PHYSICAL REVIEW D **73,** 054004 (2006)

Four-quark spectroscopy within the hyperspherical formalism

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(Received 1 February 2006; published 13 March 2006)

We present a generalization of the hyperspherical harmonic formalism to study systems made of quarks and antiquarks of the same flavor. This generalization is based on the symmetrization of the N-body wave function with respect to the symmetric group using the Barnea and Novoselsky algorithm. The formalism is applied to study four-quark systems by means of a constituent quark model successful in the description of the two- and three-quark systems. The results are compared to those obtained by means of variational approaches. Our analysis shows that four-quark systems with exotic 0^{+-} and nonexotic 2^{++} quantum numbers may be bound independently of the mass of the quark. 2^{+-} and 1^{+-} states become attractive only for larger mass of the quarks.

DOI: 10.1103/PhysRevD.73.054004 PACS numbers: 12.39.Jh, 14.40.Lb, 21.45.+v, 31.15.Ja

I. INTRODUCTION

The understanding of few-body systems relies in our capability to design methods for finding an exact or approximate solution of the N-body problem. In two-, three-, and four-body problems it is possible to obtain mathematically correct and computationally tractable equations such as the Lippmann-Schwinger, Faddeev and Yakubovsky equations describing exactly, for any assumed interaction between the particles, the motion of few-body systems [1]. However, the exact solution of these equations requires sophisticated techniques whose difficulty increases when increasing the number of particles.

There are a countless number of examples of quantum-mechanical few-body systems, from few-electron quantum dots in solid state physics to constituent quarks in subnuclear physics. The intricate feature of the few-body systems is that they develop individual characters depending on the number of constituent particles. The most important cause of these differences are the correlated motion and the Pauli principle. This individuality requires specific methods for the solution of the few-body Schrödinger equation, approximate solutions assuming restricted model spaces (as, for example, the mean field approximation) failing to describe these systems.

The solution of any few-particle system may be found in a simple and unified approach. A recent widely used method is the stochastic variational [2], a variational approach where the trial wave function is generated by a random search on an adequate basis. An alternative powerful tool is an expansion of the trial wave function in terms of hyperspherical harmonic (HH) functions. The idea is to generalize the simplicity of the spherical harmonic expansion for the angular functions of a single-particle motion to a system of particles by introducing a global length ρ, called the hyperradius, and a set of angles, Ω. For the HH expansion method to be practical, the evaluation of the potential energy matrix elements must be feasible. The main difficulty of this method is to construct HH functions of proper symmetry for a system of identical particles. This is a difficult problem that may be overcome by means of the HH formalism based on the symmetrization of the N-body wave function with respect to the symmetric group using the Barnea and Novoselsky algorithm [3]. This method, applied in nuclear physics for $N \leq 7$ [4], has only been applied to quark physics for $N = 3$ [5]. Therefore, its generalization would be ideally suited for the study of the properties of multiquark systems.

During the last few years there has been a renewed interest on the possible existence of multiquark states, specially four- (two quarks and two antiquarks) and five-quark (four quarks and one antiquark), in the low-energy hadron spectroscopy. Theoretically, the possible existence of four-quark bound states was already suggested 30 years ago, both in the light-quark sector by Jaffe [6] and in the heavy-quark sector by Iwasaki [7]. Experimentally, there were several analysis suggesting the existence of non-$q\bar{q}$ states in the high energy part of the charmonium spectrum [8]. The recent series of discoveries of new meson resonances whose properties do not fit into the predictions of the naive quark model, has reopened the interest on the possible role played by non-$q\bar{q}$ configurations in the hadron spectra [9,10]. Among them one could mention the $X(3872)$ [11], the $Y(4260)$ [12] or the new D and D_s resonances [13]. Out of the several interpretations proposed for these states, those based on four-quark states are majority [14]. There seems to be a consensus that a four-quark system containing two-light and two-heavy quarks, $qq\overline{QQ}$, is stable against dissociation into two mesons, $q\bar{Q}$, if the ratio of the mass of the heavy to the light quark is large enough [15,16]. While the conclusions agree for the $qq\bar{b}\,\bar{b}$ system, there are more discrepancies about the $qq\bar{c}\,\bar{c}$ system [17–19].

The possible existence of four-quark states in the low-lying meson spectra has been used to advance in the understanding of the light-scalar mesons, $J^{PC} = 0^{++}$. In this case, four-quark states have been justified to coexist with

$q\bar{q}$ states in the energy region below 2 GeV because they can couple to 0^{++} without orbital excitation [6], and therefore conventional $q\bar{q}$ states are expected to mix with four-quark states to yield physical mesons [10,20].

The four-heavy-quark states have not received as much attention in the last years as the ones containing light quarks. Iwasaki [7], based on the constituent quark mass value obtained within a string model for hadrons, argued that four-charm quark states could exist in the 6 GeV energy region. Using a variational method with gaussian trial radial wave functions and two-body potentials based on the exchange of color octets between quarks, Ader *et al.* [15] obtained non stable $QQ\overline{QQ}$ states. However, using a potential derived from the MIT bag model in the Born-Oppenheimer approximation the same authors concluded that the $cc\bar{c}\,\bar{c}$ state was bound by 35 MeV. Heller and Tjon [16] considered also the MIT bag model improving the Born-Oppenheimer approximation with a correct treatment of the kinetic energy, non stable $QQ\overline{QQ}$ states were found. In Ref. [21] $L = 0$ $QQ\overline{QQ}$ states were analyzed in the framework of a chromomagnetic model where only a constant hyperfine potential, not depending on radial coordinates, was retained. With these assumptions no bound $QQ\overline{QQ}$ states were found. A similar conclusion was obtained in Ref. [18] using the Bhaduri potential and solving variationally the four-body problem in a harmonic oscillator basis. Recently, Lloyd and Vary [22] investigated the $cc\bar{c}\,\bar{c}$ system using a nonrelativistic hamiltonian inspired by the one-gluon exchange potential diagonalizing the hamiltonian in a harmonic oscillator basis, obtaining several close-lying bound states.

The discussion above illustrates that the theoretical predictions for the existence of four-quark systems differ depending basically on the method used to solve the four-body problem and the interaction employed. It is our aim in this work to make a general study of four-quark systems of identical flavor in an exact way. For this purpose we will generalize the HH method [23], widely used in traditional nuclear physics for the study of few-body nuclei, to study four-quark systems. There are two main difficulties, first the simultaneous treatment of particles and antiparticles, and second the additional color degree of freedom. As a test of our formalism we will recover some results present in the literature. Finally, we will make a general study of four-quark systems of identical flavor by means of the constituent quark model of Ref. [24] that provides with a realistic framework describing in a correct way the general aspects of the meson and baryon spectra.

The paper is organized as follows. In the next section the formalism necessary to build a color singlet wave function with well-defined parity and C-parity quantum numbers is described and discussed. In Sect. III we introduce the general features of the constituent quark model used. In Sect. IV we present and analyze the results obtained for the four-quark systems. Finally, in Sect. V we resume our most important conclusions.

II. GENERAL FORMULATION OF THE PROBLEM

The system of two quarks and two antiquarks with the same flavor can be regarded as a system of four identical particles. Each particle carries a $SU(2)$ spin label and a $SU(3)$ color label. Both quarks and antiquarks are spin $\frac{1}{2}$ particles, but whereas a quark color state belongs to the $SU(3)$ fundamental representation [3], an antiquark color state is a member of the fundamental representation $[\bar{3}]$. The four-body wave function is a sum of outer products of color, spin and configuration terms

$$|\phi\rangle = |\text{Color}\rangle|\text{Spin}\rangle|R\rangle \tag{1}$$

coupled to yield an antisymmetric wave function with a set of quantum numbers that reflects the symmetries of the system. These are the total angular momentum quantum number J, its projection J^z, and the $SU(3)$ color state G (to avoid confusion with the charge conjugation quantum number the color degree of freedom will be labeled as G from now on), which by assumption must belong for physical states to the $SU(3)$ color singlet representation. Since QCD preserves parity, parity is also a good quantum number. Another relevant quantum number to the system under consideration it is the C-parity, C, i.e., the symmetry under interchange of quarks and antiquarks.

To obtain a solution of the four-body Schrödinger equation we eliminate the center of mass and use the relative, Jacobi, coordinates $\vec{\eta}_1, \vec{\eta}_2, \ldots, \vec{\eta}_{A-1}$. Then we expand the spatial part of the wave-function using the HH basis. In this formalism the Jacobi coordinates are replaced by one radial coordinate, the hyperradius ρ, and a set of $(3A-4)$ angular coordinates Ω_A. The HH basis functions are eigenfunctions of the hyperspherical part of the Laplace operator. An antisymmetric A-body basis functions with total angular momentum J_A, J_A^z, color G_A and C-parity C, are given by,

$$|nK_AJ_AJ_A^zG_AC\Gamma_A\alpha_A\beta_A\rangle = \sum_{Y_{A-1}} \frac{\Lambda_{\Gamma_A,Y_{A-1}}}{\sqrt{|\Gamma_A|}} [|K_AL_AM_A\Gamma_AY_{A-1}\alpha_A\rangle|S_AS_A^zG_AC\tilde{\Gamma}_A, \tilde{Y}_{A-1}\beta_A\rangle]^{J_AJ_A^z}|n\rangle, \tag{2}$$

where

$$\langle\rho|n\rangle \equiv R_n(\rho) \tag{3}$$

are the hyperradial basis functions, taken to be Laguerre functions.

$$\langle \Omega_A | K_A L_A M_A \Gamma_A Y_{A-1} \alpha_A \rangle \equiv \mathcal{Y}^{[A]}_{K_A L_A M_A \Gamma_A Y_{A-1} \alpha_A}(\Omega_A) \quad (4)$$

are HH functions with hyperspherical angular momentum $K = K_A$, and orbital angular momentum quantum numbers (L_A, M_A) that belong to well-defined irreducible representations (irreps) $\Gamma_1 \in \Gamma_2 \ldots \in \Gamma_A$ of the permutation group-subgroup chain $\mathcal{S}_1 \subset \mathcal{S}_2 \ldots \subset \mathcal{S}_A$, denoted by the Yamanouchi symbol $[\Gamma_A, Y_{A-1}] \equiv [\Gamma_A, \Gamma_{A-1}, \ldots, \Gamma_1]$. The dimension of the irrep Γ_m is denoted by $|\Gamma_m|$ and $\Lambda_{\Gamma_A, Y_{A-1}}$ is a phase factor [25]. Similarly, the functions

$$\langle s_1^z \ldots s_A^z, g_1 \ldots g_A | S_A S_A^z G_A C \tilde{\Gamma}_A, \tilde{Y}_{A-1} \beta_A \rangle \equiv \chi^{[A]}_{S_A S_A^z G_A \tilde{\Gamma}_A, \tilde{Y}_{A-1} \beta_A}(s_1^z \ldots s_A^z, g_1 \ldots g_A) \quad (5)$$

are the symmetrized color-spin basis functions, given in terms of the spin projections (s_i^z) and color states (g_i) of the particles. The quantum numbers α_A, and β_A are used to remove the degeneracy of the HH and color-spin states, respectively. For the construction of the symmetrized HH basis we shall use the algorithm of Barnea and Novoselsky [3], which utilizes the group of kinematic rotations. For the color-spin subspace, we will present in the following subsections a method to transform the standard basis into a symmetrized color-spin basis with well-defined color and C-parity. The calculation of the Hamiltonian matrix-elements between the antisymmetric basis functions, Eq. (2), is practically the same as in the nuclear physics case, replacing isospin by color, and the reader is referred to Barnea *et al.* [26].

A. The construction of symmetrized color-spin states

The starting point for the construction of color-spin states with well-defined permutational symmetry, color and spin quantum numbers, are the single-particle states. The spin state of the i'th particle, quark or antiquark, is given by $|s = \frac{1}{2} s^z = \pm \frac{1}{2}\rangle$. Using the Clebsh-Gordan coefficients one can successively couple the spin states to construct an A-body state with well-defined total spin

$$|S_A S_S^z S_{A-1} S_{A-2} \ldots S_1\rangle = [[[|\tfrac{1}{2}\rangle|\tfrac{1}{2}\rangle]^{S_2}|\tfrac{1}{2}\rangle]^{S_3} \ldots |\tfrac{1}{2}\rangle]^{S_A S_A^z}. \quad (6)$$

Theoretically one can adopt the same procedure for the color states replacing the $SU(2)$ Clebsh-Gordan coefficients with the appropriate $SU(3)$ ones. In practice, however, the $SU(3)$ coefficients are more involved and we shall therefore follow a different path while dealing with color. Each quark (antiquark) color state is one of the three states of the triplet representation [3] ([$\bar{3}$]) of $SU(3)$. These states are eigenstates of the operators $(\mathbf{h}_1, \mathbf{h}_2)$ which form the Cartan subalgebra of $SU(3)$. Let us denote by $|g_i\rangle$ the color state of the i'th particle, $|g_i\rangle = |1\rangle, |2\rangle, |3\rangle$ for quark states and $|\bar{1}\rangle, |\bar{2}\rangle, |\bar{3}\rangle$ for antiquark states,

$$\mathbf{h}_1 |g_i\rangle = g_i^1 |g_i\rangle; \qquad \mathbf{h}_2 |g_i\rangle = g_i^2 |g_i\rangle. \quad (7)$$

The eigenvalues (g_i^1, g_i^2) (weights) uniquely label the quark (antiquark) states, in particular, we can write

$$\begin{aligned} |1\rangle &= |(+\tfrac{1}{2}, +\tfrac{1}{3})\rangle; & |\bar{1}\rangle &= |(-\tfrac{1}{2}, -\tfrac{1}{3})\rangle \\ |2\rangle &= |(-\tfrac{1}{2}, +\tfrac{1}{3})\rangle; & |\bar{2}\rangle &= |(+\tfrac{1}{2}, -\tfrac{1}{3})\rangle \\ |3\rangle &= |(0, -\tfrac{2}{3})\rangle; & |\bar{3}\rangle &= |(0, +\tfrac{2}{3})\rangle. \end{aligned} \quad (8)$$

An A-body color state is a product of the single-particle color states. This state does not belong to a well-defined irreducible representation of the color $SU(3)$ group, but instead it is characterized by the "color projections" $G_A^1 = g_1^1 + g_2^1 + \ldots g_A^1$ and $G_A^2 = g_1^2 + g_2^2 + \ldots g_A^2$, very much the same way as the "m-scheme" states used in nuclear structure calculations which have J^z as a good quantum number but not J. From now on we shall use the notation G_A for the pair of eigenvalues (G_A^1, G_A^2). The combined color-spin single-particle state is $|s_i = \frac{1}{2} s_i^z g_i\rangle$, and the n-particle states are given by

$$|\chi_{GS}\rangle = [[[|\tfrac{1}{2} g_1\rangle|\tfrac{1}{2} g_2\rangle]^{S_2}|\tfrac{1}{2} g_3\rangle]^{S_3} \ldots |\tfrac{1}{2} g_n\rangle]^{S_n S_n^z}. \quad (9)$$

In the following we shall present a method to transform these states into basis states with well-defined permutational symmetry, color and C-parity. This method is based on an algorithm by Novoselsky, Katriel and Gilmore (NKG) [25] who devised a recursive method to split an invariant vector space into states with well-defined permutational symmetry. By now, this algorithm has been used to construct symmetrized states in broad range of problems, and therefore we shall not give a detailed account of the method but rather outline the main ideas. For a clear description of the method the interested reader is referred to [27].

The main tool of the NKG algorithm is the transposition class-sum operator. For the n-particle permutation group $\mathcal{S}_n$, this operator is the sum over all possible transpositions,

$$\mathcal{C}_2[\mathcal{S}_n] = \sum_{i<j}^{n} (i, j). \quad (10)$$

It turns out that a simultaneous eigenstate of the mutually commuting transposition class-sum operator set $\{\mathcal{C}_2[\mathcal{S}_n], \mathcal{C}_2[\mathcal{S}_{n-1}], \ldots, \mathcal{C}_2[\mathcal{S}_2]\}$ has definite symmetry with respect to the group-subgroup chain $\mathcal{S}_n \supset \mathcal{S}_{n-1} \supset \ldots \supset \mathcal{S}_2 \supset \mathcal{S}_1$. Accordingly, NKG formulated a recursive method that starts with one-particle states and constructs symmetrized A-particle states through the following procedure:

(a) Add another particle (let say a total of n).
(b) Identify the subspaces invariant under the action of the permutation group $\mathcal{S}_n$. Create the appropriate basis.
(c) In each subspace, evaluate the matrix elements of the transposition class-sum operator $\mathcal{C}[\mathcal{S}_n]$, and diagonalize it.
(d) The $\mathcal{S}_n$ symmetry of the resulting states is unambiguously identified through the eigenvalues of $\mathcal{C}_2[\mathcal{S}_n]$.

The eigenvectors of the $\mathcal{C}_2[\mathcal{S}_n]$ matrix are the transformation coefficients from the old basis (c.) to the new symmetrized one, these are the coefficients of fractional parentage.

(e) Ensure phase consistency (see Ref. [27] for a detailed account of this delicate point).

(f) If $n < A$ repeat the procedure. When this procedure is done, the original states are transformed step by step into the desired symmetrized A-particle states.

In the case under consideration, the n-particle states are labeled by a set of good quantum numbers consisting of the total spin S_n and its projection S_n^z (which we shall suppress), the color projection $G_n = (G_n^1, G_n^2)$ and the set of Young diagrams $\Gamma_n, \Gamma_{n-1}, \ldots, \Gamma_3, \Gamma_2, \Gamma_1$ which is equivalent to the Yamanouchi symbol Y_n [28]. Using these labels one can construct a complete set of states labeled by $|S_n G_n Y_n \beta_n\rangle$, where β_n is an additional label that removes the remaining degeneracies. For single particle, there is no degeneracy and one sees that

$$|S_1 G_1 \Gamma_1\rangle = |\tfrac{1}{2} g_1\rangle, \tag{11}$$

where $S_1 = \frac{1}{2}$ and $G_1 = g_1$ or equivalently $(G_1^1, G_1^2) = (g_1^1, g_1^2)$. To obtain a two-particle state labeled by the sequence of Young diagrams Γ_2, Γ_1, we first couple the spin states, and introduce the notation

$$|(S_1 G_1 \Gamma_1; g_2) S_2 G_2\rangle = [|S_1 G_1 \Gamma_1\rangle |\tfrac{1}{2} g_2\rangle]^{S_2} \tag{12}$$

to denote 2-particle states with total spin S_2 and total color projection $G_2 = G_1 + g_2$. Here, and in what follows, we have used the short hand notation $G_2 = G_1 + g_2$ for $(G_2^1, G_2^2) = (G_1^1 + g_2^1, G_1^2 + g_2^2)$. The permutation operators transform these states within an invariant subspace, each of which is characterized by the total spin S_2 and the total color projection $G_2 = (G_2^1, G_2^2)$. The linear combinations which belong to well-defined irreps of the symmetry group can be written in the form,

$$|S_2 G_2 \Gamma_2 \Gamma_1 \beta_2\rangle = \sum_{G_1 g_2; G_2 = G_1 + g_2} [S_1 G_1 \Gamma_1 g_2 | \} S_2 G_2 \Gamma_2 \beta_2] | (S_1 G_1 \Gamma_1; g_2) S_2 G_2\rangle. \tag{13}$$

Assume that we have constructed a $(n-1)$-particle states with well-defined permutational symmetry. Following the 2-particle example we first form n-particle states with well-defined total spin and total color projection,

$$|(S_{n-1} G_{n-1} \Gamma_{n-1} \beta_{n-1}; g_n) S_n G_n\rangle = [|S_{n-1} G_{n-1} \Gamma_{n-1} \beta_{n-1}\rangle |\tfrac{1}{2} g_n\rangle]^{S_n}. \tag{14}$$

Note that $G_n = G_{n-1} + g_n$. The desired, symmetrized, n-particle states can now be written in the form

$$|S_n G_n Y_n \beta_n\rangle = \sum_{S_{n-1} \beta_{n-1} G_{n-1} g_n; G_n = G_{n-1} + g_n} [S_{n-1} G_{n-1} Y_{n-1} \beta_{n-1} g_n | \} S_n G_n Y_n \beta_n] | (S_{n-1} G_{n-1} \Gamma_{n-1} \beta_{n-1}; g_n) S_n G_n\rangle. \tag{15}$$

The transformation coefficients $[S_{n-1} G_{n-1} Y_{n-1} \beta_{n-1} g_n | \} S_n G_n Y_n \beta_n]$ are the color-spin coefficients of fractional parentage, cscfps. These coefficients are the lines of an orthogonal transformation matrix. Therefore they fulfill the orthogonality and completeness relations.

The evaluation of the matrix-elements of the transposition class-sum operator, $\mathcal{C}_2[\mathcal{S}_n]$, for the nonsymmetrized n-particle states, Eq. (14), can be drastically simplified if we recast it in the form

$$\mathcal{C}_2[\mathcal{S}_n] = \mathcal{C}_2[\mathcal{S}_{n-1}] + \sum_{i=1}^{n-1} (i, n). \tag{16}$$

The first term on the right-hand side of Eq. (16) is diagonal in Γ_{n-1}. Its explicit value can be calculated exploiting the following expression for the eigenvalues of $\mathcal{C}_2[\mathcal{S}_n]$ [29],

$$\mathcal{C}_2[\Gamma_n] = \frac{1}{2} \sum_i r_i (r_i - 2i + 1), \tag{17}$$

where r_i is the number of boxes in the i'th row of Γ_n and the sum extends over all the rows. The matrix elements of the second term between states with well-defined $\mathcal{S}_{n-1}$ symmetry can be rewritten as [25],

$$\sum_{i=1}^{n-1} \langle \Gamma_{n-1} Y_{n-2} | (i, n) | \Gamma'_{n-1} Y'_{n-2}\rangle = \delta_{\Gamma_{n-1}, \Gamma'_{n-1}} \delta_{Y_{n-2}, Y'_{n-2}} \frac{n-1}{|\Gamma_{n-1}|} \sum_{Y''_{n-2} \Gamma_{n-1}} \langle \Gamma_{n-1} Y''_{n-2} | (n-1, n) | \Gamma_{n-1} Y''_{n-2}\rangle. \tag{18}$$

The $(n-1, n)$ matrix elements can be evaluated by expanding the $(n-1)$-particle states in terms of the cscfps calculated previously and by recoupling the spin states in different order,

$$\langle (S_{n-1}G_{n-1}\Gamma_{n-1}\beta_{n-1}; g_n)S_nG_n|(n-1,n)|(S'_{n-1}G'_{n-1}\Gamma_{n-1}\beta'_{n-1}; g'_n)S_nG_n\rangle$$
$$= \sum_{S_{n-2}G_{n-2}\beta_{n-2}g_{n-1}g'_{n-1}} [S_{n-2}G_{n-2}Y_{n-2}\beta_{n-2}g_{n-1}|\}S_{n-1}G_{n-1}Y_{n-1}\beta_{n-1}]$$
$$\times [S_{n-2}G_{n-2}Y_{n-2}\beta_{n-2}g'_{n-1}|\}S'_{n-1}G'_{n-1}Y_{n-1}\beta'_{n-1}] \sum_{S_{n,n-1}} \left\{ \begin{matrix} \frac{1}{2} & \frac{1}{2} & S_{n,n-1} \\ S_{n-2} & S_n & S_{n-1} \end{matrix} \right\} \left\{ \begin{matrix} \frac{1}{2} & \frac{1}{2} & S_{n,n-1} \\ S_{n-2} & S_n & S'_{n-1} \end{matrix} \right\} (-1)^{S_n+S_{n-2}}$$
$$\times (2S_{n,n-1}+1)\sqrt{(2S_{n-1}+1)(2S'_{n-1}+1)}\langle S_{n,n-1}g_{n-1}g_n|(n-1,n)|S_{n,n-1}g'_{n-1}g'_n\rangle. \quad (19)$$

Since the transposition $(n-1,n)$ commutes with the $(n-2)$-particle operators the matrix element is diagonal in the $(n-2)$-particle states. The matrix element on the right-hand side of Eq. (19) is the matrix element between two-particle states after we have decoupled them from the rest of the system. This matrix element can easily be evaluated to yield,

$$\langle S_{n,n-1}g_{n-1}g_n|(n-1,n)|S_{n,n-1}g'_{n-1}g'_n\rangle = -(-1)^{S_{n,n-1}}\delta_{g_{n-1},g'_n}\delta_{g'_{n-1},g_n}. \quad (20)$$

B. $SU(3)$ singlet states and C-parity

In the previous subsection we outlined a method to construct color-spin states with well-defined permutational symmetry Y_A, spin quantum numbers $(S_AS_A^z)$, and color projection G_A. Still, the $SU(3)$ color symmetry and C-parity of these states is not established, since a state with the same color projection can belong to numerous irreps of $SU(3)$. For example the color projection (0,0) can belong to the singlet representation [0] as well as to the octet representation, [8]. The $SU(3)$ symmetry of the A-particle states can be established using the quadratic Casimir operator, $\boldsymbol{C}_2[SU(3)]$. Any $SU(3)$ state is an eigenvector of $\boldsymbol{C}_2[SU(3)]$ with eigenvalue that depends on the specific representation of the state. In particular the states with eigenvalue 0 are singlet $SU(3)$ states. The A-particle $SU(3)$ generators

$$\lambda_a^c = \sum_{i=1}^{A} \lambda_{ai}^c, \quad (21)$$

are symmetric with respect to particle permutations. Consequently the quadratic Casimir,

$$\boldsymbol{C}_2[SU(3)] = \frac{1}{4}\vec{\lambda}^c \cdot \vec{\lambda}^c = \frac{1}{4}\sum_{a=1}^{8} \lambda_a^c\lambda_a^c, \quad (22)$$

commutes with the permutation group $\mathcal{S}_A$, and the states $\{|S_AG_AY_A\beta_A\rangle\}$ with good quantum numbers $(S_AG_AY_A)$ form an invariant subspace for $\boldsymbol{C}_2[SU(3)]$. The calculation of the matrix-elements of the quadratic Casimir can be simplified if we rewrite it as a sum of one-particle and two-particle terms

$$\boldsymbol{C}_2[SU(3)] = \frac{1}{4}\sum_{i=1}^{A} \vec{\lambda}_i^c \cdot \vec{\lambda}_i^c + \frac{1}{4}\sum_{i\neq j}^{A} \vec{\lambda}_i^c \cdot \vec{\lambda}_j^c. \quad (23)$$

The first sum on the right-hand side of Eq. (23) is just a sum of the single-particle quadratic Casimir operator, and its matrix-elements are just

$$\left\langle S_AG_AY_A\beta_A \left| \frac{1}{4}\sum_{i=1}^{A} \vec{\lambda}_i^c \cdot \vec{\lambda}_i^c \right| S_AG_AY_A\beta'_A \right\rangle = \delta_{\beta_A,\beta'_A}\frac{4}{3}A, \quad (24)$$

since the eigenvalue of the quadratic Casimir for both representations [3] and $[\bar{3}]$ equals 4/3. Because of Schur's Lemma the matrix elements of $\boldsymbol{C}_2[SU(3)]$ are independent of the particular permutational symmetry state Y_{A-1}. Exploiting this observation we can recast the second sum on the right-hand side of Eq. (23) into the form

$$\left\langle S_AG_AY_A\beta_A \left| \frac{1}{4}\sum_{i\neq j}^{A} \vec{\lambda}_i^c \cdot \vec{\lambda}_j^c \right| S_AG_AY_A\beta'_A \right\rangle = \frac{A(A-1)}{2}\frac{1}{|\Gamma_A|}\sum_{Y''_{A-1}\Gamma_A}\frac{1}{2}\langle S_AG_A\Gamma_AY''_{A-1}\beta_A|\vec{\lambda}_{A-1}^c \cdot \vec{\lambda}_A^c|S_AG_A\Gamma_AY''_{A-1}\beta'_A\rangle. \quad (25)$$

The $\vec{\lambda}_{A-1}^c \cdot \vec{\lambda}_A^c$ matrix elements can be evaluated by expanding the A-particle states in terms of the cscfps, Eq. (15), to decouple the last two particles,

$$\langle S_AG_A\Gamma_AY_{A-1}\beta_A|\vec{\lambda}_{A-1}^c \cdot \vec{\lambda}_A^c|S_AG_A\Gamma_AY_{A-1}\beta'_A\rangle$$
$$= \sum_{S_{A-1}G_{A-1}G'_{A-1}\beta_{A-1}\beta'_{A-1}g_Ag'_A} \sum_{S_{A-2}G_{A-2}\beta_{A-2}g_{A-1}g'_{A-1}} [S_{A-1}G_{A-1}Y_{A-1}\beta_{A-1}g_A|\}S_AG_AY_A\beta_A][S'_{A-1}G'_{A-1}Y_{A-1}\beta'_{A-1}g'_A|\}$$
$$\times S_AG_AY_A\beta'_A][S_{A-2}G_{A-2}Y_{A-2}\beta_{A-2}g_{A-1}|\}S_{A-1}G_{A-1}Y_{A-1}\beta_{A-1}][S_{A-2}G_{A-2}Y_{A-2}\beta_{A-2}g'_{A-1}|\}S_{A-1}G'_{A-1}Y_{A-1}\beta'_{A-1}]$$
$$\times \langle g_Ag_{A-1}|\vec{\lambda}_{A-1}^c \cdot \vec{\lambda}_A^c|g'_Ag'_{A-1}\rangle. \quad (26)$$

The last term on the right-hand side of Eq. (26) is the sum

$$\langle g_A g_{A-1}|\vec{\lambda}^c_{A-1}\cdot\vec{\lambda}^c_A|g'_A g'_{A-1}\rangle = \sum_a \langle g_{A-1}|\lambda^c_{aA-1}|g'_{A-1}\rangle \times \langle g_A|\lambda^c_{aA}|g'_A\rangle, \tag{27}$$

which can be easily evaluated using the explicit form of the $SU(3)$ generators for the [3] and $[\bar{3}]$ irreps, Eq. (8).

Turning now to establish states with good C-parity, we note that the C-parity operator, C, commutes with $\mathcal{S}_A$, the spin operator and the quadratic Casimir of $SU(3)$. Therefore the states $\{|S_A G_A Y_A \beta_A\rangle\}$ with good quantum numbers $(S_A G_A Y_A)$ that belong to the singlet irrep of $SU(3)$ form an invariant subspace for C. In order to evaluate the matrix-elements of C we first express our states in terms of the spin coupled single-particle states, Eq. (9). Then we can replace each quark state by the appropriate antiquark state and vice versa. The eigenvectors of C with eigenvalues $c = \pm 1$ that belong to the subspace of $\boldsymbol{C}_2[SU(3)]$ eigenstates with eigenvalue '0' are the desired physical color-spin basis functions.

III. A CONSTITUENT QUARK MODEL

The charmonium spectra has been thoroughly studied using constituent quark models for more than 25 years. As an almost pure coulombic system, models based on coulomb plus confinement interactions have been used to describe most of its spectroscopic properties [30]. To address the study of four-quark states we make use of a standard constituent quark model inspired in these works and originally applied to the study of the nonstrange baryon spectra and the baryon-baryon interaction [31]. This model has been generalized to all flavor sectors giving a reasonable description of the meson spectra [24], the baryon spectra [5], and the scalar mesons once four-quark configurations were included [20]. Within this model hadrons are described as clusters of constituent (massive) quarks. This is based on the assumption that the light-quark constituent quark mass appears because of the spontaneous breaking of the original $SU(3)_L \otimes SU(3)_R$ chiral symmetry at some momentum scale. In this domain of momenta, quarks interact through Goldstone boson exchange potentials. For the particular case of heavy quarks, chiral symmetry is explicitly broken and therefore the boson exchanges do not contribute to the qq interaction. Explicit expression of the boson exchange interacting potentials and a more detailed discussion can be found in Ref. [24].

Beyond the chiral symmetry breaking scale one expects the dynamics being governed by QCD perturbative effects. They are taken into account through the one-gluon-exchange (OGE) potential [32]. The nonrelativistic reduction of the OGE diagram in QCD for pointlike quarks presents a contact term that, when not treated perturbatively, leads to collapse [33]. This is why one maintains the structure of the OGE, but the δ function is regularized in a suitable way. This regularization is justified based on the finite size of the constituent quarks and should be therefore flavor dependent [34]. As a consequence, the OGE reads,

$$V_{\rm OGE}(\vec{r}_{ij}) = \frac{1}{4}\alpha_s(\vec{\lambda}^c_i\cdot\vec{\lambda}^c_j)\left\{\frac{1}{r_{ij}} - \frac{1}{6m_i m_j}\vec{\sigma}_i\cdot\vec{\sigma}_j\frac{e^{-r_{ij}/r_0(\mu)}}{r_{ij}r_0^2(\mu)}\right\}, \tag{28}$$

where λ^c are the $SU(3)$ color matrices, α_s is the quark-gluon coupling constant, and $r_0(\mu) = \hat{r}_0\mu_{nn}/\mu_{ij}$, where μ_{ij} is the reduced mass of the interacting quarks ij (n stands for the light quarks u and d) and $\hat{r}_0$ is a parameter to be determined from the data.

The strong coupling constant, taken to be constant for each flavor sector, has to be scale-dependent when describing different flavor sectors [35]. Such an effective scale dependence has been related to the typical momentum scale of each flavor sector and assimilated to the reduced mass of the system [36]. We use a strong coupling constant given by,

$$\alpha_s(\mu_{ij}) = \frac{\alpha_0}{\ln\{(\mu^2_{ij} + \mu^2_0)/\gamma^2_0\}}, \tag{29}$$

where α_0, μ_0 and γ_0 are parameters fitted within a global description of the meson spectra [24].

Finally, any model imitating QCD should incorporate confinement. Lattice calculations in the quenched approximation derived, for heavy quarks, a confining interaction linearly dependent on the interquark distance. The consideration of sea quarks apart from valence quarks (unquenched approximation) suggest a screening effect on the potential when increasing the interquark distance [37]. Creation of light-quark pairs out of vacuum in between the quarks becomes energetically preferable resulting in a complete screening of quark color charges at large distances. String breaking has been definitively confirmed through lattice calculations [38] in coincidence with the quite rapid crossover from a linear rising to a flat potential well established in $SU(2)$ Yang-Mills theories [39]. A screened potential simulating these results can be written as

$$V_{\rm CON}(\vec{r}_{ij}) = -a_c(1 - e^{-\mu_c r_{ij}})(\vec{\lambda}^c_i\cdot\vec{\lambda}^c_j). \tag{30}$$

At short distances this potential presents a linear behavior with an effective confinement strength $a = a_c\mu_c(\vec{\lambda}^c_i\cdot\vec{\lambda}^c_j)$, while it becomes constant at large distances. Such screened confining potentials provide with an explanation to the missing state problem in the baryon spectra [40], improve the description of the heavy meson spectra [41], and justify the deviation of the meson Regge trajectories from the linear behavior for higher angular momentum states [42].

IV. RESULTS AND DISCUSSION

The results for the $cc\bar{c}\,\bar{c}$ states have been obtained in the framework of the hyperspherical harmonic formalism explained above up to the maximum value of K within our computational capabilities, being $K_{\max} = 20$ for positive parity states and $K_{\max} = 21$ for negative parity states with the exception of $J^{PC} = 1^{+-}$ ($K_{\max} = 18$), 2^{++} and 2^{+-} ($K_{\max} = 22$) and 2^{--} ($K_{\max} = 23$). The two-body problem has been solved using the Numerov algorithm (see Ref. [24] for details) and all the model parameters fixed in the description of the $q\bar{q}$ spectra [20,24]. Since the only relevant two-meson decay thresholds for these four-quark systems are those formed by two $c\bar{c}$ mesons, we summarize in Table I the results obtained for the charmonium spectrum in Ref. [24] compared with the experimental data quoted by the Particle Data Group (PDG) [43]. It can be observed that the energy difference in the P-wave $c\bar{c}$ multiplets due to the noncentral terms is less than 50 MeV. Although these terms are known to play an important role in the description of the light $q\bar{q}$ states, their contribution becomes smaller as the mass of the heavy quark increases [24]. This allows to neglect them on the four-body sector, because they would only provide with a fine tune of the final results but making the solution of the four-body problem much more involved and time consuming.

To analyze the stability of these systems against dissociation through strong decay, $cc\bar{c}\,\bar{c} \rightarrow M_1(c\bar{c}) + M_2(c\bar{c})$, parity ($P$), C-parity (C), and total angular momentum (J) must be preserved. Assuming that the decay takes place in a relative S-wave for the final state, $J_{4q} = J_{M_1} \otimes J_{M_2}$ and $P_{4q} = P_{M_1} \cdot P_{M_2}$. Since the final states are also eigenstates of C-parity one has $C_{4q} = C_{M_1} \cdot C_{M_2}$. Once the meson masses have been obtained, the corresponding thresholds can be computed simply adding the mesons masses M_1 and M_2, $T(M_1, M_2) = M_1 + M_2$. In Table II we indicate the lowest two-meson decay threshold for each set of quantum numbers. Four-quark states will be stable under strong interaction if their total energy lies below all possible, and allowed, two-meson thresholds. It is useful to define

$$\Delta = M(q_1 q_2 \bar{q}_3 \bar{q}_4) - T(M_1, M_2), \tag{31}$$

in such a way that if $\Delta > 0$ the four-quark system will fall apart into two mesons, while $\Delta < 0$ will indicate that such strong decay is forbidden and therefore the decay, if allowed, must be weak or electromagnetic, being its width much narrower.

Let us first of all analyze the convergence of the expansion in terms of hyperspherical harmonics. We show in Fig. 1 the variation of the energy of the $J^{PC} = 0^{++}$ and 0^{-+} cases with K, this behavior being similar for the other quantum numbers. From this figure it can be seen that the convergence is slow, and the effective potential techniques [23] are unable to improve it. In order to obtain a more adequate value for the energy we have extrapolated it according to the expression

$$E(K) = E(K = \infty) + \frac{a}{K^b}, \tag{32}$$

where $E(K = \infty)$, a and b are fitted parameters. In Table III we show the values obtained for $E(K = \infty)$ for the two states shown in Fig. 1 as a function of the fitting range (K_0, K_f). It can be checked that the values obtained for $E(K = \infty)$ are stable within ± 10 MeV for $K_0 \geq 10$. The values obtained for the b parameter are very similar for all possible quantum numbers, being in all cases in the range 0.75–0.95, corresponding the lower limit to the positive parity states and the upper one to the negative parity states. The origin of the difficulties to obtain a better convergence can be traced back to two different aspects of these systems. On the one hand, the HH formalism is better suited to deal with bound states, however, in this particular case we will show that most of the four-quark states are above the corresponding two-meson threshold. On the other hand, the color structure of a four-quark system is

TABLE I. Charmonium spectrum in MeV. Experimental data are taken from PDG [43].

$(nL)J^{PC}$	State	CQM	Exp.
$(1S)0^{-+}$	$\eta_c(1S)$	2990	2979.6 ± 1.2
$(1S)1^{--}$	$J/\psi(1S)$	3097	3096.916 ± 0.011
$(1P)0^{++}$	$\chi_{c0}(1P)$	3443	3415.19 ± 0.34
$(1P)1^{++}$	$\chi_{c1}(1P)$	3496	3510.59 ± 0.10
$(1P)2^{++}$	$\chi_{c2}(1P)$	3525	3556.26 ± 0.11
$(1P)1^{+-}$	$h_c(1P)$	3507	3526.21 ± 0.25
$(2S)0^{-+}$	$\eta_c(2S)$	3627	3654 ± 10
$(2S)1^{--}$	$\psi(2S)$	3685	3686.093 ± 0.034
$(1D)1^{--}$	$\psi(3770)$	3776	3770 ± 2.4
$(1D)2^{--}$	$\psi(3836)$	3790	3836 ± 13
$(3S)1^{--}$	$\psi(4040)$	4050	4040 ± 10
$(2D)1^{--}$	$\psi(4160)$	4104	4159 ± 20

TABLE II. Lowest S-wave two-meson thresholds (MeV) for all J^{PC} quantum numbers.

J^{PC}	$M_1 M_2$	$T(M_1, M_2)$
0^{++}	$\eta_c(1S)\eta_c(1S)$	5980
0^{+-}	$\chi_{c1}(1P)h_c(1P)$	7003
1^{++}	$J/\psi(1S)J/\psi(1S)$	6194
1^{+-}	$J/\psi(1S)\eta_c(1S)$	6087
2^{++}	$J/\psi(1S)J/\psi(1S)$	6194
2^{+-}	$\eta_c(1S)\psi(3836)$	6780
0^{-+}	$\eta_c(1S)\chi_{c0}(1P)$	6433
0^{--}	$J/\psi(1S)\chi_{c1}(1P)$	6593
1^{-+}	$\eta_c(1S)\chi_{c1}(1P)$	6486
1^{--}	$\eta_c(1S)h_c(1P)$	6497
2^{-+}	$\eta_c(1S)\chi_{c2}(1P)$	6515
2^{--}	$J/\psi(1S)\chi_{c1}(1P)$	6593

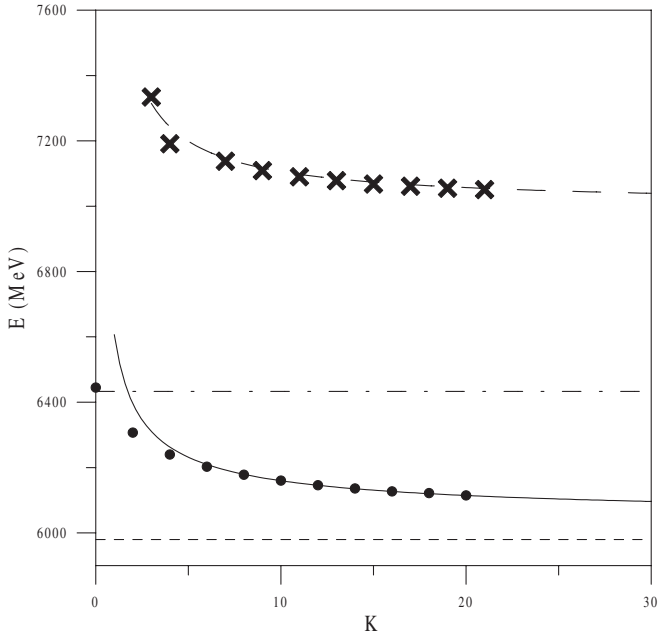

FIG. 1. Energy of the 0^{++} (circles) and 0^{-+} (crosses) four-quark states as a function of K. The solid line corresponds to the extrapolation used for the 0^{++} case while the long-dashed one corresponds to the 0^{-+} state. The thresholds for each state are denoted by a dashed line (0^{++}) and a dotted-dashed line (0^{-+}).

more involved than that of $q\bar{q}$ or qqq systems [20], with two different possibilities to obtain a color singlet. This makes possible the existence of repulsive contributions due to the color operator of the interacting potential, Eqs. (28) and (30). Although the total interaction is always confining, these deconfining terms make the convergence much slower.

We have studied all possible J^{PC} quantum numbers with $L = 0$. We show in Table IV the results obtained for $K = K_{\max}$, the maximum value of K calculated, the results obtained using the extrapolation of Eq. (32), the corresponding threshold for each set of quantum numbers, and the value of Δ, Eq. (31). A first glance to the results shows that there are two sets of quantum numbers, $J^{PC} = 0^{+-}$ and $J^{PC} = 2^{+-}$, where the four-quark configuration is clearly below the corresponding two-meson threshold for dissociation. The $J^{PC} = 1^{+-}$ and $J^{PC} = 2^{++}$ states are very close to the threshold and all the other quantum numbers are far above the corresponding threshold. Being all possible strong decays forbidden they should be narrow states with typical widths of the order of a few MeV. It is also interesting to note that the quantum numbers 0^{+-} and 2^{+-} correspond to exotic states, those whose quantum numbers cannot be obtained from a $q\bar{q}$ configuration, and therefore, if experimentally observed, they would be easily distinguished as a clear signal of pure non-$q\bar{q}$ states.

TABLE III. $E(K = \infty)$ (MeV) as a function of the fitting range (K_0, K_f) for $J^{PC} = 0^{++}$ and $J^{PC} = 0^{-+}$.

(K_0, K_f)	$E(K = \infty)[0^{++}]$	(K_0, K_f)	$E(K = \infty)[0^{-+}]$
(2, 20)	5831	(3, 21)	7018
(4, 20)	5947	(5, 21)	7007
(6, 20)	5997	(7, 21)	7004
(8, 20)	6017	(9, 21)	7003
(10, 20)	6030	(11, 21)	7000
(12, 20)	6038	(13, 21)	7000
(14, 20)	6041	(15, 21)	6993
(16, 20)	6047	(17, 21)	6990

TABLE IV. $cc\bar{c}\,\bar{c}$ masses obtained for the maximum value of K computed, $E(K_{\max})$, and using the extrapolation of Eq. (32), $E(K = \infty)$, compared with the corresponding threshold for each set of quantum numbers, $T(M_1, M_2)$, as given in Table II. The value of Δ for each state is also given. All energies are in MeV.

J^{PC}	$E(K_{\max})$	$E(K = \infty)$	$T(M_1, M_2)$	Δ
0^{++}	6115	6038	5980	+58
0^{+-}	6606	6515	7003	−488
1^{++}	6609	6530	6194	+336
1^{+-}	6176	6101	6087	+14
2^{++}	6216	6172	6194	−22
2^{+-}	6648	6586	6780	−194
0^{-+}	7051	6993	6433	+560
0^{--}	7362	7276	6593	+683
1^{-+}	7363	7275	6486	+789
1^{--}	7052	6998	6497	+501
2^{-+}	7055	7002	6515	+487
2^{--}	7357	7278	6593	+685

The energies obtained for the negative parity states are, as expected, much higher than those of positive parity and therefore all of them are far above the corresponding thresholds, $\Delta(J^{PC} = J^{-C}) > 500$ MeV. Thus, they will fall apart immediately and therefore they should be very broad and difficult to detect. Let us also note that the experimental observation stating that a unit of angular momentum (parity change) in the qqq and $q\bar{q}$ sector costs approximately 400–500 MeV of excitation energy in all flavor sectors (with some remarkable exceptions as the light-scalar mesons [20]) has an equivalent in the four-quark sector, being in this case the energy difference due to the parity change $E(J^{-C}) - E(J^{+C}) \approx 800$–900 MeV.

The existence of non-$q\bar{q}$ signals in the meson spectra has been the subject of an intensive debate [10]. To analyze whether the existence of bound states with exotic quantum numbers could be a characteristic feature of the heavy-quark sector or it is also present in the light sector we show in Fig. 2 the value of Δ as a function of the quark mass for all exotic quantum numbers and in Fig. 3 for the remaining positive parity states. Since each quantum number has a different threshold these figures should be interpreted carefully. One should notice that the value $\Delta = 0$ in both figures corresponds to $M(q_1 q_2 \bar{q}_3 \bar{q}_4) = T(M_1, M_2)$, and

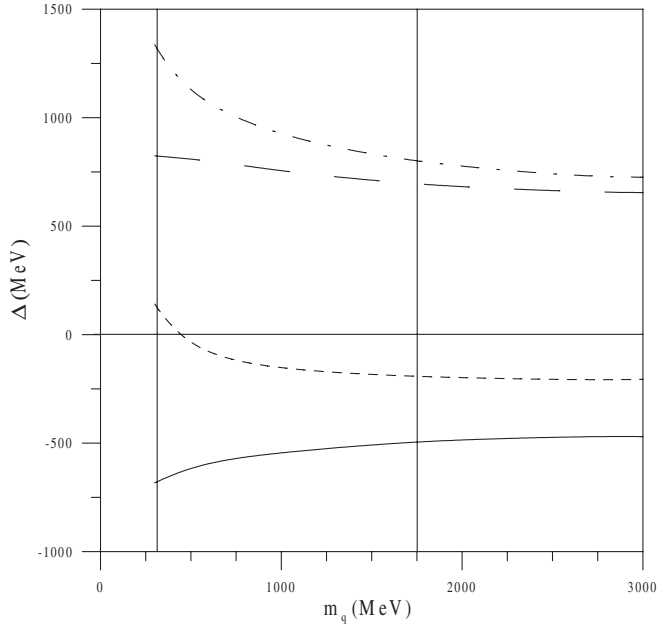

FIG. 2. Variation of the energy of the four-quark states with exotic quantum numbers with the quark mass. The solid line corresponds to the quantum numbers $J^{PC} = 0^{+-}$, dashed to 2^{+-}, long-dashed to 0^{--}, and dashed-dotted to 1^{-+}. The vertical lines correspond to the light and charm quark masses, respectively.

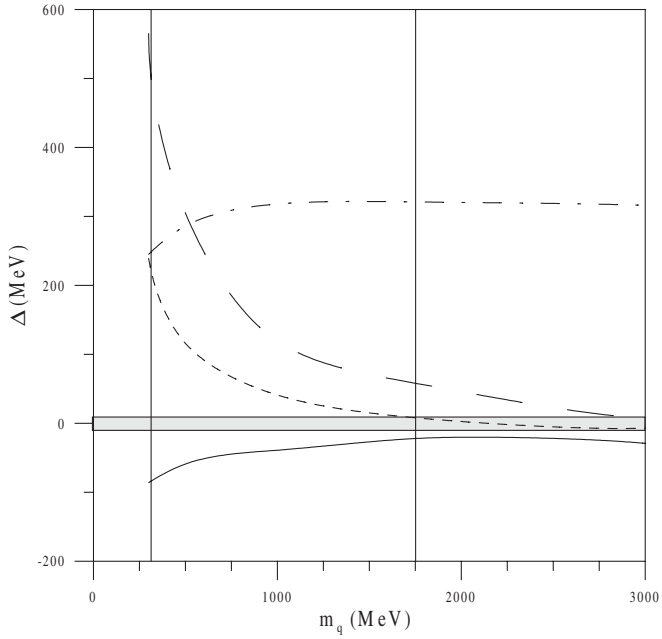

FIG. 3. Variation of the energy of the nonexotic positive parity states with the quark mass. The gray box corresponds to the ± 10 MeV uncertainty in the extrapolation. The solid line corresponds to the quantum numbers $J^{PC} = 2^{++}$, dashed to 1^{+-}, long-dashed to 0^{++}, and dashed-dotted to 1^{++}. The vertical lines correspond to the light and charm quark masses, respectively.

therefore, the fact that one state is below the others in these figures do not imply that its total mass would be smaller. Let us also note that since the heavy quarks are isoscalar states, the flavor wave function of the four heavy-quark states will be completely symmetric with total isospin equal to zero. Therefore, one should compare the results obtained in the light-quark case with a completely symmetric flavor wave function, i.e., the isotensor states.

In Fig. 3 we observe how one of the four-quark nonexotic positive parity states, the 2^{++}, becomes more bound when the quark mass is decreased, $\Delta \approx -80$ MeV for $m_q = 313$ MeV. The 1^{+-} and 0^{++} states, that were slightly above the threshold for $m_q = 1752$ MeV, increase their attraction when the quark mass is increased and only for masses above 3 GeV, close to the bottom quark mass, may be bound. With respect to the exotic quantum numbers, we observe in Fig. 2 that the 0^{--} and 1^{-+} are not bound for any value of the quark mass. The 2^{+-} state decreases its binding when the quark mass diminishes and it becomes unbound for masses of the order of 500 MeV, the strange quark mass. Only the 0^{+-} four-quark state becomes more deeply bound when the constituent quark mass decreases, and therefore only one bound state with exotic quantum numbers would remain in the light-quark sector.

Taking the experimental mass for the threshold $T(M_1, M_2)$ in the light-quark case, one can estimate the energy region where these states can be found, being $M(0^{++}) \approx 900$ MeV, $M(1^{+-}) \approx 1200$ MeV, $M(1^{++}) \approx 1900$ MeV, and $M(2^{++}) \approx 1500$ MeV. Note that the boson exchange potentials, that as explained in Sect. II are present in the description of the light-quark sector, have not been considered in this analysis. Although these terms do play a role in the description of the four-quark state total energy, it was shown in Ref. [19] that their suppression will not alter the relative order of the four-quark states. The four-quark states with quantum numbers 0^{++} and 2^{++} have been analyzed with the complete constituent quark model including boson exchanges using a variational approach [20], obtaining 1004 MeV and 1500 MeV, respectively, both in good agreement with our calculation and therefore giving confidence on our approach neglecting the boson exchange interactions in this particular analysis. The states with exotic quantum numbers would be higher in energy, being $M(0^{--}) \approx M(1^{-+}) \approx 2900$ MeV, $M(0^{+-}) \approx 1800$ MeV and $M(2^{+-}) \approx 2100$ MeV.

There are experimental evidences for three states with exotic quantum numbers in the light-quark sector. Two of them are isovectors with quantum numbers $J^{PC} = 1^{-+}$ named $\pi_1(1400)$ and $\pi_1(1600)$, and one isotensor $J^{PC} = 2^{++}$, the $X(1600)$ [43]. Based on the coupling of the $\pi_1(1400)$ to the $\eta\pi$ decay channel a possible four-quark structure has been suggested [44] while the strong $\eta'\pi$ coupling of the $\pi_1(1600)$ makes it a good candidate for a pure hybrid state [45]. The large mass obtained in our analysis for these quantum numbers makes doubtful the

identification of any of them, and, in particular, of the $\pi_1(1400)$, with a pure four-quark state, although a complete calculation with the proper flavor wave function is needed before drawing a definitive conclusion [46]. An alternative explanation of the $\pi_1(1400)$ in terms of low-energy rescattering effects has been proposed in Ref. [47] while recent reanalysis of experimental data from E852 Collaboration with improved statistics show no evidence of any exotic meson with a mass close to 1.6 GeV [48]. Concerning the $X(1600)$, being its experimental mass 1600 ± 100 MeV, a possible tetraquark configuration seems likely.

V. SUMMARY

In this work we have presented for the first time a generalization of the HH formalism to study systems made of quarks and antiquarks of the same flavor. For this purpose we made use of standard techniques widely used in nuclear physics that have been adapted to include the color degree of freedom and to treat explicitly the C-parity. This formalism opens the door to an exact study of multiquark systems up to now described by means of different techniques, being the variational methods the most standard ones. The particular color structure of four-quark systems makes the convergence slow but attainable.

The formalism has been applied to study the existence of $L = 0$ four-charm quark states. For this purpose we made use of a well-established constituent quark model that properly accounts for the charmed meson and baryon spectra. Our results suggest the possible existence of three four-quark bound states with quantum numbers 0^{+-}, 2^{+-} and 2^{++} and masses of the order of 6515, 6648, and 6216 MeV. The two states with exotic quantum numbers, clearly below their corresponding two-meson threshold, should present narrow widths and, if produced, may be easily detected.

We have analyzed the variation of our results with the constituent quark mass. In the light-quark case only the 0^{+-} and 2^{++} quantum numbers remain bound, being the 0^{+-} the lowest one with an energy close to 1800 MeV. The four-quark state with quantum numbers 1^{-+} lies around 2900 MeV, far from the experimental states $\pi_1(1400)$ and $\pi_1(1600)$, therefore supporting a hybrid configuration or a more complicated structure in terms of nonresonant structures. A possible description of the $X(1600)$ as a four-quark has also been justified.

The program we have started for and exact study of multiquark systems by means of the HH formalism will be accomplished by implementing the possibility of treating quarks of different masses. When this is done we will have at our disposal a powerful method, imported from the nuclear physics, to study in an exact way systems made of any number of quarks and antiquarks coupled to a color singlet.

ACKNOWLEDGMENTS

This work has been partially funded by Ministerio de Ciencia y Tecnología under Contract No. FPA2004-05616, by Junta de Castilla y León under Contract No. SA-104/04, and by Generalitat Valenciana under Contract No. GV05/276.

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