PHYSICAL REVIEW D **73,** 046002 (2006)

Fundamental string solutions in open string field theories

Yoji Michishita*

Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02142 USA

(Received 22 November 2005; published 9 February 2006)

In Witten's open cubic bosonic string field theory and Berkovits' superstring field theory we investigate solutions of the equations of motion with appropriate source terms, which correspond to Callan-Maldacena solution in Born-Infeld theory representing fundamental strings ending on the D-branes. The solutions are given in order by order manner, and we show some full order properties in the sense of α' expansion. In superstring case we show that the solution is $1/2$ BPS in full order.

DOI: 10.1103/PhysRevD.73.046002 PACS numbers: 11.25.Sq

I. INTRODUCTION AND SUMMARY

In Witten's cubic open string field theory [1] and its extension to superstring such as Berkovits' superstring field theory [2], it is very difficult to construct solutions with coordinate dependence. This is because string field theory is nonlocal and contains infinitely many derivatives. It prevents us from investigating behavior of higher modes and full order properties. (We consider only classical theory and do not consider string loop correction. Therefore throughout this paper "full order" means exactness in the sense of α'-expansion.)

In this paper we investigate an example of such solutions of the equations of motion with appropriate source terms, of which we can derive some full order properties: string field theory counterpart of Callan-Maldacena solution [3]. (For a related topic see [4].) This solution represents configuration of fundamental strings emanating from the D-brane. Since it is also a solution of free U(1) gauge theory, we expect that we can construct the string field theory solutions in order by order manner, starting from the linearized equation and introducing higher order source terms. In Sec. II we construct the solution in Witten's string field theory and see that it has the following properties:

(i) The coefficient of the massless component A_μ is equal to the gauge field $\tilde{A}_\mu$ in the effective action with full order correction in α'.

(ii) The solution has no tachyon component, and the massless component has no higher order correction.

(iii) Although we have no proof, we give a convincing argument that massive modes have no singularity unlike the massless component.

(iv) Energy-momentum tensor given in [5] has no contribution from massive modes, and is equal to that of free U(1) gauge theory.

In Sec. IV we construct the solution in Berkovits' superstring field theory and see that it has almost the same properties as the bosonic one. Moreover we show that it is $1/2$ BPS in full order.

*Electronic address: michishi@lns.mit.edu

II. SOLUTION IN BOSONIC STRING FIELD THEORY

Let us consider one single Dp-brane in the flat space. The bosonic quadratic part of its effective action, in both bosonic and superstring theory, is given by free U(1) gauge theory. Spacetime-filling D-brane action has only gauge field $\tilde{A}_\mu$, and lower dimensional D-brane actions are obtained from it by dropping dependence on coordinates perpendicular to the D-branes. We separate spacetime coordinates x^μ into $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$, x^i and x^I, where x^0 and x^i are directions along the Dp-brane, and x^1 and x^I are directions perpendicular to the Dp-brane. Then $\tilde{A}_1$ and $\tilde{A}_I$ are scalar fields corresponding to x^1 and x^I respectively.

Suppose $\tilde{A}_- = 0, \tilde{A}_i = 0, \tilde{A}_I = 0,$ and $\tilde{A}_+ = \tilde{A}_+(x^i)$, then the linearized equation of motion is

$$\sum_i \partial_i \partial_i \tilde{A}_+ = 0. \tag{1}$$

This is Laplace equation, and "point charge" configurations give solutions:

$$\tilde{A}_+ = \sum_n \frac{c_n}{[\sum_i (x^i - x_n^i)^2]^{(p-2)/2}}, \tag{2}$$

where c_n and x_n^i are constants. We assumed $p \geq 3$. For $p = 1$ solutions are sums of segments of linear functions and for $p = 2$ sums of $\log \sum_i (x^i - x_n^i)^2$. In these cases momentum expressions (i.e. Fourier transformations) of these solutions require introducing infrared regulators. Since in string field theory we use momentum expression, we do not consider $p = 1$ and 2 in this paper.

For this solution the right-hand side of (1) is not actually zero, but a sum of delta function sources. In [3] it has been shown that this configuration represents fundamental strings stretching along x^1 direction and ending on the D-brane at $x^i = x_n^i$, and extension of this solution to Born-Infeld theory is again given by (2), without corrections. In this interpretation the presence of the delta function sources is not a problem, because the points $x^i = x_n^i$ are not on the worldvolume of the D-brane (or are regarded to be infinitely far away).

Furthermore in superstring theory this solution is $1/2$ supersymmetric, both in linearized U(1) gauge theory [3] and Born-Infeld theory [6].

In fact this solution is an α'-exact solution as shown in [7] by computing beta function of the worldsheet sigma model.

Since leading order terms of string field theory action give free U(1) gauge theory, we expect that starting from the solution of (1) we can construct corresponding solutions of string field equation "order by order". In this section we investigate the solution in Witten's cubic bosonic string field theory.

In the bosonic string field theory the equation of motion is

$$Q\Phi + \Phi^2 = 0. \tag{3}$$

Of course the right-hand side is not actually zero. To get a right solution we have to put a source term which we will call Δ_n.

The solution is constructed by expanding Φ in some parameter g:

$$\Phi = g\Phi_0 + g^2\Phi_1 + g^3\Phi_2 + \ldots . \tag{4}$$

The equation of motion is decomposed into contributions from each order in g:

$$\Delta_0 = Q\Phi_0, \tag{5}$$

$$\Delta_1 = Q\Phi_1 + \Phi_0^2, \tag{6}$$

$$\Delta_2 = Q\Phi_2 + \Phi_0\Phi_1 + \Phi_1\Phi_0, \tag{7}$$

$$\vdots$$

$$\Delta_n = Q\Phi_n + \sum_{m=0}^{n-1}\Phi_m\Phi_{n-m-1}, \tag{8}$$

$$\vdots$$

Massless part of the lowest order Eq. (5) is equivalent to that of free U(1) gauge theory with source terms. So we take the following Φ_0 which corresponds to (2):

$$\Phi_0 = \int \frac{d^pk}{(2\pi)^p} A_+(k_i)c\partial X^+ e^{ik_iX^i}, \tag{9}$$

where coordinate expression of $A_+(k_i)$ which is given by $A_+(x^i) = \int [d^pk/(2\pi)^p]A_+(k_i)e^{ik_ix^i}$ satisfies Laplace equation with delta function source terms. Then the string field source term Δ_0 is

$$\Delta_0 = -\alpha' \int \frac{d^pk}{(2\pi)^p} k^2 A_+(k_i)c\partial c\partial X^+ e^{ik_iX^i}. \tag{10}$$

Φ_0 satisfies Siegel gauge condition: $b_0\Phi_0 = 0$. We require that at each order this condition is satisfied: $b_0\Phi_n = 0$. In addition we require that Δ_n with $n \geq 1$ also satisfy this condition: $b_0\Delta_n = 0$. Δ_0 does not satisfy it. This means that Δ_0 is the only source for physical components, and Δ_n with $n \geq 1$ are for unphysical components. This is desirable because, when we eliminate all unphysical massive modes by a gauge fixing condition and solve all equations for physical massive modes, we have to obtain equation of motion for massless modes with a simple source term to have a solution corresponding to Callan-Maldacena solution.

By acting b_0 to the equations of motion and noticing that $b_0Q\Phi_n = L_0\Phi_n$, we obtain

$$\Phi_1 = -\frac{b_0}{L_0}(\Phi_0^2), \tag{11}$$

$$\Phi_2 = -\frac{b_0}{L_0}(\Phi_0\Phi_1 + \Phi_1\Phi_0) = \frac{b_0}{L_0}\left(\Phi_0\frac{b_0}{L_0}(\Phi_0^2) + \frac{b_0}{L_0}(\Phi_0^2)\Phi_0\right), \tag{12}$$

$$\vdots$$

$$\Phi_n = -\frac{b_0}{L_0}\sum_{m=0}^{n-1}\Phi_m\Phi_{n-m-1}, \tag{13}$$

$$\vdots$$

In this manner Φ_n can be expressed by b_0/L_0 and $(n+1)$ copies of Φ_0. Since b_0 projects out some components of string fields, we have to check if there is more information extracted from the equations of motion by plugging the above solution back into them:

$$\begin{aligned}\Delta_n &= Q\Phi_n + \sum_{m=0}^{n-1}\Phi_m\Phi_{n-m-1} \\ &= -Q\frac{b_0}{L_0}\sum_{m=0}^{n-1}\Phi_m\Phi_{n-m-1} + \sum_{m=0}^{n-1}\Phi_m\Phi_{n-m-1} \\ &= \frac{b_0}{L_0}Q\sum_{m=0}^{n-1}\Phi_m\Phi_{n-m-1}.\end{aligned} \tag{14}$$

This should be regarded as determining Δ_n by lower order solutions. Notice that if lower order Φ_m in the right-hand side of the above equation satisfy equations of motion without lower order source terms, then Δ_n vanishes:

$$\begin{aligned}\Delta_n &= \frac{b_0}{L_0}\sum_{m=0}^{n-1}((Q\Phi_m)\Phi_{n-m-1} - \Phi_m(Q\Phi_{n-m-1})) \\ &= -\frac{b_0}{L_0}\left(\sum_{m=1}^{n-1}\sum_{l=0}^{m-1}\Phi_l\Phi_{m-l-1}\Phi_{n-m-1} - \sum_{m=0}^{n-2}\sum_{l=0}^{n-m-2}\Phi_m\Phi_l\Phi_{n-m-l-2}\right) \\ &= -\frac{b_0}{L_0}\left(\sum_{m=0}^{n-1}\sum_{l=0}^{m-1}\Phi_l\Phi_{m-l-1}\Phi_{n-m-1} - \sum_{l=0}^{n-1}\sum_{m=l+1}^{n-1}\Phi_l\Phi_{m-l-1}\Phi_{n-m-1}\right) \\ &= 0.\end{aligned} \tag{15}$$

For our solution source terms should not be zero, and we obtain

$$\Delta_n = \frac{b_0}{L_0} \sum_{m=0}^{n-1} [\Delta_m, \Phi_{n-m-1}], \tag{16}$$

which means that higher order source terms are induced by lower order ones.

Obviously Φ_n has no dependence on momenta along $x^{\pm}$. In addition, Φ_n has the following property: Let ω be any of the vertex operators (or Fock space states) which Φ_n consists of. Then X^I part of ω is a Virasoro descendant of the unit operator i.e. a state constructed by acting L'_{-n} $(n \geq 2)$ on $|0\rangle$, where L'_{-n} are Virasoro operators of X^I part. Moreover, $n_+(\omega) - n_-(\omega) = n + 1$, where n_+ is the number of $\partial^m X^+$ (or α^+_{-m}) in ω, and n_- is the number of $\partial^m X^-$ (or α^-_{-m}) in ω.

In summary, matter part of ω is in the following form:

$$\begin{aligned} &\prod_{l=1}^{n_+} \alpha^+_{-p_l} \prod_{l=1}^{n_-} \alpha^-_{-q_l} \prod_l L'_{-t_l} \prod_l \alpha^{i_l}_{-u_l} |k^i\rangle \\ &(n_+ - n_- = n + 1, p_l, q_l, u_l \geq 1, t_l \geq 2). \end{aligned} \tag{17}$$

The structure of X^I part represents symmetry in X^I directions.

This can be proven by induction as follows. For $n = 0$ this is obvious. Suppose $n > 0$. We take orthonormal basis of the Fock space $\{|\phi_r\rangle\}$ and its conjugate $\{\langle\phi^c_r|\}$. These satisfy $\langle\phi^c_s|\phi_r\rangle = \delta_{rs}$. Corresponding vertex operators are denoted by ϕ_r and ϕ^c_r respectively. Coefficient of $|\phi_r\rangle$ in the expansion of Φ_n by $\{|\phi_r\rangle\}$ is given by $\langle\phi^c_r|\Phi_n\rangle$:

$$\begin{aligned} \langle\phi^c_r|\Phi_n\rangle &= \left\langle \phi^c_r \,\middle|\, -\frac{b_0}{L_0} \sum_{m=0}^{n-1} \Phi_m \Phi_{n-m-1} \right\rangle \\ &= -\left\langle \frac{b_0}{L_0} \phi^c_r \,\middle|\, \sum_{m=0}^{n-1} \Phi_m \Phi_{n-m-1} \right\rangle. \end{aligned} \tag{18}$$

First we concentrate on $X^{\pm}$ sector. Since b_0 affects only on ghost part and L_0 gives a numerical factor for each level, we can neglect $\frac{b_0}{L_0}$. By the assumption of the induction, $n_+(\Phi_m) - n_-(\Phi_m)$ is $m + 1$ and $n_+(\Phi_{n-m-1}) - n_-(\Phi_{n-m-1})$ is $n - m$. There are two processes which change the number of X^+ and X^-: contraction and conformal transformations in the star product. Since X^+ has nonzero contraction only with X^- and vice versa, Both processes preserve the difference of these numbers, and the total number of X^+ and X^- in the correlator should be equal for nonzero contribution. Therefore $n_+(\phi^c_r) - n_-(\phi^c_r)$ should be $-n - 1$. This means that $n_+(\phi_r) - n_-(\phi_r)$ is $n + 1$.

Next we consider X^I sector. By the assumption of the induction, both Φ_m and Φ_{n-m-1} are Virasoro descendants of the unit operator. If ϕ^c_r is a descendant of a nontrivial primary field λ, by using the well-known procedure relating a correlator with worldsheet energy-momentum tensors to ones without it, the correlator reduces to one point function of λ, which vanishes because of its nonzero conformal dimension. This means that ϕ_r consists of Virasoro descendants of the unit operator.

Ghost part of Φ_n can also be restricted further as is explained in [8].

An immediate consequence of the above fact on the number of $X^{\pm}$ is that each coefficient of Fock space state in the solution Φ receives contribution from only one Φ_n. (Here we regard states consisting of the same oscillators with different spacetime indices as different states.) In particular, the coefficient of the massless vertex operator $c\partial X^{\mu} e^{ik_{\mu}X^{\mu}}$, which is denoted by A_{μ}, is never corrected by higher order contribution, and the coefficient of the lowest state, which represents tachyon, is zero in full order. In addition, we see that the inverses of L_0 in the expression of Φ_n with $n \geq 1$ do not cause any problem, because only massless and tachyon components, which is absent in Φ_n with $n \geq 1$, are problematic.

We can easily see that Δ_n also have the same property as Φ_n by the same argument: Matter part of Δ_n are in the form of (17), there is no more source for massless components than Δ_0, and inverses of L_0 are well-defined.

In general, A_{μ} is different from the gauge field $\tilde{A}_{\mu}$ in the effective action except at the leading order, because its gauge transformation takes different form from the standard one. They are connected by some field redefinition. In [9] it has been explained how to compute this field redefinition order by order. However, for our solution A_{μ} is equal to $\tilde{A}_{\mu}$. This is because higher order terms of the field redefinition contain two or more A_{μ} and possibly derivatives, and since $\tilde{A}_{\mu}$ has only one spacetime index, superfluous indices should be contracted with each other. Therefore higher order terms contain $A_{\mu}A^{\mu}$ or $\partial_{\mu}A^{\mu}$, which vanish for our solution. Hence our A_{μ} is also an exact solution of the effective action. This gives another proof of the fact shown in [7].

III. BEHAVIOR OF MASSIVE COMPONENTS

In this section we investigate how coefficients of massive states in our solution in the previous section behave by computing those of first and second massive states coming from Φ_1 and Φ_2, and see more full order properties suggested by it.

First we compute first massive components. It can be easily seen that $V_1(k) = c\partial X^+ \partial X^+ e^{ik_i X^i}$ is the only nonzero component and it is from Φ_1. Since its conjugate operator is $U_1(k) = -\frac{2}{(\alpha')^2} c\partial c \partial X_+ \partial X_+ e^{-ik_i X^i}$, the component is given by the following:

$$\int \frac{d^p k}{(2\pi)^p} V_1(k)\langle U_1(k)|\Phi_1\rangle = \int \frac{d^p k_{(2)}}{(2\pi)^p}\frac{d^p k_{(3)}}{(2\pi)^p} V_1(k_{(2)}+k_{(3)}) \times\left(\frac{4}{3\sqrt{3}}\right)^{2\alpha'(k_{(2)}^2+k_{(3)}^2+k_{(2)}\cdot k_{(3)})+1} \times\frac{1}{\alpha'(k_{(2)}+k_{(3)})^2+1} \times A_+(k_{(2)})A_+(k_{(3)}). \qquad (19)$$

We see that the factor $(4/3\sqrt{3})^{2\alpha'(k_{(2)}^2+k_{(3)}^2+k_{(2)}\cdot k_{(3)})}$ makes the above integral convergent, since $4/3\sqrt{3} < 1$ and $k_{(2)}^2 + k_{(3)}^2 + k_{(2)}\cdot k_{(3)} = (k_{(2)}+\frac{1}{2}k_{(3)})^2 + \frac{3}{4}k_{(3)}^2$ becomes large as $k_{(2)}, k_{(3)} \to \infty$.

In the case of one-center solution $A_+ \propto 1/k^2$, we plot $F_p(r)$, coordinate expression of the above function, defined as follows:

$$F_p(r) = (\alpha')^{p-2}\int \frac{d^p k_{(2)}}{(2\pi)^p}\frac{d^p k_{(3)}}{(2\pi)^p} e^{i(k_{(2)}+k_{(3)})_i x^i} \times\left(\frac{4}{3\sqrt{3}}\right)^{2\alpha'(k_{(2)}^2+k_{(3)}^2+k_{(2)}\cdot k_{(3)})} \times\frac{1}{\alpha'(k_{(2)}+k_{(3)})^2+1}\frac{1}{k_{(2)}^2}\frac{1}{k_{(3)}^2}, \qquad (20)$$

where $r = \sqrt{(x^i)^2/\alpha'}$. Figure 1 is the profile of $F_3(r)$. Note that F_p is real, and depends only on r because of the invariance under rotation of x^i.

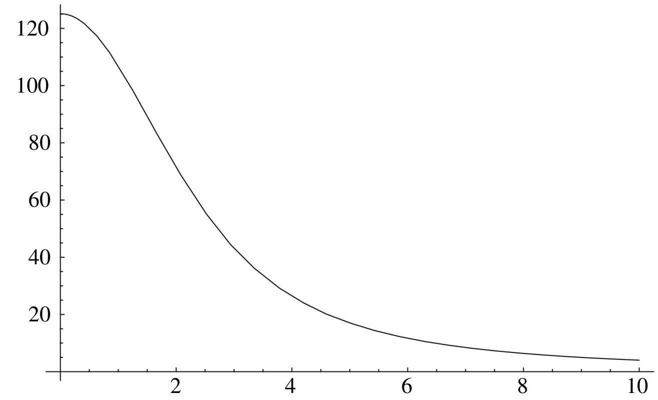

FIG. 1. $F_3(r)$

$F_p(r)$ is well-defined everywhere, in particular, at $r = 0$ unlike Φ_0. One may wonder why Φ_1, given by the product of Φ_0 which is singular at $r = 0$, is smooth. This is because of nonlocality of the string field product represented by the factor $(4/3\sqrt{3})^{2\alpha'(k_{(2)}^2+k_{(3)}^2+k_{(2)}\cdot k_{(3)})}$. The nonlocality smears off the singularity. We will see this also happens in the calculation of higher contribution.

Next we compute a coefficient of a second massive state $V_2(k) = c\partial X^+\partial X^+\partial X^+ e^{ik_iX^i}$. This is from Φ_2 and other nonzero second massive states are in Φ_1, which can be computed similarly to $V_1(k)$. The operator conjugate to $V_2(k)$ is $U_2(k) = \frac{4}{3(\alpha')^3}c\partial c\partial X_+\partial X_+\partial X_+ e^{-ik_iX^i}$. Therefore the component is

$$\int \frac{d^p k}{(2\pi)^p} V_2(k)\left\langle U_2(k)\,\middle|\,\Phi_2\right\rangle = \int \frac{d^p k}{(2\pi)^p} V_2(k)\left\langle -\frac{b_0}{L_0}U_2(k)\,\middle|\,\Phi_1\Phi_0+\Phi_0\Phi_1\right\rangle$$
$$= \int \frac{d^p k}{(2\pi)^p} V_2(k)\left(\frac{4}{3(\alpha')^3}\right)\frac{1}{\alpha' k^2+1}\langle U_2'(k)*\Phi_0 + \Phi_0 * U_2'(k)|\Phi_1\rangle$$
$$= \int \frac{d^p k}{(2\pi)^p} V_2(k)\left(-\frac{4}{3(\alpha')^3}\right)\frac{1}{\alpha' k^2+1}\left\langle U_2'(k)*\Phi_0 + \Phi_0 * U_2'(k)\,\middle|\,\frac{b_0}{L_0}\,\middle|\,\Phi_0^2\right\rangle, \qquad (21)$$

where $U_2'(k) = c\partial X_+\partial X_+\partial X_+ e^{-ik_iX^i}$. This can be computed in the same way as 4-point amplitudes by noticing that b_0/L_0 is the string field propagator. Coefficients of higher Φ_n are also given by $(n+2)$-point off-shell amplitudes. This fact was pointed out in [10] in a different context.

Technique for computation of off-shell 4-point amplitudes was developed in [11,12]. By applying it, we obtain

$$\left\langle U_2'(k)*\Phi_0 + \Phi_0 * U_2'(k)\,\middle|\,\frac{b_0}{L_0}\,\middle|\,\Phi_0^2\right\rangle = \int \frac{d^p k_{(2)}}{(2\pi)^p}\frac{d^p k_{(3)}}{(2\pi)^p}\frac{d^p k_{(4)}}{(2\pi)^p}\left[-\frac{3}{8}(\alpha')^3(2\pi)^p\delta^p(k_{(2)}+k_{(3)}+k_{(4)}-k)A_+(k_{(2)})\right.$$
$$\times A_+(k_{(3)})A_+(k_{(4)})\int_0^{\sqrt{2}-1} d\alpha \frac{8\alpha(1-\alpha^2)}{(1+\alpha^2)^3}(\kappa(\alpha))^2\left(\frac{1}{2}\frac{1+\alpha^2}{1-\alpha^2}\kappa(\alpha)\right)^{\alpha'(k^2+k_{(2)}^2+k_{(3)}^2+k_{(4)}^2)}$$
$$\left.\times\left(\frac{2\alpha}{1+\alpha^2}\right)^{2\alpha'(k_{(3)}+k_{(4)})^2}\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2\alpha'(k_{(2)}+k_{(3)})^2}\right], \qquad (22)$$

where $\kappa(\alpha)$ is defined in (A8) in the appendix.

Let us compare the above integral with on-shell Veneziano amplitude. In the computation of Veneziano amplitude we encounter the following integral:

$$\int_0^1 dy y^{\alpha'(k_{(2)}+k_{(3)})^2-2}(1-y)^{\alpha'(k_{(3)}+k_{(4)})^2-2}. \quad (23)$$

This expression is convergent around $y = 1$ if $\alpha'(k_{(3)} + k_{(4)})^2 > 1$. Divergence at $\alpha'(k_{(3)} + k_{(4)})^2 = 1$ signifies that tachyon mode propagates as an intermediate state. The integral is not well-defined beyond this point, and what we usually do is to replace the integral expression by Beta function which is well-defined except at the poles.

Going back to the expression (22), $1 - y$ corresponds to $(2\alpha/(1 + \alpha^2))^2$, and we can see (22) does not have the same problem as (23). This is because $\Phi_1 = -(b_0/L_0)\Phi_0^2$ does not have tachyon and massless components as we have shown earlier and these do not propagate as intermediate states. Therefore we can use the expression of moduli integral in (22) for any values of the momenta.

Then another question is the convergence of the integral of the momenta. Note that $0 \leq (2\alpha/(1 + \alpha^2)) < 1$ and $0 < (1 - \alpha^2)/(1 + \alpha^2) \leq 1$ in the range of α. The equality applies only at the edge of the range. Furthermore in the appendix we show that $0 < (1/2(1 + \alpha^2)/(1 - \alpha^2)\kappa(\alpha)) \leq 1$. Thus we see that these three factors makes the integral convergent.

The coordinate expression $G_p(r)$ of the above coefficient for one-center case, defined as follows, has the profile shown in Fig. 2 for $p = 3$:

$$\begin{aligned} G_p(r) = (\alpha')^{3p/2-3} \int \frac{d^p k_{(2)}}{(2\pi)^p} \frac{d^p k_{(3)}}{(2\pi)^p} \frac{d^p k_{(4)}}{(2\pi)^p} \Bigg[& e^{i(k_{(2)}+k_{(3)}+k_{(4)})_i x^i} \\ & \times \frac{1}{\alpha'(k_{(2)} + k_{(3)} + k_{(4)})^2 + 2} \frac{1}{k_{(2)}^2} \frac{1}{k_{(3)}^2} \frac{1}{k_{(4)}^2} \\ & \times \int_0^{\sqrt{2}-1} d\alpha \frac{8\alpha(1-\alpha^2)}{(1+\alpha^2)^3} (\kappa(\alpha))^2 \\ & \times \left(\frac{1}{2}\frac{1+\alpha^2}{1-\alpha^2}\kappa(\alpha)\right)^{\alpha'((k_{(2)}+k_{(3)}+k_{(4)})^2 + k_{(2)}^2 + k_{(3)}^2 + k_{(4)}^2)} \\ & \times \left(\frac{2\alpha}{1+\alpha^2}\right)^{2\alpha'(k_{(3)}+k_{(4)})^2} \left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{2\alpha'(k_{(2)}+k_{(3)})^2} \Bigg]. \quad (24) \end{aligned}$$

Higher Φ_n have properties similar to Φ_1 and Φ_2 i.e. they are related to $(n + 2)$-point off-shell amplitudes and well-defined, and have smooth profiles. The relation to off-shell amplitudes implies that integrals of moduli parameters are well-defined at any values of momenta because Φ_n do not have tachyon and massless modes, and integrals of momenta are convergent even at $r = 0$ because of the nonlocality. Convergent factors come from the following correlator:

$$\begin{aligned} & \langle f_1 \circ (e^{ik_{(1)}\cdot X})(z_1) f_2 \circ (e^{ik_{(2)}\cdot X})(z_2) \ldots f_n \circ (e^{ik_{(n)}\cdot X})(z_n)\rangle \\ & = \prod_i (f_i'(z_i))^{\alpha' k_{(i)}^2} \prod_{i\neq j} |f_i(z_i) - f_j(z_j)|^{2\alpha' k_{(i)}\cdot k_{(j)}} (2\pi)^p \delta^p\Big(\sum_i k_{(i)}\Big) \\ & \equiv \exp\Big(-\sum_{i,j} a_{ij} k_{(i)} \cdot k_{(j)}\Big)(2\pi)^p \delta^p\Big(\sum_i k_{(i)}\Big), \quad (25) \end{aligned}$$

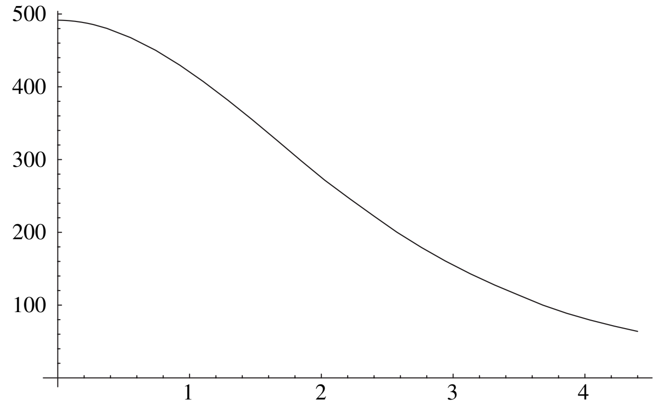

FIG. 2. $G_3(r)$

where $f_i(z)$ are conformal transformations appearing in the computation of off-shell amplitudes. Although we have no rigorous proof, we expect that $\sum_{i,j} a_{ij} k_{(i)} \cdot k_{(j)}$ is positive for spatial k_i and works as a convergent factor for integrals of momenta, because any off-shell string amplitude contains this factor and it is highly implausible that this is divergent.

The same analysis can be applied to Δ_n: Although Δ_0 is a sum of delta functions, Δ_n with $n \geq 1$ are not localized to points and have smooth profiles. This is not surprising, because the equation of motion is covariant under gauge transformation, and therefore the source term should also be covariant. So even if the source term is localized to points in some gauge, its gauge transformation is not localized due to the nonlocality of the string star product.

In [3], in the free U(1) gauge theory it was shown that the coefficient in the gauge field $\tilde{A}_\mu$ is determined by charge quantization and the energy around the singularity $r = 0$ is equal to the length times string tension.

In our case the same charge quantization is also applied to A_μ. So we expect that massive modes do not contribute to the energy. The fact that massive modes are smooth at $r = 0$ also suggests this. Therefore let us see energy-momentum tensor for our solution. For definiteness we use the energy-momentum tensor $T_{\mu\nu}$ given in [5] as Noether current of translation symmetry. Although this tensor itself is not gauge invariant, total energy and momentum computed from it are expected to be gauge invariant.[1] This tensor consists of coefficient fields in the string field and derivatives. Since this has only two spacetime indices μ and ν, superfluous indices should be contracted with each other. We have shown that nonzero component fields have one or more $+$ indices. If they are contracted with $-$ indices in the derivatives, we have vanishing contribution because our solution has no $x^\pm$ dependence. If they are contracted with $-$ indices of other fields, then the $+$ indices and $-$ indices are paired, and the excess of $+$

[1] I would like to thank A. Sen for clarifying this point.

indices should be μ and ν. Therefore difference of the number of $+$ and $-$ index in any nonzero term in $T_{\mu\nu}$ is equal to or less than two. The only term which satisfies this requirement is $\partial^i A_+ \partial_i A_+$, and T_{++} is the only nonvanishing component of $T_{\mu\nu}$.

Thus we see that not only the massless modes do not contribute to $T_{\mu\nu}$, but $T_{\mu\nu}$ is exactly equal to the energy-momentum tensor of free U(1) gauge theory. Note that the above argument can be applied to any definition of energy-momentum tensor consisting of two or more coefficient fields in the string field and derivatives.

One may wonder if the expansion (4) is meaningful. By the charge quantization g is proportional to the string coupling g_s. In addition, massive modes have no divergent point, and each coefficient in Φ receives contribution from only one Φ_n. We have seen that our solution shares some full order properties with that of [3]. These facts strongly suggest that the expansion (4) is meaningful at least in small g_s region.

IV. SOLUTION IN SUPERSTRING FIELD THEORY

In this section we investigate supersymmetric version of the solution in the previous sections. We use Berkovits' superstring field theory. The equation of motion is

$$\begin{aligned} 0 &= \eta_0(e^{-\Phi} Q e^{\Phi}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \eta_0 [\underbrace{\Phi, [\Phi, [\ldots, [\Phi,}_{n} Q\Phi]] \ldots]. \end{aligned} \tag{26}$$

As in the previous section, we expand Φ around the solution of the linearized equation Φ_0:

$$\Phi = g\Phi_0 + g^2\Phi_1 + g^3\Phi_2 + \ldots, \tag{27}$$

$$\Phi_0 = \int \frac{d^p k}{(2\pi)^p} A_+(k_i) \xi c \psi^+ e^{-\phi} e^{ik_i X^i}. \tag{28}$$

Φ_n satisfy the following equations:

$$\Delta_0 = \eta_0 Q \Phi_0, \tag{29}$$

$$\Delta_1 = \eta_0(Q\Phi_1 - \tfrac{1}{2}[\Phi_0, Q\Phi_0]), \tag{30}$$

$$\begin{aligned} \Delta_2 = \eta_0(&Q\Phi_2 + \tfrac{1}{6}[\Phi_0, [\Phi_0, Q\Phi_0]] - \tfrac{1}{2}[\Phi_0, Q\Phi_1] \\ &- \tfrac{1}{2}[\Phi_1, Q\Phi_0]), \end{aligned} \tag{31}$$

$$\vdots$$

$$\begin{aligned} \Delta_n = \eta_0 \bigg(& Q\Phi_n + \sum_{m=1}^{n} \sum_{\substack{n_1, n_2, \ldots, n_{m+1} \\ n_1+n_2+\ldots+n_{m+1}=n-m}} \frac{(-1)^m}{(m+1)!} \\ & \times [\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]] \ldots] \bigg), \end{aligned} \tag{32}$$

$$\vdots$$

where

$$\Delta_0 = \alpha' \int \frac{d^p k}{(2\pi)^p} k^2 A_+(k_i) c\partial c \psi^+ e^{-\phi} e^{ik_i X^i}. \tag{33}$$

We impose the gauge fixing conditions $b_0\Phi_n = \tilde{G}_0^- \Phi_n = 0$, and for $n \geq 1$ $b_0 \Delta_n = 0$. This condition, with $\tilde{G}_0^-$ defined as follows[13], is slightly different from the familiar one $\xi_0 \Phi_n = 0$.

$$\begin{aligned} \tilde{G}_0^- &= \left[Q, \oint \frac{dz}{2\pi i} z b \xi(z) \right] \\ &= \oint \frac{dz}{2\pi i} z (\xi T - \partial\xi bc - b e^{\phi} G_m - \eta e^{2\phi} b\partial b), \end{aligned} \tag{34}$$

where T is the total worldsheet energy-momentum tensor, and G_m is matter part of the worldsheet supercurrent. This operator is more useful than ξ_0 because of the following relations:

$$\{\eta_0, \tilde{G}_0^-\} = L_0, \qquad \{Q, \tilde{G}_0^-\} = \{b_0, \tilde{G}_0^-\} = 0, \tag{35}$$

and therefore $\tilde{G}_0^-/L_0$ is the inverse of η_0 on string fields annihilated by $\tilde{G}_0^-$. Note that Φ_0 obeys $b_0\Phi_0 = \tilde{G}_0^-\Phi_0 = 0$.

Then the equations of motion can be solved order by order:

$$\Phi_1 = \frac{1}{2} \frac{\tilde{G}_0^-}{L_0} \eta_0 \frac{b_0}{L_0} [\Phi_0, Q\Phi_0], \tag{36}$$

$$\begin{aligned} \Phi_2 = \frac{\tilde{G}_0^-}{L_0} \eta_0 \frac{b_0}{L_0} \bigg(& -\frac{1}{6} [\Phi_0, [\Phi_0, Q\Phi_0]] + \frac{1}{2} [\Phi_0, Q\Phi_1] \\ & + \frac{1}{2} [\Phi_1, Q\Phi_0] \bigg), \end{aligned} \tag{37}$$

$$\vdots$$

$$\begin{aligned} \Phi_n = -\frac{\tilde{G}_0^-}{L_0} \eta_0 \frac{b_0}{L_0} & \sum_{m=1}^{n} \sum_{\substack{n_1, n_2, \ldots, n_{m+1} \\ n_1+n_2+\ldots+n_{m+1}=n-m}} \frac{(-1)^m}{(m+1)!} \\ & \times [\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]] \ldots], \end{aligned} \tag{38}$$

$$\vdots$$

We can see that Φ_n consists of Q, η_0, $\frac{b_0}{L_0}$, $\tilde{G}_0^-/L_0$ and $(n+1)$ copies of Φ_0.

By plugging the above Φ_n back into the equations of motion, we obtain

$$\begin{aligned} \Delta_n = \eta_0 \frac{b_0}{L_0} Q & \sum_{m=1}^{n} \sum_{\substack{n_1, n_2, \ldots, n_{m+1} \\ n_1+n_2+\ldots+n_{m+1}=n-m}} \frac{(-1)^m}{(m+1)!} \\ & \times [\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]] \ldots]. \end{aligned} \tag{39}$$

As in the bosonic case, if Φ_m satisfy equations of motion with $\Delta_m = 0$ for $m < n$, then $\Delta_n = 0$. To prove this, notice the following identity:

$$Q(e^{-\Phi} Q e^{\Phi}) + (e^{-\Phi} Q e^{\Phi})^2 = 0. \tag{40}$$

Therefore

$$Q\eta_0(e^{-\Phi}Qe^{\Phi}) = [\eta_0(e^{-\Phi}Qe^{\Phi}), (e^{-\Phi}Qe^{\Phi})]. \quad (41)$$

We expand Φ in g and extract order g^{n+1} contribution of this equation. From the left-hand side,

$$Q\eta_0(e^{-\Phi}Qe^{\Phi})|_{g^{n+1}} = Q\eta_0 \sum_{m=1}^{n} \sum_{\substack{n_1,n_2,\ldots,n_{m+1} \\ n_1+n_2+\ldots+n_{m+1}=n-m}} \frac{(-1)^m}{(m+1)!} \times [\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]] \ldots]]. \quad (42)$$

Using equations of motion for lower order than g^{n+1}, the right-hand side gives

$$[\eta_0(e^{-\Phi}Qe^{\Phi}), (e^{-\Phi}Qe^{\Phi})]|_{g^{n+1}} = \sum_{l=0}^{n-1}\Bigg[\Delta_l, \sum_{m=0}^{n-l-1} \sum_{\substack{n_1,n_2,\ldots,n_{m+1} \\ n_1+n_2+\ldots+n_{m+1}=n-l-m-1}} \frac{(-1)^m}{(m+1)!} \times [\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]] \ldots]\Bigg]. \quad (43)$$

Therefore

$$\Delta_n = \frac{b_0}{L_0}\sum_{l=0}^{n-1}\Bigg[\Delta_l, \sum_{m=0}^{n-l-1} \sum_{\substack{n_1,n_2,\ldots,n_{m+1} \\ n_1+n_2+\ldots+n_{m+1}=n-l-m-1}} \frac{(-1)^m}{(m+1)!} \times [\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]] \ldots]\Bigg]. \quad (44)$$

This shows that if $\Delta_m = 0$ for $m < n$, then $\Delta_n = 0$.

Analogously to the bosonic case, Φ_n has no dependence on momenta along $x^{\pm}$, and has the following property: $n_+(\omega) - n_-(\omega) = n + 1$, where ω is any of the vertex operators (or Fock space states) of which Φ_n consists, $n_+(\omega)$ is the number of $\partial^m X^+$s and $\partial^r\psi^+$ (or α^+_{-m} and ψ^+_{-r}) in ω, and $n_-(\omega)$ is the number of $\partial^m X^-$s and $\partial^r\psi^-$ (or α^-_{-m} and ψ^-_{-r}) in ω. In addition, (X^I, ψ^I) part of ω is a super-Virasoro descendant of the unit operator. In other words, the matter part of ω is in the following form:

$$\prod_{l=1}^{N_+}\alpha^+_{-p_l}\prod_{l=1}^{M_+}\psi^+_{-q_l}\prod_{l=1}^{N_-}\alpha^-_{-r_l}\prod_{l=1}^{M_-}\psi^-_{-s_l}\prod_l L'_{-t_l}\prod_l G'_{-u_l}\prod_l \alpha^{i_l}_{-v_l}\prod_l \psi^{j_l}_{-w_l}|k_i\rangle$$
$$(N_+ + M_+ - N_- - M_- = n + 1,\ p_l, r_l, v_l \geq 1,\ t_l \geq 2,\ q_l, s_l, w_l \geq 1/2,\ u_l \geq 3/2). \quad (45)$$

where L'_n and G'_r are (X^I, ψ^I) parts of Virasoro operator and worldsheet supercharge, respectively.

This can be proven by almost the same argument as in the bosonic case. Here we have new ingredients: η_0, Q and $\tilde{G}_0^-$. η_0 does not affect the matter sector. Q and $\tilde{G}_0^-$ can replace $X^{\pm}$ by $\psi^{\pm}$ and vice versa, but preserve $n_{\pm}$. They map a super-Virasoro descendant of the unit operator to other descendants of it. Δ_n also satisfy these properties as can be seen from almost the same argument.

Therefore this solution has the same properties as in the bosonic case: each coefficient of Fock space state in the solution Φ receives contribution from only one Φ_n. In particular, the coefficient A_μ of the massless mode $\xi c\psi^+ e^{-\phi}e^{ik_iX^i}$ is never corrected by higher order contribution. The inverses of L_0 in the expression of Φ_n with $n \geq 1$ do not cause any problem. A_μ is equal to the gauge field in the effective action. This gives another proof of the fact shown in [7]. Massive modes are convergent even at the singular points of the massless mode. Energy-momentum tensor as Noether current of translation symmetry is equal to that of free U(1) gauge theory.

A new property which is not in bosonic theory is supersymmetry. Therefore let us investigate supersymmetry of this solution. Supersymmetry transformation of R-sector string field Ψ is given by [14]

$$\delta(\eta_0\Psi) = -\eta_0 s(e^{-\Phi}(Qe^{\Phi})), \quad (46)$$

where

$$s = \oint \frac{dz}{2\pi i} e^{i\pi/4}\bar{\epsilon}_A \xi(z) e^{-\phi(z)/2}\Sigma^A(z), \quad (47)$$

$\bar{\epsilon}_A$ is a constant ten-dimensional Majorana-Weyl spinor, and $\Sigma^A(z)$ is a spin operator. $e^{-\phi(z)/2}\Sigma^A(z)$ is regarded as Grassmann odd. The action of s on a string field is defined as the contour integral of (47) around it.

It is easy to see that the linearized solution Φ_0 is 1/2 supersymmetric at the linearized level, since on-shell linearized transformation for massless fields is the same as that of the U(1) gauge theory. Because of $A_- = A_i = A_I = 0$ and $A_+ = A_+(k_i)$, the transformation of gaugino $\psi^A(k)$ is

$$\delta\psi^A(k) = ik_iA_+(k_i)(\Gamma^{i+}\epsilon)^A. \quad (48)$$

We see that the unbroken supersymmetry parameter is given by $\Gamma^+\epsilon = 0$.

In fact, the full solution is also 1/2 supersymmetric with the same unbroken parameter. This can be shown as follows. First, notice that when $\Gamma^+\epsilon = 0$, Φ_0 satisfies

$$s\Phi_0 = s\eta_0\Phi_0 = sQ\Phi_0 = s\eta_0Q\Phi_0 = 0, \quad (49)$$

and s commutes with $\tilde{G}_0^-/L_0$ and b_0/L_0. Then by plugging our solution, $e^{-\Phi}(Qe^{\Phi})$ is expressed by Φ_0, $(\tilde{G}_0^-/L_0)\eta_0$, Q and $\frac{b_0}{L_0}$. Using Leibniz rule for Q and η_0, and $\{Q, \frac{b_0}{L_0}\} = \{\eta_0, \tilde{G}_0^-/L_0\} = 1$, we can rewrite $e^{-\Phi}(Qe^{\Phi})$ in such a form that any Q and η_0 act directly on one of Φ_0. Since s also satisfies Leibniz rule when it acts on products of string fields, we can again rewrite $se^{-\Phi}(Qe^{\Phi})$ in such a form that s acts directly on one of Φ_0, $\eta_0\Phi_0$, $Q\Phi_0$ or $\eta_0 Q\Phi_0$. Thus we can see $se^{-\Phi}(Qe^{\Phi}) = 0$ and therefore $\delta(\eta_0\Psi) = 0$.

V. DISCUSSION

We have shown that our solutions have various full order properties in the sense of α' expansion. Among them, the fact that massive modes have no singularity lacks a rigorous proof for third and higher massive states coming from Φ_n with $n \geq 3$. It is desirable to give a proof of it, because this fact is important for not only our solutions, but also general structure of off-shell amplitudes.

We have constructed higher order source terms for unphysical modes along with higher order contributions to the solutions, and have seen that those are not localized to points. This is natural in a sense, because full order string theory is a nonlocal theory unlike its low energy effective theory. Although this is expected not to affect the equation of motion for massless modes obtained after integrating out all the massive modes, it is better to give other evidences that our source terms really correspond to endpoints of fundamental strings.

Readers might wonder why massive modes do not contribute to the energy-momentum tensor, in spite of the fact that they satisfy Siegel gauge condition and therefore they are physical excitations. It may be useful to consider if this fact has any deep meaning for physical properties of massive modes.

The order by order method employed here can be applied to other systems e.g. closed string field theory. It is interesting to construct solutions corresponding to, for example, macroscopic fundamental string solution or pp-wave solution, which are also known as α'-exact solutions in supergravity. We can expect to derive some full order properties of those solutions by the same method as in this paper.

ACKNOWLEDGMENTS

The author wishes to thank S. Iso, Y. Okawa, and A. Sen for useful discussions, and especially B. Zwiebach for reading the manuscript and giving helpful comments. This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DF-FC02-94ER40818, and by the Nishina Memorial Foundation.

APPENDIX A

In this appendix we show that the momentum integral of (22) is convergent, by seeing that the factor $\frac{1}{2}(1 + \alpha^2/1 - \alpha^2)\kappa(\alpha)$ is less than or equal to 1. First we give the definition of $\kappa(\alpha)$.

4-point amplitudes can be computed by mapping four vertex operators on four upper half planes by $w = h_i(Z_i)$, defined as follows,

$$h_1(Z) = h_2(Z) = \ln Z - \frac{\tau}{2}, \tag{A1}$$

$$h_3(Z) = h_4(Z) = -\ln Z + \pi i + \frac{\tau}{2}, \tag{A2}$$

and the Giddings map $z = z(w)$ [11], defined implicitly as follows,

$$w = \frac{\tau}{2} + N \int_{+0}^{z} d\zeta \frac{\sqrt{\zeta^2 + \gamma^2}\sqrt{\zeta^2 + \gamma^{-2}}}{(\zeta^2 - \alpha^2)(\zeta^2 - \alpha^{-2})}, \tag{A3}$$

$$N = \frac{2\alpha(\alpha^{-2} - \alpha^2)}{\sqrt{\alpha^2 + \gamma^2}\sqrt{\alpha^2 + \gamma^{-2}}}, \tag{A4}$$

to one single upper half plane, on which the four vertex operators are at $z = \pm\alpha$ and $z = \pm\alpha^{-1}$.

τ and γ are functions of α, and implicitly determined by the following equations.

$$\frac{\pi}{2} = N \int_0^{\gamma} d\zeta \frac{\sqrt{\gamma^2 - \zeta^2}\sqrt{\gamma^{-2} - \zeta^2}}{(\zeta^2 + \alpha^2)(\zeta^2 + \alpha^{-2})}, \tag{A5}$$

$$\tau = N \int_{\gamma}^{\gamma^{-1}} d\zeta \frac{\sqrt{\zeta^2 - \gamma^2}\sqrt{\gamma^{-2} - \zeta^2}}{(\zeta^2 + \alpha^2)(\zeta^2 + \alpha^{-2})}. \tag{A6}$$

γ is a monotonously increasing function, and $0 \leq \gamma \leq 1$ as can be seen from the fact that $z = i\gamma$ and $z = i\gamma^{-1}$ are where two of the four strings meet, and therefore $z = i\gamma$ is always below $z = i\gamma^{-1}$ on the imaginary axis. τ is a modulus to be integrated over $0 \leq \tau \leq \infty$ which corresponds to $\alpha_0 \equiv \sqrt{2} - 1 \geq \alpha \geq 0$. Near $\alpha = 0$, $\gamma \sim \sqrt{3}\alpha$, and $\gamma = 1$ only at $\alpha = \alpha_0$. Figure 3 is the profile of γ.

The above conformal mappings for the vertex operators give the following factor, which appears in (22):

$$(\alpha^{-1}\kappa(\alpha))^{\alpha' k^2}(\alpha^{-1}\kappa(\alpha))^{\alpha' k_{(2)}^2}(\alpha\kappa(\alpha))^{\alpha' k_{(3)}^2}(\alpha\kappa(\alpha))^{\alpha' k_{(4)}^2}, \tag{A7}$$

where[2]

$$\kappa(\alpha) = \exp(I(\alpha)), \tag{A8}$$

$$\begin{aligned} I(\alpha) &= \int_0^{\alpha} d\zeta \left[N \frac{\sqrt{\zeta^2 + \gamma^2}\sqrt{\zeta^2 + \gamma^{-2}}}{(\zeta^2 - \alpha^2)(\zeta^2 - \alpha^{-2})} + \frac{1}{\zeta - \alpha} \right] \\ &= \int_0^1 d\zeta \left[N\alpha \frac{\sqrt{\alpha^2\zeta^2 + \gamma^2}\sqrt{\alpha^2\zeta^2 + \gamma^{-2}}}{(1 - \zeta^2)(1 - \alpha^4\zeta^2)} + \frac{1}{\zeta - 1} \right]. \end{aligned} \tag{A9}$$

[2]Although this looks different from Eq. (3.13) in [12], this is equal to it as can be seen by partial integration and replacing $\ln(1 - w)$ by $\int^w d\zeta \frac{1}{\zeta - 1}$.

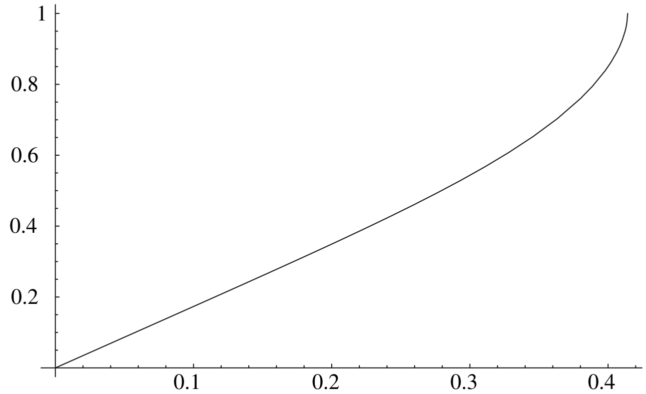

FIG. 3. $\gamma(\alpha)$

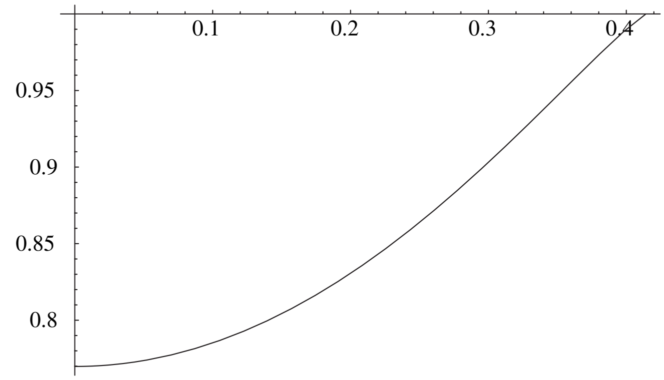

FIG. 4. $\frac{1}{2}((1+\alpha^2)/(1-\alpha^2))\kappa(\alpha)$

The two terms of the integrand are divergent at $\zeta = \alpha$, but their sum is not. Though it is difficult to perform this integral at generic α, it is possible at the edges of the range of α:

$$I(0) = \frac{8}{3\sqrt{3}}, \tag{A10}$$

$$I(\alpha_0) = \ln\sqrt{2}. \tag{A11}$$

To show $0 < \frac{1}{2}((1+\alpha^2)/(1-\alpha^2))\kappa(\alpha) \leq 1$, we add some extra terms to the integrand which sum up to zero:

$$\begin{aligned} I(\alpha) = &\int_0^1 d\zeta\bigg[N\alpha\frac{\sqrt{\alpha^2\zeta^2+\gamma^2}\sqrt{\alpha^2\zeta^2+\gamma^{-2}}}{(1-\zeta^2)(1-\alpha^4\zeta^2)} \\ &- 2(1-\alpha^2)\frac{1+\alpha^2\zeta^2}{(1-\zeta^2)(1-\alpha^4\zeta^2)}\bigg] \\ &+ \int_0^1 d\zeta\bigg[2(1-\alpha^2)\frac{1+\alpha^2\zeta^2}{(1-\zeta^2)(1-\alpha^4\zeta^2)} \\ &- 2(1-\alpha_0^2)\frac{1+\alpha_0^2\zeta^2}{(1-\zeta^2)(1-\alpha_0^4\zeta^2)}\bigg] \\ &+ \int_0^1 d\zeta\bigg[2(1-\alpha_0^2)\frac{1+\alpha_0^2\zeta^2}{(1-\zeta^2)(1-\alpha_0^4\zeta^2)} + \frac{1}{\zeta-1}\bigg]. \end{aligned} \tag{A12}$$

The third integral is equal to $I(\alpha_0)$, and the second integral can be explicitly done because the integrand is a rational function:

$$\begin{aligned} &\int_0^1 d\zeta\bigg[2(1-\alpha^2)\frac{1+\alpha^2\zeta^2}{(1-\zeta^2)(1-\alpha^4\zeta^2)} - 2(1-\alpha_0^2) \\ &\quad\times\frac{1+\alpha_0^2\zeta^2}{(1-\zeta^2)(1-\alpha_0^4\zeta^2)}\bigg] = \ln\Big(\sqrt{2}\frac{1-\alpha^2}{1+\alpha^2}\Big). \end{aligned} \tag{A13}$$

The sum of two terms of the integrand in the first integral is not singular at $\zeta = 1$. The final result of this manipulation is

$$\begin{aligned} I(\alpha) = &\ln\Big(2\frac{1-\alpha^2}{1+\alpha^2}\Big) - \frac{2\alpha^2(1-\alpha^2)(1-\gamma^2)}{(\alpha^2+\gamma^2)(1+\alpha^2\gamma^2)} \\ &\times\int_0^1 d\zeta\bigg[(1+\alpha^2)\sqrt{\frac{(\alpha^2\zeta^2+\gamma^2)(1+\alpha^2\gamma^2\zeta^2)}{(\alpha^2+\gamma^2)(1+\alpha^2\gamma^2)}} \\ &+ 1 + \alpha^2\zeta^2\bigg]^{-1}. \end{aligned} \tag{A14}$$

Then we obtain the following expression of $\frac{1}{2}\times((1+\alpha^2)/(1-\alpha^2))\kappa(\alpha)$:

$$\begin{aligned} \frac{1}{2}\frac{1+\alpha^2}{1-\alpha^2}\kappa(\alpha) = &\exp\bigg(-\frac{2\alpha^2(1-\alpha^2)(1-\gamma^2)}{(\alpha^2+\gamma^2)(1+\alpha^2\gamma^2)} \\ &\times\int_0^1 d\zeta\bigg[(1+\alpha^2) \\ &\times\sqrt{\frac{(\alpha^2\zeta^2+\gamma^2)(1+\alpha^2\gamma^2\zeta^2)}{(\alpha^2+\gamma^2)(1+\alpha^2\gamma^2)}} \\ &+ 1 + \alpha^2\zeta^2\bigg]^{-1}\bigg). \end{aligned} \tag{A15}$$

It is easy to see that the exponent of the right-hand side is always negative, and zero only at $\alpha = \alpha_0$ (where $\gamma = 1$). Thus $0 < \frac{1}{2}((1+\alpha^2)/(1-\alpha^2))\kappa(\alpha) \leq 1$, and the momentum integral of (22) is convergent. Figure 4 is the profile of $\frac{1}{2}\frac{1+\alpha^2}{1-\alpha^2}\kappa(\alpha)$.

[1] E. Witten, Nucl. Phys. **B268**, 253 (1986).
[2] N. Berkovits, Nucl. Phys. **B450**, 90 (1995).
[3] C. G. Callan, Jr. and J. M. Maldacena, Nucl. Phys. **B513**, 198 (1998).
[4] L. Bonora, C. Maccaferri, R. J. Scherer Santos, and D. D. Tolla, Phys. Lett. B **619**, 359 (2005).
[5] A. Sen, J. High Energy Phys. 08 (2004) 034.
[6] S. Lee, A. Peet, and L. Thorlacius, Nucl. Phys. **B514**, 161 (1998).
[7] L. Thorlacius, Phys. Rev. Lett. **80**, 1588 (1998).
[8] B. Zwiebach, hep-th/0010190.
[9] E. Coletti, I. Sigalov, and W. Taylor, J. High Energy Phys. 09 (2003) 050.
[10] T. Kugo and B. Zwiebach, Prog. Theor. Phys. **87**, 801 (1992).
[11] S. Giddings, Nucl. Phys. **B278**, 242 (1986).
[12] S. Samuel, Nucl. Phys. **B308**, 285 (1988).
[13] N. Berkovits and C. T. Echevarria, Phys. Lett. B **478**, 343 (2000).
[14] I. Kishimoto and T. Takahashi, J. High Energy Phys. 11 (2005) 051.