High-accuracy critical exponents of $O(N)$ hierarchical sigma models

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(Received 18 November 2005; published 8 February 2006)

We perform high-accuracy calculations of the critical exponent γ and its subleading exponent for the $3D$ $O(N)$ Dyson's hierarchical model for N up to 20. We calculate the critical temperatures for the nonlinear sigma model measure $\delta(\vec{\phi}.\vec{\phi} - 1)$. We discuss the possibility of extracting the first coefficients of the $1/N$ expansion from our numerical data. We show that the leading and subleading exponents agree with the Polchinski equation and the equivalent Litim equation, in the local potential approximation, with at least 4 significant digits.

DOI: 10.1103/PhysRevD.73.047701 PACS numbers: 11.15.Pg, 11.10.Hi, 64.60.Fr

The large N limit and the $1/N$ expansion [1–3] appear prominently in recent developments in particle physics, condensed matter and string theory [4–7]. For sigma models, the basic gap equation can be obtained by using the method of steepest descent for the functional integral [1,8]. For N large and negative, the maxima of the action dominate instead of the minima and the radius of convergence of the $1/N$ expansion should be zero. In order to turn a $1/N$ expansion into a *quantitative* tool, we need to: (1) understand the large order behavior of the series, (2) locate the singularities of the Borel transform, and (3) compare the accuracy of various procedures with numerical results for given values of N. Calculating the series or obtaining accurate numerical results at fixed N are difficult tasks and we do not know any model where this program has been completed. For instance for the critical exponents in three dimensions, we are only aware of calculation up to order $1/N^2$ in Refs. [9–11]. Several results related to the possibility (or impossibility) of resumming particular $1/N$ expansions are known [12–14]. Overall, it seems that there is a rather pessimistic impression regarding the possibility of using the $1/N$ expansion for low values of N. For this reason, it would be interesting to discuss the three questions enumerated above for a model where we have good chances to obtain definite answers. Dyson's hierarchical model [15,16] is a good candidate for this purpose.

In this paper, we provide high-accuracy numerical values for the critical exponent γ, the subleading exponent Δ and the critical parameter β_c for the $3D$ $O(N)$ hierarchical nonlinear sigma models. These quantities appear in the magnetic susceptibility near β_c in the symmetric phase as

$$\chi = (\beta_c - \beta)^{-\gamma}(A_0 + A_1(\beta_c - \beta)^{\Delta} + \cdots). \quad (1)$$

The method of calculation of the critical exponents used here is an extension of one of the methods described at length in the case of $N = 1$ [17] and will only be sketched briefly. On the other hand, the accuracy of the approximations used depend non trivially on N as we shall discuss later. The renormalization group (RG) transformation can be constructed as a blockspin transformation followed by a rescaling of the field. For Dyson's hierarchical model, the block spin transformation affects only the local measure. The RG transformation can be expressed conveniently in terms of the Fourier transform (denoted R hereafter) of this local measure. In the following, we keep the $O(N)$ symmetry unbroken and the Fourier transform will depend only on $\vec{k}.\vec{k} \equiv u$. Here $\vec{k}$ is a source conjugated to the local field variable $\vec{\phi}$. Replacing k by u and the second derivative by the N-dimensional Laplacian in Eq. (2.5) of Ref. [17], we obtain the RG transformation for the Fourier transform of the local measure

$$R_{n+1,N}(u) \propto e^{[-(\beta/2)(4u(\partial^2/\partial u^2)+2N(\partial/\partial u))]}(R_{n,N}(cu/4))^2, \quad (2)$$

where $c = 2^{1-2/D}$ in order to reproduce the scaling of a Gaussian massless field in D dimensions. $D = 3$ hereafter. We fix the normalization constant by imposing $R_{n,N}(0) = 1$ so that $R_{n,N}(k)$ has a simple probabilistic interpretation [17]. In the following, the calculations will be performed using polynomial approximations of degree $l_{\max}$:

$$R_{n,N}(k) \simeq 1 + a_{n,1}u + a_{n,2}u^2 + \cdots + a_{n,l_{\max}}u^{l_{\max}}. \quad (3)$$

The finite volume susceptibility for 2^n sites is related to the first coefficient by the relation $\chi_n = -2a_{n,1}(2/c)^n$. The truncated recursion formula for the $a_{n,m}$ reads

$$a_{n+1,m} = \frac{\sum_{l=m}^{2l_{\max}}(\sum_{p+q=l} a_{n,p}a_{n,q})B_{m,l}}{\sum_{l=0}^{2l_{\max}}(\sum_{p+q=l} a_{n,p}a_{n,q})B_{0,l}}, \quad (4)$$

with

$$B_{m,l} = \frac{\Gamma(l+1)\Gamma(l+N/2)}{\Gamma(m+1)\Gamma(m+N/2)} \frac{1}{(l-m)!}\left(\frac{c}{4}\right)^l(-2\beta)^{l-m}. \quad (5)$$

We emphasize that in the above formula and in our nu-

*Electronic address: yannick-meurice@uiowa.edu merical calculations, no truncation is applied after squaring and so the sum in Eq. (4) does extend up to $2l_{\rm max}$. Since the derivatives appear to arbitrarily large order in Eq. (2) and can lower the degree of a polynomial of order larger than $l_{\rm max}$, this affects all the coefficients of order less than $l_{\rm max}$. This procedure has been discussed and justified in Ref. [18].

The critical exponents appearing in Eq. (1) are obtained by calculating the eigenvalues $\lambda_1, \lambda_2, \cdots$ of the matrix $\partial a_{n+1,l}/\partial a_{n,m}$ at the nontrivial fixed point. The exponents γ and Δ, can be expressed as

$$\gamma = \frac{\ln(2/c)}{\ln(\lambda_1)}, \qquad \Delta = \left|\frac{\ln(\lambda_2)}{\ln(\lambda_1)}\right|. \tag{6}$$

The critical exponents are universal and, within numerical errors, independent of the manner that we approach the nontrivial fixed point. In the following, we have mostly started with the local measure of the nonlinear sigma model $\delta(\vec{\phi}.\vec{\phi} - 1)$. The corresponding Fourier transform reads

$$R_{0,N}(u) = \sum_{l=0}^{\infty} \frac{(-1)^l u^l \Gamma(\frac{N}{2})}{2^{2l} l! \Gamma(\frac{N}{2} + l)}. \tag{7}$$

A motivation for this choice is that, as we will explain below, the value of β_c can be calculated in the large N limit. Other measures have also been used in order to check the universal values of the two exponents.

The asymptotic behavior of the ratio $a_{n+1,1}/a_{n,1}$ allows us to decide unambiguously if we are in the symmetric phase (where the ratio approaches $c/2 \simeq 0.63$) or in the broken phase (where the ratio approaches c). Using a binary search, one can determine the critical value of β with great accuracy. As this critical value depends on $l_{\rm max}$, we denote it $\beta_c(l_{\rm max})$. When $l_{\rm max} \to \infty$, $\beta_c(l_{\rm max}) \to \beta_c$. The rate at which this limit is reached depends on N. This is illustrated in Fig. 1 where we see that in order to reach β_c with a given accuracy, we need to increase $l_{\rm max}$ when N increases. In Fig. 2, we give the minimum $l_{\rm max}$ necessary for $\beta_c(l_{\rm max})$ to share 20 significant digits with β_c. $l_{\rm max} \simeq 22 + 6.2N^{0.7}$ is a good fit for Fig. 2.

The nontrivial fixed point *for a given value of* $l_{\rm max}$ can be constructed by iterating sufficiently many times the RG map at values sufficiently close to $\beta_c(l_{\rm max})$. In order to get an accuracy ϵ for the fixed point for that value of $l_{\rm max}$, we need to iterate n times the map until

$$\lambda_2^n \sim \epsilon, \tag{8}$$

in order to get rid of the irrelevant directions. At the same time, we want the growth in the relevant direction to be limited, in other words,

$$|\beta - \beta_c(l_{\rm max})|\lambda_1^n < \epsilon. \tag{9}$$

Combining these two requirements together with Eq. (6) we obtain

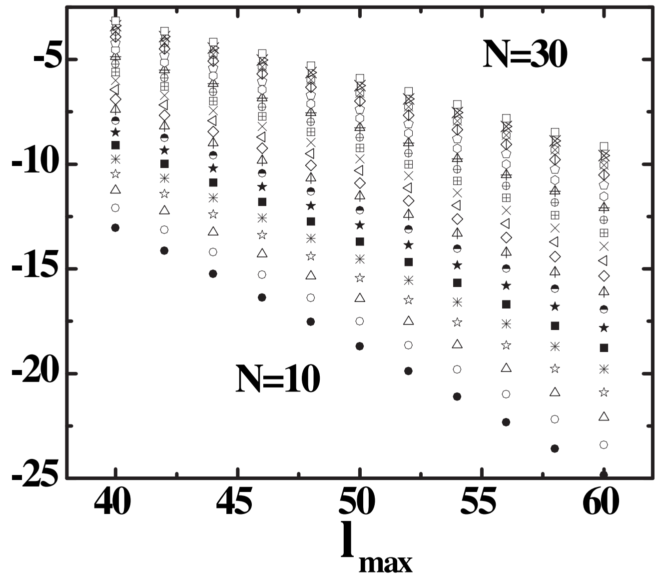

FIG. 1. $\log_{10}\frac{|\beta(l_{\rm max})-\beta_c|}{\beta_c}$ calculated for $l_{\rm max} = 40$ to $l_{\rm max} = 60$ for $N = 10$ (filled circles), $N = 11$ (empty circles), $N = 12$ (empty triangles) up to $N = 30$ (empty squares).

$$|\beta - \beta_c(l_{\rm max})| \simeq \epsilon^{1+1/\Delta} \tag{10}$$

This is an order magnitude estimate, however it works well except for $N = 1$ where we need to pick β slightly closer to the critical value. By "working well," we mean that if we go closer to the critical value, changes smaller than ϵ are observed in the first two eigenvalues. The numerical results for $\epsilon = 10^{-10}$ and N up to 20, are given in the Tables I and II for the values of $l_{\rm max}$ of Fig. 2. Errors of 1 or less in the last printed digit should be understood in all the tables.

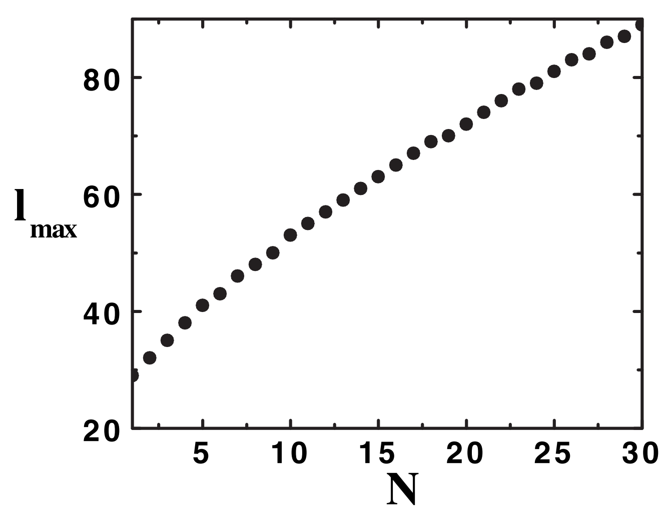

FIG. 2. Minimal value of $l_{\rm max}$ in order to have $\log_{10}\frac{|\beta_c(l_{\rm max})-\beta_c(\infty)|}{\beta_c(\infty)} = -20$ versus N.

TABLE I. β_c and the first two eigenvalues for $N = 1 \cdots 20$.

N	β_c	λ_1	λ_2
1	1.179 030 170 446 269 732 5	1.427 172 478	0.859 411 649 2
2	2.473 526 575 291 985 400 0	1.385 743 490	0.856 340 906 6
3	3.827 382 033 357 339 767 1	1.354 668 326	0.850 694 515 0
4	5.211 161 563 553 365 616 5	1.332 749 866	0.844 052 295 6
5	6.610 415 346 285 506 843 5	1.317 578 283	0.837 643 674 7
6	8.018 111 405 370 672 594 1	1.306 955 396	0.832 034 502 2
7	9.430 709 644 742 779 688 2	1.299 321 025	0.827 337 817 2
8	10.846 330 737 925 124 699	1.293 666 393	0.823 467 678 5
9	12.263 918 029 354 988 652	1.289 354 227	0.820 283 344 9
10	13.682 844 072 802 585 664	1.285 978 489	0.817 648 546 1
11	15.102 717 572 108 367 579	1.283 274 741	0.815 449 265 2
12	16.523 283 812 777 939 366	1.281 066 141	0.813 595 313 7
13	17.944 370 719 047 342 283	1.279 231 192	0.812 016 855 5
14	19.365 858 255 947 423 937	1.277 684 252	0.810 660 096 3
15	20.787 660 334 686 062 513	1.276 363 511	0.809 483 485 7
16	22.209 713 705 054 412 233	1.275 223 389	0.808 454 715 0
17	23.631 970 906 283 518 487	1.274 229 622	0.807 548 444 0
18	25.054 395 659 078 177 206	1.273 356 000	0.806 744 610 7
19	26.476 959 772 907 788 848	1.272 582 158	0.806 027 179 3
20	27.899 641 020 779 716 433	1.271 892 050	0.805 383 211 6

As N increases, the values displayed in Table II seem to slowly approach asymptotic values. This is expected. Using the general formulation of Refs. [2,8] together with the particular form of the propagator [19] for the model considered here, one finds the leading terms

$$\gamma \simeq 2 + a_1/N + \cdots \tag{11}$$

$$\Delta \simeq 1 + b_1/N + \cdots$$

$$\beta_c/N \simeq (2-c)/(2(c-1)) + c_1/N + \cdots. \tag{12}$$

The magnitude of the coefficients a_1, b_1, c_1 of the leading $1/N$ corrections can be estimated by subtracting the asymptotic value and multiplying by N. The results are shown in Table III. They indicate that $a_1 \simeq -1.6$, $b_1 \simeq -2.0$, $c_1 \simeq -0.57$. It seems possible to improve the accuracy by estimating the next to leading order corrections and so on. However, the stability of this procedure is more delicate and remains to be studied with simpler examples.

We now compare the exponents calculated here with those calculated with three other RG transformations [20–22]. As we proceed to explain, the exponents should be the same in the four cases (including ours). The change of coordinates that relates the RG transformation considered here and the one studied in Ref. [22] is given in the introduction of [23] (for $L = 2^{1/3}$). The fact that the limit $L \to 1$ in the formulation of Ref. [22] yields the Polchinski equation in the local potential approximation studied in Ref. [21] is explained in Ref. [24]. Consequently, these two RG transformations should be the same in the *linear* approximation. Finally, Litim [20,25] proposed an optimized version of the exact RG transformation and suggested [26] that it was equivalent to the Polchinski equation in the local potential approximation. The equivalence was subsequently proved by Morris [27].

To facilitate the comparison, we display $\nu = \gamma/2$ (since $\eta = 0$ here) and $\omega = \Delta/\nu$ in Table IV. Our results coincide with the 4 digits given in column (2) of Table 3 (for ν)

TABLE II. γ, Δ and β_c/N for $N = 1 \cdots 20$.

N	γ	Δ	β_c/N
1	1.299 140 73	0.425 946 859	1.179 030 170
2	1.416 449 96	0.475 380 831	1.236 763 288
3	1.522 279 70	0.532 691 965	1.275 794 011
4	1.608 728 17	0.590 232 008	1.302 790 391
5	1.675 510 51	0.642 369 187	1.322 083 069
6	1.726 177 03	0.686 892 637	1.336 351 901
7	1.764 798 63	0.723 880 426	1.347 244 235
8	1.794 692 74	0.754 352 622	1.355 791 342
9	1.818 271 05	0.779 508 505	1.362 657 559
10	1.837 222 91	0.800 424 484	1.368 284 407
11	1.852 726 36	0.817 977 695	1.372 974 325
12	1.865 610 92	0.832 855 522	1.376 940 318
13	1.876 469 98	0.845 589 221	1.380 336 209
14	1.885 735 62	0.856 588 705	1.383 275 590
15	1.893 728 12	0.866 171 682	1.385 844 022
16	1.900 689 03	0.874 586 271	1.388 107 107
17	1.906 803 38	0.882 027 998	1.390 115 936
18	1.912 215 07	0.888 652 409	1.391 910 870
19	1.917 037 52	0.894 584 429	1.393 524 199
20	1.921 361 21	0.899 925 325	1.394 982 051
∞	2	1	$\frac{2-c}{2(c-1)} = 1.42366..$

TABLE III. $N(2-\gamma)$, $N(1-\Delta)$ and $N(\frac{2-c}{2(c-1)} - \frac{\beta_c}{N})$ for $N = 1 \cdots 20$.

N	$N(2-\gamma)$	$N(1-\Delta)$	$N(\frac{2-c}{2(c-1)} - \frac{\beta_c}{N})$
1	0.7009	0.5741	0.2446
2	1.167	1.049	0.3738
3	1.433	1.402	0.4436
4	1.565	1.639	0.4835
5	1.622	1.788	0.5079
6	1.643	1.879	0.5239
7	1.646	1.933	0.5349
8	1.642	1.965	0.5430
9	1.636	1.984	0.5490
10	1.628	1.996	0.5538
11	1.620	2.002	0.5576
12	1.613	2.006	0.5606
13	1.606	2.007	0.5632
14	1.600	2.008	0.5654
15	1.594	2.007	0.5673
16	1.589	2.007	0.5689
17	1.584	2.006	0.5703
18	1.580	2.004	0.5715
19	1.576	2.003	0.5726
20	1.573	2.001	0.5736

TABLE IV. ν, ω and α for $N = 1 \cdots 20$.

N	$\nu = \gamma/2$	$\omega = \Delta/\nu$	$\alpha = 2 - 3\nu$
1	0.649 570	0.655 736	0.051 289
2	0.708 225	0.671 229	−0.124 675
3	0.761 140	0.699 861	−0.283 420
4	0.804 364	0.733 787	−0.413 092
5	0.837 755	0.766 774	−0.513 266
6	0.863 089	0.795 854	−0.589 266
7	0.882 399	0.820 355	−0.647 198
8	0.897 346	0.840 648	−0.692 039
9	0.909 136	0.857 417	−0.727 407
10	0.918 611	0.871 342	−0.755 834

and 4 (for ω) in [21]. They coincide with the six digits for ν given in the line $d = 3$ of Table 8 of [22] for $N = 1, 2, 3, 5$ and 10. However, we found discrepancies of order 1 in the fifth digit of ν and slightly larger for ω with the values found in Table 1 of [20]. Our estimated errors are of order 1 in the 9th digit. For $N = 1$, this is confirmed by an independent method [17]. For $N = 2$, 3, 5, and 10, this is confirmed up to the sixth digit [22]. Consequently, a discrepancy in the 5th digit cannot be explained by our numerical errors. Note also that for $N \geq 2$, α is more negative than for nearest neighbor models [11].

In summary, we have provided high-accuracy data for γ, Δ and β_c for N up to 20. It seems likely that a few terms of the $1/N$ expansion for these three quantities can be estimated from this data. Work is in progress to calculate these expansions independently by semianalytical methods and learn about the asymptotic behavior of the series and their accuracy. The discrepancy with the 5th digit of Ref. [20] remains to be explained.

We thank G. 't Hooft, J. Zinn-Justin and G. Parisi for valuable comments on related topics. This research was supported in part by the Department of Energy under Contract No. FG02-91ER40664. M. B. Oktay has been supported by SFI Grant No. 04/BRG/P0275.

[1] H. E. Stanley, Phys. Rev. **176**, 718 (1968).
[2] S. K. Ma, Phys. Lett. A **43**, 475 (1973).
[3] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974); **B75**, 461 (1974).
[4] M. Moshe and J. Zinn-Justin, Phys. Rep. **385**, 69 (2003).
[5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
[6] R. Narayanan and H. Neuberger, hep-lat/0501031.
[7] M. Teper, hep-lat/0509019.
[8] F. David, D. A. Kessler, and H. Neuberger, Nucl. Phys. **B257**, 695 (1985).
[9] Y. Okabe, M. Oku, and R. Abe, Prog. Theor. Phys. **59**, 1825 (1978).
[10] J. A. Gracey, J. Phys. A **24**, L197 (1991).
[11] A. Pelissetto and E. Vicari, Phys. Rep. **368**, 549 (2002).
[12] B. de Wit and G. 't Hooft, Phys. Lett. **69B**, 61 (1977).
[13] J. Avan and H. J. de Vega, Phys. Rev. D **29**, 2904 (1984).
[14] J. L. Kneur and D. Reynaud, J. High Energy Phys. 01 (2003) 014.
[15] F. Dyson, Commun. Math. Phys. **12**, 91 (1969).
[16] G. Baker, Phys. Rev. B **5**, 2622 (1972).
[17] J. Godina, Y. Meurice, and M. Oktay, Phys. Rev. D **59**, 096002 (1999).
[18] Y. Meurice and S. Niermann, J. Stat. Phys. **108**, 213 (2002).
[19] Y. Meurice, Phys. Rev. D **67**, 025006 (2003).
[20] D. F. Litim, Nucl. Phys. **B631**, 128 (2002).
[21] J. Comellas and A. Travesset, Nucl. Phys. **B498**, 539 (1997).
[22] J. Gottker-Schnetmann, cond-mat/9909418.
[23] H. Koch and P. Wittwer, Commun. Math. Phys. **138**, 537 (1991).
[24] G. Felder, Commun. Math. Phys. **111**, 101 (1987).
[25] D. F. Litim, Phys. Lett. B **486**, 92 (2000).
[26] D. F. Litim, J. High Energy Phys. 07 (2005) 005.
[27] T. R. Morris, J. High Energy Phys. 07 (2005) 027.