PHYSICAL REVIEW D **73,** 064028 (2006)

Chaplygin traversable wormholes

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(Received 1 November 2005; published 22 March 2006)

The generalized Chaplygin gas (GCG) is a candidate for the unification of dark energy and dark matter, and is parametrized by an exotic equation of state given by $p_{\rm ch} = -A/\rho_{\rm ch}^{\alpha}$, where A is a positive constant and $0 < \alpha \leq 1$. In this paper, exact solutions of spherically symmetric traversable wormholes supported by the GCG are found, possibly arising from a density fluctuation in the GCG cosmological background. To be a solution of a wormhole, the GCG equation of state imposes the following generic restriction $A < (8\pi r_0^2)^{-(1+\alpha)}$, where r_0 is the wormhole throat radius, consequently violating the null energy condition. The spatial distribution of the exotic GCG is restricted to the throat neighborhood, and the physical properties and characteristics of these Chaplygin wormholes are further analyzed. Four specific solutions are explored in some detail, namely, that of a constant redshift function, a specific choice for the form function, a constant energy density, and finally, isotropic pressure Chaplygin wormhole geometries.

DOI: 10.1103/PhysRevD.73.064028 PACS numbers: 04.20.Gz, 04.20.Jb, 98.80.Es

I. INTRODUCTION

The nature of the energy content of the Universe is a fundamental issue in cosmology, and a growing amount of observational evidence currently favors an accelerating flat Friedmann-Robertson-Walker model, constituted of $\sim 1/3$ of baryonic and dark matter and $\sim 2/3$ of a negative pressure dark energy component. The dark matter content was originally inferred from spiral galactic rotation curves, which showed a behavior that is significantly different from the predictions of Newtonian mechanics, and was later used to address the issue of large scale structure formation. On the other hand, it has been suggested that dark energy is a possible candidate for the present accelerated cosmic expansion [1]. The dark energy models are parametrized by an equation of state $\omega = p/\rho < -1/3$, where p is the spatially homogeneous negative pressure and ρ is the dark energy density. The range for which $\omega < -1$ has been denoted phantom energy, and possesses peculiar properties, such as, an infinitely increasing energy density [2], resulting in a "big rip," negative temperatures [3], and the violation of the null energy condition, thus providing a natural scenario for the existence of wormholes [4–6]. In fact, recent fits to supernovae, cosmic microwave background radiation, and weak gravitational lensing data favor an equation of state with the dark energy parameter crossing the phantom divide $\omega = -1$ [7,8]. Note that $\omega = -1$ corresponds to the presence of a cosmological constant. It has also been shown, in a cosmological setting, that the transition into the phantom regime, for a single scalar field [8] is probably physically implausible, so that a mixture of various interacting nonideal fluids is necessary.

An alternative model is that of the Chaplygin gas, also denoted as quartessence, based on a negative pressure fluid, which is inversely proportional to the energy density [9,10]. The equation of state representing the generalized Chaplygin gas (GCG) is given by

$$p_{\rm ch} = -\frac{A}{\rho_{\rm ch}^{\alpha}}, \tag{1}$$

where A and α are positive constants, and the latter lies in the range $0 < \alpha \leq 1$. The particular case of $\alpha = 1$ corresponds to the Chaplygin gas. Within the framework of a flat Friedmann-Robertson-Walker cosmology the GCG equation of state, after being inserted in the energy conservation equation, $\dot{\rho} = -3\dot{a}(\rho + p)/a$, yields the following evolution of the energy density:

$$\rho_{\rm ch} = \left[A + \frac{B}{a^{3(1+\alpha)}}\right]^{1/(1+\alpha)}, \tag{2}$$

where a is the scale factor, and B is normally considered to be a positive integration constant, as to ensure the dominant energy condition. However, it is also possible to consider $B < 0$, consequently violating the dominant energy condition, and one verifies that the energy density will be an increasing function of the scale function [11]. An attractive feature of this model is that, at early times, the energy density behaves as matter, $\rho_{\rm ch} \sim a^{-3}$, and as a cosmological constant at a later stage, $\rho_{\rm ch} = \text{const}$. It has also been suggested that these two stages are intermediated by a phase described by a mixture of vacuum energy density and a soft matter equation of state $p = \alpha\rho$ with $\alpha \neq 1$ [10]. This dual behavior is responsible for the interpretation that the GCG model is a candidate of a unified model of dark matter and dark energy [12], and probably contains some of the key ingredients in the dynamics of the Universe for early and late times. In Ref. [13], a new model for describing the unification of dark energy and dark matter was proposed, which further generalizes the GCG model, and was thus dubbed the new generalized Chaplygin gas (NGCG) model. The equation of state of the NGCG is given by $p = -\tilde{A}(a)/\rho^{\alpha}$, where a is the scale factor and $\tilde{A}(a) = -\omega A a^{-3(1+\omega)(1+\alpha)}$, and the

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interaction between dark energy and dark matter is characterized by the constant α.

The GCG model has been confronted successfully with a wide variety of observational tests, namely, supernovae data [14], cosmic microwave background radiation [15], gravitational lensing [16], gamma-ray bursts [17], and other observational data [18], which have placed constraints on the free parameters. A variable Chaplygin gas model, with $p = -A(a)/\rho$, where $A(a)$ is a positive function of the scale factor has also been analyzed, and it was found to be consistent with several observational data for a broad range of parameters [19]. In the context of the NGCG model, it was shown that the analysis of the observational data also provides tight constraints on the parameters of the model [13]. The GCG scenario has also been analyzed in some detail in a modified gravity approach [20], and another interesting aspect of the Chaplygin gas is its connection with branes in the context of string theory. The Chaplygin gas equation of state, with $\alpha = 1$, is associated with the parametrization invariant Nambu-Goto d-brane action in a $d + 2$ spacetime [10,21], which leads to the action of a Newtonian fluid, in the light-cone parametrization. Thus, in this context, the Chaplygin gas corresponds to a gas of d-branes in a $d + 2$ spacetime.

It has been argued that a flaw exists in the GCG model, as it produces oscillations or an exponential blowup of the matter power spectrum, which is inconsistent with observations. However, it has been counterargued [22] that, due to the fact that the GCG is a unique mixture of an interacting dark matter component and a cosmological dark energy, a flow of energy exists from the former to the latter. This energy flow is vanishingly small at early times, but became significant only recently, leading to a dominance of the dark energy component. It was also shown that the epoch of the dark energy dominance occurs when the dark matter perturbations start deviating from its linear behavior, and that the Newtonian equations for small scale perturbations for dark matter do not involve any k-dependent term. Therefore, it was concluded that neither oscillations nor a blowup in the power spectrum should develop [22].

As emphasized above, recent fits to observational data favor an evolving equation of state with ω crossing the phantom divide -1. If confirmed in the future, this behavior holds important implications to the model construction of dark energy, and thus excludes the cosmological constant and models with a constant parameter. In this context, note that the Chaplygin gas in the dark energy limit cannot cross the phantom divide, however in Ref. [23] it was shown that an interaction term between the Chaplygin gas, which in this model plays the role of dark energy, and dark matter can achieve the phantom crossing. In Ref. [24], by considering an extension of the Chaplygin gas, it was shown that the phantom divide $\omega = -1$ can also be realized. A further generalization of the GCG was also carried out in Ref. [25] to allow for the case where ω lies in the phantom regime.

The phantom regime is rather significant, as it violates the null energy condition (NEC), providing a natural scenario for the existence of traversable wormholes [4–6]. An interesting feature is that, due to the fact of the accelerated expansion of the Universe, macroscopic wormholes could naturally be grown from the submicroscopic constructions that originally pervaded the quantum foam. In Ref. [26] the evolution of wormholes and ringholes embedded in a background accelerating Universe driven by dark energy was analyzed. An interesting feature is that the wormhole's size increases by a factor which is proportional to the scale factor of the Universe, and still increases significantly if the cosmic expansion is driven by phantom energy. The accretion of dark and phantom energy onto Morris-Thorne wormholes [27,28] was further explored, and it was shown that this accretion gradually increases the wormhole throat which eventually overtakes the accelerated expansion of the universe, consequently engulfing the entire Universe, and becomes infinite at a time in the future before the big rip. This process was dubbed the "big trip" [27,28]. It was shown that using k-essence dark energy also leads to the big rip [29], although, in an interesting article [30], considering a generalized Chaplygin gas the big rip may be avoided altogether. In this paper, we shall be primarily interested in considering the possibility that traversable wormholes may be supported by the GCG equation of state. In this context, and related to the features of the big trip, the accretion of a generalized Chaplygin gas onto wormholes was explored in Ref. [31]. Several cases were extensively analyzed. Imposing the dominant energy condition, it was found that the evolution of the wormhole mass decreases with cosmic time. Considering the violation of the dominant energy condition, i.e., with $B < 0$, as the wormhole accretes Chaplygin phantom energy, the wormhole mass increases from an initial value, and reaches a plateau as time tends to infinity. In fact, a wide region of the Chaplygin parameters were found where the big trip is avoided.

Despite the fact that the GCG in the dark energy regime describes a spatially homogeneously distributed fluid, it has been pointed out that the GCG equation of state is that of a polytropic gas with a negative polytropic index [32], and thus inhomogeneous structures, such as a GCG dark energy star may arise from a density fluctuation in the GCG cosmological background. Other astrophysical implications of the model have also been analyzed. In Ref. [33], a gravitational vacuum star solution (gravastar) was constructed by replacing the interior de Sitter solution, with the Chaplygin gas equation of state in the phantom regime, which was dubbed a Born-Infeld phantom gravastar. A generalization of the gravastar picture (see Ref. [34] and references therein), with the inclusion of an interior solution governed by the equation of state $\omega = p/\rho < -1/3$ was also analyzed in Ref. [34].

In this work, we shall construct static and spherically symmetric traversable wormhole geometries, satisfying the GCG equation of state, which we denote "Chaplygin wormholes," by considering a matching of these geometries to an exterior vacuum spacetime, and further analyze the physical properties and characteristics of these solutions. We find that the spatial distribution of the exotic GCG is restricted to the throat neighborhood. We shall also consider specific solutions, and explore the traversability conditions [35,36] and apply the "volume integral quantifier" [37], which amounts to measuring the amount of averaged null energy condition violating matter, to particular cases.

This paper is outlined in the following manner. In Sec. II, we present a general solution of a traversable wormhole supported by a generalized Chaplygin gas, with a cutoff of the stress-energy tensor at a junction interface. In Sec. III, specific wormhole geometries are analyzed and several of their physical properties and characteristics are explored in some detail, namely, that of a constant redshift function, a specific choice for the form function, a constant energy density, and finally, isotropic pressure Chaplygin wormhole geometries. Finally, in Sec. IV, we conclude.

II. CHAPLYGIN WORMHOLES

A. Metric and field equations

The spacetime metric representing a spherically symmetric and static wormhole is given by

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate, r, denoted as the redshift function, and the form function, respectively [35]. The radial coordinate has a range that increases from a minimum value at r_0, corresponding to the wormhole throat, to infinity. One may also consider a cutoff of the stress-energy tensor at a junction radius a.

A fundamental property of a wormhole is that a flaring out condition of the throat, given by $(b - b'r)/b^2 > 0$, is imposed [35,36]. The latter may be deduced from the mathematics of embedding, and from this we verify that at the throat $b(r_0) = r = r_0$, the condition $b'(r_0) < 1$ is imposed to have wormhole solutions. Another condition that needs to be satisfied is $1 - b(r)/r > 0$, i.e., $b(r) < r$. For the wormhole to be traversable, one must demand that there are no horizons present, which are identified as the surfaces with $e^{2\Phi} \to 0$, so that $\Phi(r)$ must be finite everywhere.

Using the Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$ (with $c = G = 1$), we obtain the following relationships:

$$b' = 8\pi r^2 \rho, \tag{4}$$

$$\Phi' = \frac{b + 8\pi r^3 p_r}{2r^2(1 - b/r)}, \tag{5}$$

$$p_r' = \frac{2}{r}(p_t - p_r) - (\rho + p_r)\Phi', \tag{6}$$

where the prime denotes a derivative with respect to the radial coordinate, r. $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure, and $p_t(r)$ is the lateral pressure measured in the orthogonal direction to the radial direction. Equation (6) may be obtained using the conservation of the stress-energy tensor, $T^{\mu\nu}{}_{;\nu} = 0$, which can be interpreted as the hydrostatic equation for equilibrium for the material threading the wormhole.

Another fundamental property of wormholes is the violation of the null energy condition (NEC), $T_{\mu\nu}k^\mu k^\nu \geq 0$, where k^μ is *any* null vector [35,38]. From Eqs. (4) and (5), considering an orthonormal reference frame with $k^{\hat{\mu}} = (1, 1, 0, 0)$, so that $T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = \rho + p_r$, one verifies

$$\rho(r) + p_r(r) = \frac{1}{8\pi}\left[\frac{b'r - b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r}\right]. \tag{7}$$

Evaluated at the throat, r_0, and considering the flaring out condition and the finite character of $\Phi(r)$, we have $\rho + p_r < 0$. Matter that violates the NEC is denoted *exotic matter.*

B. GCG equation of state

The equation of state representing the generalized Chaplygin gas (GCG) is given by $p_{\rm ch} = -A/\rho_{\rm ch}^\alpha$, where A and α are positive constants, and the latter lies in the range $0 < \alpha \leq 1$. The particular case of $\alpha = 1$ corresponds to the Chaplygin gas. An attractive feature of this model, as mentioned in the Introduction, is that at early times the energy density behaves as matter, $\rho_{\rm ch} \sim a^{-3}$, and as a cosmological constant at a later stage, $\rho_{\rm ch} = \text{const}$. In a cosmological context, at a late stage dominated by an accelerated expansion of the Universe, the cosmological constant may be given by $8\pi A^{1/(1+\alpha)}$. This dual behavior is responsible for the interpretation that the GCG model is a candidate of a unified model of dark matter and dark energy. It has been shown that GCG can be algebraically decomposed into a dark matter and a dark energy component [22], in which there exists a transference of energy from the former to the latter.

It was noted in Ref. [32] that the GCG equation of state is that of a polytropic gas with a negative polytropic index, and thus suggested that one could analyze astrophysical implications of the model. In this context, we shall explore the construction of traversable wormholes, possibly from a density fluctuation in the GCG cosmological background. As in Refs. [4,5], we will consider that the pressure in the GCG equation of state is a radial pressure, and the tangential pressure can be determined from the Einstein equations, namely, Eq. (6). Thus, taking into account the GCG equation of state in the form $p_r = -A/\rho^\alpha$, and using Eqs. (4) and (5), we have the following condition:

$$\Phi'(r) = \left[-A(8\pi)^{1+\alpha}\frac{r^{2\alpha+1}}{2(b')^\alpha} + \frac{b}{2r^2}\right]\Big/\left(1-\frac{b}{r}\right). \tag{8}$$

Solutions of the metric (3) satisfying Eq. (8) shall be denoted "Chaplygin wormholes."

We now have a system of four equations, namely, Eqs. (4)–(6) and (8), with five unknown functions of r, i.e., the stress-energy components, $\rho(r)$, $p_r(r)$, and $p_t(r)$, and the metric fields, $b(r)$ and $\Phi(r)$. To construct specific solutions, we may adopt several approaches, and in this work we shall mainly use the strategy of considering restricted choices for $b(r)$ and $\Phi(r)$, in order to obtain solutions with the properties and characteristics of wormholes. One may also impose a specific form for the stress-energy components and through the field equations and Eq. (8) determine $b(r)$ and $\Phi(r)$. Throughout this paper, we shall consider the cases that the energy density is positive $\rho > 0$, which implies that only form functions of the type $b'(r) > 0$ are considered.

As shown above, to be a wormhole solution, the condition $b'(r_0) < 1$ is imposed. Now, using the GCG equation of state, evaluated at the throat, and taking into account Eq. (5), we verify that the energy density at r_0 is given by $\rho(r_0) = (8\pi r_0^2 A)^{1/\alpha}$. Finally, using Eq. (4), and the condition $b'(r_0) < 1$, we verify that for Chaplygin wormholes the following condition is imposed:

$$A < (8\pi r_0^2)^{-(1+\alpha)}. \tag{9}$$

It is a simple matter to show that this condition necessarily violates the NEC at the wormhole throat. However, for the GCG cosmological models it is generally assumed that the NEC is satisfied, i.e., $p + \rho \geq 0$, which implies $\rho \geq A^{1/(1+\alpha)}$. The NEC violation is a fundamental ingredient in wormhole physics, and it is in this context that we shall explore the construction of traversable wormholes, i.e., for $\rho < A^{1/(1+\alpha)}$. Note that, as emphasized in Refs. [11,31], considering a negative integration constant, $B < 0$, in the evolution of the energy density, Eq. (2), one also deduces that $\rho_{\rm ch} < A^{1/(1+\alpha)}$. This condition violates the dominant energy condition, and is consistent with the analysis outlined in this paper, proving the compatibility of both works.

Note that the velocity of sound has been interpreted as $v_s^2 = \partial p/\partial\rho = A\alpha/\rho^{1+\alpha}$. Thus, from the condition that the latter should not exceed the speed of light, i.e., $v_s \leq 1$, and from the violation of the NEC, $\rho + p < 0$, we have the additional constraints $1 < A/\rho^{1+\alpha} \leq 1/\alpha$. The latter evaluated at the throat takes the form $\alpha^\alpha \leq A(8\pi r_0^2)^{(1+\alpha)} < 1$ for $\alpha < 1$. However, it is worth pointing out that, in the presence of exotic matter, one cannot naively interpret $\partial p/\partial\rho$ as the speed of sound, as a detailed microphysical model describing the physics of exotic matter is still lacking. Therefore, one cannot *a priori* impose $0 < \partial p/\partial\rho \leq 1$, and it is worth noting that there are several known examples of exotic $\partial p/\partial\rho < 0$ behavior, namely, the Casimir effect and the false vacuum. (See Ref. [39] for a detailed analysis.)

C. Stress-energy tensor cutoff

We can construct asymptotically flat spacetimes, in which $b(r)/r \to 0$ and $\Phi \to 0$ as $r \to \infty$. However, one may also construct solutions with a cutoff of the stress-energy, by matching the interior solution of metric (3) to an exterior vacuum spacetime, at a junction interface. If the junction contains surface stresses, we have a thin shell, and if no surface stresses are present, the junction interface is denoted a boundary surface. The solutions analyzed in this work are not asymptotically flat, where the spatial distribution of the exotic GCG is restricted to the throat neighborhood, so that the dimensions of these Chaplygin wormholes are not arbitrarily large.

For simplicity, consider that the exterior vacuum solution is the Schwarzschild spacetime, given by the following metric:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{10}$$

Note that the matching occurs at a radius greater than the event horizon $r_b = 2M$, i.e., $a > 2M$. The Darmois-Israel formalism [40] then provides the following expressions for the surface stresses of a dynamic thin shell [6,41]:

$$\sigma = -\frac{1}{4\pi a}\left(\sqrt{1-\frac{2M}{a}+\dot{a}^2} - \sqrt{1-\frac{b(a)}{a}+\dot{a}^2}\right), \tag{11}$$

$$\mathcal{P} = \frac{1}{8\pi a}\left[\frac{1-\frac{M}{a}+\dot{a}^2+a\ddot{a}}{\sqrt{1-\frac{2M}{a}+\dot{a}^2}} - \frac{(1+a\Phi')(1-\frac{b}{a}+\dot{a}^2)+a\ddot{a}-\frac{\dot{a}^2(b-b'a)}{2(a-b)}}{\sqrt{1-\frac{b(a)}{a}+\dot{a}^2}}\right], \tag{12}$$

where the overdot denotes a derivative with respect to the proper time, τ. σ and $\mathcal{P}$ are the surface energy density and the tangential surface pressure, respectively. The static case is given by taking into account $\dot{a} = \ddot{a} = 0$ [42].

III. SPECIFIC SOLUTIONS

A. Constant redshift function

Consider, for instance, a constant redshift function, $\Phi'(r) = 0$, so that from Eq. (8) one determines the following form function:

$$b(r) = r_0\left[(3A)^{1/\alpha}\left(\frac{8\pi}{3r_0}\right)^{(1+\alpha)/\alpha} \times (r^{3(\alpha+1)/\alpha} - r_0^{3(\alpha+1)/\alpha}) + 1\right]^{\alpha/(1+\alpha)}. \quad (13)$$

For the particular case of the Chaplygin gas, $\alpha = 1$, the form function reduces to

$$b(r) = r_0\left[\frac{64}{3}\frac{A\pi^2}{r_0^2}(r^6 - r_0^6) + 1\right]^{1/2}. \quad (14)$$

To be a solution of a wormhole, the condition $b'(r_0) < 1$ is imposed. Thus, from the latter condition and Eq. (14), we deduce the restriction $A < (8\pi r_0^2)^{-2}$. For instance, considering $A = \beta(8\pi r_0^2)^{-2}$, with $0 < \beta < 1$, the form function is given by

$$b(r) = r_0\left\{\frac{\beta}{3}\left[\left(\frac{r}{r_0}\right)^6 - 1\right] + 1\right\}^{1/2}. \quad (15)$$

Note that this does not correspond to an asymptotically flat spacetime; however, one may match this solution to an exterior vacuum geometry, so that the dimensions of this specific Chaplygin wormhole cannot be arbitrarily large. To be a solution of a wormhole, the condition $b(r) < r$ is also imposed. Note that $b(r) = r$ has two real and positive roots given by $r_- = r_0$ and $r_+ = r_0\{[(12/\beta - 3)^{1/2} - 1]/2\}^{1/2}$, so that, to be a solution of a wormhole, r lies in the range

$$r_0 < r < r_0\left\{\frac{1}{2}\left[\left(\frac{12}{\beta} - 3\right)^{1/2} - 1\right]\right\}^{1/2}. \quad (16)$$

This restriction is shown graphically in Fig. 1 for the values of $\beta = 1/3$ and $\beta = 1/4$. Note that the wormhole dimensions increase for decreasing values of β.

It is also of interest to consider the traversability conditions required for a human being to journey through the wormhole. Rather than reproduce the results here, we refer the reader to Refs. [5,35,36]. We shall consider, for simplicity, a constant and nonrelativistic traversal velocity for the traveler. Note that, for the specific solutions considered in this subsection, the conditions required that the acceleration felt by the traveler, and that the radial tidal acceleration should not exceed Earth's gravity, $g_\oplus$, are readily satisfied (see Refs. [5,35,36] for details). From the condition that the lateral tidal acceleration should not exceed Earth's gravitational acceleration, evaluated at the wormhole throat r_0, one obtains the inequality

$$v \leq r_0\sqrt{\frac{2g_\oplus}{(1-\beta)|\eta^{\hat{2}'}|}}, \quad (17)$$

where $\eta^{\hat{2}'}$ is the separation between two arbitrary parts of the traveler's body, measured along the lateral direction in the traveler's reference frame, and, for simplicity, we shall assume that $|\eta^{\hat{2}'}| \approx 2$ m [5,35].

Now, as in Ref. [5], considering the equality case, with $\beta = 1/2$, and assuming that the wormhole throat is given by $r_0 \approx 10^2$ m, the traversal velocity takes the following value $v \approx 4 \times 10^2$ m/s. Considering that the junction radius is given by $a \approx 10^4$ m, then one obtains the traversal times of $\Delta\tau \approx \Delta t \approx 50$ s, as measured by the traveler and for the observers that remain at rest at the space stations situated at a, respectively (see Ref. [5] for details). It is interesting to note that these traversability conditions are identical to the specific case of an asymptotically flat phantom wormhole spacetime analyzed in Ref. [5].

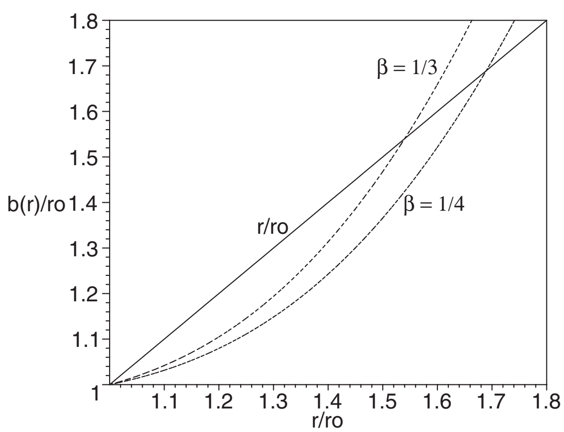

FIG. 1. To be a solution of a wormhole, the condition $b(r) < r$ is imposed, so that only the interval below the solid line r/r_0 provide wormhole solutions. The plot depicts the values of $\beta = 1/3$ and $\beta = 1/4$, and one verifies that the wormhole dimensions increase for decreasing values of β. See the text for details.

B. $b(r) = r_0\sqrt{r/r_0}$

Consider $b(r) = r_0\sqrt{r/r_0}$, so that from Eq. (8), considering the Chaplygin gas with $\alpha = 1$, one determines the following redshift function:

$$\Phi(r) = -128A\pi^2 r_0^4\left[\ln\left(\sqrt{\frac{r}{r_0}} - 1\right) + \sum_{n=1}^{9}\frac{1}{n}\left(\frac{r}{r_0}\right)^{n/2}\right] + \ln\left(1 - \sqrt{\frac{r_0}{r}}\right) + C. \quad (18)$$

The constant of integration, C, may be determined from the boundary conditions, $\Phi(a)$, at the junction interface. Note that this solution reflects a nontraversable wormhole as it possesses an event horizon at the throat $r = r_0$, as may be readily verified from the first term in square brackets in the right-hand side of Eq. (18). However, imposing the condition $A = (128\pi^2 r_0^4)^{-1}$, Eq. (18) reduces to

$$\Phi(r) = -\sum_{n=1}^{9}\frac{1}{n}\left(\frac{r}{r_0}\right)^{n/2} - \ln\left(\sqrt{\frac{r}{r_0}}\right) + C. \quad (19)$$

As in the previous example, this solution is not asymptotically flat, however, one may match the latter to an exterior vacuum spacetime at a junction radius a. Note that the constant C is given by

$$C = \Phi(a) + \sum_{n=1}^{9} \frac{1}{n}\left(\frac{a}{r_0}\right)^{n/2} + \ln\left(\sqrt{\frac{a}{r_0}}\right). \tag{20}$$

This solution now reflects a traversable wormhole, as the redshift function is finite in the range $r_0 \leq r \leq a$.

Using the "volume integral quantifier," which provides information about the "total amount" of averaged null energy condition violating matter in the spacetime (see Ref. [37] for details), given by $I_V = \int[\rho(r) + p_r(r)]dV$, with a cutoff of the stress-energy at a, we have

$$\begin{aligned} I_V &= \left[r\left(1-\frac{b}{r}\right)\ln\left(\frac{e^{2\Phi}}{1-b/r}\right)\right]_{r_0}^{a} \\ &\quad - \int_{r_0}^{a}(1-b')\left[\ln\left(\frac{e^{2\Phi}}{1-b/r}\right)\right]dr \\ &= \int_{r_0}^{a}(r-b)\left[\ln\left(\frac{e^{2\Phi}}{1-b/r}\right)\right]' dr. \end{aligned} \tag{21}$$

Now, using the form and redshift functions provided above, and evaluating the integral, one finally ends up with the following simplified expression for the "volume integral quantifier":

$$I_V = r_0\left[\sqrt{\frac{a}{r_0}} - \frac{2}{11}\left(\frac{a}{r_0}\right)^{11/2} - \frac{9}{11}\right]. \tag{22}$$

By taking the limit $a \to r_0$, one readily verifies that $I_V \to 0$. This proves that, as in the specific case of phantom wormholes [5], one may theoretically construct a wormhole with arbitrarily small amounts of a Chaplygin gas. As emphasized in Ref. [5], this result is not unexpected. However, it is interesting to note the relative ease that one may theoretically construct wormholes supported by infinitesimal amounts of exotic fluids used in cosmology to explain the present accelerated cosmic expansion.

C. Constant energy density

Considering a constant energy density, $\rho = \rho_0$, we verify from Eq. (4) that the form function is given by

$$b(r) = C(r^3 - r_0^3) + r_0, \tag{23}$$

with the definition $C = 8\pi\rho_0/3$. The condition $b'(r_0) < 1$ imposes the following restriction: $3Cr_0^2 < 1$. Consider $C = \beta(3r_0^2)^{-1}$, with $0 < \beta < 1$, so that Eq. (23) takes the form

$$b(r) = r_0\left\{\frac{\beta}{3}\left[\left(\frac{r}{r_0}\right)^3 - 1\right] + 1\right\}. \tag{24}$$

To be a wormhole solution the condition $b(r) < r$ is also imposed. Note that $b(r) = r$ has two real positive roots given by $r_- = r_0$ and $r_+ = r_0(\sqrt{12/\beta - 3} - 1)/2$, so that r lies in the range

$$r_0 < r < \frac{r_0}{2}\left(\sqrt{\frac{12}{\beta} - 3} - 1\right). \tag{25}$$

From Eq. (8), the redshift function is formally given by

$$\Phi(r) = -\frac{1}{2}\int \frac{\beta(r^3 - r_0^3) - 24\pi A(8\pi r_0^2/\beta)^\alpha r_0^2 r^3 + 3r_0}{r(r-r_0)[r^2 + r_0 r - (3-\beta)r_0^2]}dr + C_1. \tag{26}$$

The constant of integration, C_1, may be determined from the boundary conditions, at the junction interface, a. For the particular case of the Chaplygin gas, $\alpha = 1$, Eq. (26) is integrated to provide

$$\begin{aligned} \Phi(r) = &-\frac{1}{2\beta(1-\beta)}\left\{(64A\pi^2 r_0^4 - \beta)\ln(r - r_0) \right.\\ &+ \left[\frac{\beta}{2} - \frac{32A\pi^2 r_0^4(3-2\beta)}{\beta}\right]\ln[\beta r^2 + \beta r_0 r \\ &- (3-\beta)r_0^2] - \frac{3(\beta^2 - 64A\pi^2 r_0^4)}{\sqrt{3\beta(4-\beta)}} \\ &\left.\times \operatorname{arctanh}\left[\frac{\beta(2r + r_0)}{r_0\sqrt{3\beta(4-\beta)}}\right]\right\} - \frac{1}{2}\ln(r) + C_1. \end{aligned} \tag{27}$$

One verifies, as in the previous case, that this is a solution of a nontraversable wormhole, as an event horizon exists at r_0. However, imposing the condition $A = \beta(64\pi^2 r_0^4)^{-1}$ with $0 < \beta < 1$, the redshift function reduces to

$$\begin{aligned} \Phi(r) = &-\frac{1}{2}\ln(r) + \frac{3}{4\beta}\ln[\beta r^2 + \beta r_0 r - (3-\beta)r_0^2] \\ &+ \frac{1}{2\sqrt{3\beta(4-\beta)}}\operatorname{arctanh}\left[\frac{\beta(2r + r_0)}{r_0\sqrt{3\beta(4-\beta)}}\right] + C_1. \end{aligned} \tag{28}$$

It is a simple matter to show that $\Phi(r)$ given by Eq. (28) is finite in the range (25), so that this solution now reflects a traversable wormhole. This interior wormhole solution is now matched to an exterior vacuum spacetime within the range of the inequality (25). Note that the condition $A = \beta(64\pi^2 r_0^4)^{-1}$ satisfies the restriction of the inequality (9).

D. Isotropic pressure

Consider an isotropic pressure, $p_r = p_t$, so that from Eq. (6) we have the differential equation

$$\frac{A\alpha\rho'}{\rho(A - \rho^{\alpha+1})} = \Phi'(r), \tag{29}$$

which has the following solution:

$$\rho(r) = [A^{-1} + e^{-[(\alpha+1)/\alpha]\Phi(r)}C_1]^{-[1/(1+\alpha)]}. \tag{30}$$

From the GCG equation of state, $p_r = -A/\rho^\alpha$, and using Eq. (5), we verify that the energy density, evaluated at the throat, reduces to $\rho(r_0) = (8\pi r_0^2 A)^{1/\alpha}$, so that the constant

C_1 is given by

$$C_1 = [(8\pi r_0^2 A)^{-(1+\alpha)/\alpha} - A^{-1}]e^{[(\alpha+1)/\alpha]\Phi(r_0)}. \tag{31}$$

Taking into account Eq. (4), and considering a specific choice for the redshift function, for instance,

$$\Phi(r) = \ln\left(\frac{r}{r_0}\right), \tag{32}$$

we have

$$b'(r) = 8\pi r^2\left[A^{-1} + C_1\left(\frac{r}{r_0}\right)^{-(1+\alpha)/\alpha}\right]^{-1/(1+\alpha)}. \tag{33}$$

Equation (33) may be integrated to provide

$$b(r) = C_2 + (8\pi r^3/3)A^{1/(1+\alpha)}\text{hypergeom}\left(\left[\frac{1}{1+\alpha}, \frac{-3\alpha}{1+\alpha}\right], \left[\frac{1-2\alpha}{1+\alpha}\right], -\left(\frac{r_0}{r}\right)^{1+(1/\alpha)} C_1 A\right). \tag{34}$$

For the particular case of $\alpha = 1$, the hypergeometric function takes the form

$$\text{hypergeom}\left([1/2, -3/2], [-1/2], -\frac{r_0^2 C_1 A}{r^2}\right) = r^{-3}(r^2 - 2r_0^2 C_1 A)\sqrt{r^2 + r_0^2 C_1 A}, \tag{35}$$

so that the form function reduces to

$$b(r) = \frac{8\pi\sqrt{A}}{3}\sqrt{r^2 + r_0^2 C_1 A}(r^2 - 2r_0^2 C_1 A) + C_2. \tag{36}$$

Note that the constant C_1, for this case, may be obtained from Eq. (31), and takes the form

$$C_1 = [(8\pi r_0^2 A)^{-2} - A^{-1}]. \tag{37}$$

Taking the radial derivative of the form function, or simply using Eq. (33), and inserting the expression deduced for C_1, we have

$$b'(r) = \frac{8\pi\sqrt{A} r^3}{\sqrt{r^2 - r_0^2 + (8\pi r_0^2)^{-2} A^{-1}}}, \tag{38}$$

which, evaluated at the throat, reduces to $b'(r_0) = (8\pi r_0^2)^2 A$. From the condition $b'(r_0) < 1$, we verify once again $A < (8\pi r_0^2)^{-2}$, which is consistent with the generic wormhole restriction given by Eq. (9), with $\alpha = 1$. Note that considering $A = \beta(8\pi r_0^2)^{-2}$, with $0 < \beta < 1$, the constant C_1 takes the form $C_1 = (8\pi r_0^2)^2(1-\beta)/\beta^2$. The constant C_2, may be evaluated from the condition $b(r_0) = r_0$, and is given by $C_2 = 2r_0/(3\beta)$. The form function is finally given by the following expression:

$$b(r) = \frac{r_0}{3\beta}\left\{\sqrt{\beta\left(\frac{r}{r_0}\right)^2 + (1-\beta)}\left[\beta\left(\frac{r}{r_0}\right)^2 - 2(1-\beta)\right] + 2\right\}. \tag{39}$$

To be a solution of a wormhole, the condition $b(r) < r$ is imposed, as emphasized above. This restriction is shown graphically in Fig. 2 for the values of $\beta = 1/3$ and $\beta = 1/4$. One verifies that the wormhole dimensions increase for decreasing values of β, so that the dimensions of this specific Chaplygin wormhole cannot be arbitrarily large.

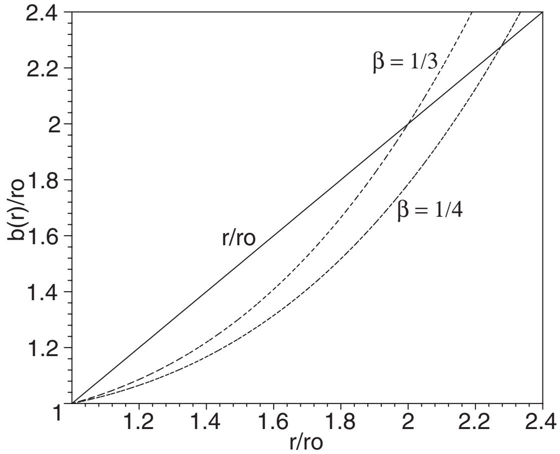

FIG. 2. To be a solution of a wormhole, the condition $b(r) < r$ is imposed, so that only the interval below the solid line r/r_0 provide wormhole solutions. The plot depicts the values of $\beta = 1/3$ and $\beta = 1/4$, and one verifies that the wormhole dimensions increase for decreasing values of β. See the text for details.

IV. SUMMARY AND CONCLUSION

The generalized Chaplygin gas (GCG) is a possible candidate for the unification of dark energy, responsible for the present accelerated cosmic expansion, and of dark matter, inferred, for instance, from galactic rotation curves. The GCG is parametrized by an exotic equation of state given by $p_{\rm ch} = -A/\rho_{\rm ch}^\alpha$, where A and α are positive constants, with $0 < \alpha \leq 1$. The GCG models, in a cosmological context, have at least two significant features that stand out. First, they describe a smooth transition from a decelerated expansion of the Universe to a present epoch of a cosmic acceleration. Second, they provide a unified macroscopic phenomenological description of dark matter and of dark energy.

In this paper, we have studied the possibility that traversable wormholes may be supported by the GCG. We have found that, to be a generic solution of a wormhole, the GCG equation of state imposes the following restriction $A < (8\pi r_0^2)^{-(1+\alpha)}$, consequently violating the NEC condition, which is a necessary ingredient in wormhole physics. We analyzed the physical properties and characteristics of these Chaplygin wormholes, studying in some detail four specific solutions. The first is that of a constant redshift function, and specific wormhole dimensions, the traversal velocity, and traversal time were deduced from the traversability conditions for this particular geometry. The second solution is that of a specific choice for the form function, and the theoretical construction of this spacetime with infinitesimal amounts of averaged null energy condition violating Chaplygin gas was also explored. The third case analyzed is that of a constant energy density, and finally isotropic pressure Chaplygin traversable wormhole solutions were also presented. The solutions found are not asymptotically flat, where the spatial distribution of the exotic GCG is restricted to the throat vicinity, so that the dimensions of these Chaplygin wormholes are not arbitrarily large.

In concluding, it is noteworthy the relative ease with which one may theoretically construct traversable wormholes with the exotic fluid equations of state used in cosmology to explain the present accelerated expansion of the Universe. As for phantom energy traversable wormholes [5], these Chaplygin variations have far-reaching physical and cosmological implications, namely, apart from being used for interstellar shortcuts, an absurdly advanced civilization may convert them into time machines [36,43,44], probably implying the violation of causality.

ACKNOWLEDGMENTS

I thank José Jiménez Madrid for pointing out specific details of the constant $B < 0$ and the respective violation of the dominant energy condition, and Neven Bilić and Xin Zhang for pointing out several references.

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