Permutation symmetry S_3 and vacuum expectation value structure of flavor-triplet Higgs scalars

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A model with flavor-triplet Higgs scalars ϕ_i ($i = 1, 2, 3$) is investigated under a permutation symmetry S_3 and its symmetry breaking. A possible S_3 breaking form of the Higgs potential whose vacuum expectation values $v_i = \langle\phi_i\rangle$ satisfy a relation $v_1^2 + v_2^2 + v_3^2 = \frac{2}{3}(v_1 + v_2 + v_3)^2$ is investigated, because if we suppose a seesaw-like mass matrix model $M_e = mM^{-1}m$ with $m_{ij} \propto \delta_{ij}v_i$ and $M_{ij} \propto \delta_{ij}$, such a model can lead to the well-known charged lepton mass relation $m_e + m_\mu + m_\tau = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2$.

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I. INTRODUCTION

Of the observed mass spectra of the fundamental particles, quarks and leptons, the charged lepton mass spectrum seems to give a promising clue to the unified understanding of quarks and leptons, because the observed charged lepton masses satisfy a very simple mass relation [1–3],

$$m_e + m_\mu + m_\tau = \tfrac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2, \qquad (1.1)$$

with remarkable precision. The mass formula (1.1) can give an excellent prediction of the tau lepton mass value

$$m_\tau = 1776.97\ \text{MeV}, \qquad (1.2)$$

from the observed electron and muon mass values [4], $m_e = 0.510\,998\,92$ and $m_\mu = 105.658\,369$ MeV (cf. the observed value [4] $m_\tau = 1776.99^{+0.29}_{-0.26}$ MeV). This excellent agreement seems to be beyond a matter of accidental coincidence, so that we should consider the origin of the mass formula (1.1) seriously. Several authors [5–7] have challenged to give an explanation of the mass formula (1.1) from a geometrical point of view. However, up to the present, the theoretical basis of the mass formula (1.1) is still not clear. (For a review, for example, see Ref. [8,9].)

The charged lepton mass formula (1.1) has the following peculiar features:

(a) The mass formula is described in terms of the root squared masses $\sqrt{m_{ei}}$.
(b) The formula is well satisfied at a low energy scale rather than at a high energy scale.
(c) The mass formula is invariant under the exchanges $\sqrt{m_{ei}} \leftrightarrow \sqrt{m_{ej}}$.

Feature (a) suggests that the charged lepton mass spectrum is given by a bilinear form on the basis of some mass-generation mechanism. For example, in Refs. [3,10–12], the formula (1.1) has been discussed on the basis of a seesaw-like mechanism [13]:

$$M_e = mM_E^{-1}m^\dagger, \qquad (1.3)$$

where M_E is a heavy charged lepton mass matrix $M_E \propto \text{diag}(1, 1, 1)$, and m is given by $m \propto \text{diag}(v_1, v_2, v_3)$ [v_i are vacuum expectation values (VEVs) of flavor-triplet scalars ϕ_i]. This idea that mass spectrum is due not to the structure of the Yukawa coupling constants, but to the VEV structure of Higgs scalars ϕ_i at a low energy scale, is very attractive as an explanation of feature (b). We have to seek for a model where the VEVs v_i satisfy the following relation:

$$v_1^2 + v_2^2 + v_3^2 = \tfrac{2}{3}(v_1 + v_2 + v_3)^2. \qquad (1.4)$$

Feature (c) suggests that the Higgs potential is invariant under a permutation symmetry S_3. Such attempts to understand the charged lepton mass spectrum from the VEV structure of flavor-triplet scalars are found in Refs. [3,11,12]. The basic idea is as follows: We consider the following S_3 invariant Higgs potential,

$$V = \mu^2\sum_i(\bar{\phi}_i\phi_i) + \frac{1}{2}\lambda_1\left[\sum_i(\bar{\phi}_i\phi_i)\right]^2 + \lambda_2(\bar{\phi}_\sigma\phi_\sigma)(\bar{\phi}_\pi\phi_\pi + \bar{\phi}_\eta\phi_\eta), \qquad (1.5)$$

where $(\bar{\phi}_i\phi_i) = \phi_i^-\phi_i^+ + \bar{\phi}_i^0\phi_i^0$ ($i = 1, 2, 3$), and (ϕ_π, ϕ_η) and ϕ_σ are a doublet and a singlet in the real basis of S_3, respectively:

$$\phi_\pi = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2),\quad \phi_\eta = \frac{1}{\sqrt{6}}(\phi_1 + \phi_2 - 2\phi_3),\quad \phi_\sigma = \frac{1}{\sqrt{3}}(\phi_1 + \phi_2 + \phi_3). \qquad (1.6)$$

Although, in Refs. [3,11,12], we have regarded the fields ϕ_i as $SU(2)_L$ doublet scalars, we can also consider another model which gives the seesaw form (1.3). For example, by considering a model shown in Fig. 1, we can regard the fields ϕ_i as $SU(2)_L$ singlet scalars. Hereafter, the $SU(2)_L$ structure in the Higgs potential will be neglected. The fields ϕ_i denote both cases, a case of the $SU(2)_L$ doublets and a case of the $SU(2)_L$ singlets.

The conditions that the potential (1.5) takes the minimum lead to the relation for the VEVs $v_i \equiv \langle\phi_i^0\rangle$:

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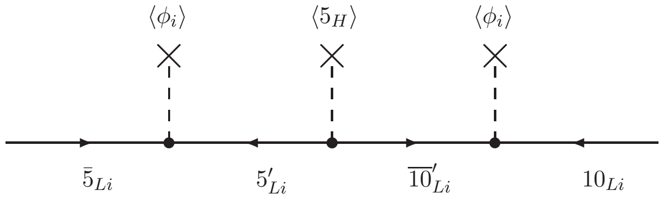

FIG. 1. Seesaw mass generation of the charged leptons, where flavor-triplet scalars ϕ_i are singlets of SU(5).

$$|v_\sigma|^2 = |v_\pi|^2 + |v_\eta|^2 = \frac{-\mu^2}{2\lambda_1 + \lambda_2}. \tag{1.7}$$

Therefore, from the relation

$$\overline{\phi}_1\phi_1 + \overline{\phi}_2\phi_2 + \overline{\phi}_3\phi_3 = \overline{\phi}_\pi\phi_\pi + \overline{\phi}_\eta\phi_\eta + \overline{\phi}_\sigma\phi_\sigma, \tag{1.8}$$

we obtain

$$|v_1|^2 + |v_2|^2 + |v_3|^2 = |v_\pi|^2 + |v_\eta|^2 + |v_\sigma|^2 = 2|v_\sigma|^2 = 2\left(\frac{v_1 + v_2 + v_3}{\sqrt{3}}\right)^2. \tag{1.9}$$

Thus, we obtain the relation (1.4).

However, note that (i) the Higgs potential (1.5) which is invariant under the permutation symmetry S_3 is not a general form of the S_3 invariant Higgs potential, and (ii) it cannot give a relation between $v_\pi = \langle\phi_\pi\rangle$ and $v_\eta = \langle\phi_\eta\rangle$, because the S_3 invariant Higgs potential (1.5) is also invariant under the permutation between ϕ_π and ϕ_η (we cannot determine the values v_π and v_η individually). We have to investigate a potential term which violates a $\phi_\pi \leftrightarrow \phi_\eta$ symmetry (also breaks the S_3 symmetry) but keeps the relation (1.4).

In order to obtain the charged lepton mass relation (1.1), we have to build a model with a seesaw-type mass matrix (1.3). A recent attempt to build such a model will be found in Ref. [14]. However, the purpose of the present paper is not to investigate such a seesaw mass matrix model. The purpose of the present paper is to discuss the Higgs potential form which gives the structure (1.4).

II. S_3 SYMMETRIC HIGGS POTENTIAL

In this section, we discuss a general form of the S_3 symmetric Higgs potential. However, for mass terms, we confine ourselves to the case $\mu^2 \sum \bar{\phi}_i\phi_i$ as given in Eq. (1.5). We consider an S_3-invariant general form only for the dimension-four terms. In general, we have two scalars $(\bar{\phi}_\sigma\phi_\sigma)$ and $(\bar{\phi}_\pi\phi_\pi + \bar{\phi}_\eta\phi_\eta)$ and one pseudoscalar $(\bar{\phi}_\pi\phi_\eta - \bar{\eta}_\eta\phi_\eta)$ [15], so that we write the general form of the S_3-invariant Higgs potential as follows:

$$\begin{aligned} V = {}& \mu^2(\bar{\phi}_1\phi_1 + \bar{\phi}_2\phi_2 + \bar{\phi}_3\phi_3) + \tfrac{1}{2}\lambda_\sigma(\bar{\phi}_\sigma\phi_\sigma)^2 \\ & + \tfrac{1}{2}\lambda_+(\bar{\phi}_\pi\phi_\pi + \bar{\phi}_\eta\phi_\eta)^2 + \tfrac{1}{2}\lambda_-(\bar{\phi}_\pi\phi_\eta - \bar{\phi}_\eta\phi_\pi)^2 \\ & + \lambda_2(\bar{\phi}_\sigma\phi_\sigma)(\bar{\phi}_\pi\phi_\pi + \bar{\phi}_\eta\phi_\eta) \\ & + \lambda_3[(\bar{\phi}_\pi\phi_\pi)(\bar{\phi}_\eta\phi_\sigma) + (\bar{\phi}_\pi\phi_\eta)(\bar{\phi}_\pi\phi_\sigma) \\ & + (\bar{\phi}_\eta\phi_\pi)(\bar{\phi}_\pi\phi_\sigma) - (\bar{\phi}_\eta\phi_\eta)(\bar{\phi}_\eta\phi_\sigma) + \text{H.c.}]. \end{aligned} \tag{2.1}$$

Here, for simplicity, we have denoted the case that ϕ_i are $SU(2)_L$ singlets. If the fields ϕ_i are $SU(2)_L$ doublets, the general form includes, for example, $(\bar{\phi}_\sigma\phi_\pi)(\bar{\phi}_\pi\phi_\sigma) + (\bar{\phi}_\sigma\phi_\eta)(\bar{\phi}_\eta\phi_\sigma)$ in addition to $(\bar{\phi}_\sigma\phi_\sigma)(\bar{\phi}_\pi\phi_\pi + \bar{\phi}_\eta\phi_\eta)$, and so on.

The Higgs potential (2.1) cannot, in general, lead to the relation (1.4). Especially, the λ_3-terms badly spoil the relation (1.4). Only when $\lambda_3 = 0$, the potential (2.1) leads to a simple relation,

$$v_\pi^2 + v_\eta^2 = \frac{\lambda_2 - \lambda_\sigma}{\lambda_2 - \lambda_+} v_\sigma^2, \tag{2.2}$$

so that we get the relation

$$v_1^2 + v_2^2 + v_3^2 = \tfrac{1}{3}K(v_1 + v_2 + v_3)^2, \tag{2.3}$$

where

$$K = 1 + \frac{\lambda_2 - \lambda_\sigma}{\lambda_2 - \lambda_+}. \tag{2.4}$$

Thus, the case with

$$\lambda_3 = 0, \qquad \lambda_\sigma = \lambda_+ \neq \lambda_2 \tag{2.5}$$

can give the relation (1.4).

The Higgs potential (2.1) with the conditions (2.5) is essentially identical with the Higgs potential (1.5). [Hereafter, we will use the expression (1.5) as the S_3-invariant Higgs potential which can give the relation (1.4).] The characteristic of the Higgs potentials (1.5) [and also (2.1) with the constraints (2.5)] which can give the relation (1.4) is that it is invariant under the replacement

$$(\bar{\phi}_\sigma\phi_\sigma) \leftrightarrow (\bar{\phi}_\pi\phi_\pi + \bar{\phi}_\eta\phi_\eta). \tag{2.6}$$

As we stated in Sec. I, the potential (1.5) cannot give the difference between v_π and v_η and it generates a massless scalar, because the potential (1.5) is invariant under an SU(2)-flavor symmetry for the basis (ϕ_π, ϕ_η). We have to introduce a symmetry breaking of the SU(2)-flavor.

III. S_3 SYMMETRY BREAKING IN THE HIGGS POTENTIAL

In this section, we investigate an S_3 symmetry breaking term which does not spoil the relation (1.4).

By the way, when we define parameters z_i by $v_i = z_i v$ with the normalization condition $z_1^2 + z_2^2 + z_3^2 = 1$, the parameters z_i, which satisfy the relation

$$z_1^2 + z_2^2 + z_3^2 = 1 = \tfrac{2}{3}(z_1 + z_2 + z_3)^2, \qquad (3.1)$$

are explicitly expressed as follows [11]:

$$z_1 = \frac{1 - \sqrt{1-\varepsilon}}{\sqrt{6}}, \qquad z_2 = \frac{2 + \sqrt{1-\varepsilon} - \sqrt{3}\sqrt{1+\varepsilon}}{2\sqrt{6}},$$
$$z_3 = \frac{2 + \sqrt{1-\varepsilon} + \sqrt{3}\sqrt{1+\varepsilon}}{2\sqrt{6}}, \qquad (3.2)$$

where the expression (3.2) has been so taken as to give $m_e \to 0$ in the limit of $\varepsilon \to 0$, and the value of ε is

$$\varepsilon = 0.079\,072, \qquad (3.3)$$

from the observed values of the charged lepton masses. Also, for the parameters $z_a = v_a/v$ $(a = \pi, \eta, \sigma)$, we obtain

$$z_\pi = -\frac{1}{4}(\sqrt{3}\sqrt{1-\varepsilon} - \sqrt{1+\varepsilon}),$$
$$z_\eta = -\frac{1}{4}(\sqrt{3}\sqrt{1+\varepsilon} + \sqrt{1-\varepsilon}), \qquad z_\sigma = \frac{1}{\sqrt{2}}. \qquad (3.4)$$

We must to seek for the potential form which gives $z_\pi \neq z_\eta$ keeping the relation (3.1).

As an example of such an S_3 symmetry breaking term, in Ref. [12], a term

$$V_{SB} = \lambda_{SB}[\xi_\pi(\bar{\phi}_\pi \phi_\pi) - \xi_\eta(\bar{\phi}_\eta \phi_\eta)]^2 \qquad (3.5)$$

with $\xi_\pi \neq \xi_\eta$ has been suggested. However, in this paper, we would like to consider a case that the symmetry is softly broken. Therefore, in this section, we consider a symmetry breaking in the mass term in the Higgs potential (1.5).

The Higgs potential is invariant under the SU(2)-flavor symmetry for the basis (ϕ_π, ϕ_η), i.e. under the transformation

$$\begin{pmatrix} \phi'_\pi \\ \phi'_\eta \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} \phi_\pi \\ \phi_\eta \end{pmatrix}, \qquad (3.6)$$

where $s = \sin\theta$ and $c = \cos\theta$. Now, we want to fix the mixing (3.6) to a special basis. Therefore, we take a symmetry breaking form

$$V_{SB} = \mu_{SB}^2(\bar{\phi}'_\pi \phi'_\pi) = \mu_{SB}^2(c\bar{\phi}_\pi - s\bar{\phi}_\eta)(c\phi_\pi - s\phi_\eta). \qquad (3.7)$$

This does not mean that the SU(2)-flavor invariant potential (1.5) is also given in terms of the new basis $(\phi'_\pi, \phi'_\eta, \phi_\sigma)$. We assume that the potential V, Eq. (1.5) (hereafter, we call it V_0), is still given in terms of the basis $(\phi_\pi, \phi_\eta, \phi_\sigma)$, while, only for V_{SB} it is given by the form (3.7). Therefore, we regard the Higgs potential

$$V = V_0 + V_{SB} \qquad (3.8)$$

as a function of the fields $(\phi_\pi, \phi_\eta, \phi_\sigma)$, and the mixing parameter θ as a fundamental parameter in the present model.

Then, we obtain

$$[\mu^2 + \lambda_1(|v_\pi|^2 + |v_\eta|^2 + |v_\sigma|^2) + \lambda_2|v_\sigma|^2]v_\pi + \mu_{SB}^2 c(cv_\pi - sv_\eta) = 0, \qquad (3.9)$$

$$[\mu^2 + \lambda_1(|v_\pi|^2 + |v_\eta|^2 + |v_\sigma|^2) + \lambda_2|v_\sigma|^2]v_\eta - \mu_{SB}^2 s(cv_\pi - sv_\eta) = 0, \qquad (3.10)$$

$$[\mu^2 + \lambda_1(|v_\pi|^2 + |v_\eta|^2 + |v_\sigma|^2) + \lambda_2(|v_\pi|^2 + |v_\eta|^2)]v_\sigma = 0, \qquad (3.11)$$

from the conditions $\partial V/\partial\bar{\phi}_\pi = 0$, $\partial V/\partial\bar{\phi}_\eta = 0$, and $\partial V/\partial\bar{\phi}_\sigma = 0$, respectively. Therefore, for $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, and $\mu_{SB}^2 \neq 0$, we obtain the relations

$$|v_\pi|^2 + |v_\eta|^2 = |v_\sigma|^2, \qquad (3.12)$$

and

$$\frac{s}{c} = \frac{v_\pi}{v_\eta}. \qquad (3.13)$$

The relation (3.12) leads to the relation (1.4). The relation (3.13) means

$$\tan\theta = \frac{\sqrt{3}(z_2 - z_1)}{2z_3 - z_2 - z_1} = \frac{\sqrt{3}(\sqrt{m_\mu} - \sqrt{m_e})}{2\sqrt{m_\tau} - \sqrt{m_\mu} - \sqrt{m_e}}, \qquad (3.14)$$

which gives a numerical result

$$\theta = 12.7324°, \qquad \text{i.e. } \sin\theta = 0.220\,398. \qquad (3.15)$$

Note that the mixing parameter value (3.15) is in excellent agreement with the observed Cabibbo mixing angle, i.e. $|V_{us}| = 0.2200 \pm 0.0026$ [4]. At present, this is an accidental coincidence, because we have not yet discussed a model of the quark mixing. [A formula for the Cabibbo angle similar to Eq. (3.14) is found in Ref. [16].] However, this coincidence will become a hint for seeking for the quark mixing model. In the present stage, the value (3.15) of θ is only a phenomenological result from the observed charged lepton mass spectrum.

IV. CONCLUDING REMARKS

In conclusion, we have investigated a Higgs potential which gives the VEV structure (1.4). We have considered the following scenario for the Higgs potential:

(i) First, we consider SU(3)-flavor symmetric Higgs potential.
(ii) The SU(3) symmetric Higgs potential is broken by the λ_2-terms as shown in Eq. (1.5), but it is still invariant under the S_3 flavor symmetry.
(iii) The S_3 symmetric potential (1.5) is softly broken by the term (3.7) with a phenomenological mixing parameter θ.

Then, we can obtain a realistic charged lepton mass spectrum for the parameter value (3.15). We consider that those symmetry breakings are explicitly broken at a high energy scale. In the present low energy phenomenology, we do not discuss the origin of those symmetry breakings.

The present model is a multi-Higgs model, so that the model basically induces the flavor-changing neutral currents (FCNC). However, the Higgs scalars ϕ_i in the present model do not couple to the quarks and leptons directly. Symbolically speaking, in the seesaw mass matrix model $M_f = m_L M_F^{-1} m_R$, the FCNC effects through the exchange of the Higgs scalars ϕ_{Li} are suppressed by the order of $(M_F^{-1} m_R)^2$. Therefore, we will be able to avoid the FCNC problem from the present seesaw model.

Finally, we would like to emphasize that the relation (1.4) can be obtained independently of the explicit values of the parameters λ_1, λ_2, and μ_{SB}^2 in the Higgs potential. The relation (1.4) is determined only by the form (parameter-independent structure) of the Higgs potential. The explicit values of z_i are dependent only on the value of ε (i.e. on the value of the mixing parameter θ). The magnitude of the parameter ε of the S_3 violation should be understood from more fundamental theory in the future. We believe that the charged lepton mass spectrum will be described only in terms of fundamental constants without adjustable parameters, while quark and neutrino mass matrices will be described in terms of such fundamental constants and some phenomenological parameters.

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