Analytical solution of $t\bar{t}$ dilepton equations

Lars Sonnenschein*

LPNHE, Universités Paris VI et VII, 4, Place Jussieu, Tour 33, RdC, 75252 Paris Cedex 05, France

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The top quark antiquark production system in the dilepton decay channel is described by a set of equations which is nonlinear in the unknown neutrino momenta. Its most precise and least time consuming solution is of major importance for measurements of top quark properties like the top quark mass and $t\bar{t}$ spin correlations. The initial system of equations can be transformed into two polynomial equations with two unknowns by means of elementary algebraic operations. These two polynomials of multidegree two can be reduced to one univariate polynomial of degree four by means of resultants. The obtained quartic equation is solved analytically.

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I. INTRODUCTION

In 1992, Dalitz and Goldstein published a numerical method based on geometrical considerations to solve the system of equations describing the kinematics of the $t\bar{t}$ decay in the dilepton channel [1]. In 2004 an approximation of the system of equations—assuming that the transverse momentum of the $t\bar{t}$ system can be neglected—has been solved analytically [2] by means of computer algebra software such as [3]. Meanwhile the transverse momentum constraint has been omitted while the solution is still derived by means of computer algebra and its accuracy does not reach real precision [4]. In 2005, the system of equations could be solved algebraically to real precision free of any singularity [5]. The analytical solution introduced here is based on a new Ansatz which minimises the amount of intermediate steps to derive the solution. This approach makes the need of computer algebra superfluous. In addition it provides more transparency and control over singularities which are intrinsic to the analytical solution. Further the accuracy achieved is—as already in the algebraic approach [5]—of real precision. Important improvements in terms of robustness, code volume and time consumption with respect to the algebraic approach make this method more convenient for applications in practice. Other solution methods can compare their performance to the reference method described here. It should be mentioned that different approaches leading to analytical solutions, without giving a complete algebraic derivation and without rigorous discussion of reducible and irreducible singularities exist in the literature [6,7].

In the next section the system of $t\bar{t}$ dilepton equations is introduced, followed by the derivation of the analytical solution including a rigorous discussion of the reducible and irreducible singularities of the analytical solution. Subsequently, the performance of the method is elaborated.

*Electronic address: sonne@in2p3.fr

II. $t\bar{t}$ DILEPTON KINEMATICS

The system of equations describing the kinematics of $t\bar{t}$ dilepton events can be expressed by the two linear and six non linear equations

$$
\begin{aligned}
\not{E}_x &= p_{\nu_x} + p_{\bar{\nu}_x},\\
\not{E}_y &= p_{\nu_y} + p_{\bar{\nu}_y},\\
E_\nu^2 &= p_{\nu_x}^2 + p_{\nu_y}^2 + p_{\nu_z}^2,\\
E_{\bar{\nu}}^2 &= p_{\bar{\nu}_x}^2 + p_{\bar{\nu}_y}^2 + p_{\bar{\nu}_z}^2,\\
m_{W^+}^2 &= (E_{\ell^+} + E_\nu)^2 - (p_{\ell_x^+} + p_{\nu_x})^2 - (p_{\ell_y^+} + p_{\nu_y})^2\\
&\quad - (p_{\ell_z^+} + p_{\nu_z})^2,\\
m_{W^-}^2 &= (E_{\ell^-} + E_{\bar{\nu}})^2 - (p_{\ell_x^-} + p_{\bar{\nu}_x})^2 - (p_{\ell_y^-} + p_{\bar{\nu}_y})^2\\
&\quad - (p_{\ell_z^-} + p_{\bar{\nu}_z})^2,\\
m_t^2 &= (E_b + E_{\ell^+} + E_\nu)^2 - (p_{b_x} + p_{\ell_x^+} + p_{\nu_x})^2\\
&\quad - (p_{b_y} + p_{\ell_y^+} + p_{\nu_y})^2 - (p_{b_z} + p_{\ell_z^+} + p_{\nu_z})^2,\\
m_{\bar{t}}^2 &= (E_{\bar{b}} + E_{\ell^-} + E_{\bar{\nu}})^2 - (p_{\bar{b}_x} p_{\ell_x^-} + p_{\bar{\nu}_x})^2\\
&\quad - (p_{\bar{b}_y} + p_{\ell_y^-} + p_{\bar{\nu}_y})^2 - (p_{\bar{b}_z} + p_{\ell_z^-} + p_{\bar{\nu}_z})^2. \quad (1)
\end{aligned}
$$

The z-axis is here assumed to be parallel orientated to the beam axis while the x- and y-coordinates span the transverse plane. The first two equations relate the projection of the missing transverse energy onto one of the transverse axes (x or y) to the sum of the neutrino and antineutrino momentum components belonging to the same projection. The next two equations relate the energy of the neutrino and antineutrino, which are assumed to be massless in good approximation, with their momenta. Finally four non linear equations describe the W boson and top quark (antiquark) mass constraints by relating the invariant masses to the energy and momenta of their decay particles via relativistic 4-vector arithmetics.

III. ANALYTICAL SOLUTION

The system of Eq. (1) can be subdivided in two entangled sets of equations. One set of equations, describing the $t \to bW^+ \to b\ell^+\nu_\ell$ parton branch of the event, depends on p_{ν_z} while the other pair of equations, describing the $\bar{t} \to \bar{b}W^- \to \bar{b}\ell^-\bar{\nu}_\ell$ parton branch of the event, depends on $p_{\bar{\nu}_z}$.

The equation describing the invariance of the W boson mass can be expressed in the following way

$$\begin{aligned} m_{W^+}^2 &= (E_{\ell^+} + E_\nu)^2 - (\vec{p}_{\ell^+} + \vec{p}_\nu)^2 \\ &= E_{\ell^+}^2 + 2E_{\ell^+}E_\nu + E_\nu^2 - \vec{p}_{\ell^+}^2 - 2\vec{p}_{\ell^+}\vec{p}_\nu - \vec{p}_\nu^2 \\ &= m_{\ell^+}^2 + 2E_{\ell^+}E_\nu - 2\vec{p}_{\ell^+}\vec{p}_\nu \end{aligned} \tag{2}$$

which can be rewritten as

$$E_\nu = \frac{m_{W^+}^2 - m_{\ell^+}^2 + 2\vec{p}_{\ell^+}\vec{p}_\nu}{2E_{\ell^+}}. \tag{3}$$

The equation describing the invariance of the top quark mass can be transformed in the same way leading to

$$E_\nu = \frac{m_t^2 - m_b^2 - m_{\ell^+}^2 - 2E_bE_{\ell^+} + 2\vec{p}_b\vec{p}_{\ell^+} + 2(\vec{p}_b + \vec{p}_{\ell^+})\vec{p}_\nu}{2(E_b + E_{\ell^+})} \tag{4}$$

where additional terms emerge due to the fact that quantities which depended in Eq. (3) only on the lepton depend now also on the b quark. Next the unknown E_ν can be eliminated by subtracting Eq. (4) from (3), leading to an equation of the form

$$0 = a_1 + a_2p_{\nu_x} + a_3p_{\nu_y} + a_4p_{\nu_z} \tag{5}$$

where the coefficients a are constants given in the appendix A This equation is linear in the three neutrino momentum components. Since the unknown p_{ν_z} does only appear in the top quark parton branch it is mandatory to eliminate this variable with a linear independent equation of the top quark parton branch to obtain finally together with the equations of the antitop quark branch two equations of the two unknowns p_{ν_x} and p_{ν_y}.

To eliminate the unknown p_{ν_z} it is straight forward to use Eq. (3) (for convenience multiplied by the denominator $2E_{\ell^+}$). The neutrino energy E_ν can be expressed in terms of the three neutrino momenta components in substituting it with the third equation of (1). To obtain a polynomial equation the squared of this equation is being considered in the following. The terms squared in the longitudinal neutrino momentum cancel out accidentally. It is exactly this cancellation which permits to eliminate the neutrino momentum p_{ν_z} by means of the linear Eq. (5). The resulting equation of the form

$$\begin{aligned} 0 = c_{22} + c_{21}p_{\nu_x} + c_{11}p_{\nu_y} + c_{21}p_{\nu_x}^2 + c_{10}p_{\nu_x}p_{\nu_y} \\ + c_{00}p_{\nu_y}^2 \end{aligned} \tag{6}$$

is a multivariate polynomial of multidegree two which depends only on the transverse neutrino momenta p_{ν_x} and p_{ν_y}. The coefficients are again constants which can be expressed in terms of the former derived constants a and are given in the Appendix A

In the same way can be proceeded for the equations describing the antitop quark parton branch. The equivalent of Eq. (5) reads

$$0 = b_1 + b_2p_{\bar{\nu}_x} + b_3p_{\bar{\nu}_y} + b_4p_{\bar{\nu}_z} \tag{7}$$

and the counter part of polynomial (6) can be written as

$$\begin{aligned} 0 = d'_{22} + d'_{21}p_{\bar{\nu}_x} + d'_{11}p_{\bar{\nu}_y} + d'_{21}p_{\bar{\nu}_x}^2 + d'_{10}p_{\bar{\nu}_x}p'_{\bar{\nu}_y} \\ + d'_{00}p_{\bar{\nu}_y}^2. \end{aligned} \tag{8}$$

The two equations linear in the three (anti-)neutrino momenta (5) and (7) build the minimal Ansatz used here. In contrast the Ansatz made in [2,4] is based on two equations linear in the four unknowns $p_{\bar{\nu}_x}$, $p_{\bar{\nu}_y}$, $p_{\bar{\nu}_z}$, p_{ν_z}.

To reduce Eqs. (6) and (8) to two polynomial equations of two unknowns the transverse antineutrino momenta of Eq. (8) can be expressed by the transverse neutrino momenta with help of the missing transverse energy relations of the system of Eq. (1). Since these relations are linear in the neutrino and antineutrino momenta the substitution leads again to a polynomial of the form

$$\begin{aligned} 0 = d_{22} + d_{21}p_{\nu_x} + d_{11}p_{\nu_y} + d_{21}p_{\nu_x}^2 + d_{10}p_{\nu_x}p_{\nu_y} \\ + d_{00}p_{\nu_y}^2 \end{aligned} \tag{9}$$

with multidegree two whose coefficients are given in the appendix A To solve these two polynomials without loss of generality to p_{ν_x} the resultant with respect to the neutrino momentum p_{ν_y} is computed as follows. The coefficients and monomials of the two polynomials (6) and (9) are rewritten in such a way that they are ordered in powers of p_{ν_y} like

$$c = c_0p_{\nu_y}^2 + c_1p_{\nu_y} + c_2, \tag{10}$$

$$d = d_0p_{\nu_y}^2 + d_1p_{\nu_y} + d_2 \tag{11}$$

where c and d are polynomials of the remaining unknowns p_{ν_x}, p_{ν_y} and the coefficients c_m, d_n are univariate polynomials of p_{ν_x} only. The resultant can then be obtained by computing the determinant of the Sylvester matrix

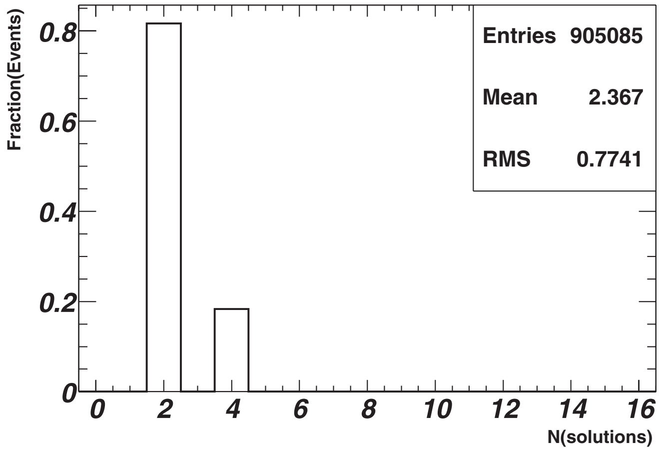

FIG. 1. Number of solutions per event for particles before any radiation. t quark and W boson masses are assumed to be known exactly.

$$\mathrm{Res}\,(p_{\nu_y}) = \det\begin{pmatrix} c_0 & & d_0 & \\ c_1 & c_0 & d_1 & d_0 \\ c_2 & c_1 & d_2 & d_1 \\ & c_2 & & d_2 \end{pmatrix} = 0 \tag{12}$$

which is equated to zero. The omitted elements of the matrix are identical to zero. The resultant is a univariate polynomial of the form

$$h_0 p_{\nu_x}^4 + h_1 p_{\nu_x}^3 + h_2 p_{\nu_x}^2 + h_3 p_{\nu_x} + h_4 \tag{13}$$

which contains the remaining unknown p_{ν_x}. It is of degree four and can be solved analytically. The coefficients h are given in the appendix A This result shows that there is at most a four fold ambiguity. In Fig. 1 the distribution of the number of solutions per event is plotted. Here it has been assumed that the 4-vectors of the particles and the top quark and W boson masses which enter into the system of equations are known exactly. Under these conditions there are two solutions in about 80% of cases and four solutions else. In the next section it will be investigated how this distribution changes under more realistic conditions when the assumption of exactness between particles and reconstructed objects is not valid anymore. Once the solution of a neutrino momentum p_{ν_x} has been found the other neutrino and antineutrino momentum components have to be determined. The antineutrino momentum $p_{\bar{\nu}_x}$ can be immediately obtained by the linear transverse missing energy relation of the initial system of Eq. (1). To derive the neutrino momentum p_{ν_y} Eq. (10) is multiplied by d_0 and Eq. (11) is multiplied by c_0 so that their difference yields a linear equation in the neutrino momentum

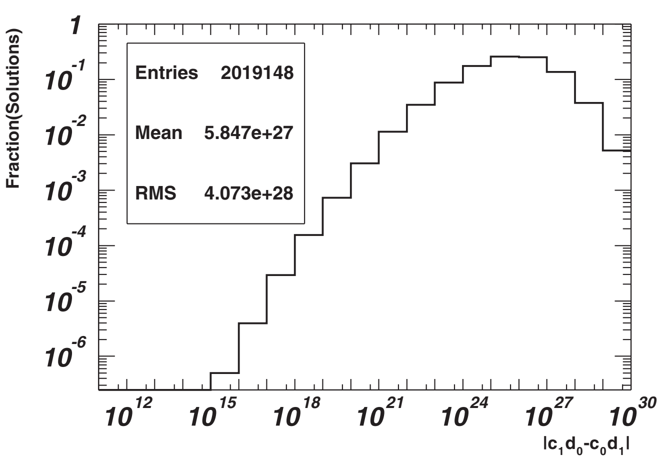

FIG. 2. Distribution of the expression $c_1 d_0 - c_0 d_1$ which appears in the denominator in the solution of p_{ν_y}. Since the distribution is symmetric around zero the module of the expression is plotted. As can be seen the values assumed by the expression are far away from zero which would cause a singularity in the solution.

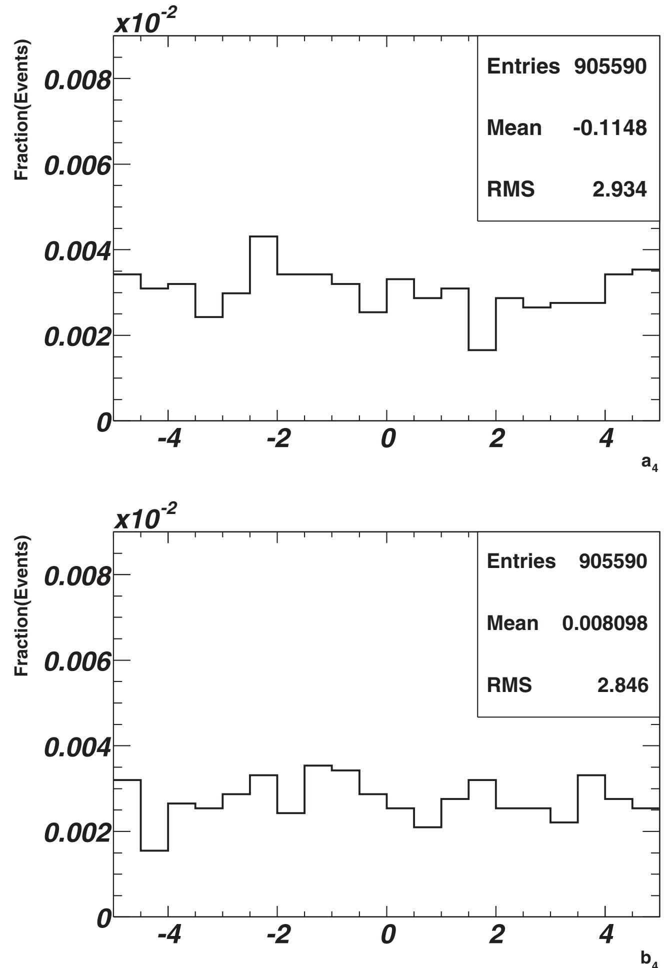

FIG. 3. Distributions of the coefficients a_4 (top) and b_4 (bottom). The coefficients are flat distributed over the whole phase space including the value zero where an irreducible singularity of their reciprocal resides.

p_{ν_y} which can then be isolated as

$$p_{\nu_y} = \frac{c_0 d_2 - c_2 d_0}{c_1 d_0 - c_0 d_1}. \tag{14}$$

Again the antineutrino momentum $p_{\bar{\nu}_y}$ can be immediately obtained by the corresponding linear transverse missing energy relation of the initial system of equations. As shown in Fig. 2 the coefficient in the denominator of Eq. (14) does not acquire values which are even close to the singularity at zero. Thus it is ensured that the neutrino momenta p_{ν_y} and $p_{\bar{\nu}_y}$ can be computed accurately over the whole phase space of possible solutions.

Finally the longitudinal (anti-)neutrino momenta p_{ν_z} and $p_{\bar{\nu}_z}$ can be easily obtained by the linear Eqs. (5) and (9) assuming that the coefficients a_4 and b_4 are different from zero since they appear as a product together with the longitudinal (anti-)neutrino momenta themselves. The distributions of the coefficients are shown in Fig. 3. The fraction of solutions close to the singularity—irreducible in the analytical solution—is below the per mill level and may be neglected for practical purposes. From a theoretical point of view this singularity can be circumvented in solving the neutrino momenta p_{ν_z} and $p_{\bar{\nu}_z}$ analytically with the Eqs. (2) and (3) of the algebraic approach [5] which does not contain any singularity. It has been verified that the longitudinal (anti-)neutrino momentum does not typically vanish together with the coefficient a_4 (b_4) simultaneously, which would cause the singularity to disappear.

IV. PERFORMANCE OF THE METHOD

The performance studies discussed here are assuming Tevatron proton antiproton collider settings with a center of mass energy of 1.96 TeV which has been set up in the Monte Carlo event generator PYTHIA 6.220 [8]. Cross-checks at a center of mass energy of 14 TeV assuming the LHC proton collider environment confirm the independence of the method of particular collider settings.

The quartic equation in p_{ν_x} (13) is typically flat around the neutrino momenta of interest where solutions are to be expected. Figure 4 shows the function for a given event. Only in the bottom plot where the function is zoomed out the roots can be recognized. Computationally the solutions are robust. The generated (anti-)neutrino momenta coincide with one of the solutions to real precision assuming that the 4-vectors of the two leptons and the b, $\bar{b}$ quarks and the masses of the (anti-)top quarks and W bosons which are entered into the quartic equation are known exactly. The fraction of events where no solution can be found or no solution coincides with the generated (anti-)neutrino momenta to real precision is below the per mill level. If the W boson mass is generated off-shell while its pole mass is assumed in the solution the efficiency drops to 89%. Relaxing the same assumption for the top quark mass

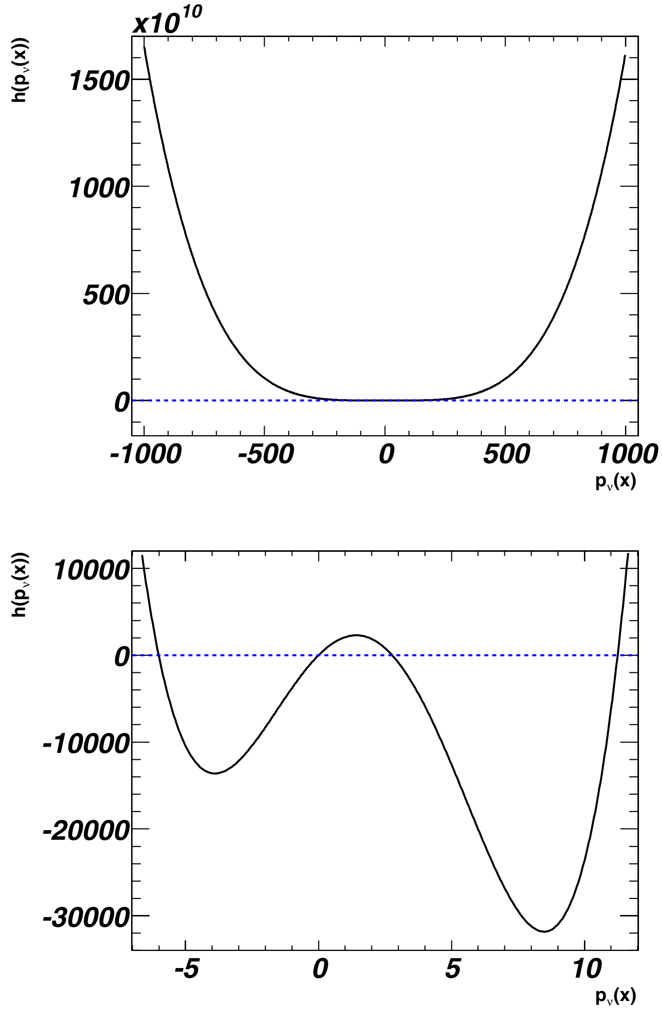

FIG. 4 (color online). A typical quartic equation whose real roots in p_{ν_x} are solutions of the initial system of equations describing the $t\bar{t}$ dilepton kinematics. The bottom plot is zoomed around the interesting p_{ν_x} range of the abscissa where the analytical solution becomes singular.

results into a further decrease of efficiency to 84%. Beyond, an infrared-safe cone algorithm [9] with cone size $R = 0.5$ in the space spanned by pseudorapidity and azimuthal angle has been applied to the hadronic final state particles. Two reconstructed jets, two leptons and missing transverse energy are required for an event to be selected. The jets are accepted as b-tagged if they coincide within $\Delta R < 0.5$ with the generated b quarks. The solution efficiency drops to 71% and can be re-established at 81% in solving both b quark jet permutations. Smearing the leptons and jets with the energy resolution of the D0 detector [10] decreases the efficiency to 75%. In practice, a given event can be solved repeatedly, with the energy of the particles and objects smeared randomly within the detector resolution, once each iteration. These observations are consistent with the findings of the algebraic approach [5]. This confirms on one hand the reliability of the algebraic approach and rises on the other hand the question what

TABLE I. Number of solutions, fractions and statistical quantities for events which have been solved ($N_{sol} > 0$). The left column shows the fraction of events having exactly two solutions. In the center the average number of solutions per solved event is given. To the right the RMS of this distribution is shown.

	$\frac{N_{sol}=2}{N_{sol}>0}$	$\langle N_{sol}^{>0}\rangle$	RMS($N_{sol}^{>0}$)
t, W masses known exactly	0.82	2.37	0.77
W mass known exactly t pole mass assumed	0.84	2.32	0.74
t, W pole mass assumed	0.85	2.31	0.72
t, W pole mass assumed both $b\bar{b}$ permutations	0.59	3.00	1.35
reconstructed b-jets (parton matched)	0.79	2.42	0.82
wrong b-jet permutation (parton matched)	0.82	2.36	0.77
both b-jet permutations (parton matched)	0.52	3.22	1.48
both b-jet permutations (parton matched, jets + leptons smeared)	0.54	3.19	1.47
both b-jet permutations (parton matched, jets + leptons smeared), reconstructed objects 100 × resolution smeared	0.0072	7.96	4.72

numerical methods with a superior solution efficiency are actually solving.

Considering only events which could be solved it is important to investigate the number of solutions in dependence of the experimental settings since this number is directly proportional to the ambiguities of the solved and reconstructed events which in turn determines the significance of the solutions and any observable making use of it. In Table I the fraction of solved events having exactly two solutions, the average number of solutions and its RMS is given for different experimental settings. The first four lines describe the evolution of these quantities derived from the particle final state. Relaxing the amount of assumption about the top quark and W boson masses increases the fraction of solved events with exactly two solutions while the average number of solutions and its RMS decrease slightly. Allowing both b quark jet permutations—assuming that the charge of the quarks can not be determined with adequate certainty—the fraction of events having exactly two solutions drops considerably in favor of a higher solution multiplicity with larger RMS. The table items below show the number of solutions for reconstructed objects, first for right, wrong and both b quark jet permutations then energy resolution smearing is applied to the reconstructed objects and finally 100 solution iterations have been accomplished to take into account the uncertainty in the measured energy of the reconstructed objects. The general tendency is that the fraction of solved events with exactly two solutions decreases with less accurate knowledge about the particles and objects while the solution multiplicity and its RMS does increase.

V. CONCLUSIONS

The analytical solution of the system of equations describing the $t\bar{t}$ dilepton kinematics has been presented. The Ansatz of formulating two equations linear in the three neutrino and antineutrino momentum components leads after substitution of the longitudinal (anti-)neutrino momenta to two multivariate polynomials of two unknowns with multidegree two. It turns out that each of these two polynomials has a singularity which can be removed. In contrast there are two irreducible singularities in the linear equations described above which can be circumvented in exploiting the analytical Ansatz of the algebraic approach [5] to determine the longitudinal (anti-)neutrino momenta. The two multivariate polynomials can be reduced to a univariate polynomial of degree four by means of resultants. The obtained quartic equation is solved analytically. The solution could be derived without any use of computer algebra software. The fraction of events without any solution or with no solution matching the generated (anti-)neutrino momenta with real precision are below the per mill level assuming that the particle momenta and masses inserted into the analytical solution are known exactly. Consistent with the observations of the algebraic approach [5] little deviations of the inserted particle momenta and masses from their true values drop the solution efficiency and purity considerably. At the same time the solution multiplicity increases. This raises the question what more efficient numerical methods are actually solving. General solution methods can compare their performance with the analytical solution described here.

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APPENDIX

1. Polynomial coefficients

The coefficients of Eq. (5) are given by

$$\begin{aligned} a_1 &= (E_b + E_{\ell^+})(m_W^2 - m_{\ell^+}^2) - E_{\ell^+}(m_t^2 - m_b^2 - m_{\ell^+}^2) \\ &\quad + 2E_b E_{\ell^+}^2 - 2E_{\ell^+}\vec{p}_b\vec{p}_{\ell^+}, \\ a_2 &= 2(E_b p_{\ell_x^+} - E_{\ell^+} p_{b_x}), \\ a_3 &= 2(E_b p_{\ell_y^+} - E_{\ell^+} p_{b_y}), \\ a_4 &= 2(E_b p_{\ell_z^+} - E_{\ell^+} p_{b_z}) \end{aligned}$$

where it is important that the coefficient a_4 does not vanish since Eq. (5) has to be divided by it to isolate the unknown p_{ν_z}. As explained in Sec. III this irreducible singularity can be circumvented in solving for p_{ν_z} with the analytical Ansatz made in the algebraic approach [5].

The equivalent equation of the antitop quark parton branch is

$$0 = b_1 + b_2 p_{\bar{\nu}_x} + b_3 p_{\bar{\nu}_y} + b_4 p_{\bar{\nu}_z} \tag{A1}$$

with the coefficients

$$\begin{aligned} b_1 &= (E_{\bar{b}} + E_{\ell^-})(m_W^2 - m_{\ell^-}^2) - E_{\ell^-}(m_t^2 - m_{\bar{b}}^2 - m_{\ell^-}^2) \\ &\quad + 2E_{\bar{b}} E_{\ell^-}^2 - 2E_{\ell^-}\vec{p}_{\bar{b}}\vec{p}_{\ell^-}, \\ b_2 &= 2(E_{\bar{b}} p_{\ell_x^-} - E_{\ell^-} p_{\bar{b}_x}), \\ b_3 &= 2(E_{\bar{b}} p_{\ell_y^-} - E_{\ell^-} p_{\bar{b}_y}), \\ b_4 &= 2(E_{\bar{b}} p_{\ell_z^-} - E_{\ell^-} p_{\bar{b}_z}). \end{aligned}$$

Again there is a singularity in case of vanishing coefficient b_4. The coefficients of Eq. (6) are given by

$$\begin{aligned} c_{22} &= (m_{W^+}^2 - m_{\ell^+}^2)^2 - 4(E_{\ell^+}^2 - p_{\ell_z^+}^2)(a_1/a_4)^2 - 4(m_{W^+}^2 - m_{\ell^+}^2)p_{\ell_z^+} a_1/a_4, \\ c_{21} &= 4(m_{W^+}^2 - m_{\ell^+}^2)(p_{\ell_x^+} - p_{\ell_z^+} a_2/a_4) - 8(E_{\ell^+}^2 - p_{\ell_z^+}^2)a_1 a_2/a_4^2 - 8p_{\ell_x^+} p_{\ell_z^+} a_1/a_4, \\ c_{20} &= -4(E_{\ell^+}^2 - p_{\ell_x^+}^2) - 4(E_{\ell^+}^2 - p_{\ell_z^+}^2)(a_2/a_4)^2 - 8p_{\ell_x^+} p_{\ell_z^+} a_2/a_4, \\ c_{11} &= 4(m_{W^+}^2 - m_{\ell^+}^2)(p_{\ell_y^+} - p_{\ell_z^+} a_3/a_4) - 8(E_{\ell^+}^2 - p_{\ell_z^+}^2)a_1 a_3/a_4^2 - 8p_{\ell_y^+} p_{\ell_z^+} a_1/a_4, \\ c_{10} &= -8(E_{\ell^+}^2 - p_{\ell_z^+})a_2 a_3/a_4^2 + 8p_{\ell_x^+} p_{\ell_y^+} - 8p_{\ell_x^+} p_{\ell_z^+} a_3/a_4 - 8p_{\ell_y^+} p_{\ell_z^+} a_2/a_4, \\ c_{00} &= -4(E_{\ell^+}^2 - p_{\ell_y^+}^2) - 4(E_{\ell^+}^2 - p_{\ell_z^+}^2)(a_3/a_4)^2 - 8p_{\ell_y^+} p_{\ell_z^+} a_3/a_4. \end{aligned}$$

Similar the coefficients d' of the antitop quark branch depend on the coefficients b in the following way

$$\begin{aligned} d'_{22} &= (m_{W^-}^2 - m_{\ell^-}^2)^2 - 4(E_{\ell^-}^2 - p_{\ell_z^-}^2)(b_1/b_4)^2 - 4(m_{W^-}^2 - m_{\ell^-}^2)p_{\ell_z^-} b_1/b_4, \\ d'_{21} &= 4(m_{W^-}^2 - m_{\ell^-}^2)(p_{\ell_x^-} - p_{\ell_z^-} b_2/b_4) - 8(E_{\ell^-}^2 - p_{\ell_z^-}^2)b_1 b_2/b_4^2 - 8p_{\ell_x^-} p_{\ell_z^-} b_1/b_4, \\ d'_{20} &= -4(E_{\ell^-}^2 - p_{\ell_x^-}^2) - 4(E_{\ell^-}^2 - p_{\ell_z^-}^2)(b_2/b_4)^2 - 8p_{\ell_x^-} p_{\ell_z^-} b_2/b_4, \\ d'_{11} &= 4(m_{W^-}^2 - m_{\ell^-}^2)(p_{\ell_y^-} - p_{\ell_z^-} b_3/b_4) - 8(E_{\ell^-}^2 - p_{\ell_z^-}^2)b_1 b_3/b_4^2 - 8p_{\ell_y^-} p_{\ell_z^-} b_1/b_4, \\ d'_{10} &= -8(E_{\ell^-}^2 - p_{\ell_z^-}^2)b_2 b_3/b_4^2 + 8p_{\ell_x^-} p_{\ell_y^-} - 8p_{\ell_x^-} p_{\ell_z^-} b_3/b_4 - 8p_{\ell_y^-} p_{\ell_z^-} b_2/b_4, \\ d'_{00} &= -4(E_{\ell^-}^2 - p_{\ell_y^-}^2) - 4(E_{\ell^-}^2 - p_{\ell_z^-}^2)(b_3/b_4)^2 - 8p_{\ell_y^-} p_{\ell_z^-} b_3/b_4. \end{aligned}$$

The remaining unknowns in these equations—which are the transverse antineutrino momenta—are substituted by the missing transverse energy relations of the system of Eq. (1) to obtain finally the set of equations

$$\begin{aligned}
d_{22} &= d'_{22} + \not{E}_x{}^2 d'_{20} + \not{E}_y{}^2 d'_{00} + \not{E}_x \not{E}_y d'_{10} + \not{E}_x d'_{21} + \not{E}_y d'_{11},\\
d_{21} &= -d'_{21} - 2\not{E}_x d'_{20} - \not{E}_y d'_{10},\\
d_{20} &= d'_{20},\\
d_{11} &= -d'_{11} - 2\not{E}_y d'_{00} - \not{E}_x d'_{10}\\
d_{10} &= d'_{10},\\
d_{00} &= d'_{00},
\end{aligned}$$

which depends merely on the transverse neutrino momenta p_{ν_x} and p_{ν_y}.

The resultant expressed in terms of the multivariate polynomials c_{jk} and d_{mn} are given by

$$\begin{aligned}
h_4 = {}& c_{00}^2 d_{22}^2 + c_{11} d_{22}(c_{11} d_{00} - c_{00} d_{11})\\
&+ c_{00} c_{22}(d_{11}^2 - 2d_{00} d_{22}) + c_{22} d_{00}(c_{22} d_{00} - c_{11} d_{11}),\\
h_3 = {}& c_{00} d_{21}(2c_{00} d_{22} - c_{11} d_{11}) + c_{00} d_{11}(2c_{22} d_{10} + c_{21} d_{11})\\
&+ c_{22} d_{00}(2c_{21} d_{00} - c_{11} d_{10}) - c_{00} d_{22}(c_{11} d_{10} + c_{10} d_{11})\\
&- 2c_{00} d_{00}(c_{22} d_{21} + c_{21} d_{22}) - d_{00} d_{11}(c_{11} c_{21} + c_{10} c_{22})\\
&+ c_{11} d_{00}(c_{11} d_{21} + 2c_{10} d_{22}),\\
h_2 = {}& c_{00}^2(2d_{22} d_{20} + d_{21}^2) - c_{00} d_{21}(c_{11} d_{10} + c_{10} d_{11})\\
&+ c_{11} d_{20}(c_{11} d_{00} - c_{00} d_{11}) + c_{00} d_{10}(c_{22} d_{10} - c_{10} d_{22})\\
&+ c_{00} d_{11}(2c_{21} d_{10} + c_{20} d_{11}) + (2c_{22} c_{20} + c_{21}^2) d_{00})\\
&- 2c_{00} d_{00}(c_{22} d_{20} + c_{21} d_{21} + c_{20} d_{22})\\
&+ c_{10} d_{00}(2c_{11} d_{21} + c_{10} d_{22}) - d_{00} d_{10}(c_{11} c_{21} c_{10} c_{22})\\
&- d_{00} d_{11}(c_{11} c_{20} + c_{10} c_{21}),\\
h_1 = {}& c_{00} d_{21}(2c_{00} d_{20} - c_{10} d_{10}) - c_{00} d_{20}(c_{11} d_{10} + c_{10} d_{11})\\
&+ c_{00} d_{10}(c_{21} d_{10} + 2c_{20} d_{11})\\
&- 2c_{00} d_{00}(c_{21} d_{20} + c_{20} d_{21})\\
&+ c_{10} d_{00}(2c_{11} d_{20} + c_{10} d_{21})\\
&- c_{20} d_{00}(2c_{21} d_{00} - c_{10} d_{11}) - d_{00} d_{10}(c_{11} c_{20} + c_{10} c_{21}),\\
h_0 = {}& c_{00}^2 d_{20}^2 + c_{10} d_{20}(c_{10} d_{00} - c_{00} d_{10})\\
&+ c_{20} d_{10}(c_{00} d_{10} - c_{10} d_{00}) + c_{20} d_{00}(c_{20} d_{00} - 2c_{00} d_{20}).
\end{aligned}$$

To avoid singularities which arise in the case of vanishing factors a_4 or b_4 the coefficients c_{jk}, d_{mn} of the polynomials (6) and (8) have been multiplied with the least common multiple of the denominators which are a_4^2 and b_4^2 respectively. These factors are constant for a given event and thus do not alter the position of the real roots which correspond to the neutrino momenta p_{ν_x}.

2. Quartic equation

The quartic equation can be solved analytically in reducing it to a cubic equation. There are several ways to achieve this. Here the method of Ferrari [9]—who was the first to develop an algebraic technique for solving the general quartic equation—is being used.

First the leading coefficient h_0 of the quartic polynomial (13) is normalized to one (in the case the leading coefficient vanishes the problem is already reduced to a cubic equation). If the constant h_4 vanishes the quartic polynomial can be factorized into p_{ν_x} times a cubic equation. In this case one root namely $p_{\nu_x} = 0$ is already known. The substitution $p_{\nu_x} = p'_{\nu_x} - h_1/4$ leads to the simplified equation

$$0 = p'^4_{\nu_x} + k_1 p'^2_{\nu_x} + k_2 p'_{\nu_x} + k_3$$

with the coefficients

$$k_1 = h_2 - 3h_1^2/8, \qquad k_2 = h_3 + h_1^3/8 - h_1 h_2/2,$$
$$k_3 = h_4 - 3h_1^4/256 + h_1^2 h_2/16 - h_1 h_3/4.$$

If the coefficient k_3 vanishes again the equation can be factorized into p'_{ν_x} times a cubic polynomial. If the coefficient k_2 vanishes the quartic polynomial in p'_{ν_x} can be expressed as a quadratic equation in $p'^2_{\nu_x}$. In the general case where all three coefficients k_1, k_2 and k_3 are different from zero the quartic polynomial can be factorized into the product of two quadratic polynomials as follows

$$p'^4_{\nu_x} + k_1 p'^2_{\nu_x} + k_2 p'_{\nu_x} + k_3 = (p'^2_{\nu_x} + t_1 p'_{\nu_x} + t_2) \times (p'^2_{\nu_x} - t_1 p'_{\nu_x} + k_3/t_2). \tag{A2}$$

Once the new coefficients t_1 and t_2 have been determined the quadratic polynomials can be easily solved. Comparison of the coefficients yields

$$k_1 = k_3/t_2 + t_2 - t_1^2$$

and

$$k_2 = t_1(k_3/t_2 - t_2).$$

It is ensured that t_2 which appears in the denominator does not vanish since the coefficient k_3 has been assumed to be different from zero and k_1, k_2 are finite. Eliminating t_2 in the two nonlinear equations above yields to a cubic equation in t_1^2. To achieve this the two equations above are rewritten in the following form

$$k_3/t_2 + t_2 = k_1 + t_1^2 + k_2/t_1, \qquad k_3/t_2 - t_2 = k_2/t_1.$$

Adding and subtracting them leads to

$$2k_3/t_1 = k_1 + t_1^2 + k_2/t_1, \tag{A3}$$

$$2t_2 = k_1 + t_1^2 - k_2/t_1 \tag{A4}$$

whose product can finally be written as

$$0 = t_1^6 + 2k_1 t_1^4 + (k_1^2 - 4k_3)t_1^2 - k_2^2$$

which is a cubic equation in t_1^2. Any positive root of t_1^2 can be used to derive all real solutions of the initial quartic equation (negative roots would lead to imaginary values of $\pm t_1$. Either sign can be used to solve the factorized quartic equation. Changing the sign corresponds to swapping the coefficients between the first and the second quadratic polynomial in Eq. (A2). Descartes' Sign Rule [12] can be exploited to ensure that there is always at least one positive root. According to the rule the number of sign changes of the consecutive polynomial coefficients is the maximal number of positive roots. Now one can substitute t_1^2 by $-t_1^2$ to determine the maximal number of negative roots. Since k_2 is real the constant coefficient $-k_2^2$ is negative. The leading monomial has also a negative coefficient. Thus there can be two or zero sign changes. A cubic equation with real coefficients has always either one or three real roots. In the case of two or zero negative roots there must conclusively be at least one positive root. Once this root has been determined, t_1 can be inserted into Eq. (A4) above to determine t_2 and subsequently the quadratic polynomials (A2) of the quartic equation.

3. Cubic equation

There are several ways to solve the cubic equation [11]. Here the approach of [14] has been adopted. The cubic equation

$$0 = z^3 + s_1 z^2 + s_2 z + s_3$$

is assumed to have real coefficients. First the two variables

$$q = \frac{s_1^2 - 3s_2}{9}$$

and

$$r = \frac{2s_1^3 - 9s_1 s_2 + 27s_3}{54}$$

are determined. If $r^2 < q^3$ the cubic equation has three real roots which can be found by computing

$$\theta = \arccos r/\sqrt{q^3}.$$

The three roots are then given by

$$z_1 = -2\sqrt{q}\cos\left(\frac{\theta}{3}\right) - \frac{s_1}{3},$$

$$z_1 = -2\sqrt{q}\cos\left(\frac{\theta + 2\pi}{3}\right) - \frac{s_1}{3},$$

$$z_1 = -2\sqrt{q}\cos\left(\frac{\theta - 2\pi}{3}\right) - \frac{s_1}{3}.$$

Their first appearance goes back to François Vièta who published them in 1615. In the case of $r^2 \geq q^3$ there is only one real solution and defining the auxiliary variables

$$u = (-r + \sqrt{r^2 - q^3})^{1/3}$$

and

$$v = (-r - \sqrt{r^2 - q^3})^{1/3}$$

allows to express the real solution simply in terms of u and v as

$$z_1 = u + v - \frac{s_1}{3}.$$

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