensitive to the specific case under consideration. The calculation using the radiative self-field is the least restrictive; allowing all $\mathcal{E} \lesssim 1$. One could technically let the electromagnetic radius exceed the physical one in this case, although we take the point of view that this would be too unphysical. Regardless, taking into account the full retarded self-field shows that there exists a significant class of charge distributions where the self-torque will become too large whenever $\mathcal{E} \gg \epsilon^2$. In scenarios where the charge-current coupling is negligible, the relative self-energy can be of order ϵ. Still, there are certain special cases such as spherical symmetry where the restriction on $\mathcal{E}$ can be completely relaxed.

Despite appearances, these assumptions are really no different than the standard slow-motion approximation. Using it, the radiative and retarded self-fields in a neighborhood of W were shown to be approximated by (5.8) and (5.32) respectively. These expressions are quite general, and although the retarded field is written using the peculiar constructions of Dixon's theory, it can easily be translated into a more conventional notation by using (3.7) and (3.8). (4.17), (4.18), (4.19), (4.20), and (4.21) also provide a convenient starting point for this.

These expressions for the fields were then combined with the exact forms of Dixon's equations of motion to yield the results contained in Sec. V. All of the approximations adopted here were therefore applied only to compute the fields. In the (unrealistic) case that the body was assumed to couple only to the radiative component of its self-field, the center-of-mass line was found to evolve according to (5.12), (5.13), (5.14), and (5.15). The external forces and torques that appear in these expressions are given exactly by (A61) and (A62), or more intuitively by (A63) and (A64).

In order to investigate the Lorentz-Dirac limit, it was first necessary to require the particle's radius to be sufficiently small that the dipole and higher contributions to the external force could be considered negligible without having to throw away the Lorentz-Dirac component of the self-force. This idea was summarized in the condition (5.16), which transformed the equations of motion in the radiative case into (5.17), (5.18), (5.19), (5.20), and (5.21). Recovering the Lorentz-Dirac equation was then seen to require bounding the angular momentum and dipole moment by (5.27) and (5.28) respectively.

Since this limit is meant to describe a nonspinning particle, the bound on the angular momentum is hardly surprising. (5.28) is rather different, though. It shows that even when ignoring the "singular" portion of the self-field, the Lorentz-Dirac equation does not apply to all small charge distributions. If a problem were to involve very large (external) field strengths and small self-energies, the class of charges which move like a point particle would in fact become extremely small. Oddly enough, this problem disappears for large self-energies. In these cases, the restriction on the dipole moment is no stronger than was required for self-consistency of the initial slow-motion assumptions.

This issue is made considerably more complex when the body is allowed to interact with its full retarded self-field. Without placing any restrictions on the spin or dipole moment, $\dot{n}^\alpha$ was found to be given by (5.35). This involves an effective mass $m = M + M_{\rm em}$, which involves an electromagnetic contribution given by (5.36). It does not generally remain constant, but rather evolves according to (5.38). There are also self-torques given by (5.39) and (5.40). Combining these expressions with (A57) and (3.34) recovers the full equations of motion. These are our central result. They provide a considerable generalization of the Lorentz-Dirac equation.

They are also much more complicated than the equations derived using only the radiative portion of the self-field. The majority of this complexity arises from interactions between φ and H^α, which—given (3.7) and (3.8)—can be viewed as charge-current couplings in the center-of-mass frame. Predictably then, our results are considerably simplified when only one of these quantities is significant. φ will (generically) drop out of the equations of motion only if it is of order $\epsilon^2 H$ or less. The reverse is also true. Both of these cases were discussed in detail in Sec. VI. It should be emphasized that these restrictions on the relative magnitudes of the charge and current densities are exceptionally strong. Simply saying that one of these quantities is much larger than the other without any further qualification is not sufficient to remove the coupling terms.

Even in these cases, though, the previous restrictions on λ, S^{ab}, and Q^{ab} must be supplemented by either (6.9) or (6.15) in order to recover the Lorentz-Dirac equation. Intuitively, these relations ensure that the 'center-of-electromagnetic mass' is not too far from $r = 0$. This is roughly what was already required by (5.28), although the two conditions are not quite the same.

Note that the assumptions we have given to be able to derive the Lorentz-Dirac equation are not exhaustive. They were the most obvious choices obtained by examining our general equations of motion, but are slightly more restrictive than necessary. Still, they seem to be reasonably effective for most systems that are not finely-tuned.

B. Discussion

In summary, it was shown that the Lorentz-Dirac limit is rather delicate, and that the radiative self-field is rarely an adequate replacement for the retarded one. It is now interesting to speculate how these results might generalize in curved spacetime. One might be interested in the motion of a charged particle moving in a background spacetime, an uncharged body allowed to generate its own gravitational field, or even the general case of a massive charge. With the exception of this last possibility (which does not necessarily follow from the two simpler calculations), such systems have been considered in the past by a number of authors [4,13,18,20,30,37,38]. All of these calculations have either applied a point particle ansatz, or considered only very special (though usually astrophysically motivated) classes of extended bodies. Both such methods have agreed with each other, although it is clear that *some* extended bodies must exist which do not move like point particles.

Finding the scope of the existing MiSaTaQuWa equation [13]—as well as its generalization—would likely require an analysis similar to the one given here. The methods used in this paper have in fact been chosen specifically for their ability to be easily applied to fully dynamic spacetimes (except for those used to expand the self-field). To see this, it should first be mentioned that all of the advantages of Dixon's formalism are retained without approximation in full general relativity [27,28].

This provides two essential results. The first of these is a natural notion of a center-of-mass line. This definition has been proven to satisfy nearly all of the intuitive notions that one might expect of such an object [33], so it can be considered a reasonable measure of a particle's "average" position. Just as importantly, it is exactly determined by a finite number of ordinary differential equations [28,31]. The analogous equations here were (3.32), (3.33), (3.34), and (A57). These expressions are only slightly complicated by the transition to curved spacetime.

The second contribution of Dixon's formalism is the set of stress-energy moments itself. These are all contained in the 'stress-energy skeleton' $\hat{T}^{ab}(r, s)$. Here r is a coordinate in the tangent space of $z(s)$, which is an essential point that was not obvious in flat spacetime. Importantly, the constraints on $\hat{T}^{ab}$ implied by the generalized form of (2.4) are exactly the same in all spacetimes. This obviously includes the flat case summarized in the appendix. Just as (A24) gave the general form for any current skeleton $\hat{J}^a$ satisfying the constraints, one can also find all possible forms of $\hat{T}^{ab}$ [27].

As with current skeleton, not all of these possibilities are physically reasonable. Some will be singular, and others will have supports extending to spatial infinity. We conjecture that such cases can be removed exactly as they were in the appendix for the current moments. This would leave a set of "reduced moment potentials" that could be arbitrarily generated from some simple recipe. An automatic byproduct of this reduction process would be a relatively straightforward method of generating T^{ab} from the potentials. The analogous results in this paper were that the current skeleton was determined by the freely-specifiable functions φ and $\bar{H}^{\alpha\beta}$. These in turn generated J^a via (3.7), (3.8), and (3.16).

In the end, such reduced potentials for the stress-energy moments would serve (in combination with solutions of the ODE's for the linear and angular momenta) as a nearly background-independent way of specifying physically reasonable conserved stress-energy tensors. Besides the intrinsic elegance of such a construction, it would also solve several problems that did not arise in the present paper. The most important of these is the location of the center-of-mass. At any s, this is clearly determined by T^{ab} on $\Sigma(s)$. The definition is highly implicit, however, and it is almost always impractical to apply in practice. Specifying the matter via $\hat{T}^{ab}$ would avoid this problem. Constructing it in the natural way would *start* with the center-of-mass position. It would also incorporate p^a and S^{ab} automatically. These quantities obviously must solve certain differential equations, and writing down a T^{ab} on each time slice with the correct momenta would be extremely difficult with any generality. These issues were the main motivations for discussing the current moments in so much detail in the appendix, as well as the use of φ and H^α throughout this paper instead of J^a. These choices slightly obscure the results here, but would be essential in any generalization.

With this formalism in place, one would then need to compute the metric to find the body's motion. As in the electromagnetic case, this is the most difficult step. Of course, Einstein's equation is considerably more complicated than Maxwell's, so it may be prohibitive to carry out the calculation by hand (except in special cases). Regardless, the above procedure could be integrated into existing numerical relativity codes. This would then allow one to rigorously study spacetimes with nonsingular matter fields without having to solve the conventional elasticity (or Navier-Stokes) equations. The most convenient such systems in this formalism may not represent the most astrophysically interesting types of matter, although there is no shortage of important problems in numerical relativity where the details of the matter distributions are not a primary concern.

Whatever the results of such inquiries, our results in electromagnetism can be used to speculate how point particle methods might break down in gravitational self-force problems. For this, it is useful to give an intuitive explanation for such failures in Maxwell's theory. The simplest of these derives from the fact that the center-of-mass does not generally correspond to anything that could be called a 'center-of-charge.' In these cases, electromagnetic self-forces effectively act through a lever arm. This induces a torque, which in turn affects the particle's overall motion. The situation is made particularly complicated when the self-force is broken up into several pieces (as is natural). Each such piece tends to act through a different point, and there is no particular reason that any of them should coincide with the center-of-mass (although the initial restriction (4.37) on the magnitude of the dipole moment does help).

In contrast, one might expect that gravitational self-forces would always act through the center-of-mass; effectively removing this type of effect. This is not necessarily true, though. The center-of-mass definition only requires that $t^{abc}n_b n_c = 0$, which does not necessarily mean that t^{abc} completely vanishes. The portion of the self-force acting as an effective mass could also produce a significant torque. (A47) shows, however, that t^{abc} does vanish whenever $S^{ab} \to 0$. This is the only case in which the MiSaTaQuWa equation can be reasonably expected to hold, so imposing it at the start should remove any problems. Since the dipole moment here is entirely dependent on S^{ab} (unlike in the electromagnetic case) it should be relatively easy to arrange for an initially-vanishing angular momentum to remain small. These sorts of effects might therefore be expected to be less troublesome in gravity than in electromagnetism, although they will probably still exist.

This type of mechanism did not lead to most of the complications found in this paper, however. These were instead due to couplings between the charge and 3-current densities as viewed in the center-of-mass frame. In gravity, the situation could be even worse. Similar interactions might exist between mass, 3-momentum, and stress densities. Intuitively, though, it would appear that adopting appropriate energy conditions should considerably soften these interactions.

ACKNOWLEDGMENTS

I would like to thank Pablo Laguna for reading over this manuscript, and for many helpful discussions leading up to the results presented here. I also thank Amos Ori for useful comments. This work was supported in part by NSF Grant No. PHY-0244788 and the Center for Gravitational Wave Physics funded by the National Science Foundation under Cooperative Agreement PHY-0114375.

APPENDIX A: DIXON'S FORMALISM

1. Current Multipoles

This portion of the appendix reviews Dixon's decomposition of the electromagnetic current vector into multipole moments. It also derives natural ''potential functions'' that can be used to generate sets of moments for all physically interesting current vectors satisfying (2.3). These objects are shown to determine J^a in a simple way. The main reasons for these constructions are explained in Sec. VII B, although some secondary points are also mentioned in Sec. III. In short, analogs of these steps would become essential in the gravitational self-force problem, so we include them here to allow a relatively straightforward generalization.

The notation here will be that defined in Sec. III. Using it, we can define multipole moments for the current vector. Such objects are usually constructed by integrating a source function against a suitable number of radius vectors. Defining $r^a = x^a - z^a(s)$ for $x \in \Sigma(s)$, one might therefore expect that for $n \geq 0$, the $2n$-pole moment could be given by

$$Q^{b_1\cdots b_n a}(s) := \int \mathrm{d}^4x\ r^{b_1}\cdots r^{b_n}\hat{J}^a(x - z(s), s), \tag{A1}$$

where we have assumed the existence of some distribution $\hat{J}^a(x, s)$ which is related to $J^a(x)$, but has compact support in x. This will be called the current skeleton. Note that (A1) automatically implies that

$$Q^{b_1\cdots b_n a} = Q^{(b_1\cdots b_n)a} \tag{A2}$$

for all $n \geq 1$.

It is rather cumbersome to keep track of each $Q^{\cdots}$ directly, so we instead define a generating function

$$G^a(k, s) := Q^a(s) + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}k_{b_1}\cdots k_{b_n}Q^{b_1\cdots b_n a}(s). \tag{A3}$$

An arbitrary multipole moment can now be extracted from this in the usual way:

$$Q^{b_1\cdots b_n a}(s) = i^n(\partial^{b_1}\cdots\partial^{b_n}G^a(k, s))|_{k=0}. \tag{A4}$$

G^a is therefore completely equivalent to the set $\{Q^a, Q^{ba}, \ldots\}$.

Although this is a useful property, the definition of G^a is not simply a mathematical convenience. Using (A1) and (A3), it can be shown that

$$G^a(k, s) = \int \mathrm{d}^4r\ \hat{J}^a(r, s)e^{-ik\cdot r}. \tag{A5}$$

The Fourier transform of G^a,

$$\tilde{G}^a(r, s) := \int \mathrm{d}^4k\ G^a(k, s)e^{ik\cdot r}, \tag{A6}$$

is therefore proportional to $\hat{J}^a$:

$$\tilde{G}^a(r, s) = (2\pi)^4\hat{J}^a(r, s). \tag{A7}$$

This shows that if J^a and $\hat{J}^a$ equivalent in an appropriate sense, the set of moments can be used to completely reconstruct the current vector.

In order to relate the current to its skeleton, it is convenient to think of $\hat{J}^a$ as a linear functional on the space of all C^∞ test functions with compact support (as is typically done in distribution theory). In particular, knowing

$$\langle\hat{J}^a(r, s), \phi_a(x)\rangle := \int \mathrm{d}^4\ x\hat{J}^a(r, s)\phi_a(x) \tag{A8}$$

for all suitable test functions $\phi_a(x)$ can be used to define $\hat{J}^a$. An analogous statement can also be made for J^a. These two objects can then be related to each other by writing $\langle J^a, \phi_a \rangle$ in terms of $\langle \hat{J}^a, \phi_a \rangle$. The latter expression depends on s, while the former one does not. It is therefore most straightforward to link the two by simply integrating out the s-dependence:

$$\langle J^a(x), \phi_a(x) \rangle = \int \mathrm{d}s \langle \hat{J}^a(r, s), \phi_a(x) \rangle, \tag{A9}$$

$$= \frac{1}{(2\pi)^4} \int \mathrm{d}s \langle G^a(k, s), \tilde{\phi}_a(k) e^{-ik\cdot z(s)} \rangle. \tag{A10}$$

Following [26], we take this (along with (A3)) to *define* what is meant by saying that the $Q^{\cdots}$'s are 'multipole moments of J^a.'

This is not unique definition, however. To remove the remaining freedom in a useful way, we simply state Dixon's results [24,26,27]. Let

$$n_{b_1} Q^{b_1 \cdots b_{n-1}[b_n a]} = 0, \tag{A11}$$

$$Q^{(b_1 \cdots b_n)} = 0, \tag{A12}$$

for all $n \geq 2$. Also assume that the monopole moment has the special form

$$Q^a = q v^a, \tag{A13}$$

where q is the total charge as it is usually defined.

Now choose a test function of the form $\phi_a(x) = \partial_a \phi(x)$, with $\phi(x)$ itself also a test function. Then (A4), (A10), and (A12) can be used to show that

$$\begin{aligned}\langle \partial_a J^a, \phi \rangle &= -\langle J^a, \partial_a \phi \rangle, \\ &= \frac{i}{(2\pi)^4} \int \mathrm{d}s \ \langle k_a G^a(k, s), \tilde{\phi}(k) e^{-ik\cdot z(s)} \rangle, \\ &= -\int \mathrm{d}s \ q v^a(s) \partial_a \phi(z(s)), \\ &= -\int \mathrm{d}s \ q \frac{\mathrm{d}}{\mathrm{d}s} \phi(z(s)).\end{aligned} \tag{A14}$$

This must vanish for all ϕ, which can be ensured by simply requiring that

$$\dot{q} = 0. \tag{A15}$$

This (trivial) evolution equation is the only one implied by (2.3). It can be shown that moments satisfying (A11)–(A13) and (A15) describe any J^a with the given properties in a uniquely simple way [24,26,27]. A precise statement of the theorem that was proven is contained in [27].

These conditions allow a great deal of freedom in choosing different moments. The fact that there is only one evolution equation implied by (2.3) does not mean that the higher moments necessarily remain constant. Rather, they can be given an almost arbitrary time dependence. This is a reflection of the fact that we have not yet chosen to model any particular type of matter. Specifying how the higher moments change in time is essentially equivalent to choosing an equation of state. Although this must be given in order to have a well-defined initial value problem, self-forces and self-torques can be written down without ever having to explicitly evaluate the time derivatives of the current moments. This allows us to derive equations of motion valid for a very large class of systems.

In order to do so, we first need to pick out sets of moments (or equivalently their generating functions) which represent physically reasonable current vectors. Despite appearances, this cannot be done arbitrarily. Dixon's theorem ensures that any nonsingular current vector with support W can be described by a set of moments with the given properties, although the reverse is not necessarily true. Extra conditions need to be imposed in order to ensure that the J^a associated with any particular G^a (or $\hat{J}^a$) has the correct smoothness and support properties.

To gain some insight into this, fix some test function ϕ_a. From this, construct a second test function ϕ'_a which agrees with ϕ_a everywhere except in an infinitesimal neighborhood of Z. Assume that the support of ϕ'_a does not include Z itself. (A3) and (A10) then show that any finite number (and only finite number) of multipoles can be changed without affecting $\langle J^a, \phi'_a \rangle$. The same cannot be said for $\langle J^a, \phi_a \rangle$. But for any physical current vector, $\langle J^a, \phi'_a \rangle \simeq \langle J^a, \phi_a \rangle$. This shows that any finite subset of an admissible collection of moments is completely determined by its complement.

Because of this, it is not reasonable to impose conditions directly on the individual moments to ensure that their associated current vector is physically acceptable. Such restrictions are most easily stated in terms of $\tilde{G}^a$ or $\hat{J}^a$, but doing so first requires finding how $\hat{J}^a$ is affected by (A11)–(A13) and (A15). This is now done by finding the general form of G^a, and then taking its Fourier transform.

It is shown in [27] that the constraint equations are precisely equivalent to requiring that the generating function have the form

$$G^a(k, s) = G^a_{(1)}(h \cdot k, s) + (n \cdot k) G^a_{(2)}(h \cdot k, s), \tag{A16}$$

where $n \cdot k := n_a k^a$, $(h \cdot k)^a = h^a_b k^b$, and $n_a G^a_{(2)} = 0$. Also,

$$k_a G^a(k, s) = q k_a v^a(s), \tag{A17}$$

$$\partial^{[a} G^{b]}_{(2)}(h \cdot k, s) = 0. \tag{A18}$$

It is now useful to take Fourier transforms of these equations to find their equivalent forms when representing the moments by $\hat{J}^a$ $(= \tilde{G}^a/(2\pi)^4)$. $\tilde{G}^a_{(1)}$ does not take on any special form, although (A19) shows that

$$\langle r^c h^{[a}_c \tilde{G}^{b]}_{(2)}(r, s), \phi(r) \rangle = 0. \tag{A19}$$

This equation is solved by any $\tilde{G}^a_{(2)}$ of the form

$$\tilde{G}^{a}_{(2)}(r,s) = h^a_b(s) r^b \tilde{G}_{(2)}(r,s), \tag{A20}$$

for all functions $\tilde{G}_{(2)}$ (Despite the notation, it will shortly be clear that the inverse Fourier transform of $\tilde{G}_{(2)}$ does not exist in general). Note that this is not the most general solution of (A19). A term of the form $g^a(n \cdot r, s)\delta^3(h \cdot r)$ may also be added to $\tilde{G}^a_{(2)}$, although we choose to ignore this possibility.

In any case, (A16) and (A20) give an explicit form for $\tilde{G}^a$

$$\tilde{G}^{a}(k,s) = \tilde{G}^a_{(1)} - i h^a_b r^b n^c \partial_c \tilde{G}_{(2)}. \tag{A21}$$

Since $G^a_{(1)}$ and $G^a_{(2)}$ are independent of $n \cdot k$, their (four-dimensional) Fourier transforms must be proportional to $\delta(n \cdot r)$. It is therefore natural to define quantities A, B^a, and C such that

$$\tilde{G}^{a}_{(1)}(r,s) = (2\pi)^4 \delta(n \cdot r)(A(h \cdot r, s) n^a + B^a(h \cdot r, s)), \tag{A22}$$

$$\tilde{G}_{(2)}(r,s) = -i(2\pi)^4 \delta(n \cdot r) C(h \cdot r, s), \tag{A23}$$

where $B^a n_a = 0$. Combining these expressions with (A21), $\hat{J}^a$ is found to have the form

$$\hat{J}^a(r,s) = \delta(n \cdot r)(A(h \cdot r, s) n^a(s) + B^a(h \cdot r, s)) - \delta'(n \cdot r) C(h \cdot r, s) h^a_b(s) r^b. \tag{A24}$$

This is further restricted by (A17), the Fourier transform of which becomes

$$\begin{aligned} \langle \partial_a \hat{J}^a(x,s), \phi(x) \rangle &= (2\pi)^{-4} i \langle k_a G^a(k,s), \tilde{\phi}(k) \rangle, \\ &= -(2\pi)^{-4} q v^a(s) \langle \tilde{1}, \partial_a \phi(r) \rangle, \\ &= q v^a(s) \langle \partial_a \delta^4(r), \phi(r) \rangle. \end{aligned} \tag{A25}$$

Comparing this to the divergence of (A24) shows that

$$q\delta^3(h \cdot r) = A(h \cdot r, s) - h^a_b \partial_a (r^b C(h \cdot r, s)), \tag{A26}$$

$$q v^a \partial_a \delta^3(h \cdot r) = \partial_a B^a(h \cdot r, s). \tag{A27}$$

The solution to the "homogeneous" analog of (A26),

$$h^a_b \partial_a (r^b C_{(0)}(h \cdot r, s)) = -q\delta^3(h \cdot r), \tag{A28}$$

is

$$C_{(0)}(h \cdot x, s) = -\frac{q}{4\pi |r|^3}, \tag{A29}$$

where $|r|^2 := -h_{ab} r^a r^b \geq 0$. For future convenience, we now define new functions $\varphi(r,s)$ and $q_0(s)$ such that,

$$C = N\left(\varphi - \frac{q_0(s)}{4\pi |r|^3}\right). \tag{A30}$$

N is just the lapse, as given in (3.5).

In terms of q_0 and φ, (A26) now becomes

$$A(h \cdot r, s) = h^a_b \partial_a (r^b N \varphi(h \cdot r, s)) + (q - q_0(s))\delta^3(h \cdot r) + \frac{q_0}{4\pi} \frac{\dot{n}^a r_a}{|r|^3}. \tag{A31}$$

$\hat{J}^a$ is therefore given by (A24) with A and C having the respective forms (A31) and (A30). We also have that B^a satisfies (A27). This effectively parameterizes all sets of moments satisfying Dixon's constraints.

(A9) can now be used to relate $\hat{J}^a$ to J^a. It is convenient to do this in the (r^α, τ) coordinates defined in Sec. III. Using them, (A9) becomes

$$\langle J^a, \phi_a \rangle = \int \mathrm{d}s \int \mathrm{d}^3 r \int \mathrm{d}\tau \, N(x) \hat{J}^a(x - z(s), s) \phi_a(x). \tag{A32}$$

Applying (A24), the τ-integral in this equation can be carried out explicitly:

$$\langle J^a, \phi_a \rangle = \int \mathrm{d}s \int \mathrm{d}^3 r \Big\{ [A n^a + B^a + N^{-2} e^a_\alpha r^\alpha \dot{n}_\beta v^\beta C + N^{-1} e^a_\alpha v^\beta \partial_\beta (r^\alpha C)] \phi_a + N^{-1} C e^a_\alpha r^\alpha \frac{\partial \phi_a}{\partial s} \Big\}, \tag{A33}$$

where all quantities are evaluated at s.

Commuting the s-integral with the spatial ones and integrating the last term by parts,

$$\langle J^a, \phi_a \rangle = \int \mathrm{d}^3 r \int \mathrm{d}s \Big\{ n^a \Big[A + \frac{C}{N} \dot{n}_\beta r^\beta \Big] + e^a_\alpha \Big[B^\alpha + v^\beta \partial_\beta \Big(r^\alpha \frac{C}{N} \Big) - r^\alpha \frac{\partial}{\partial s} \Big(\frac{C}{N} \Big) \Big] \Big\} \phi_a. \tag{A34}$$

But we also have that $\langle J^a, \phi_a \rangle = \int \mathrm{d}^3 r \int \mathrm{d}s N J^a \phi_a$, which gives us an obvious way to explicitly write J^a in terms of A, B^a, and C.

The charge density with respect to the tetrad frame takes on the particularly simple form

$$\rho := n_a J^a, \tag{A35}$$

$$= \partial_\alpha (r^\alpha \varphi) + N^{-1}(q - q_0)\delta^3(r). \tag{A36}$$

Physically, $\rho(r^\alpha, s)$ must be nonsingular and have support W. Without loss of generality, we can therefore choose $q_0 = q$. ρ will then be admissible if $\varphi(r^\alpha, s)$ is a continuous function.

$\hat{J}^a$ was originally introduced as something with compact support (in r^a), so A, B^a, and C must also have compact support (in r^α). Using this along with the requirement that ρ vanish outside W implies that $\varphi = q/4\pi|r|^3$ in this region. So any choice of φ which satisfies this "boundary condition" and is continuous will generate a physically reasonable charge density.

For the 3-current $j^\alpha := e^\alpha_a J^a$, the (necessarily regular) portions contributed by φ can be temporarily ignored to show that

$$j^\alpha = N^{-1}\left[B^\alpha + (\ldots) - v^\beta \partial_\beta\left(\frac{qr^\alpha}{4\pi|r|^3}\right)\right]. \tag{A37}$$

The divergence of this last term is equal to $-qv^\alpha\partial_\alpha\delta^3(r)$, which exactly cancels the divergence of B^α given in (A27). It is therefore useful to write B^α in the form

$$B^\alpha = H^\alpha + v^\beta\partial_\beta\left(\frac{qr^\alpha}{4\pi|r|^3}\right), \tag{A38}$$

with H^α an arbitrary piecewise continuous vector field satisfying $\partial_\alpha H^\alpha = 0$. This ensures that j^α is nonsingular.

Writing out the current in full,

$$j^\alpha = N^{-1}(H^\alpha + v^\beta\partial_\beta(r^\alpha\varphi) - r^\alpha\dot{\varphi}), \tag{A39}$$

it is clear that j^α will have support W if

$$H^\alpha = -v^\beta\partial_\beta\left(\frac{qr^\alpha}{4\pi|r|^3}\right) \tag{A40}$$

outside W. This also guarantees that $\mathrm{supp}(B^\alpha) = W$, as required.

This completes our study of the current moments. Essentially all physically interesting expansions can be extracted from a $\hat{J}^a$ of the form (A24). A, B^a and C are all functions of (r^α, s), and have support W. C has the form (A30), where φ is an arbitrary continuous function equalling $q/4\pi|r|^3$ outside of W. (A31) shows that A is also derived from φ (with $q_0 = q$). B^a has the form (A38), where H^α is a piecewise continuous (3-) vector field satisfying $\partial_\alpha H^\alpha = 0$. The relevant portions of these results are also summarized in Sec. III.

2. Stress-energy moments

Let the multipole moments of T^{ab} be denoted by the set $\{t^{bc}, t^{abc}, \ldots, t^{a_1\cdots a_n bc}, \ldots\}$, where $t^{(a_1\cdots a_n)bc} = t^{a_1\cdots a_n(bc)} = t^{a_1\cdots a_n bc}$. As with the current moments, it is convenient to keep track of this collection with a generating function

$$G^{ab}(k, s) := \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}k_{c_1}\cdots k_{c_n}t^{c_1\cdots c_n ab}(s). \tag{A41}$$

(2.4) inextricably links T^{ab} and J^a, so an expression for $\langle T^{ab}, \phi_{ab}\rangle$ as simple as the one for $\langle J^a, \phi_a\rangle$ is not possible while retaining simple constraint and evolution equations. Still, one might expect that $\langle T^{ab}, \phi_{ab}\rangle$ should at least be proportional to $\int ds\langle\hat{T}^{ab}, \phi_{ab}\rangle$ (where $\hat{T}^{ab} := (2\pi)^4\tilde{G}^{ab}$). Define a distribution $\tilde{\Phi}^{ab} = \tilde{\Phi}^{(ab)}$ to make up the difference:

$$\langle T^{ab}, \phi_{ab}\rangle = \int ds\langle\hat{T}^{ab}(r, s) + \tilde{\Phi}^{ab}(r, s), \phi_{ab}(x)\rangle. \tag{A42}$$

As before, we call all sets $\{t^{\cdots}\}$ satisfying (A41) and (A42) "multipole moments of T^{ab}." This is not a unique definition, however (even if $\tilde{\Phi}^{ab}$ were given). Following [24,27], we will now pick out a set which very simply and naturally implies (2.4).

Using (A14) as a guide, $\langle\partial_a T^{ab}, \phi_b\rangle$ can be found by substituting a test function of the form $\phi_{ab} = \partial_a\phi_b$ into (A42). It is clear that the resulting expression depends on $k_a G^{ab}$, which is analogous to what happened when computing $\partial_a J^a$. In that case, (A12) showed that $k_a G^a$ depended only on Q^a. (2.3) is usually interpreted as an expression of global charge conservation, so it was not unreasonable that it only restricted the monopole moment. In the case of the stress-energy tensor, we expect that (2.4) should have something to say about both the linear and angular momenta of the body (i.e. its monopole and dipole moments). We therefore suppose that $k_a G^{ab}$ involves only t^{ab} and t^{abc}. This can be accomplished by letting

$$t^{(a_1\cdots a_n b)c} = 0 \tag{A43}$$

for all $n \geq 2$.

Using this constraint implies that

$$\langle\partial_b T^{ab}, \phi_a\rangle \propto \frac{1}{(2\pi)^4}\int ds\langle ik_b G^{ab}(k, s), \tilde{\phi}_a(k)e^{-ik\cdot z(s)}\rangle, \tag{A44}$$

$$= \int ds[t^{ab}\partial_a\phi_b(z(s)) + t^{abc}\partial_a\partial_b\phi_c(z(s))]. \tag{A45}$$

Dixon found that the first two moments can be given the special forms [24,27]

$$t^{ab} = p^{(a}v^{b)}, \tag{A46}$$

$$t^{abc} = S^{a(b}v^{c)}, \tag{A47}$$

where we call p^a the linear momentum, and $S^{ab} = S^{[ab]}$ the angular momentum. Using these expressions,

$$\langle\partial_b T^{ab}, \phi_a\rangle \propto \int ds[\dot{p}^a\phi_a(z(s)) + \tfrac{1}{2}(\dot{S}^{ab} - 2p^{[a}v^{b]})\partial_a\phi_b(z(s))]. \tag{A48}$$

In the absence of an electromagnetic field (or a current), the left-hand side of this equation vanishes, and the proportionality sign becomes an equality (since $\tilde{\Phi}^{ab}$ vanishes in this case). Varying ϕ_a then recovers the standard equations of motion for a free particle:

$$\dot{p}^a = 0, \tag{A49}$$

$$\dot{S}^{ab} = 2p^{[a}v^{b]}. \tag{A50}$$

The situation is of course much more complicated when a field is present. Writing out $\langle\partial_a T^{ab}, \phi_b\rangle$ in full, (2.4) implies that

$$\int ds\{[\dot{p}^a\phi_a + \tfrac{1}{2}(\dot{S}^{ab} - 2p^{[a}v^{b]})\partial_a\phi_b] + \langle ik_b\Phi^{ab}, \tilde{\phi}_a e^{-ik\cdot z(s)}\rangle\} = -\langle F^{ab}J_b, \phi_a\rangle. \tag{A51}$$

The right hand side of this equation can now be written in terms of the current moments. Using (A10), the Fourier convolution theorem, and a Taylor series, it can be shown to be [24]

$$\langle F^{ab} J_b, \phi_a \rangle = \frac{1}{(2\pi)^4} \int \mathrm{d}s \bigg\langle \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} k_{c_1} \cdots k_{c_n} \times \Psi^{c_1 \cdots c_n a}, \tilde{\phi}_a(k) e^{-ik\cdot z(s)} \bigg\rangle, \tag{A52}$$

$$\Psi^{c_1\cdots c_n a}(s) := \frac{1}{(2\pi)^4} \bigg\langle \sum_{p=0}^{\infty} \frac{(-i)^p}{p!} l_{d_1} \cdots l_{d_p} \times Q^{d_1\cdots d_p c_1 \cdots c_n b}, \tilde{F}^a{}_b(l) e^{-il\cdot z(s)} \bigg\rangle. \tag{A53}$$

Here, $\tilde{F}^{ab}$ does not quite represent the Fourier transform of F^{ab}, which is not well-defined. It is instead equal to the Fourier transform of some function $^*F^{ab}$ which coincides with the field in some neighborhood of $\Sigma(s) \cap W$, but has compact support. If $^*F^{ab}$ is just as smooth as F^{ab}, its precise form is irrelevant [24].

(A52) now makes it natural to interpret the $\Psi^{\cdots}$'s as multipole moments of the force density exerted on the body. We might therefore expect the net force to be proportional to Ψ^a, and the net torque to $\Psi^{[ab]}$. This identification can be made if Φ^{ab} has the form [24]

$$\Phi^{ab}(k,s) = \frac{1}{(2\pi)^4}\bigg\{\Psi^{(ab)} - ik_c\bigg[\Psi^{c(ab)} - \frac{1}{2}\Psi^{abc}\bigg] + \sum_{n=2}^{\infty} \frac{(-i)^n}{n!n} k_{c_1}\cdots k_{c_n}\bigg[2\Psi^{c_1\cdots c_n(ab)} - \frac{n+2}{n+1}\Psi^{(c_1\cdots c_n ab)}\bigg]\bigg\}. \tag{A54}$$

Combining this with (A51) shows that

$$\int \mathrm{d}s[(\dot{p}^a + \Psi^a)\phi_a + \tfrac{1}{2}(\dot{S}^{ab} - 2p^{[a}v^{b]} + 2\Psi^{[ab]})\partial_a \phi_b] = 0. \tag{A55}$$

Given that this must hold for all possible choices of ϕ_a, it follows that

$$\dot{p}^a = -\Psi^a, \tag{A56}$$

$$\dot{S}^{ab} = 2(p^{[a}v^{b]} - \Psi^{[ab]}). \tag{A57}$$

By construction, these are the only evolution equations implied by (2.4). If we impose one more constraint equation:

$$n_{a_1} t^{a_1\cdots a_{n-2}[a_{n-1}[a_n b]c]} = 0 \tag{A58}$$

for $n \geq 3$, the chosen moments are unique in an appropriate sense. They are also sufficiently general to describe all physically interesting stress-energy tensors [27]. Note that Φ^{ab} only depends on J^a and F^{ab}, and that the constraint Eqs. (A43) and (A58) are independent of these quantities. Portions of the stress-energy tensor which depend on the current have therefore been completely isolated from those which are not.

Changes in the higher moments may once again be interpreted as "equation of state" (this identification is actually more direct in this case). Their evolution is not completely arbitrary, however. Besides respecting the constraint equations, they must also be chosen so that T^{ab} remains physically reasonable. These extra restrictions would be provided by analogs of (A36) and (A39). Although these will not be derived here, they can probably be constructed in a similar way. The presence of the field makes their derivation more complicated, but it should still be possible to repeat all of the steps carried out with the current moments.

It suffices to note that for our purposes, this procedure has resulted in particularly natural definitions for the stress-energy moments – most importantly the linear and angular momenta. It can be shown that the choices made here imply that these momenta are given by (3.1) and (3.2) [24]. Interestingly, the net force and torque do not depend on G^{ab} in any necessary way. They are apparently as independent of the details of the body's internal structure as possible.

Before moving on, we can gain some insight into the force moments defined by (A53). Only the first two of these are important here, and it is straightforward to show that

$$\Psi^a = \langle \hat{J}_b(r,s), F^{ab}(x)\rangle, \tag{A59}$$

$$\Psi^{[ab]} = \langle \hat{J}_c(r,s), r^{[a}F^{b]c}(x)\rangle. \tag{A60}$$

Applying (A24), these expressions take the more explicit forms

$$\Psi^a = \int \mathrm{d}^3 r \bigg\{ N J_b F^{ab} + \frac{\partial}{\partial s}\bigg[\bigg(\varphi - \frac{q}{4\pi|r|^3}\bigg) e^\beta_b r_\beta F^{ab}\bigg]\bigg\}, \tag{A61}$$

$$\Psi^{[ab]} = \int \mathrm{d}^3 r \bigg\{ N J_c r^\alpha e^{[a}_\alpha F^{b]c} + \bigg(\varphi - \frac{q}{4\pi|r|^3}\bigg) e^\gamma_c r_\gamma v^{[a}F^{b]c} + \frac{\partial}{\partial s}\bigg[\bigg(\varphi - \frac{q}{4\pi|r|^3}\bigg) e^\gamma_c r_\gamma r^\alpha e^{[a}_\alpha F^{b]c}\bigg]\bigg\}, \tag{A62}$$

which were derived by a different method in Sec. III.

Although (A61) and (A62) are exact, they are rather difficult to interpret. Their meaning is made considerably more transparent if the field can be expanded in a Taylor series inside $\Sigma(s) \cap W$. Then (A59) and (A60) together with (A1) show that

$$\Psi^a(s) \simeq \sum_{\ell=0}^{L} \frac{1}{n!} Q^{c_1\cdots c_\ell b}(\partial_{c_1} \cdots \partial_{c_\ell} F^a{}_b)_{z(s)}, \quad \text{(A63)}$$

$$\Psi^{ab}(s) \simeq \sum_{\ell=0}^{L} \frac{1}{n!} Q^{d_1\cdots d_\ell ac}(\partial_{d_1} \cdots \partial_{d_\ell} F^b{}_c)_{z(s)}. \quad \text{(A64)}$$

When F^{ab} is approximately constant throughout the charge, we recover the Lorentz force law, $\dot{p}^a = -\Psi^a \simeq -qF^{ab}v_b$. Unfortunately, these series are not useful when the field varies considerably over $\Sigma(s) \cap W$. And this is exactly what the self-field does.

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