Quark loop contribution to $\pi^0 \to 4\gamma$

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We find the contribution of constituent quark-loop mechanism to the branching ratio of $\pi^0 \to 4\gamma$ process to be $B_{4\gamma}^{\mathrm{hadr}} = \Gamma_{\pi^0 \to 4\gamma}/\Gamma_{\pi^0 \to 2\gamma} \approx 5.45 \cdot 10^{-16}$ for the reasonable choice of constituent quark mass $m \approx 280$ MeV. This result is in agreement with the vector-dominance approach result obtained years ago. Thus the main contribution arises from the QED mechanism $\pi^0 \to \gamma(\gamma^*) \to \gamma(3\gamma)$ including light-light scattering block with electron loop. This contribution was investigated in the paper of one of us and gave $B_{4\gamma}^{\mathrm{QED}} \sim 2.6 \cdot 10^{-11}$.

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I. INTRODUCTION

In the program of investigation of CP-violating effects one of the directions is the search of the "forbidden" process [1,2]:

$$\pi^0 \to 3\gamma. \tag{1}$$

An important background to it is the allowed decay:

$$\pi^0 \to 4\gamma. \tag{2}$$

A realistic estimation of the differential width of it is needed. Two types of contributions—hadronic and electromagnetic—as a relevant mechanisms of $\pi^0 \to 4\gamma$ must be considered.

It was suggested in [3] (1972) and confirmed in [4] (1995) that the main contribution have an electromagnetic nature:

$$B_{4\gamma}^{\mathrm{QED}} = \frac{\Gamma_{\pi^0 \to 4\gamma}}{\Gamma_{\pi^0 \to 2\gamma}} \sim 2.6 \cdot 10^{-11}. \tag{3}$$

As for the hadronic part—two quite different predictions were done: paper [5] gave $B_{4\gamma}^{\mathrm{hadr}} \sim 10^{-9}$ and the resulty of paper [3] is $B_{4\gamma}^{\mathrm{hadr}} \sim 10^{-14}$. Significantly lower bound for branching $B_{4\gamma}^{\mathrm{hadr}} \sim 7.1 \cdot 10^{-18}$ was quite recently obtained in frames of chiral perturbation theory in [6].

This contradiction is the motivation of our investigation.

It is known the duality property in the description of strong-interaction phenomena:

(i) The meson and barion (hadrons) approach;

(ii) Quark-gluon approach.

In this paper we estimate (using the quark-gluon approach) the $\pi^0 \to 4\gamma$ width in the model with constituent quark loop.

II. APPROACH BASED ON PCAC AND VECTOR-DOMINANCE HYPOTHESES

Considering the process $\pi^0 \to 4\gamma$ within the Vector-Dominance model (VDM) leads to the following Feynman diagram (FD) Fig. 1. The amplitude which corresponds to this FD is [3]:

$$\begin{aligned} M_{4\gamma} = \frac{M_{2\gamma}}{M}\bigg[& \frac{g_{\rho\pi\gamma}^2}{(k_2+k_3+k_4)^2 - m_\rho^2} \\ & + \frac{g_{\omega\pi\gamma}^2}{(k_2+k_3+k_4)^2 - m_\omega^2}\bigg] \\ & \times \frac{1}{(k_3+k_4)^2 - M^2}\varepsilon_{\mu\nu\sigma\lambda} e_3^\mu k_3^\nu e_4^\sigma k_4^\lambda \\ & \cdot \varepsilon_{\alpha\beta\gamma\delta}(k_2+k_3+k_4)^\beta e_1^\gamma k_1^\delta \\ & \cdot \varepsilon^{\alpha}_{\beta'\gamma'\delta'}(k_2+k_3+k_4)^{\beta'} e_2^{\gamma'} k_2^{\delta'} \\ & + 11 \text{ terms obtained by permutations,} \end{aligned} \tag{4}$$

where $\{e_i, k_i\}$—is the polarization vector and momentum of i-th photon; M—is the mass of pion and $M_{2\gamma}$—is the amplitude of $\pi^0 \to 2\gamma$ decay. The coupling constants are fixed by VDM to be:

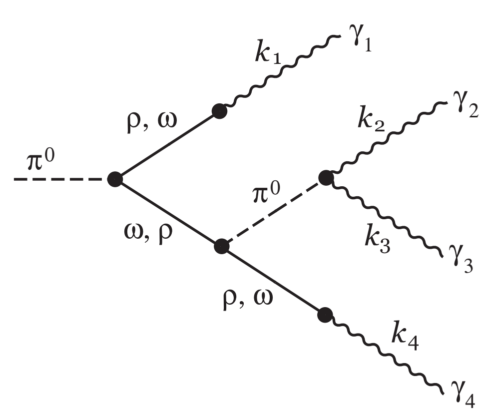

FIG. 1. Diagrams of hadronic contribution within Vector-Dominance model (VDM).

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$$g_{v\pi\gamma} = \frac{g_{\pi\omega\rho}\lambda_v}{M^3} \tag{5}$$

where $\lambda_\rho = g_\rho$, $\lambda_\omega = g_\omega/\sqrt{2}$, $g_\rho^2 = 2F_\pi^2 m_\rho^2$, $g_{\pi\omega\rho}^2/4\pi = 0.51$ (see [7]), $F_\pi = 94$ MeV and:

$$g_\omega^2 = \begin{cases} 0.43 g_\rho^2, & \text{Das, Mathur, Okubo} \\ 0.23 g_\rho^2 & \text{Oakes, Sakurai} \end{cases} \tag{6}$$

[8,9], respectively. Estimating the branching ratio contribution which comes from amplitude (4), one gets [3]:

$$B_{4\gamma}^{\mathrm{hadr}} \le \begin{cases} 8.6 \cdot 10^{-16}, & \text{Das, Mathur, Okubo.} \\ 7.0 \cdot 10^{-16}, & \text{Oakes, Sakurai.} \end{cases} \tag{7}$$

In paper [5] the photons' identity was not taken into account (thus destructive interferences were lost). And this gave the strongly overestimated result $B_{4\gamma}^{\mathrm{hadr}} \sim 10^{-9}$.

III. APPROACH OF CONSTITUENT (HEAVY) QUARK LOOP

In this paper we consider the mechanism with the constituent quark loop. First we need to determine the pion-quark coupling constant g. To do this we write out the $\pi^0 \to 2\gamma$ amplitude within quark-loop approach assuming the pion wave function to be

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}). \tag{8}$$

This gives [see Fig. 2(a)]:

$$M_{2\gamma} = 2ig\frac{\alpha m}{\pi M^2}\frac{N_2}{\sqrt{2}}F(z)(e_1 e_2 k_1 k_2), \tag{9}$$

where $(e_1 e_2 k_1 k_2) \equiv \varepsilon_{\mu\nu\alpha\beta} e_1^\mu e_2^\nu k_1^\alpha k_2^\beta$, $\alpha = 1/137$—is fine-structure constant, the color-charge factor is $N_2 = 3 \cdot ((\frac{2}{3})^2 - (\frac{1}{3})^2) = 1$, m, M—constituent quark mass and neutral pion mass correspondingly, $z = M^2/m^2$ and $F(z) = \int_0^1 \frac{dx}{x} \ln(1 - zx(1-x))$. We use constituent quark mass equal to $m = 280$ MeV (according to the analysis performed in paper [10]), thus $z \approx 0.23$ and $F(z) \approx -0.118$.

Comparing (9) with current-algebra (CA) result:

$$M_{2\gamma}^{\mathrm{CA}} = \frac{\alpha}{2\pi F_\pi}(e_1 e_2 k_1 k_2), \qquad \Gamma_{2\gamma} = \frac{\alpha^2}{2^6 \pi^3}\frac{M^3}{F_\pi^2} \approx 7.4 \text{ eV} \tag{10}$$

with $F_\pi = 94$ MeV, we obtain for pion-quark coupling constant $g = 2.065$.

Now we start to estimate the hadronic contribution to $\pi^0 \to 4\gamma$ decay within the quark-loop approach which gives us the set of FDs presented in Fig. 2(b). The amplitude for the first FD has the form:

$$M_{4\gamma}^{(1)} = -g\alpha^2 \frac{N_4}{\sqrt{2}} \cdot I, \qquad I = \int \frac{d^4 k}{i\pi^2} \frac{\mathrm{Sp}}{A_1 A_2 A_3 A_4 A_5}, \tag{11}$$

where $A_1 = k^2 - m^2$, $A_2 = (k + k_2)^2 - m^2$, $A_3 = (k - k_3)^2 - m^2$, $A_4 = (k - k_3 - k_4)^2 - m^2$, $A_5 = (k + k_1 + k_2)^2 - m^2$; k, m—are the momentum and mass of quark in the loop ($m = 280$ MeV). $\mathrm{Sp} \equiv \mathrm{Sp}[\gamma_5(\hat{k} - \hat{k}_3 - \hat{k}_4 + m)\hat{e}_4(\hat{k} - \hat{k}_3 + m)\hat{e}_3(\hat{k} + m)\hat{e}_2(\hat{k} + \hat{k}_2 + m)\hat{e}_1(\hat{k} + \hat{k}_1 + \hat{k}_2 + m)]$. The color-charge factor for this case is $N_4 = 3 \cdot ((\frac{2}{3})^4 - (\frac{1}{3})^4) = \frac{5}{9}$. Next we join the denominators of loop fermions propagators A_i:

$$\frac{1}{A_1 \cdots A_5} = 4! \int_0^1 \frac{d\tau}{(A_1 x_1 + \ldots + A_5 x_5)^5}, \tag{12}$$

where $d\tau = dx_1 \ldots dx_5 \delta(x_1 + \ldots + x_5 - 1)$, and get $A_1 x_1 + \ldots + A_5 x_5 = (k + b)^2 - d$, where $b = (k_1 + k_2)x_5 + k_2 x_2 - k_3 x_3 - (k_3 + k_4)x_4$, $d = m^2 - M^2 a^2$, $a^2 = x_4 x_5 + x_3 x_5(t_{13} + t_{23}) + t_{23} x_2 x_3 + x_2 x_4(t_{23} + t_{24}) + t_{34} x_4(x_1 + x_2) + t_{12} x_5(x_1 + x_3)$, $t_{ij} \equiv 2(k_i k_j)/M^2$. Performing a shift of loop momentum ($k \to \chi + b$) and calculating the loop trace $\mathrm{Sp}[\ldots]$ we can cancel even powers of new integration momenta χ with denominators $(\chi^2 - d)$. After that we may use a well-known formula of integration:

$$\int \frac{d^4\chi}{i\pi^2} \frac{1}{(\chi^2 - d)^n} = \frac{(-1)^n}{(n-1)(n-2)} \frac{1}{d^{n-2}}. \tag{13}$$

To perform the integration over Feynman x's we will expand the obtained expression for amplitude over $z = M^2/m^2 \approx 0.23$, i.e. use formula:

$$d^{-n} = m^{-2n}\Big(1 + na^2 z + \frac{1}{2!}n(n+1)(a^2 z)^2 + \frac{1}{3!}n(n+1)(n+2)(a^2 z)^3 + \ldots\Big). \tag{14}$$

This makes the integration over x's trivial:

$$\int d\tau x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} = \frac{n_1! n_2! n_3! n_4!}{(n_1 + n_2 + n_3 + n_4 + 4)!}. \tag{15}$$

Thus we obtain the amplitude for first FD:

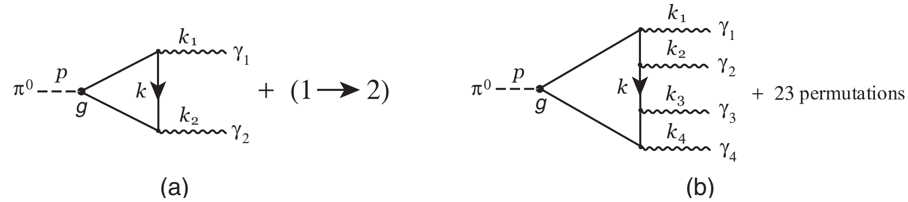

FIG. 2. Diagrams of hadronic contribution within constituent quark-loop approach.

$$M_{4\gamma}^{(1)} = -g\alpha^2 \frac{N_4}{\sqrt{2}} \cdot I, \qquad I = \frac{1}{m}(B_0 + zB_1 + z^2 B_2 + z^3 B_3 + \ldots), \tag{16}$$

where B_i—some rather complicated but known functions of e_i and t_{ij}. The total amplitude $M_{4\gamma}$ is the sum of 24 terms obtained from (16) by permutations of final photon legs (i.e. pairs $\{e_i, k_i\}$):

$$M_{4\gamma} = \sum_{\text{perm}} M_{4\gamma}^{(1)}. \tag{17}$$

Performing this summation we see that:

$$\sum_{\text{perm}} B_0 = \sum_{\text{perm}} B_1 = \sum_{\text{perm}} B_2 = 0, \qquad \sum_{\text{perm}} B_3 \neq 0. \tag{18}$$

This can be obtained by applying Schouten identity:

$$(p_1 p_2 p_3 p_4) Q_\mu = (\mu p_2 p_3 p_4)(Qp_1) + (p_1 \mu p_3 p_4)(Qp_2) + (p_1 p_2 \mu p_4)(Qp_3) + (p_1 p_2 p_3 \mu)(Qp_4), \tag{19}$$

to antisymmetric Levi-Civita tensors $(\alpha\beta\gamma\delta)$ in functions B_i. Here we should notice that the total amplitude $M_{4\gamma}$ fulfills the requirements of gauge invariance and Bose symmetry:

$$k_1^\mu M_{\mu\nu\rho\sigma}^{4\gamma} = k_2^\nu M_{\mu\nu\rho\sigma}^{4\gamma} = k_3^\rho M_{\mu\nu\rho\sigma}^{4\gamma} = k_4^\sigma M_{\mu\nu\rho\sigma}^{4\gamma} = 0,$$
$$M_{\mu\nu\rho\sigma}^{4\gamma}(k_1, k_2, k_3, k_4) = M_{\nu\mu\rho\sigma}^{4\gamma}(k_2, k_1, k_3, k_4) = \ldots.$$

After squaring the amplitude $M_{4\gamma}$ and summing over photons polarizations $\{\lambda_1\lambda_2\lambda_3\lambda_4\}$ we have:

$$\sum_{\lambda_1\lambda_2\lambda_3\lambda_4} |M_{4\gamma}|^2 = \left(-g\alpha^2 \frac{N_4}{\sqrt{2}}\right)^2 \frac{1}{225} \frac{z^6}{m^2} P, \qquad \text{with} \quad P = \sum_{\text{perm}} S_2, \tag{20}$$

where:

$$S_2 = \frac{1}{2} t_{12}(8t_{13}(t_{14}^2(t_{12}t_{13} + t_{23}(-2t_{14} + 3t_{23})) + (3t_{12}t_{13} + 4t_{23}(-2t_{14} + t_{23}))t_{24}^2 + 6t_{13}t_{24}^3) + 4t_{13}t_{14}(t_{14} - t_{23})t_{24}t_{34} - 32t_{13}t_{24}^2 t_{34}^2 + (3t_{12}^2 - 40t_{13}t_{24})t_{34}^3 + 13t_{12}t_{34}^4). \tag{21}$$

The $\pi^0 \to 4\gamma$ decay width $d\Gamma_{4\gamma}^{\text{hadr}}$ has a form:

$$d\Gamma_{4\gamma}^{\text{hadr}} = \frac{1}{(2\pi)^8 2M} \int \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} |M_{4\gamma}|^2 d\phi_4, \tag{22}$$

where the phase volume $d\phi_4$ is considered in Appendix A. To evaluate the hadronic contribution to total decay width $\Gamma_{4\gamma}^{\text{hadr}}$ we perform the numerical phase-space integration and finally obtain:

$$\Gamma_{4\gamma}^{\text{hadr}} = \frac{M}{2^{11}\pi^6} g^2 \alpha^4 \frac{N_4^2}{225} z^7 \mathcal{J}, \qquad \text{where } \mathcal{J} = \int d\tilde{\phi}_4 P \approx 0.097, \tag{23}$$

where the dimensionless phase volume $d\tilde{\phi}_4$ is defined by the relation $d\phi_4 = \frac{\pi^2}{2} M^4 d\tilde{\phi}_4$. Thus our result for hadronic contribution to total decay width is $\Gamma_{4\gamma}^{\text{hadr}} \approx 4 \cdot 10^{-15}$ eV.

IV. CONCLUSION

Using the result listed above (see (23)) we estimate the hadronic contribution to branching of $\pi^0 \to 4\gamma$ decay as:

$$B_{4\gamma}^{\text{hadr}} = \frac{\Gamma_{4\gamma}^{\text{hadr}}}{\Gamma_{2\gamma}} \approx 5.45 \cdot 10^{-16}. \tag{24}$$

This value has the same order of magnitude as the one obtained in [3]. It also is in a good agreement with the recent result [11] obtained within the meson and barion approach. But it contradicts the result of paper [6] obtained by using chiral perturbation theory.

The result obtained in [12] $B_{4\gamma}^{\text{hadr}} \leq 10^{-9}$ is in strong contradiction to our estimate and QED result [4]. As for the result of paper [5] it presumably needs a serious revision as amplitude interferences were not taken into account.

We should notice that our considerations were performed within QED. As for QCD corrections they may be parametrized in the variation of constituent quark mass m. For instance if uncertainty of m is about 5 MeV it can result in $|\delta B_{4\gamma}/B_{4\gamma}| \sim 10\%$. The (rather rough) estimation of the next term of expansion on $z = M^2/m^2$ gives $\approx -10\%$. So for lower mass of quark $m = 275$ MeV these corrections mainly compensate for each other and our result turns to be correct within several percents. For $m = 285$ MeV the resulting correction is negative and about -20%.

Finally we may conclude that the main contribution to $\pi^0 \to 4\gamma$ decay has the QED nature, i.e. mechanism with light-light scattering block with electron loop: $\pi^0 \to \gamma(\gamma^*) \to \gamma(3\gamma)$. This contribution was investigated in a paper of one of us [4] and appeared to be equal to

$$B_{4\gamma}^{\text{QED}} \approx 2.6 \cdot 10^{-11}. \tag{25}$$

APPENDIX: PHASE VOLUME

The phase volume $d\phi_4$ for decay $\pi^0 \to 4\gamma$ is:

$$d\phi_4 = \frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\frac{d^3k_3}{2\omega_3}\frac{d^3k_4}{2\omega_4}\delta^4(p - k_1 - k_2 - k_3 - k_4), \qquad p^2 = M^2, \qquad k_i^2 = 0. \tag{A1}$$

Being expressed in terms of energy fractions of photons $y_i = \omega_i/M$ (which satisfy the identity $y_1 + y_2 + y_3 + y_4 = 1$) and cosines of photons momenta orientations

$C_{ij} = \cos(\widehat{\vec{k}_i \vec{k}_j})$ this phase volume reads to be:

$$\int d\phi_4 \ldots = \frac{\pi^2}{2} M^4 \int_0^{1/2} dy_1 dy_2 \int_0^{2\pi} d\phi_3 \int_{-1}^{1} dC_{12} dC_{13} \frac{y_1 y_2 y_3}{A} \ldots, \tag{A2}$$

where $A = 1 - y_1(1 - C_{13}) - y_2(1 - C_{23})$. The region of integration is determined by conditions:

$$\begin{aligned} &0 < y_{1,2} < \frac{1}{2}, \qquad -1 < C_{ij} < 1, \qquad 0 < y_3 = \frac{1}{2A}(1 - 2(y_1 + y_2) + 2y_1 y_2(1 - C_{12})) < \frac{1}{2}, \\ &0 < y_4 = \sqrt{y_1^2 + y_2^2 + y_3^2 + 2y_1 y_2 C_{12} + 2y_1 y_3 C_{13} + 2y_2 y_3 C_{23}} < \frac{1}{2}, \end{aligned} \tag{A3}$$

while $C_{23} = C_{12} C_{13} + S_{12} S_{13} \cos(\phi_3)$, and $S_{ij} = \sin(\widehat{\vec{k}_i \vec{k}_j})$. The kinematical variables t_{ij} can be expressed as:

$$\begin{aligned} &t_{12} = 2y_1 y_2(1 - C_{12}), \qquad t_{23} = 2y_2 y_3(1 - C_{23}), \qquad t_{13} = 2y_1 y_3(1 - C_{13}), \qquad t_{24} = 2y_2(y_2 + y_4 + y_1 C_{12} + y_3 C_{23}), \\ &t_{14} = 2y_1(y_1 + y_4 + y_2 C_{12} + y_3 C_{13}), \qquad t_{34} = 2y_3(y_3 + y_4 + y_1 C_{13} + y_2 C_{23}). \end{aligned} \tag{A4}$$

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