w and w' of scalar field models of dark energy

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Important observables to reveal the nature of dark energy are the equation of state w and its time derivative in units of the Hubble time w'. Recently, it was shown that the simplest scalar field models of dark energy (quintessence) occupy rather narrow regions in the $w - w'$ plane. We extend the $w - w'$ plane to $w < -1$ and derive bounds on w' as a function of w for tracker phantom dark energy. We also derive bounds on tracker k-essence. The observational window for w' for $w < -1$ is not narrow, $\sigma(w') \lesssim 6|1 + w|$.

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I. INTRODUCTION

The equation of state $w = p/\rho$ of dark energy is a key observable to reveal the nature of dark energy which accelerates the Universe. $w = -1$ for the cosmological constant and w is in general a function of time for scalar field models of dark energy (quintessence). w and its time derivative in units of the Hubble time, $w' = dw/d\ln a$, are currently constrained from the distance measurements of SNIa (assuming a prior on Ω_m) as $w_0 = -1.31\pm^{0.22}_{0.28}$, $w'_0 = -1.48\pm^{0.90}_{0.81}$ (at 95% confidence level) [1]. The important question is precisely how much we should determine the equation of state observationally.

In this respect, Caldwell and Linder have recently attempted an observation-oriented phase space analysis of quintessence [2]. Namely, instead of the scalar field and its time derivative, they numerically studied the dynamics in the $w - w'$ plane and found that "phase space" of quintessence in the $w - w'$ plane is narrow and that a desired measurement resolution should be $\sigma(w') \lesssim (1 + w)$. More recently, Scherrer analytically derived a tighter lower bound on w' [3]. In this paper, after reviewing the limits (Sec. II) and slightly updating the result in [3], we extend these results to phantom dark energy (Sec. III) and k-essence (Sec. IV).

II. LIMITS OF QUINTESSENCE REVISITED

Firstly, we review the limit of tracker quintessence [3] to introduce the notation. Then we obtain a lower bound on w' for tracker quintessence models.

A. Generic bound

We consider a flat universe consisting of (nonrelativistic) matter and scalar field dark energy ϕ (quintessence). The equation of motion of the quintessence field ϕ is

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi}(\phi) = 0, \tag{2.1}$$

where $V_{,\phi} = \delta V/\delta\phi$. The equation of state w is given by

$$w = \frac{\dot{\phi}^2/2 - V}{\dot{\phi}^2/2 + V}. \tag{2.2}$$

Equation(2.2) suggests that the equation of motion (2.1) may be rewritten by using $w' = dw/d\ln a$. In fact, it is rewritten as [4]

$$\mp\frac{V_{,\phi}}{V} = \sqrt{\frac{3\kappa^2(1 + w)}{\Omega_\phi}}\left(1 + \frac{x'}{6}\right), \tag{2.3}$$

where the minus sign corresponds to $\dot{\phi} > 0 (V_{,\phi} < 0)$ and the plus sign to the opposite, $\kappa^2 = 8\pi G$, and Ω_ϕ is the density parameter of dark energy. x is defined by

$$x = \ln\left(\frac{1 + w}{1 - w}\right), \tag{2.4}$$

and x' is the derivative of x with respect to $\ln a$ and is related with w' as

$$x' = \frac{2w'}{(1 - w)(1 + w)}. \tag{2.5}$$

Since the left-hand side of Eq. (2.3) is positive, $1 + x'/6 > 0$. In terms of w' by the use of Eq. (2.5), we obtain [3]

$$w' > -3(1 - w)(1 + w). \tag{2.6}$$

This bound applies to a more general class of quintessence field which monotonically rolls down the potential.

B. Tracker quintessence

The bound can be tightened for tracker fields which have nearly constant w initially and eventually evolve toward $w = -1$. Tracker fields have attractorlike solutions in the sense that a very wide range of initial conditions rapidly converges to a common cosmic evolutionary track [4]. Taking the derivative of Eq. (2.3) with respect to ϕ, we obtain [3,5]

$$\Gamma - 1 = \frac{3(w_B - w)(1 - \Omega_\phi)}{(1 + w)(6 + x')} - \frac{(1 - w)x'}{2(1 + w)(6 + x')} - \frac{2x''}{(1 + w)(6 + x')^2}, \quad (2.7)$$

where $\Gamma = VV_{,\phi\phi}/V_{,\phi}^2$, w_B is the equation of state of background matter, and x'' is the second derivative of x with respect to $\ln a$. Since w is a constant for tracker fields and Ω_ϕ is initially negligible, w is written in terms of Γ as

$$w = \frac{w_B - 2(\Gamma - 1)}{2(\Gamma - 1) + 1}. \quad (2.8)$$

Since $w' \leq 0$ for tracker fields, $x' \leq 0$. However, since w asymptotically approaches toward -1, x' eventually stops decreasing and then increases toward zero. The minimum of x', x'_m gives the minimum of w' via Eq. (2.5). To find x'_m, we put $x'' = 0$ in Eq. (2.7) and find that

$$x'_m = -6\frac{w(1 - \Omega_\phi) + 2(1 + w)(\Gamma - 1)}{(1 - w) + 2(1 + w)(\Gamma - 1)} > -6\frac{2(1 + w)(\Gamma - 1)}{(1 - w) + 2(1 + w)(\Gamma - 1)}. \quad (2.9)$$

Since x_m is an increasing function of $w(< -1)$, a lower bound is given by w of the tracker solution Eq. (2.8)

$$x'_m > \frac{6w}{1 - 2w}. \quad (2.10)$$

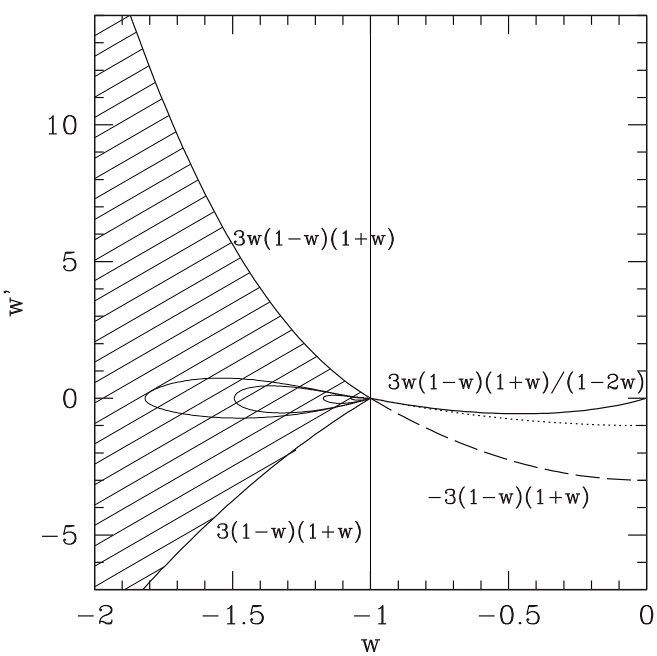

FIG. 1. Bounds on w' as a function of w. For $w > -1$, the curves are lower bounds: The solid curve is our lower bound while the dotted curve is from Ref. [3]. The dashed curve is the generic lower bound Eq. (2.6). The shaded region is bounded by lower and upper bounds for the phantom. The loops in the shaded region are the trajectories for $V = V_0 \log(\kappa\phi)$.

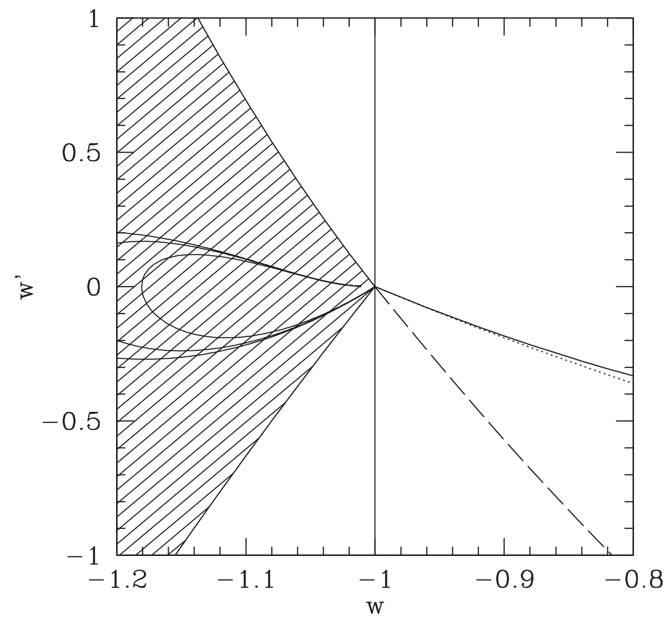

FIG. 2. Same as Fig. 1 but for narrower range of w. The loops on the shaded region are the trajectories for $V = V_0\sqrt{\kappa\phi}$.

From Eq. (2.5), in terms of w', we obtain a lower bound on w',

$$w' > \frac{3w}{1 - 2w}(1 - w)(1 + w) \geq -(1 - w)(1 + w). \quad (2.11)$$

The last inequality is the limit derived by Scherrer [3]. In [3], $\Gamma > 1$ is used to derive the final inequality. However, for tracker quintessence, $\Gamma = 1 - w/2(1 + w)(>1)$ and thus a slightly stronger bound is obtained. These bounds are shown in Figs. 1 and 2.

III. LIMITS OF PHANTOM

We extend the range of w to $w < -1$. The phantom is scalar field dark energy with $w < -1$ [6]. Although there are several models of phantom, we consider a simple scalar field model with the wrong sign of the kinetic term (ghost) [6]. We understand that there are several serious obstacles (e.g. rapid gravitational decay of vacuum) for such a ghost field to be a cosmologically relevant field [7]. Our intention here is more observation-oriented and to provide observables related with the dynamics of the field which behaves like a phantom. After giving the generic lower bound for phantom dark energy, we derive an upper bound on w' for tracker phantom models.

A. Generic bound

In this model, the energy density and the pressure of dark energy is given by $\rho = -\dot{\phi}^2/2 + V$, $p = -\dot{\phi}^2/2 - V$, respectively. We consider a non-negative $V(\phi)$. The equation of motion is given by

$$\ddot{\phi} + 3H\dot{\phi} - V_{,\phi} = 0. \tag{3.1}$$

Therefore, the scalar field rolls down the inverted potential $-V$ (or rolls up the potential V). Similar to quintessence, the phantom equation of motion can be rewritten as

$$\mp \frac{V_{,\phi}}{V} = \sqrt{\frac{-3\kappa^2(1+w)}{\Omega_\phi}}\left(-1 + \frac{x'}{6}\right), \tag{3.2}$$

where the minus sign for $\dot{\phi} > 0 (V_{,\phi} > 0)$ and the plus sign for the opposite. Since the left-hand side is positive, we have an upper bound on x', $x' < 6$, which in turn gives a lower bound of w':

$$w' > 3(1-w)(1+w). \tag{3.3}$$

An interesting feature about this lower bound may be that it is not connected smoothly with the quintessential bound Eq. (2.6) since the kinetic term changes its sign across $w = -1$.

B. Tracker phantom

The tracker equation for phantom field is given by

$$\Gamma - 1 = \frac{3(w_B - w)(1-\Omega_\phi)}{(1+w)(6-x')} - \frac{(1-w)x'}{2(1+w)(6-x')} + \frac{2x''}{(1+w)(6-x')^2}. \tag{3.4}$$

Therefore, for the tracker solution for which w is nearly constant and $\Omega_\phi \to 0$, w is given by

$$w = \frac{w_B - 2(\Gamma - 1)}{2(\Gamma - 1) + 1}. \tag{3.5}$$

Thus $\Gamma < 1/2$ is required for tracking phantom $w < -1$.

Another issue to be addressed is the stability of the tracker solution against perturbation. In the case of quintessence, positive effective mass squared $V_{,\phi\phi} > 0$ is required for the stability. For the phantom the opposite condition $V_{,\phi\phi} < 0$ is required since the phantom rolls down the inverted potential. For constant w, to which a tracker solution corresponds, it can be shown that

$$V_{,\phi\phi} = -\frac{9}{4}H^2(1-w)((1+\Omega_\phi)w + 2). \tag{3.6}$$

Thus, $V_{,\phi\phi} < 0$ implies $-2 < w < -1$ and in terms of Γ the condition is $\Gamma < 0$.

As tracker phantom models, we consider a solution in which w is initially nearly constant and then it evolves toward -1. Such solutions are obtained for convex V, $V_{,\phi\phi} < 0$ (a simple example is $V \propto \sqrt{\phi}$). This is because if $V_{,\phi\phi} < 0$, then $|V_{,\phi}|$ decreases as ϕ climbs up the potential and $|V_{,\phi}|$ becomes negligible but H increases as the phantom becomes dominated and the dynamics of ϕ is eventually dominated by the Hubble friction ($\ddot{\phi} \simeq -3H\dot{\phi}$) and ϕ ceases to move.

Since $w' \geq 0$ for these tracker solutions, $x' \leq 0$. As the phantom is attracted toward $w = -1$, x' stops decreasing and then increases back to a value near zero, being similar to tracker quintessence [3]. The minimum value of x', x'_m, gives an *upper bound* on w' through Eq. (2.5).

To find x'_m, we put $x'' = 0$ in Eq. (3.4) and find that

$$x'_m = -6\frac{w(1-\Omega_\phi) + 2(1+w)(\Gamma - 1)}{(1-w) - 2(1+w)(\Gamma - 1)}. \tag{3.7}$$

Since x_m is an increasing function of $w(< -1)$, a lower bound is given by w of the tracker solution Eq. (3.5):

$$x'_m > 6w\Omega_\phi > 6w. \tag{3.8}$$

From Eq. (2.5), in terms of w', we obtain an upper bound on w':

$$w' < 3w(1-w)(1+w). \tag{3.9}$$

The bound is shown in Figs. 1 and 2. The trajectories of (w, w') for a logarithmic potential $V = V_0 \exp(-\kappa^2\phi^2)$ (Fig. 1) and for $V = V_0\sqrt{\kappa\phi}$ (Fig. 2) are also shown for several initial conditions.

IV. LIMITS OF TRACKER K-ESSENCE

We finally give bounds on w' for k-essence with $w > -1$. K-essence is a scalar field model of dark energy which has a noncanonical kinetic term [8–10]. The pressure of the scalar field ϕ, p, is given by the Lagrangian density $p(\phi, X)$ itself where $X = -\partial_\mu\phi\partial^\mu\phi/2$ and is equal to $\dot{\phi}^2/2$ for the Friedmann model. The energy density ρ is given by $\rho = 2X\partial p/\partial X - p \equiv 2Xp_X - p$.

The equation of motion of the scalar field is given by

$$\ddot{\phi}(p_X + \dot{\phi}^2 p_{XX}) + 3Hp_X\dot{\phi} + p_{X\phi}\dot{\phi}^2 - p_\phi = 0, \tag{4.1}$$

where $p_\phi = \partial p/\partial\phi$, for example. For the factorized form of

$$p(\phi, X) = V(\phi)W(X), \tag{4.2}$$

we can express the equation of motion of ϕ in an alternative form similar to quintessence [11]:

$$\mp \frac{V_{,\phi}}{V^{3/2}} = \frac{\kappa}{2}\sqrt{\frac{(1+w)W_X}{3\Omega_\phi}}(6 + Ax'), \tag{4.3}$$

$$A = \frac{(XW_X - W)(2XW_{XX} + W_X)}{XW_X^2 - WW_X - XWW_{XX}} = \frac{1-w}{c_s^2 - w}, \tag{4.4}$$

where the minus (plus) sign corresponds to $\dot{\phi} > 0$ (< 0), respectively. c_s^2 is the speed of sound of k-essence defined by [12]

$$c_s^2 = \frac{\delta p}{\delta \rho} = \frac{p_X}{p_X + 2Xp_{XX}} = \frac{W_X}{W_X + 2XW_{XX}}. \tag{4.5}$$

A. Tracker K-essence

Similar to quintessence, we define a dimensionless function Γ by $\Gamma = VV_{,\phi\phi}/V_{,\phi}^2$. Taking the time derivative of Eq. (4.3), we obtain [11]

$$\Gamma - \frac{3}{2} = -\frac{1}{(1+w)(6+Ax')}\Bigg[3(w-w_B)(1-\Omega_\phi) + \frac{(1-w)^2}{2(c_s^2-w)}x' + \frac{2(1-w)(c_s^2-w)x'' + 2(\dot{w}(1-c_s^2) - (c_s^2)^{\cdot}(1-w))x'/H}{(6+Ax')(c_s^2-w)^2}\Bigg]. \tag{4.6}$$

Equation (4.6) might be called the k-essential counterpart of the tracker equation. Therefore for the tracker solution (assuming $\Gamma \simeq$ const and $\Omega_\phi \ll 1$) we can write w in terms of Γ:

$$w = \frac{w_B - 2(\Gamma - 3/2)}{2(\Gamma - 3/2) + 1} \simeq \text{const.} \tag{4.7}$$

For tracker k-essence models with $w > -1$, w is nearly constant, so X is also constant since w only depends on X for the factorized $p(\phi, X)$, Eq. (4.2). Then the evolution of energy density depends only on $V(\phi)$, so $\dot{\phi} > 0$ (< 0) corresponds to $V_{,\phi} < 0$ (> 0) and the left-hand side of Eq. (4.4) is positive. This implies $6 + Ax' > 0$, which is written in terms of w' as

$$w' > -3(c_s^2 - w)(1+w). \tag{4.8}$$

This bound is similar to Eq. (2.6). However, in deriving it, we restrict ourselves to tracker k-essence models.

Similar to tracker quintessence, we can sharpen the bound by considering the dynamics more carefully. Since $w' \leq 0$ for tracker fields, $x' \leq 0$. However, since w asymptotically approaches toward -1, x' eventually stops decreasing and then increases toward zero. The minimum of x', x'_m gives the minimum of w' via Eq. (2.5). The analysis is the same as that of tracker quintessence and we only give the final result:

$$w' > \frac{3w}{1-2w}(c_s^2 - w)(1+w). \tag{4.9}$$

This is the k-essential counterpart of the lower bound on w'. If we impose the upper bound on the sound speed as $c_s^2 \leq 1$, then the above bound is reduced to that of tracker quintessence:

$$w' > \frac{3w}{1-2w}(1-w)(1+w). \tag{4.10}$$

The sound speed of dark energy [13–15] is currently difficult to measure (see, for example, [16]). Therefore, it seems difficult to distinguish quintessence from k-essence from the measurements of w and w'.

V. SUMMARY

In this paper, we have extended and generalized bounds on dark energy models in the $w - w'$ plane. First, we have slightly improved the lower bound for tracker quintessence. Second, we have derived both a lower bound for the phantom and an upper bound for the tracker phantom. Finally, we have obtained two lower bounds for k-essence. While the required observational accuracy of w' is similar for quintessence and k-essence, $\sigma(w') \lesssim (1+w)$, the windows of w' for the phantom may not be so narrow, $\sigma(w') \lesssim 6|1+w|$. Although the fate of the universe with $w < -1$ would be disastrous (the future big rip singularity and the disintegration of bound objects) [17,18], be aware of the possibility.

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