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Embedding of the Kerr-Newman black hole surface in Euclidean space

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We obtain a global embedding of the surface of a rapidly rotating Kerr-Newman black hole in an Euclidean 4-dimensional space.

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I. INTRODUCTION

In this paper we discuss the problem of isometric embedding of the surface of a rapidly rotating black hole in a flat space.

It is well known that intrinsically defined Riemannian manifolds can be isometrically embedded in a flat space. According to the Cartan-Janet [1,2] theorem, every analytic Riemannian manifold of dimension n can be locally real analytically isometrically embedded into $\mathbb{E}^N$ with $N = n(n+1)/2$. The so called Fundamental Theorem of Riemannian geometry (Nash, 1956 [3]) states that every smooth Riemannian manifold of dimension n can be globally isometrically embedded in a Euclidean space $\mathbb{E}^N$ with $N = (n+2)(n+3)/2$.

The problem of isometric embedding of 2D manifolds in $\mathbb{E}^3$ is well studied. It is known that any compact surface embedded isometrically in $\mathbb{E}^3$ has at least one point of positive Gauss curvature. Any 2D compact surface with positive Gauss curvature is always isometrically embeddable in $\mathbb{E}^3$, and this embedding is unique up to rigid rotations. (For general discussion of these results and for further references, see e.g. [4]). It is possible to construct examples when a smooth geometry on a 2D ball with negative Gauss curvature cannot be isometrically embedded in $\mathbb{E}^3$ (see e.g. [5,6]). On the other hand, it is easy to construct an example of a global smooth isometric embedding for a surface of the topology S^2 which has both, positive and negative Gauss curvature ball-regions, separated by a closed loop where the Gauss curvature vanishes. An example of such an embedding is shown in Fig. 1 [7].

The surface geometry of a charged rotating black hole and its isometric embedding in $\mathbb{E}^3$ was studied long time ago by Smarr [8]. He showed that when the dimensionless rotation parameter $\alpha = J/M^2$ is sufficiently large, there are two regions near poles of the horizon surface where the Gauss curvature becomes negative. Smarr proved that these regions cannot be isometrically embedded (even locally) in $\mathbb{E}^3$ as a revolution surface, but such local embedding is possible in a 3D Minkowsky space. More recently different aspects of the embedding of a surface of a rotating black hole and its ergosphere in $\mathbb{E}^3$ were discussed in [9,10]. A numerical scheme for construction of the isometric embedding for surfaces with spherical topology was proposed in [10]. The surface geometry of a rotating black hole in an external magnetic field and its embedding in $\mathbb{E}^3$ was studied in [11–13].

The purpose of this paper is to obtain the global isometric embedding of a surface of a rapidly rotating black hole in $\mathbb{E}^4$. In Sec. II we discuss general properties of 2D axisymmetric metrics and prove that if the Gauss curvature is negative at the fixed points of the rotation group it is impossible to isometrically embed a region containing such a fixed point in $\mathbb{E}^3$. In Section III we demonstrate that such surfaces can be globally embedded in $\mathbb{E}^4$. We obtain the embedding of surfaces of rapidly rotating black holes in $\mathbb{E}^4$ in an explicit form in Sec. IV. Section V contains a brief summary and discussions.

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II. GEOMETRY OF 2D AXISYMMETRIC DISTORTED SPHERES

Let us consider an axisymmetric deformation S of a unit sphere S^2. Its metric can be written in the form

$$dl^2 = h(x)dx^2 + f(x)d\phi^2. \tag{1}$$

Here $\xi = \partial_\phi$ is a Killing vector field with closed trajectories. Introducing a new coordinate $\mu = \int dx\sqrt{hf}$ one can rewrite (1) in the form

$$dl^2 = f(\mu)^{-1}d\mu^2 + f(\mu)d\phi^2. \tag{2}$$

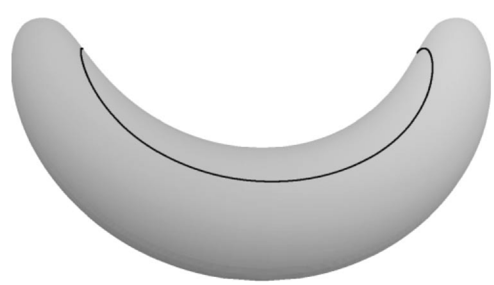

FIG. 1. This picture shows the ‘‘croissant’’ surface. A solid line separates two regions with opposite signs of the Gauss curvature. Each of these regions has the topology of a 2D ball. The Gauss curvature is negative in the upper ball-region.

We assume that the function f is positive inside the interval (μ_0, μ_1) and vanishes at its ends. We choose $\mu_0 = -1$. The surface area of S is $2\pi(\mu_1 + 1)$. By multiplying the metric (1) by a constant scale factor $(\mu_1 + 1)^{-1/2}$ one can always put $\mu_1 = 1$, We shall use this choice, for which the surface area of S is 4π and the fixed points of ξ are located at $\mu = \pm 1$.

The metric (2) is regular (no conical singularities) at the points $\mu = \pm 1$ (where $r = 0$) if $f'(\pm 1) = \mp 2$. (Here and later $(\ldots)' = d(\ldots)/dz$.) The Gaussian curvature for the metric (2) is

$$K = -\tfrac{1}{2} f''. \tag{3}$$

Let us introduce a new coordinate $r = \sqrt{f(\mu)}$. The metric (2) in these coordinates is

$$dl^2 = V(r)dr^2 + r^2 d\phi^2, \qquad V = \frac{4}{f'^2}. \tag{4}$$

We denote by K_0 the value of K at a fixed point of rotations. Then in the vicinity of this point one has

$$V \approx 1 + \tfrac{1}{2} K_0 r^2 + \ldots. \tag{5}$$

For $K_0 < 0$ the region in the vicinity of the fixed point cannot be embedded as a revolution surface in a Euclidean space $\mathbb{E}^n$ for any $n \geq 3$. Indeed, consider a space $\mathbb{E}^n$ with the metric

$$dS^2 = dX^2 + dY^2 + \sum_{i=3}^{n} dZ_i^2. \tag{6}$$

For the surface of revolution $X = r\cos\phi$, $Y = r\sin\phi$, $Z_i = Z_i(r)$ the induced metric is

$$dl^2 = V(r)dr^2 + r^2 d\phi^2, \qquad V(r) = 1 + \sum_{i=3}^{n} (dZ_i/dr)^2. \tag{7}$$

For a regular surface $V(0) = 1$ and $V(r) \geq 1$ in the vicinity of $r = 0$. According to (5) this is impossible when $K_0 < 0$.

We show now that if $K_0 < 0$ then a ball-region near the fixed point p_0 of axisymmetric 2D geometry cannot be isometrically embedded in $\mathbb{E}^3$. Let us assume that such an embedding (not necessarily as a revolution surface) exists. One can choose coordinates (X^1, X^2, Z) in $\mathbb{E}^3$ so that $X^1 = X^2 = 0$ at p_0, and in its vicinity

$$Z = \tfrac{1}{2}(k_1 X^{12} + k_2 X^{22}) + \ldots, \tag{8}$$

where k_a $(a = 1, 2)$ are principal curvatures at p_0. Here and later "dots" denote omitted higher order terms. The metric on this surface induced by its embedding is

$$dl^2 = (1 + k_1^2 X^{12})dX^{12} + (1 + k_2^2 X^{22})dX^{22} + 2k_1 k_2 X^1 X^2 dX^1 dX^2 + \ldots. \tag{9}$$

In the vicinity of p_0 the Killing vector ξ generating rotations has the form

$$\xi = p^a \partial_a, \tag{10}$$

where p^a $(a = 1, 2)$ are regular functions of (X^1, X^2) vanishing at $(0, 0)$. Their expansion near p_0 has the form

$$p^a = P^a_b X^b + P^a_{bc} X^b X^c + P^a_{bcd} X^b X^c X^d + \ldots. \tag{11}$$

Consider the Taylor expansion near p_0 of the Killing equation

$$\xi_{a:b} = \xi_{a,b} - \Gamma_{ab}{}^c \xi_c = 0 \tag{12}$$

in the metric (9). Since the expansion of both Γ^c_{ab} and ξ_c starts with a linear in X^a terms, the Eq. (12) can be used to obtain restrictions on the coefficients P^a_b, P^a_{bc}, and P^a_{bcd} in (11). Simple calculations give

$$P^1_1 = P^2_2 = 0, \qquad P^1_2 = -P^2_1 = q, \qquad P^a_{bc} = P^a_{bcd} = 0, \tag{13}$$

$$qk_1(k_1 - k_2) = qk_2(k_1 - k_2) = 0. \tag{14}$$

If the Killing vector does not vanish identically then $q \neq 0$ and the Eqs. (14) imply that $k_1 = k_2$. This contradicts to the assumption of the existence of the embedding with $K_0 = k_1 k_2 < 0$.

III. EMBEDDING OF A 2D SURFACE WITH $K_0 < 0$ IN $\mathbb{E}^4$

Increasing the number of dimensions of the flat space from 3 to 4 makes it possible to find an isometric embedding of 2D manifolds with $K_0 < 0$. Denote by (X, Y, Z, R) Cartesian coordinates in $\mathbb{E}^4$ and determine the embedding by equations

$$X = \frac{r}{\Phi_0}\xi(\psi), \qquad Y = \frac{r}{\Phi_0}\eta(\psi), \qquad Z = \frac{r}{\Phi_0}\zeta(\psi), \tag{15}$$

$$R = R(r), \tag{16}$$

where $0 \leq \psi \leq 2\pi$, and functions ξ, η and ζ obey the condition

$$\xi^2(\psi) + \eta^2(\psi) + \zeta^2(\psi) = 1. \tag{17}$$

In other words, $\mathbf{n} = (\xi, \eta, \zeta)$ as a function of ψ is a line on a unit sphere S^2. We require that this line is a smooth closed loop $(\mathbf{n}(0) = \mathbf{n}(2\pi))$ without self-intersections. Since a loop on a unit sphere allows continuous deformations preserving its length, there is an ambiguity in the choice of functions (ξ, η, ζ).

We denote $\Phi = (\xi_{,\psi}^2 + \eta_{,\psi}^2 + \zeta_{,\psi}^2)^{1/2}$ then

$$2\pi\Phi_0 = \int_0^{2\pi} d\psi \Phi(\psi). \tag{18}$$

is the length of the loop. Instead of the coordinate ψ it is convenient to use a new angle coordinate ϕ which is proportional to the proper length of a curve $r = \text{const}$

$$\phi = \Phi_0^{-1} \int_0^{\psi} d\psi' \Phi(\psi'). \tag{19}$$

The coordinate ϕ is a monotonic function of ψ and for $\psi = 0$ and $\psi = 2\pi$ it takes values 0 and 2π, respectively.

Eqs. (15) give the embedding in $\mathbb{E}^3$ of a linear surface formed by straight lines passing through $r = 0$. This surface has $K = 0$ outside the point $r = 0$ where, in a general case, it has a conelike singularity with the angle deficit $2\pi(1 - \Phi_0)$.

We shall use the embedding (15) and (16) for the case when the angle deficit is negative. In this case one can use, for example, the following set of functions

$$\xi = \cos\psi/F, \qquad \eta = \sin\psi/F, \qquad \zeta = a\sin(2\psi)/F, \tag{20}$$

$$F = \sqrt{1 + a^2x^2}, \qquad x = \sin^2(2\psi). \tag{21}$$

This embedding for the functions (ξ, η, ζ) defined by (20) and (21) is shown in Fig. 2.

For this choice

$$\Phi = (1 + 4a^2 - 3a^2x)^{1/2}(a^2x + 1)^{-1}, \tag{22}$$

$$\Phi_0 = \frac{1}{\pi}\int_0^1 \frac{dx(1 + 4a^2 - 3a^2x)^{1/2}}{\sqrt{x(1 - x)}(a^2x + 1)}. \tag{23}$$

Calculations give

$$\Phi_0 = \frac{8}{\pi\sqrt{1 + 4a^2}}[(1 + a^2)\Pi(-a^2, k) - 3/4K(k)], \tag{24}$$

FIG. 2 (color online). The surface shown at this picture is formed by straight lines passing through a point $r = 0$. Its Gauss curvature vanishes. The surface has a cone singularity at $r = 0$ with a negative angle deficit.

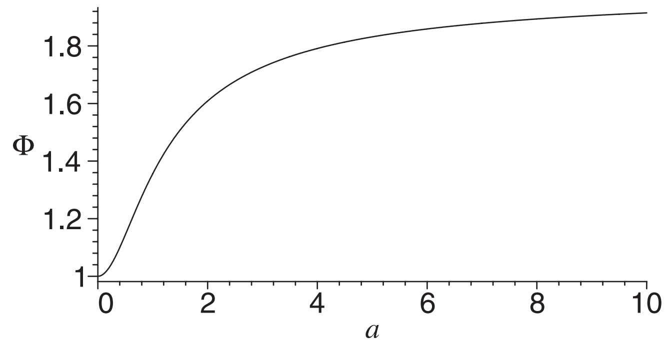

FIG. 3. Φ_0 as a function of the parameter a.

$$k = \sqrt{3}a/(1 + 4a^2). \tag{25}$$

Here $K(k)$ and $\Pi(\nu, k)$ are complete elliptic integrals of the first and third kind, respectively. The function Φ_0 monotonically increases from 1 (at $a = 0$) to 2 (at $a \to \infty$) (see Fig. 3).

The induced metric for the embedded 2D surface defined by (15) and (16) is

$$dl^2 = [\Phi_0^{-2} + (dR/dr)^2]dr^2 + r^2d\phi^2. \tag{26}$$

If the angle deficit is positive ($\Phi_0 < 1$), the pole point $r = 0$ in the metric (15) remains a cone singular point for any $R(r)$. For $\Phi > 1$ (the negative angle deficit), the pole-point $r = 0$ in the metric (26) is regular if $(dR/dr)_0^2 = 1 - \Phi_0^2$. By comparing (4) and (26) one obtains

$$r = f^{1/2}, \qquad (dR/dr)^2 = (V - \Phi_0^{-2}). \tag{27}$$

This relation gives the following equation relating $R(\mu)$ with $f(\mu)$

$$R' = (1 - f'^2/(4\Phi_0^2))^{1/2}f^{-1/2}. \tag{28}$$

It is easy to check that $R'' = 0$ at points where $f' = 0$. In order R' to be real, the following condition must be valid $\Phi_0 \geq \frac{1}{2}\max_{\mu\in(-1,1)}|f'(\mu)|$. At a point where $|f'|$ reaches its maximum the quantity $f'' = -2K$ vanishes. Thus it is sufficient to require that Φ_0 is greater or equal to the values of $|f'|$ calculated at the points separating regions with the positive and negative Gauss curvature.

IV. EMBEDDING OF THE SURFACE OF THE KERR-NEWMAN HORIZON IN $\mathbb{E}^4$

The surface geometry of the Kerr-Newman black hole is described by the metric $ds^2 = N^2dl^2$, where

$$dl^2 = (1 - \beta^2\sin^2\theta)d\theta^2 + \sin^2\theta[1 - \beta^2\sin^2\theta]^{-1}d\phi^2, \tag{29}$$

$$N = (r_+^2 + a^2)^{1/2}, \qquad \beta = a(r_+^2 + a^2)^{-1/2}. \tag{30}$$

Here $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The metric dl^2 is normalized so that the area of the surface with this metric

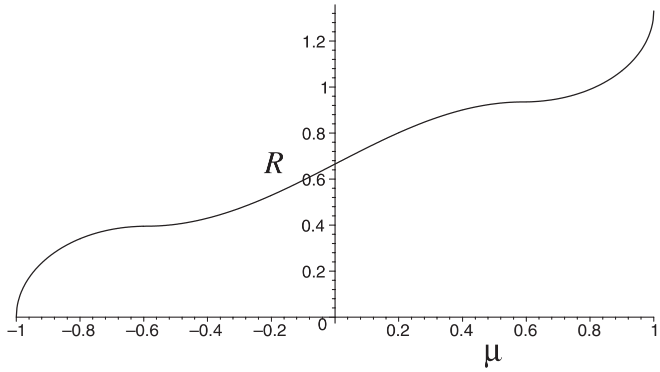

FIG. 4. Plot for R as a function of μ for $\beta = 0.7$.

is 4π. In the coordinates $\mu = \cos\theta$ the metric (29) takes the form (2) with

$$f(\mu) = (1 - \mu^2)[1 - \beta^2(1 - \mu^2)]^{-1}. \quad (31)$$

For the black hole with mass M, charge Q and the angular momentum $J = Ma$

$$r_+ = M - (M^2 - a^2 - Q^2)^{1/2}. \quad (32)$$

The rotation parameter a and mass M can be written in terms of the distortion parameter β as follows

$$a = \beta N, \qquad M = \tfrac{1}{2}N(1 - \beta^2)^{-1/2}(1 + Q^2/N^2). \quad (33)$$

The condition $M^2 \geq a^2 + Q^2$ for given parameters N and β requires that [8]

$$0 \leq Q \leq N(1 - \beta^2)^{1/2}, \quad (34)$$

$$\tfrac{1}{2}N(1 - \beta^2)^{-1/2} \leq M \leq N(1 - \beta^2)^{1/2}. \quad (35)$$

The distortion parameter has its maximal value $\beta_{\max} = 1/\sqrt{2}$ for $Q = 0$. The Gauss curvature of the surface with the metric (29) is

$$K = [1 - \beta^2(1 + 3\mu^2][1 - \beta^2(1 - \mu^2)]^{-3}. \quad (36)$$

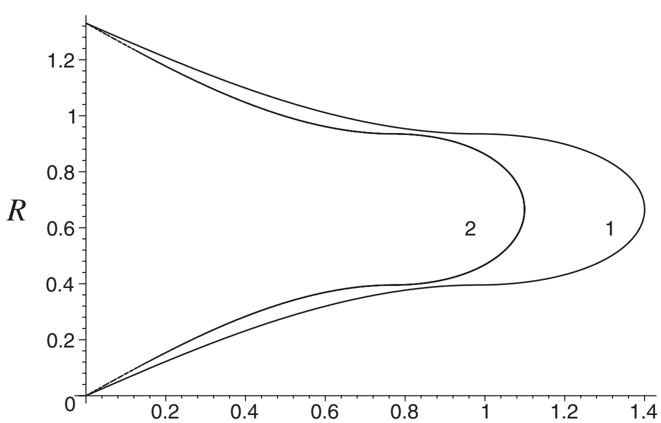

FIG. 5. Plot for R as a function of r (curve 1) and ρ (curve 2) for $\beta = 0.7$.

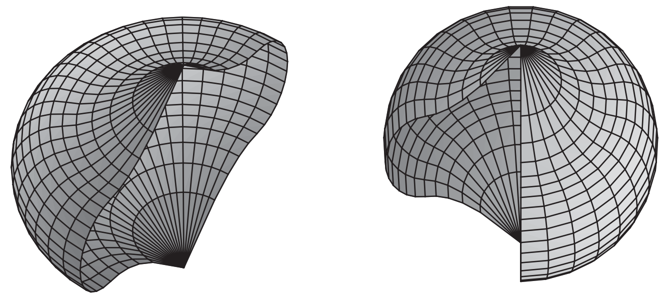

FIG. 6. By gluing these two figures along their edges one obtains a 2D surface without angle deficits and isometric to the surface of a rotating black hole ($\beta = 0.7$.)

For $\frac{1}{2} < \beta \leq \frac{1}{\sqrt{2}}$ the Gauss curvature is negative in the vicinity of poles in the region $\mu_c \leq |\mu| \leq 1$

$$\mu_c = (1 - \beta^2)^{1/2}(\sqrt{3}\beta)^{-1}. \quad (37)$$

At $|\mu| = \mu_c$ the Gauss curvature vanishes. As it was shown earlier, at this point $|f'|$ has its maximum

$$|f'|_{\max} = |f'|_{\mu_c} = \frac{3\sqrt{3}}{8\beta(1 - \beta^2)^{3/2}}, \quad (38)$$

and one must choose the parameter Φ_0 so that $\Phi_0 \geq \frac{1}{2}|f'|_{\max}$. Simplest possible choice is

$$\Phi_0 = \tfrac{1}{2}|f'|_{\mu_c}. \quad (39)$$

Using (39) and integrating the Eq. (28) one determines R as a function of μ. A plot of this function for $\beta = 0.7$ is shown in Fig. 4. Plot 1 at Fig. 5 shows R as a function of r for the same values of β.

The metric (26) can also be written in the form

$$dl^2 = [1 + (dR/d\rho)^2]d\rho^2 + \Phi_0^2\rho^2 d\phi^2, \quad (40)$$

where $\rho = r/\Phi_0$. Plot 2 at Fig. 5 shows R as a function of ρ for $\beta = 0.7$. The metric (40) coincides locally with the metric on the revolution surface determined by the equation $R = R(\rho)$ in $\mathbb{E}^3$. This does not give a global isometric embedding since the period of the angle coordinate is $2\pi\Phi_0$. This surface can be obtained by gluing two figures shown in Fig. 6 along their edges. For the left figure ϕ changes from 0 to π, while for the right one it changes from π to $2\pi\Phi_0$.

V. CONCLUDING REMARKS

We demonstrated that a surface of a rapidly rotating black hole, which cannot be isometrically embedded in $\mathbb{E}^3$, allows such a global embedding in $\mathbb{E}^4$. To construct this embedding one considers first a 2D surface in $\mathbb{E}^3$ formed by straight lines passing through one point ($r = 0$) which has a cone singularity at $r = 0$ with negative angle deficit. Its Gauss curvature outside $r = 0$ vanishes. Next element of the construction is finding a function $R(r)$. The revolution surface for this function in $\mathbb{E}^3$ has a positive angle deficit at $r = 0$. By combining these two maps in such a way that positive and negative angle deficits cancel one another, one obtains a regular global embedding in $\mathbb{E}^4$. This construction can easily be used to find the embedding in $\mathbb{E}^4$ of surfaces of rapidly rotating stationary black holes distorted by an action of external forces or fields, provided the axial symmetry of the spacetime is preserved. An interesting example is a case of a rotating black hole in a homogeneous at infinity magnetic field directed along the axis of the rotation (see e.g. [11–13].

ACKNOWLEDGMENTS

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