

Scaling violation and gauge/string duality

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We explore the possibilities for scaling violation in gauge theories that have string duals. Like in perturbative QCD, short-distance behavior yields logarithms that violate the scaling.

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I. INTRODUCTION

As is well known, the QCD analysis of large-momentum-transfer processes results in power behavior. In particular, the dimensional scaling laws [1]

$$\mathcal{A} \sim p^{4-n} \left[1 + O\left(\frac{1}{p}\right) \right] \quad (1.1)$$

for the asymptotic behavior of fixed-angle scattering hold for a broad range of processes in the physics of hadrons. Here, p is a large momentum scale and n is a minimum total number of hadronic constituents (valence quarks). In the early days of string theory (dual resonance models), there was an outstanding problem to recover these laws.

Recently, the high-energy behavior of superstring amplitudes was studied in the case of warped spacetime geometries which are the products of AdS_5 with some compact five-manifolds [2–4]. One of the most important results is that of Polchinski and Strassler [2]. They proposed a scheme of evaluating high-energy fixed-angle string amplitudes and recovered the dimensional scaling laws. Thus, the long-standing problem on the way to a string theory description of hadronic processes was solved.

In fact, what was proposed in [2] is to integrate string tree-level amplitudes in flat four-dimensional space over an additional parameter which is nothing else but the fifth coordinate of AdS_5 .¹ Using this idea, one can readily extend the scheme to string loop amplitudes and, moreover, introduce a perturbation series by assuming that perturbation theory in question is a topological expansion. It was claimed in [4] that as long as the theory is finite, the amplitudes exactly fall as powers of momentum. Certainly, this is not the case in the real world where logarithms violate the scaling. According to QCD, these logarithms are due to short-distance processes. For such processes asymptotic freedom makes perturbative series to be valuable at least as far as we are satisfied with a few terms in the series [5].

As follows from above, it is highly desirable to recover the logarithms in a string theory dual. Moreover, the finite-

ness of superstring loops in flat space may be regarded as a consequence of their soft short-distance behavior seen, for example, in the exponential drop of fixed-angle scattering. Since the short-distance behavior is no longer soft for strings on warped spacetime geometries, it seems plausible to raise the issue of the logarithms.

There are two main obstacles on the way. First, full control of superstring theory on curved backgrounds like AdS_5 is beyond our grasp at present. Second, the string theory dual to QCD is unknown. The standard lore is that the background metric is given by

$$ds^2 = \frac{R^2}{r^2} (f(r)\eta_{\mu\nu}dx^\mu dx^\nu + dr^2) + d\Omega^2, \quad (1.2)$$

where $\eta_{\mu\nu}$ is a four-dimensional Minkowski metric. We take $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Since $f(r) \approx 1$ in the region of small r , the metric behaves asymptotically as $\text{AdS}_5 \times \text{X}$.

Fortunately, there is also a piece of good news. First, the logarithms of QCD have a short-distance origin. On the string theory side it corresponds to small r that is the region where the most important piece of the metric is known. Second, as follows from a discussion of [4] the nonzero modes of \mathbf{r} and Ω 's do not play a crucial role in the derivation of the scaling behavior.² What turns out to be crucial is a warped geometry and a zero mode of \mathbf{r} . So, we are bound to learn something if we succeed.

In this paper we address this issue within the simplified model of [2,4]. Our aim is to gain some understanding of the singular behavior and, as a consequence, scaling violation by doing simple examples.

Before proceeding to the detailed analysis, we conclude this section by setting the basic framework for computing matrix elements and amplitudes within our model. The part of the worldsheet action which is most appropriate for our purposes is simply³

$$S_0 = \frac{1}{4\pi\alpha_r} \int_{\Sigma} d^2z d^2\theta \eta_{\mu\nu} \bar{D}\mathbf{X}^\mu D\mathbf{X}^\nu, \quad \alpha_r = \alpha' \frac{r^2}{R^2}. \quad (1.3)$$

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¹One can also include an integration over coordinates of the compact five-manifold but it does not matter to the scaling laws.

² \mathbf{X} , \mathbf{r} , and Ω are taken to be sigma model fields on a string worldsheet. x , r , and Ω are their zero modes, respectively.

³Since we consider only zero modes of \mathbf{r} and Ω , the kinetic terms are due to \mathbf{X} 's. It is convenient to introduce a function α_r and use the superspace notations of [6].

Here \mathbf{X} is a two-dimensional superfield and Σ is a two-dimensional Riemann surface. We take a constant dilaton and, unfortunately, discard a RR background. We will give some remarks on it in conclusion.

As noted above, we assume that perturbation theory at hand is a topological expansion. Schematically, a g -loop amplitude with n external legs is of the form

$$A_n^{(g)} = \int_0^\infty [dr] \int_{\mathcal{M}_{g,n}} [d\tau] F_n^{(g)}, \quad (1.4)$$

where $F_n^{(g)}$ is an integrand of the "standard" g -loop string amplitude in Minkowski space with α' replaced by α_r , $[d\tau]$ is an integration measure for the moduli space $\mathcal{M}_{g,n}$ of the Riemann surface with n punctures, and $[dr]$ is an integration measure for the zero mode of the fifth dimension. Since we follow the line of QCD, we will look for divergences of these integrals at $r = 0$. However, before starting to discuss examples in detail we need to choose a method of regularization to deal with the infinite integrals. Both the integrals might be divergent, so they should be regulated in a consistent way. We will use the method of dimensional regularization that allows us to do so. This means that the background to be considered is given by $\text{AdS}_{d+1} \times X$.

II. VECTOR CURRENTS

The simplest example to discuss is a correlator of two vector currents. We compute it by following the lines of the first-quantized string theory. To this end, we build the corresponding vertex operator. Then, we define the correlator as an expectation value of the two vertex operators.

Following [7], we take an operator of naive dimension one half $D\mathbf{X}^\mu$ and dress it with an operator of naive dimension zero which is a function. In the simplest case, the vertex operator integrated over the worldsheet boundary takes the form

$$J^\mu(q) = \oint_C dz d\theta D\mathbf{X}^\mu e^{iq \cdot \mathbf{X}} \Psi(\mathbf{r}), \quad (2.1)$$

where \mathbf{X} is a restriction of the superfield on the boundary, $\mathbf{X}^\mu(z, \theta) = X^\mu(z) + \sqrt{\alpha'} \theta \psi^\mu(z)$, and $q \cdot \mathbf{X} \equiv q_\mu \mathbf{X}^\mu$. Ψ is a solution to the linearized Yang-Mills equation on AdS_{d+1}

$$[r^2 \partial_r^2 - (d-3)r \partial_r - q \cdot qr^2] \Psi(r) = 0. \quad (2.2)$$

For a solution to be nonzero at $r = 0$ for $d = 4$, we choose

$$\Psi(qr) = q^\nu r^\nu K_\nu(qr), \quad \nu = \frac{d}{2} - 1, \quad (2.3)$$

where $q = \sqrt{q \cdot q}$.

⁴Since we discard the nonzero modes of \mathbf{r} and Ω 's, we set the corresponding YM connections to be zero. However, in such an approximation there is gauge invariance $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$. It is fixed by $\partial \cdot A = 0$.

A couple of comments are in order:

(i) Since the current is conserved, it obeys $q \cdot J = 0$. In the approximation we are using this is obvious. Indeed,

$$q \cdot J = -i\Psi(qr) \oint_C dz d\theta D e^{iq \cdot \mathbf{X}} = 0$$

as a total derivative. This is the reason why we consider the worldsheet with boundaries or, equivalently, a spacefilling brane.

(ii) There is a subtle point. The use of this approximation is legitimate only for $q^2 = 0$. Certainly, this is not what we need. However, it seems that it is safe to go off shell, at least for rather small q^2 , as it follows from stringy calculations of the renormalization constants [8]. We will return to this issue below.

Now that we have the vertex operators for the currents, we can focus on the correlator. On general grounds, it takes the form

$$i \int \frac{d^d k}{(2\pi)^d} \langle J^\mu(q) J^\nu(k) \rangle = (q^\mu q^\nu - \eta^{\mu\nu} q^2) \text{Tr}(\lambda_1 \lambda_2) \Pi(q^2), \quad (2.4)$$

where $\Pi(q^2)$ is given by a perturbative series. Since we assume that the *flavor* symmetry group is $U(N_f)$, we include a Chan-Paton factor $\text{Tr}(\lambda_1 \lambda_2)$.

It seems natural to try a unit disk (upper half plane) as the worldsheet to leading order. We take a covariant measure for the zero modes $\sqrt{-g} d^{d+6} \xi$. As to the nonzero modes of \mathbf{X} 's, they are quantized in an ordinary way as follows from the worldsheet action (1.3)⁵

$$\langle \mathbf{X}^\mu(z_1, \theta_1) \mathbf{X}^\nu(z_2, \theta_2) \rangle = \alpha_r \eta^{\mu\nu} (G(z_{12}) - \theta_1 \theta_2 K(z_{12})), \quad (2.5)$$

$$z_{12} = z_1 - z_2.$$

It is clear that the integration over the zero modes of \mathbf{X} 's produces a delta-function, while that of Ω 's gives a volume of the compact space X . The integration over the nonzero modes of \mathbf{X} 's also does not require much work, so we find

$$\begin{aligned} \Pi^{(0)}(q^2) &= \frac{1}{2} \mathcal{N}^2 V_x \mu^{4-d} \int_0^1 dz [(\partial_z G(z))^2 - K^2(z)] \\ &\times \int_0^\infty dr \left(\frac{R}{r}\right)^{d+1} \Psi^2(qr) \alpha_r^2 e^{\alpha_r q^2 G(z)}, \end{aligned} \quad (2.6)$$

where \mathcal{N} is a normalization factor which will be fixed shortly. As usual in dimensional regularization, we have introduced an arbitrary scale μ to account for the right dimension. The translational invariance is fixed by setting the second vertex operator at the origin. As a consequence, a factor $\frac{1}{2}$ is accounted to the right hand side of (2.6).

⁵We use the point splitting method to regulate the propagators [9].

Using the integral representation

$$K_\nu(\eta) = \frac{1}{2} \eta^{-\nu} \int_0^\infty d\rho \frac{e^{-(1/2)(\eta^2 \rho + (1/\rho))}}{\rho^{\nu+1}} \quad (2.7)$$

and keeping only the singular term and a q -dependent piece of the finite term, we find at $d = 4 - 2\varepsilon$

$$\mu^{4-d} \int_0^\infty dr r^{3-d} \Psi^2(qr) e^{\alpha, q^2 G(z)} = \frac{1}{2\varepsilon} + \frac{1}{2} \ln \frac{\mu^2}{q^2} + \dots \quad (2.8)$$

We are now in a position to perform the integral over z . It is trivial as one can see by substituting the explicit expressions for the propagators

$$G(z_{12}) = -2 \ln(4 \sin \pi z_{12}), \quad K(z_{12}) = \frac{2\pi}{\sin \pi z_{12}}. \quad (2.9)$$

Finally, we use minimal subtraction to get rid of the pole term $\frac{1}{\varepsilon}$. Thus

$$\Pi^{(0)}(q^2) = -\frac{1}{4\pi^2} \ln \frac{q^2}{\mu^2}. \quad (2.10)$$

We have fixed the normalization by setting $\mathcal{N} = \frac{1}{2\pi^2 \alpha'} \times (RV_x)^{-1/2}$. Although this looks like the desired QCD result [10], one more step is needed.⁶ On the right hand side of (2.10) q is small, while we are looking for the asymptotic behavior for large q . What saves the day is the known fact from the renormalization of the vacuum polarization that the leading logarithmic behavior for $q \rightarrow \infty$ is related with the pole $\frac{1}{\varepsilon}$ exactly as in (2.8). Thus, Eq. (2.10) holds for large q .

At this point, a few remarks are in order.

(i) Rather than using dimensional regularization, we could use a regularization scheme with a short-distance cutoff. It is very easy to see a relation between ε and the cutoff. Evaluating the integral (2.8) near $r = 0$, we find $\int_a^l \frac{dx}{x} = \frac{1}{2} \ln \frac{l^2}{a^2}$ at $d = 4$. Thus, $\frac{1}{\varepsilon} = \ln \frac{l^2}{a^2}$.

(ii) In perturbative QCD the correlator is given by a series of Feynman diagrams. A dominant diagram looks like that in Fig. 1.

The evaluation of the diagram results in an expression of the form

$$\begin{aligned} \Pi(q^2) &\sim \mu^{4-d} \int_0^1 d\alpha \alpha(1-\alpha) \\ &\times \int_0^\infty dt t^{1-(d/2)} e^{-t(m^2 + \alpha(1-\alpha)q^2)}, \end{aligned} \quad (2.11)$$

where m is a fermion mass. t and α are the Schwinger and Feynman parameters, respectively.

⁶Of course, the number of colors is missing. We will return to this issue in Sec. IV.

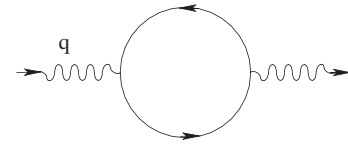


FIG. 1. A Feynman diagram for the vector current two-point correlator (vacuum polarization). Solid and wavy lines denote fermions and gauge bosons, respectively.

Let us now compare Eqs. (2.6) and (2.11). Although there is some similarity, these formulae are not the same. Nevertheless, we can reach an agreement if we set $m^2 = 0$ in (2.11) and restrict ourselves to the singular and q -dependent pieces. Then up to numerical factors both the integrals are reduced to

$$\Gamma\left(2 - \frac{d}{2}\right) \left(\frac{q^2}{\mu^2}\right)^{(d/2)-2}.$$

Equivalently, we could look for the singularities at the lower limits and reach an agreement by identifying the variables $r^2 = t$ and $z = \alpha$. We discard momentum independent constants because they are irrelevant for large q .

(iii) At present, it is not clear whether the representation we are using can be taken literally but it seems to us quite safe. Let us mention that in the literature a curious way of converting loop diagrams into trees was discussed a long time ago [11]. This line of thought has recently attracted some attention in the context of the AdS/CFT correspondence. In particular, it was argued that these trees are nothing else but tree diagrams in AdS space [12]. From this point of view it seems that there is no a paradox: we represent the one-loop diagram of QCD as the tree-level diagram of string theory in AdS background.⁷

(iv) It is worth mentioning that the asymptotic behavior (2.10) has been derived within an effective 5-dimensional Yang-Mills theory in AdS₅ space [14]. Unlike that, our approach admits a straightforward extension to higher orders of perturbation theory.

Let us now see how it works in the next-to-leading order. Again, our aim is to derive the leading asymptotics. To this end, we take a cylinder C_2 as the worldsheet. We describe C_2 as the region

$$0 \leq \Re z \leq 1, \quad z = z + i\tau \quad (2.12)$$

on the complex plane whose metric is $ds^2 = dzd\bar{z}$.

Since we are interested in the case of AdS _{$d+1$} , the problem is to extend the modular measure of open string to d flat dimensions. This was much studied to compute perturbative field theory amplitudes via string theory in the $\alpha' \rightarrow 0$ limit (see, e.g., [8]). In making our further analysis,

⁷From the viewpoint of string theory with a constant tension it looks like a resummation of the perturbation series. This was also observed by investigating the Regge limit [13].

we will adopt the proposal of [15] according to which the interpretation of each factor is transparent as it follows from the point-particle results. The computation of the correlator is very similar to what we have already done at the tree level. Thus, we obtain⁸

$$\begin{aligned} \Pi^{(1)}(q^2) &= \frac{i}{2} \mathcal{N}^2 V_x g^2 N_f \mu^{8-2d} \int_0^\infty dr \left(\frac{R}{r}\right)^{d+1} \\ &\times \Psi^2(qr) \alpha_r^{4-(d/2)} \int_0^\infty d\tau \tau^{-(d/2)} [\eta(i\tau)]^{2-d} \\ &\times \sum_{(\alpha,\beta) \neq (1,1)} (-1)^{\alpha+\beta} \left[\frac{\theta_{\alpha\beta}(0, i\tau)}{\eta(i\tau)} \right]^{(d-2)/2} \\ &\times \int_0^{i\tau} dz [(\partial_z G(z))^2 - K_{\alpha\beta}^2(z)] e^{\alpha, q^2 G(z)}, \quad (2.13) \end{aligned}$$

where η is the Dedekind eta function. We exclude the sector (1, 1) from the sum over the spin structures, so the fermions ψ 's have no zero modes. Again the translational invariance is fixed by setting the second vertex operator at the origin and taking into account a factor $\frac{i\tau}{2}$. Since the worldsheet is the cylinder, we need to insert a relative factor g^2 . As usual in string computations using dimensional regularization [8], g^2 is accompanied with $(\alpha' \mu^2)^{2-(d/2)}$ that in the problem at hand becomes $(\alpha_r \mu^2)^{2-(d/2)}$.

As discussed in introduction, we are interested in the region of small r or, equivalently, large momenta. As a consequence, α_r is small. Thus, the integral over τ may be studied along the lines of the old days $\alpha' \rightarrow 0$ limit which is now $\alpha_r \rightarrow 0$. It is well known that in this case only the neighborhood of $\tau = \infty$ contributes to divergences. At this point, it is worth saying that this is *not* the supergravity approximation. α'/R^2 may be large but what is really small is $r^2 \alpha'/R^2$.

To proceed further, we set both the vertex operators on a single boundary $\Re z = 0$. A little algebra shows that the sector (1, 0) dominates for large τ . At this point, it is useful to define new variables

$$\varphi = -iz/\tau, \quad t = 2\pi\alpha_r\tau. \quad (2.14)$$

Taking the leading asymptotics for the propagators

$$G(\varphi_{12}) = 2\pi\tau(\varphi_{12}^2 - \varphi_{12}), \quad K_{10}(\varphi_{12}) = 2\pi\tau,$$

and the theta functions, we get

$$\begin{aligned} \Pi^{(1)}(q^2) &= -\left(\frac{4}{\pi}\right)^{d/2} g^2 N_f \left(\frac{\pi R}{\mu^2}\right)^{d-4} \int_0^1 d\varphi (\varphi^2 - \varphi) \\ &\times \int_0^\infty \frac{dt}{t} t^{2-d/2} e^{tq^2(\varphi^2 - \varphi)} \int_0^\infty dr r^{3-d} \Psi^2(qr). \quad (2.15) \end{aligned}$$

⁸Note that it vanishes at $d = 10$. For a discussion, see, e.g., [16].

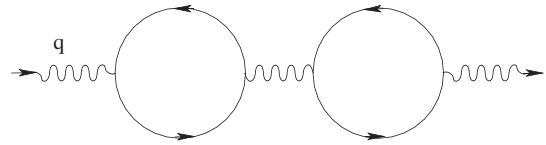


FIG. 2. A reducible diagram dominating in the next-to-leading order.

The result has the factorization property. It is not so surprising since we can anticipate from the QCD side that the leading asymptotics is given by a diagram shown in Fig. 2. Indeed, the first factor is what QCD analysis provides in the case of massless fermions (2.11), while the second is not exactly the left hand side of (2.8). The difference is due to the exponential $e^{\alpha, q^2 G(z)}$. Fortunately, it is irrelevant for the region of small r and, moreover, for the terms of interest as those on the right hand side of Eq. (2.8). Note also that the integral converges for large r because of the exponential falloff of $\Psi(qr)$.

It is straightforward to find the double pole term $\frac{1}{\epsilon^2}$ at $d = 4 - 2\epsilon$. The evaluation of the first factor results in $\frac{1}{6} \times (-\frac{1}{\epsilon} + \ln \frac{q^2}{\mu^2})$, while that of the second— $\frac{1}{2}(\frac{1}{\epsilon} - \ln \frac{q^2}{\mu^2})$. Again, we keep only the singular terms and q -dependent pieces of the finite terms. It is well-known that the renormalization program is formulated for one-particle irreducible diagrams that in the case of interest means that we have to remove the poles in each term. The use of minimal subtraction yields

$$\Pi^{(1)}(q^2) = \frac{2}{3} g^2 N_f \ln^2 \frac{q^2}{\mu^2}. \quad (2.16)$$

By the same argument we used in the discussion of $\Pi^{(0)}(q^2)$, the result holds for large q .

III. LIGHT MESONS

Perhaps more significant theories on AdS₅ can be applied with some success to the physics of mesons [14,17]. This motivates us to consider the vector mesons.

We proceed, as before, by first building the corresponding vertex operator. In fact, we have already done much work by doing so for the vector current. The discussion differs in two respects:

(i) We introduce a polarization vector $\xi_\mu(p)$. As a result, the vertex operator (2.1) takes the form

$$\mathcal{O}(p, \xi) = \oint dz d\theta \xi \cdot D \mathbf{X} e^{ip \cdot \mathbf{X}} \Psi(\mathbf{r}). \quad (3.1)$$

In the approximation we are using it is invariant under $\xi_\mu \rightarrow \xi_\mu + \phi(p)p_\mu$ as such a shift leads to a total derivative. We fix this gauge degree of freedom by setting $\xi \cdot p = 0$;

(ii) We choose a solution to Eq. (2.2) with $q \cdot q = -m^2$ such that it vanishes at $r = 0$ for $d = 4$. Thus

$$\Psi(mr) = m^\nu r^\nu J_\nu(mr), \quad \nu = \frac{d}{2} - 1, \quad (3.2)$$

with m being a meson mass. Here we should emphasize a subtle point. The use of the approximation is legitimate only for $m^2 = 0$ unless one uses a consistent prescription to go off shell, e.g., as in [8]. It seems safe at least for small deviations. We will therefore consider the light mesons.

Let us now turn to a matrix element of the two meson operators. As in the preceding example, we define it as an expectation value of the vertex operators. We have

$$\langle \mathcal{O}(p_1, \xi_1) \mathcal{O}(p_2, \xi_2) \rangle = \text{Tr}(\lambda_1 \lambda_2) (\xi_1 \cdot \xi_2 F + \dots) \times \delta^{(d+1)}(p_1 + p_2), \quad (3.3)$$

where F is given by a perturbation series. The three dots represent other terms that are higher order in m^2 and hence do not make a significant contribution as long as we consider the light mesons.

It seems plausible to try a unit disk as the worldsheet to leading order. Then $F^{(0)}$ can be read from Eqs. (2.4), (2.6), and (2.9). We find

$$F^{(0)} = 1. \quad (3.4)$$

It is important to remember that there is a great difference between the current and meson vertex operators. For the mesons the integral over r is convergent that allows us to include it into the corresponding normalization factor. Thus we have set

$$\mathcal{N} = \frac{1}{2\pi\alpha' m} \left(2^d \pi^{d+1} V_x \int_0^\infty dr \left(\frac{R}{r} \right)^{d-3} \Psi^2(mr) \right)^{-1/2}.$$

We can go further and consider the next-to-leading order. In doing so, we extend the formalism of the previous section to the case of interest. We take the cylinder C_2 as the worldsheet. Then the correction can be seen directly from Eq. (2.16). Keeping only the pole term $\frac{1}{\epsilon}$ results in

$$F^{(1)} = \frac{8}{3} g^2 N_f \frac{1}{\epsilon}. \quad (3.5)$$

This must be renormalized. One possible way to deal with the problem at hand is to renormalize the operator $\mathcal{O}(p)$. Again, we choose minimal subtraction. This gives

$$\mathcal{O}_0 = \sqrt{Z} \mathcal{O}, \quad Z = 1 - \frac{8}{3} g^2 N_f \frac{1}{\epsilon}, \quad (3.6)$$

with \mathcal{O}_0 being the bare operator.

To complete this part of the story, it is worth writing down the anomalous dimension of \mathcal{O} . Since it is simply related to the residue in (3.6), we find

$$\gamma_{\mathcal{O}} = \frac{8}{3} g^2 N_f. \quad (3.7)$$

Even though the approximation we used is not very good, we can still benefit from it. The point is that the vertex operators (3.1) are gauge invariant. As in gauge

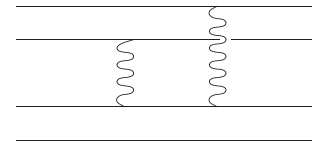


FIG. 3. A typical Born diagram for pion elastic scattering.

theory, this makes getting the corresponding β -function easier. As an example, consider elastic scattering of mesons. The amplitude is defined as an expectation value of the four vertex operators

$$\langle \mathcal{O}(p_1, \xi_1) \dots \mathcal{O}(p_4, \xi_4) \rangle = \mathcal{A}_4 \delta^{(4)}(p_1 + \dots + p_4). \quad (3.8)$$

To keep things as simple as possible, we specialize to a convenient structure in the kinematic factor.⁹ It is $\xi_1 \cdot \xi_3 \xi_2 \cdot \xi_4$. Thus, we have

$$\mathcal{A}_4 = \text{Tr}(\lambda_1 \lambda_2 \lambda_3 \lambda_4) (\xi_1 \cdot \xi_3 \xi_2 \cdot \xi_4 \mathcal{A} + \dots), \quad (3.9)$$

where \mathcal{A} is given by a perturbation series. For the mesons the integral over r is finite. It means that $\mathcal{A}^{(0)}$ is finite. On the other hand, $\mathcal{A}^{(1)}$ is divergent as the integral over τ diverges. Now, because of gauge invariance, it immediately comes to mind to relate the divergent part of $\mathcal{A}^{(1)}$ with $\mathcal{A}^{(0)}$ in the same manner as above. Keeping only the pole term $\frac{1}{\epsilon}$ gives

$$\mathcal{A}^{(1)} = \frac{8}{3} g^2 N_f \frac{1}{\epsilon} \mathcal{A}^{(0)}. \quad (3.10)$$

Assuming that $\mathcal{A}^{(0)}$ is proportional to g^2 and involving minimal subtraction, we are in a position to write down the β -function. It is given by

$$\beta(g) = \frac{8}{3} g^3 N_f. \quad (3.11)$$

In fact, we have been a little cavalier here. It is worth noting that the problem of interest differs from renormalization of the Yang-Mills theory in two respects:

(i) In general, $\mathcal{A}^{(0)}$ grows more rapidly than g^2 . For example, it may go like g^4 as it is clear from a Born diagram shown in Fig. 3. The bare coupling is now defined by $g_0 = \frac{g}{\sqrt{Z}}$. This leads to a correction factor $\frac{1}{2}$ in the expression [8].

(ii) There is some combinatorics that also impacts the numerical prefactor of the β -function. In the example we are considering the two diagrams shown in Fig. 4 are equivalent. This increases the prefactor via the corresponding increase of the residue. We will not dig deeper into these issues. Our reasoning is that they impact the absolute value of the prefactor rather than its sign. So, the bottom line is that the model is not asymptotically free.

⁹See, e.g., [18].



FIG. 4. Two possibilities for virtual gluons.

A final remark: having derived the anomalous dimension and β -function, it seems to be time to involve the Callan-Simanik equation to resum logarithms. We will not do so because there is no asymptotic freedom.

IV. CONCLUDING COMMENTS

There is a large number of open problems associated with the circle ideas exploited in this paper. In this section we list a few.

Here we used the simplified model without any RR background as we still lack the description of such a background within the NSR formalism. In particular, the solution of string theory on AdS_5 or on its nonconformal deformation that could help us is unknown. So, it is not a big surprise that the number of colors N_c is missing in all the expressions we obtained. From this point of view we are in a situation very similar to that in the Skyrme model when the model describes color singlets. To include N_c , one has to add an extra term into the action. For example, this was done by adding the Wess-Zumino term into the original Skyrme action [19]. The common wisdom is that the worldsheet theory with a warped background metric is conformal by virtue of a proper RR background. It seems that this is not the whole story. Our hope is that it might also help to get a right correction to the β -function (see Eq. (3.11)) and, as a consequence, asymptotic freedom.

A related problem is to understand the nonzero modes of \mathbf{r} and even the transverse fields Ω . It is clear that this would

allow us to define vertex operators for arbitrary large q^2 and avoid the problem with off shell continuation. On the other hand, there are some claims that the nonzero modes of \mathbf{r} and even those of \mathbf{X} 's might be effectively irrelevant in the worldsheet path integral [12]. What really happens still remains to be seen.

In this paper we pursued the line of thought borrowed from QCD. Another possible way is to deform the background metric for *small* r or even somehow cut AdS space to get the desired scaling violation.¹⁰ It would be interesting to see whether the QCD line of thought is equivalent to that or not.

Finally, one of the possible interpretations for the models of [20] is that they may be thought of as string theory in spacetime whose fifth coordinate is latticized. If one tries to do some computations along the lines of Sec. II, this will require the use of lattice regularization for consistency. It suggests that for the models to be well defined, all the coordinates X must be latticized after the Wick rotation to Euclidean spacetime. We believe that this issue is worthy of future study.

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¹⁰Note that for the AdS geometry truncated at some large value of r a rough estimate of [4] gives an exponential correction which violates the scaling already to leading order.

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