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Loop equation in $D = 4$, $\mathcal{N} = 4$ super Yang-Mills theory and string field equation on $AdS_5 \times S^5$

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We consider the loop equation in four-dimensional $\mathcal{N} = 4$ SYM, which is a functional differential equation for the Wilson loop $W(C)$ and expresses the propagation and the interaction of the string C. Our $W(C)$ consists of the scalar and the gaugino fields as well as the gauge field. The loop C is specified by six bosonic coordinates $y^i(s)$ and two fermionic coordinates $\zeta(s)$ and $\eta(s)$ besides the four-dimensional spacetime coordinates $x^\mu(s)$. We have successfully determined, to quadratic order in ζ and η, the parameters in $W(C)$ and the loop differential operator so that the equation of motion of SYM can be correctly reproduced to give the nonlinear term of $W(C)$. We extract the most singular and linear part of our loop equation and compare it with the Hamiltonian constraint of the string propagating on $AdS_5 \times S^5$ background.

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I. INTRODUCTION

The large N dualities between string theories and gauge theories are of great importance for the understanding of both theories. The AdS/CFT correspondence [1–5] is one of the most interesting examples of such dualities. On the string theory side of this correspondence, we consider the type IIB superstring on $AdS_5 \times S^5$ geometry. On the other hand, the corresponding gauge theory is the four-dimensional $\mathcal{N} = 4$ SU(N) super Yang-Mills theory (SYM). Since this correspondence was first conjectured, a many of the aspects of it have been studied. Among them, the correspondences associated with the Wilson loop operator of SYM [6–12] seem to be very important, because the proposed counterpart in the string theory side is nothing but the fundamental string. The standard argument of the correspondence begins with considering the following Wilson loop operator $W(C)$ defined on the loop C:

$$W(C) = \mathrm{Tr}\,\mathrm{P}\exp\biggl(i\int_0^l ds(A_\mu(x(s))\dot{x}^\mu(s) + A_{3+i}(x(s))\dot{y}^i(s))\biggr), \tag{1.1}$$

where A_μ ($\mu = 0, \ldots, 3$) and A_{3+i} ($i = 1, \ldots, 6$) are the gauge field and the six scalar fields, respectively, and P denotes the path ordering. The loop C is defined by ten coordinates, $x^\mu(s)$ and $y^i(s)$: x^μ are the coordinates of the four-dimensional spacetime in which the gauge theory lives, and this Wilson loop depends also on the additional six "coordinates" y^i. The corresponding object to this Wilson loop in the string theory side is the string world sheet whose boundary is specified by the loop C. Then the conjectured relation [6–12] is

$$\exp(-A_{\text{world sheet}}) = \langle W(C)\rangle_{\text{SYM}}, \tag{1.2}$$

where $A_{\text{world sheet}}$ is the area of the classical solution of the string world sheet, and $\langle W(C)\rangle_{\text{SYM}}$ is the expectation value of the Wilson loop operator.

On the other hand, there is another interesting correspondence between the Wilson loop operator and the fundamental string in the context of the string/gauge duality; the correspondence between the Wilson loop operator and the string field. In [13] the loop equation of the Wilson loop operator in type IIB matrix model was investigated and they argued that the light cone Hamiltonian of the string field can be derived from the loop equation. If there is a similar correspondence between the Wilson loop operator $W(C)$ in four-dimensional $\mathcal{N} = 4$ SYM and the string field $\Psi[X(s)]$, it is natural to expect that the string field lives in a curved geometry, i.e., in $AdS_5 \times S^5$. Hence, the loop equation of $W(C)$ would have the same information as the Hamiltonian of the string field on $AdS_5 \times S^5$. Although the construction of string field theory (SFT) on $AdS_5 \times S^5$ spacetime is still a challenge, there have been lots of developments in understanding the SFT on the pp-wave background [14–22] which is obtained by taking the Penrose limit of $AdS_5 \times S^5$ geometry. Connections between the pp-wave string states and the local operators in the gauge theory, i.e., the Berenstein-Maldacena-Nastase (BMN) operators, are also studied intensively [23] (see also [24–26] and references therein). Recently, one of the present authors has shown that these BMN operators emerge in the expansion of the Wilson loop operator with respect to the fluctuations $\delta C = \{\delta x^\mu, \delta y^i\}$ of the loop C [27]:

$$W(C) \sim \sum_J \frac{1}{J!}\biggl\{\mathcal{O}^J_{\text{ground}} + \delta x_0^\mu \mathcal{O}^J_{\mu,0} + \delta\dot{y}_0^p \mathcal{O}^J_{4+p,0} + \sum_n \delta x^\mu_{-n}\delta x^\nu_n \mathcal{O}^{J-1}_{\mu\nu,n} + \sum_n \delta\dot{y}^p_{-n}\delta x^\mu_n \mathcal{O}^{J-1}_{4+p\,\mu,n} + \sum_n \delta\dot{y}^p_{-n}\delta\dot{y}^q_n \mathcal{O}^{J-1}_{4+p\,4+q,n} + \cdots\biggr\}, \tag{1.3}$$

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where $\mathcal{O}^J_{\text{ground}}$, $\mathcal{O}^J_{M,0}$, and $\mathcal{O}^J_{MN,n}$ are the BMN operators and the indices p and q run from 1 to 4.[1] It is quite interesting that this expansion resembles the expansion of a string field with respect to the string states:

$$\Psi[X(s)] = \sum_A \langle \delta X | A \rangle \psi_A(X_0), \tag{1.4}$$

where $\{|A\rangle\}$ is a complete set of first-quantized string states, and $\psi_A(X_0)$, which is a function of the center-of-mass coordinate X_0 of the string, is the local field corresponding to the string state $|A\rangle$.

Hence, it is a very interesting theme to investigate the loop equation of the Wilson loop operator in four-dimensional $\mathcal{N} = 4$ SYM with the expectation that it would have the same information as the equation of motion of the string field on the $AdS_5 \times S^5$ background and on the pp-wave background as well. However, the loop equation in $\mathcal{N} = 4$ SYM including its fermionic part has not been completely established.[2] The purpose of this paper is to construct the loop equation in four-dimensional $\mathcal{N} = 4$ SYM as a first step of such an investigation. We will also carry out a (partial) analysis of the loop equation toward the identification of the Wilson loop operator with the string field.

For explaining the problems in constructing the loop equation in $\mathcal{N} = 4$ SYM, let us recapitulate the derivation of the loop equation in bosonic Yang-Mills theory. In bosonic Yang-Mills theory, the Wilson loop operator is given simply by

$$W(C) = \operatorname{Tr} \mathrm{P} \exp\left(i \int_0^l ds A_\mu(x(s)) \dot{x}^\mu(s) \right) \equiv \operatorname{Tr} \mathcal{W}_0^l(C). \tag{1.5}$$

Here, $\mathcal{W}_{u_1}^{u_2}(C)$ expresses the Wilson line defined on the portion of the loop C with the parameter region $[u_1, u_2]$. In the rest of this paper we often omit the argument C of $\mathcal{W}_{u_1}^{u_2}(C)$ when it causes no confusion. The starting point of deriving the loop equation is the following formula for the functional derivative acting on $\mathcal{W}_{u_1}^{u_2}$:

$$\begin{aligned} \frac{\delta}{\delta x^\mu(s)} \mathcal{W}_{u_1}^{u_2} &= i \int_{u_1}^{u_2} du \mathcal{W}_{u_1}^{u} [\partial_\mu A_\nu(x(u)) \dot{x}^\nu(u) \delta(u-s) + A_\mu(x(u)) \dot{\delta}(u-s)] \mathcal{W}_u^{u_2} \\ &= \mathcal{W}_{u_1}^{s} i(F_{\mu\nu}\dot{x}^\nu)_s \mathcal{W}_s^{u_2} + \mathcal{W}_{u_1}^{u_2} i(A_\mu)_{u_2} \delta(s-u_2) - i(A_\mu)_{u_1} \mathcal{W}_{u_1}^{u_2} \delta(s-u_1), \end{aligned} \tag{1.6}$$

where $(F_{\mu\nu}\dot{x}^\nu)_s$, for example, is the abbreviation of $F_{\mu\nu}(x(s))\dot{x}^\nu(s)$. Using this formula twice we get

$$\begin{aligned} \frac{\delta}{\delta x^\mu(s_2)} \frac{\delta}{\delta x_\mu(s_1)} W(C) &= \frac{\delta}{\delta x^\mu(s_2)} \operatorname{Tr}[i(F^\mu{}_\nu \dot{x}^\nu)_{s_1} \mathcal{W}_{s_1}^{s_1+l}] \\ &= \operatorname{Tr}[i(F^\mu{}_\nu \dot{x}^\nu)_{s_1} \mathcal{W}_{s_1}^{s_2} i(F_{\mu\rho}\dot{x}^\rho)_{s_2} \mathcal{W}_{s_2}^{s_1+l}] + \delta(s_1-s_2) \operatorname{Tr}[i(D_\mu F^\mu{}_\nu \dot{x}^\nu)_{s_1} \mathcal{W}_{s_1}^{s_1+l}]. \end{aligned} \tag{1.7}$$

We call the first term in the final form of (1.7) the $\not{\delta}$-term and the second term the δ-term. It is important that the δ-term is proportional to the left-hand side (LHS) of the equation of motion (EOM), $D_\mu F^\mu{}_\nu = 0$. Let us consider the expectation value of (1.7) or that of the product of (1.7) and other Wilson loop operators. Then the δ-term can be evaluated as follows:

$$\begin{aligned} \int \mathcal{D}A_\mu \operatorname{Tr}[it^a (D_\mu F^{\mu\nu} \dot{x}_\nu)^a_{s_1} \mathcal{W}_{s_1}^{s_1+l}](\cdots) e^{iS} &= g^2 \int \mathcal{D}A_\mu \operatorname{Tr}\left[t^a \left(\dot{x}_\nu \frac{\delta}{\delta A_\nu^a} e^{iS} \right)_{s_1} \mathcal{W}_{s_1}^{s_1+l} \right](\cdots) \\ &= -g^2 \int \mathcal{D}A_\mu \operatorname{Tr}\left[t^a \dot{x}_\nu(s_1) \frac{\delta}{\delta A_\nu^a(x(s_1))} \mathcal{W}_{s_1}^{s_1+l} \right](\cdots) e^{iS}, \end{aligned} \tag{1.8}$$

where t^a ($a = 1, \ldots, N^2-1$) are the generators of the $SU(N)$ gauge group, dots $(\cdots)$ express the possible other Wilson loop operators, and g is the Yang-Mills coupling constant. We have performed functional integration by parts in obtaining the final expression.[3] The functional derivative with respect to $A_\nu^a(x(s_1))$ divides the Wilson loop into two parts and we have, in functional integration,

[1] Note that we have expanded the Wilson loop operator with respect to the fluctuations of the four coordinates δx^μ and the four "winding number density" $\delta \dot{y}^p$.

[2] Loop equations in four-dimensional $\mathcal{N} = 4$ SYM were studied to check the correspondence (1.2) [8–11]. In [8,9], they studied a special class of loops satisfying the condition $\dot{x}^2 + \dot{y}^2 = 0$ or its fermionic extension. In [10,11], they considered the simple Wilson loop (1.5) and argued that the contribution of the scalars and gauginos is irrelevant for their analysis. A manifestly supersymmetric formulation of the loop equation in $\mathcal{N} = 1$ SYM is given in [28,29].

[3] If some operators in $(\cdots)$ lie on the loop C, there are other contributions to the right-hand side (RHS) of (1.8) which arise when the functional derivative acts on such operators. Here we just neglect such situations.

$$\frac{\delta}{\delta x^\mu(s_2)} \frac{\delta}{\delta x_\mu(s_1)} W(C) = \mathrm{Tr}[i(F^\mu{}_\nu \dot{x}^\nu)_{s_1} \mathcal{W}_{s_1}^{s_2} i(F_{\mu\rho}\dot{x}^\rho)_{s_2} \mathcal{W}_{s_2}^{s_1+l}] - i\frac{g^2}{2}\delta(s_1 - s_2)\int_{s_1}^{s_1+l} ds\, \delta^{(4)}(x(s) - x(s_1))\dot{x}^\nu(s_1)\dot{x}_\nu(s) W(C_1) W(C_2), \tag{1.9}$$

where the loop C_1 (C_2) is the part of the loop C with the parameter region $[s_1, s]$ ($[s, s_1 + l]$).[4] We call (1.7) and (1.9) "loop equation" in this paper. The loop equation in bosonic Yang-Mills theory has been used to study the area-law property of the Wilson loop (see [30,31] and references therein).

We would like to extend the above derivation of the loop equation to the four-dimensional $\mathcal{N} = 4$ SYM. Concretely, we have to give the SYM extension of both the Wilson loop operator and the quadratic functional derivative with respect to the loop coordinates in such a way that the δ-term which is multiplied by $\delta(s_1 - s_2)$ is proportional to the EOM and hence it gives the nonlinear term in the Wilson loop. Our Wilson loop operator in $\mathcal{N} = 4$ SYM is given by modifying (1.1) to include the gaugino fields. Accordingly, the loop C is specified by two fermionic spinor coordinates, $\zeta(s)$ and $\eta(s)$, as well as $4 + 6$ bosonic coordinates, $x^\mu(s)$ and $y^i(s)$. The coordinate $\zeta(s)$ has already appeared in the literature [8,9]. Its mass-dimension is $-1/2$ and it has the same chirality as the gaugino. On the other hand, another fermionic coordinate $\eta(s)$ has mass-dimension $-2/3$ and the opposite chirality to that of $\zeta(s)$ and gaugino. Therefore, we can consider the quadratic functional derivative $(\delta/\delta\eta(s_2))(\delta/\delta\bar{\zeta}(s_1))$ which has the same mass-dimension 2 as $(\delta/\delta x^\mu(s_2)) \times (\delta/\delta x_\mu(s_1))$ and $(\delta/\delta y^i(s_2))(\delta/\delta y^i(s_1))$. By taking as the total quadratic functional derivative for the loop equation a suitable linear combination of the above three, we have succeeded in determining the dependence of $W(C)$ on the fermionic coordinates so that the δ-term may vanish to quadratic order in ζ and η if we use the EOM of $\mathcal{N} = 4$ SYM.

In this way we can obtain the $\mathcal{N} = 4$ SYM version of the loop equation (1.9). For our application of the loop equation to the analysis of the AdS/CFT correspondence, in particular, the identification of the Wilson loop operator as the string field on $\mathrm{AdS}_5 \times \mathrm{S}^5$, we have to consider the coincident limit $s_1 \to s_2$ of the quadratic functional derivative. This limit is singular and needs some kind of regularization. In this paper, we adopt the regularization of replacing the massless free propagator $1/x^2$ by $1/(x^2 + \epsilon^2)$, and extract the most singular part of order $1/\epsilon^4$ in the loop equation. We find that the resulting equation for the Wilson loop resembles the Hamiltonian constraint of bosonic string on $\mathrm{AdS}_5 \times \mathrm{S}^5$ if we identify the UV regularization parameter ϵ with the radial coordinate of AdS_5.

[4] In this step we use the following formulas:

$$\frac{\delta}{\delta A^a_\mu(x(s))}\mathcal{W}_{u_1}^{u_2} = \int_{u_1}^{u_2} du\, \mathcal{W}_{u_1}^{u} i t^a \dot{x}^\mu(u)\delta^{(4)}(x(u) - x(s))\mathcal{W}_{u}^{u_2},$$

and

$$(t^a)_{ij}(t^a)_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right).$$

In (1.9) and throughout this paper, we neglect the $1/N$ term in the second formula.

The rest of this paper is organized as follows. In Secs. II and III, we consider the loop equation in $\mathcal{N} = 4$ SYM at the lowest order in the fermionic coordinates: We derive the loop equation in Sec. II, and then in Sec. III we pick up the most singular and linear part of the loop equation and compare it with the Hamiltonian constraint of the bosonic string on the $\mathrm{AdS}_5 \times \mathrm{S}^5$. In Sec. IV we extend our loop equation to the quadratic order in the fermionic coordinates. Section V is devoted to the conclusion and discussions. Our notations and conventions are summarized in Appendix A. Details of the calculations used in Sec. IV are given in Appendix B. In Appendix C we calculate the most singular and linear part of the loop equation to quadratic order in fermionic coordinates. In Appendix D we consider more general functional derivatives than those we consider in Secs. III and IV.

II. LOOP EQUATION IN $\mathcal{N} = 4$ SYM I: THE LOWEST ORDER IN ζ AND η

As we explained in the previous section, the loop equation in four-dimensional $\mathcal{N} = 4$ SYM depends on the fermionic loop coordinates $\zeta(s)$ and $\eta(s)$ as well as the bosonic coordinates $x^\mu(s)$ and $y^i(s)$. In this section, we will derive the loop equation at the lowest order in $\zeta(s)$ and $\eta(s)$; namely, we consider the loop equation by putting $\zeta(s) = \eta(s) = 0$ from outside (i.e., after functional differentiations with respect to the loop coordinates). Extension to quadratic order in $\zeta(s)$ and $\eta(s)$ is given in Sec. IV.

First, our notations for the four-dimensional $\mathcal{N} = 4$ SYM are as follows. The field content of this theory is one gauge field, six scalar fields, and four gauginos which are four-dimensional Weyl spinors. In this paper we adopt the ten-dimensional $\mathcal{N} = 1$ notation: the gauge field and the scalar fields are expressed by A_μ ($\mu = 0, \ldots, 3$) and A_{3+i} ($i = 1, \ldots, 6$), respectively, and we combine four Weyl spinors to make one ten-dimensional Majorana-Weyl spinor Ψ. We have summarized our notations in Appendix A. Our action of the four-dimensional $\mathcal{N} = 4$ SYM is

$$\mathcal{L} = -\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{2}F_{MN}F^{MN} + i\bar{\Psi}\Gamma^M D_M \Psi\right), \tag{2.1}$$

with M and N running from 0 to 9. We have defined the field strengths and the covariant derivatives as follows:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \\ F_{\mu,3+i} &= -F_{3+i,\mu} = D_\mu A_{3+i}, \\ F_{3+i,3+j} &= i[A_{3+i}, A_{3+j}], \\ D_\mu \mathcal{O} &= \partial_\mu \mathcal{O} + i[A_\mu, \mathcal{O}], \\ D_{3+i}\mathcal{O} &= i[A_{3+i}, \mathcal{O}]. \end{aligned} \tag{2.2}$$

The ten-dimensional Dirac matrices Γ_M satisfy the following Clifford algebra:

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}, \tag{2.3}$$

where $\eta_{MN} = \mathrm{diag}(-1, 1, \ldots, 1)$ is the ten-dimensional flat metric. We will also use the four-dimensional flat metric $\eta^{(4)}_{MN} = \mathrm{diag}(-1, 1, 1, 1, 0, \ldots, 0)$. The action (2.1) is invariant under the SUSY transformation δ_ζ:

$$\delta_\zeta A_M = -i\bar{\zeta}\Gamma_M\Psi, \qquad \delta_\zeta\Psi = \tfrac{1}{2}F_{MN}\Gamma^{MN}\zeta, \tag{2.4}$$

with $\Gamma^{MN} = (1/2)[\Gamma^M, \Gamma^N]$. In (2.4), ζ is the fermionic variable with the same chirality as that of Ψ.

Let us start constructing the loop equation. As we stated in the previous section, we have to give the $\mathcal{N} = 4$ SYM extension of both the Wilson loop operator and the quadratic functional derivative with respect to the loop coordinates. The guiding principle of this extension is that the δ-term, namely, the term multiplied by $\delta(s_1 - s_2)$, be proportional to the EOM of $\mathcal{N} = 4$ SYM [see (1.7)]. The EOM of four-dimensional $\mathcal{N} = 4$ SYM is given by

$$D_M F^M{}_N - \bar{\Psi}\Gamma_N\Psi = 0, \tag{2.5}$$

$$\not{D}\Psi = 0, \tag{2.6}$$

with $\not{D} = \Gamma^M D_M$. Note that (2.5) expresses the EOM of the scalar fields as well as the gauge field. The first term of (2.5) can be obtained by considering the Wilson loop operator (1.1):

$$W(C) = \mathrm{Tr}\,\mathrm{P}\exp\left(i\int_0^l ds A_M(x(s))\dot{X}^M(s)\right) \equiv \mathrm{Tr}\,\mathcal{W}_0^l, \tag{2.7}$$

where we have introduced the ten-dimensional loop coordinates X^M with $X^\mu = x^\mu$ and $X^{3+i} = y^i$. By performing the ten-dimensional functional differentiation and repeating the derivation of (1.7), we obtain

$$\frac{\delta}{\delta X_M(s_2)}\frac{\delta}{\delta X^M(s_1)}W(C) = \mathrm{Tr}[(iF^M{}_N\dot{X}^N)_{s_1}\mathcal{W}_{s_1}^{s_2}(iF_{MP}\dot{X}^P)_{s_2}\mathcal{W}_{s_2}^{s_1+l}] + \delta(s_1 - s_2)\,\mathrm{Tr}[(iD_M F^M{}_N\dot{X}^N)_{s_1}\mathcal{W}_{s_1}^{s_1+l}]. \tag{2.8}$$

We see that the δ-term of (2.8) contains correctly the first term of (2.5). It is obvious that, in order to reproduce completely the LHS of (2.5) including the gaugino current, we have to introduce the fermionic fields in the Wilson loop. Previously, the following fermionic extension of the Wilson loop operator obtained as the "SUSY transformation" of (2.7) has been considered [8,9]:

$$W(C) = \mathrm{Tr}\,\mathrm{P}\exp\left(i\int_0^l ds\,\mathcal{A}_M(x(s), \zeta(s))\dot{X}^M(s)\right), \tag{2.9}$$

where $\mathcal{A}_M$ is the finite SUSY transformation of A_M:

$$\begin{aligned} \mathcal{A}_M(x, \zeta) &= A_M + \delta_\zeta A_M + \frac{1}{2}\delta_\zeta^2 A_M + \cdots \\ &= A_M - i\bar{\zeta}\Gamma_M\Psi - \frac{i}{4}F_{NP}\bar{\zeta}\Gamma_M\Gamma^{NP}\zeta + \cdots. \end{aligned} \tag{2.10}$$

In (2.9), the parameter ζ is promoted to a s-dependent fermionic loop coordinate $\zeta(s)$. However, it seems hard to reproduce the complete EOM of (2.5) by adopting this type of Wilson loop and a simple quadratic functional derivative.

In this paper we consider another type of Wilson loop operator by introducing an additional fermionic coordinate η. Our motivation of introducing such a coordinate is the dimension of the functional differential operator. The quadratic functional derivative on the LHS of (2.8) has mass-dimension 2, and we want a differential operator with respect to fermionic coordinates whose mass-dimension is also 2. Because the mass-dimension of ζ is $-1/2$, we are lead to the idea of introducing an additional fermionic variable η which carries mass-dimension $-3/2$ and therefore allows us to consider the following differential operator with mass-dimension 2:

$$\frac{\delta}{\delta\eta_\alpha(s_2)}\frac{\delta}{\delta\bar{\zeta}_\alpha(s_1)}, \tag{2.11}$$

where the α is the spinor index. The chirality of η must be opposite to that of ζ and Ψ.

Next we must fix the dependence of the Wilson loop operator on these two fermionic coordinates ζ and η. We have already given a well motivated way of introducing the coordinate ζ, i.e., through SUSY transformation (2.10).[5] On the other hand, we do not know such an origin of the variable η. In any case, the dependence of the Wilson loop on η should be determined from the requirement that the

[5] In Sec. IV we will find that the coefficients of the terms $\delta_\zeta^n A_M$ in (2.10) need to be modified for $n \geq 2$. Here we need only the terms with $n = 0$ and 1.

functional derivative (2.11) acting on the Wilson loop supply the needed gaugino current term in (2.5). Actually, if we set $\zeta = \eta = 0$ from outside, this requirement can be fulfilled by considering the following operator:

$$W(C) = \mathrm{Tr}\mathrm{P}\exp\bigg(i\int_0^l ds(A_M(x(s))\dot{X}^M(s) - i\bar{\zeta}(s)\Gamma_M\Psi(x(s))\dot{X}^M(s) + \overline{\Psi}(x(s))\dot{\eta}(s))\bigg). \tag{2.12}$$

Let us consider $K_{\beta_1}W(C)|_{\zeta=\eta=0}$ with quadratic functional derivative K_{β_1} defined by

$$K_{\beta_1} = \frac{\delta}{\delta X_M(s_2)}\frac{\delta}{\delta X^M(s_1)} + \beta_1\frac{\delta}{\delta\eta(s_2)}\frac{\delta}{\delta\bar{\zeta}(s_1)}, \tag{2.13}$$

and $W(C)$ given by (2.12). In (2.13), β_1 is a numerical coefficient to be determined below. Similarly to (1.6), the first derivatives of the Wilson line of (2.12) are given by

$$\frac{\delta}{\delta X^M(s)}\mathcal{W}_{u_1}^{u_2} = \mathcal{W}_{u_1}^s(iF_{MN}\dot{X}^N + \bar{\zeta}\Gamma_{[N}D_{M]}\Psi\dot{X}^N + iD_M\overline{\Psi}\,\dot{\eta} - \dot{\bar{\zeta}}\Gamma_M\Psi)_s\mathcal{W}_s^{u_2} + \mathcal{W}_{u_1}^s[\bar{\zeta}\Gamma_M\Psi, \bar{\zeta}\Gamma_N\Psi\dot{X}^N + i\overline{\Psi}\,\dot{\eta}]_s\mathcal{W}_s^{u_2} + \mathcal{W}_{u_1}^{u_2}(iA_M + \bar{\zeta}\Gamma_M\Psi)_{u_2}\delta(s-u_2) - (iA_M + \bar{\zeta}\Gamma_M\Psi)_{u_1}\mathcal{W}_{u_1}^{u_2}\delta(s-u_1), \tag{2.14}$$

$$\frac{\delta}{\delta\bar{\zeta}(s)}\mathcal{W}_{u_1}^{u_2} = \mathcal{W}_{u_1}^s(\Gamma_M\Psi\dot{X}^M)_s\mathcal{W}_s^{u_2}, \tag{2.15}$$

$$\frac{\delta}{\delta\eta(s)}\mathcal{W}_{u_1}^{u_2} = \mathcal{W}_{u_1}^s(iD_M\overline{\Psi}\dot{X}^M + i[\bar{\zeta}\Gamma_M\Psi\dot{X}^M + i\overline{\Psi}\,\dot{\eta}, \overline{\Psi}])_s\mathcal{W}_s^{u_2} + \mathcal{W}_{u_1}^{u_2}(-i\overline{\Psi})_{u_2}\delta(s-u_2) - (-i\overline{\Psi})_{u_1}\mathcal{W}_{u_1}^{u_2}\delta(s-u_1). \tag{2.16}$$

Using these formulas, we obtain

$$K_{\beta_1}W(C)|_{\zeta=\eta=0} = \mathrm{Tr}\,\mathrm{P}[((iF_{MN}\dot{X}^N)_{s_1}(iF^M{}_P\dot{X}^P)_{s_2} - \beta_1(\Gamma_N\Psi\dot{X}^N)_{s_1}(iD_P\overline{\Psi}\dot{X}^P)_{s_2})\mathcal{W}_0^l] + \delta(s_1 - s_2)\,\mathrm{Tr}[i((D_MF^M{}_N - 2\beta_1\overline{\Psi}\Gamma_N\Psi)\dot{X}^N)_{s_1}\mathcal{W}_{s_1}^{s_1+l}]. \tag{2.17}$$

We find that the last term of (2.17) contains the LHS of the EOM (2.5) if we set

$$\beta_1 = \tfrac{1}{2}. \tag{2.18}$$

Then carrying out the functional integration by parts as we did in (1.8), we get

$$K_{\beta_1=1/2}W(C)|_{\zeta=\eta=0} = \dot{X}^N(s_1)\dot{X}^P(s_2)\,\mathrm{Tr}\,\mathrm{P}\bigg[\bigg((iF_{MN})_{s_1}(iF^M{}_P)_{s_2} - \frac{1}{2}(\Gamma_N\Psi)_{s_1}(iD_P\overline{\Psi})_{s_2}\bigg)\mathcal{W}_0^l\bigg] - i\delta(s_1 - s_2)\frac{g^2}{2}\int_{s_1}^{s_1+l}ds\,\delta^{(4)}(x(s) - x(s_1))\dot{X}_N(s)\dot{X}^N(s_1)W(C_1)W(C_2). \tag{2.19}$$

In this way we have derived the loop equation by setting $\zeta = \eta = 0$ from outside.

Before closing this section, we will make some comments on the last δ-term of (2.19). Recall that C_1 (C_2) is the part of the loop C with its parameter region $[s_1, s]$ ($[s, s_1 + l]$). The existence of four-dimensional delta function $\delta^{(4)}(x(s) - x(s_1))$ implies that the integration with respect to s has contributions only from points satisfying $x(s) = x(s_1)$. There are two types of such contributions. One is the contribution from the point $s = s_1$, and this exists for any loop C. The other kind of contribution arises if the loop has self-intersecting points and if $x^\mu(s_1)$ is just one of these points. For the former contribution, either of the two loops C_1 and C_2 becomes trivial and we have

$$W(C_1)W(C_2) = \mathrm{Tr}[1]W(C) = NW(C). \tag{2.20}$$

For the latter contribution, none of the two loops become trivial and we should regard this term as the interacting part of the loop equation.

III. LOOP EQUATION AND HAMILTONIAN CONSTRAINT

In this section we will consider the limit $s_1 \to s_2$ in the loop equation (2.19). In this limit, (2.19) has some singularities. We will extract the most singular contribution to the terms linear in the Wilson loop. Namely, we neglect the contribution from the self-intersecting points of C in the last term of (2.19). We compare the resulting linear equation for the Wilson loop with the Hamiltonian constraint of bosonic string on $AdS_5 \times S^5$. Recall that we call the first and the second term on the RHS of (2.19) $\not{\partial}$-term and δ-term, respectively.

A. Linear and the most singular part of the loop equation

First we will consider the singular part of the $\not{\partial}$-term of (2.19) in the limit $s_1 \to s_2$. Singularities arise when two operators at s_1 and s_2 collide with each other. We evaluate these singularities by taking the contraction of the two operators by using the following UV regularized free propagators:

$$\overbrace{A^a_M(x)A^b_N}(\tilde{x}) = \frac{g^2}{4\pi^2}\frac{\delta^{ab}\eta_{MN}}{(x-\tilde{x})^2+\epsilon^2}, \qquad \overbrace{\Psi^a_\alpha(x)\overline{\Psi}^b_\beta}(\tilde{x}) = \frac{ig^2}{4\pi^2}\not{\partial}_{\alpha\beta}\frac{\delta^{ab}}{(x-\tilde{x})^2+\epsilon^2}, \tag{3.1}$$

where ϵ is the short distance cutoff parameter. Here, we consider only the leading order terms in the SYM coupling constant g. Using (3.1) we have

$$\partial_M \overbrace{A^a_N(x)\partial_P A^b_Q}(\tilde{x})\Big|_{x=\tilde{x}} = \frac{g^2}{4\pi^2}\frac{2\delta^{ab}\eta^{(4)}_{MP}\eta_{NQ}}{\epsilon^4}, \quad \partial_M \overbrace{\Psi^a_\alpha(x)\overline{\Psi}^b_\beta}(\tilde{x})\Big|_{x=\tilde{x}} = \frac{ig^2}{4\pi^2}\frac{-2(\Gamma^N)_{\alpha\beta}\eta^{(4)}_{MN}\delta^{ab}}{\epsilon^4}. \tag{3.2}$$

Using the first equation of (3.2) and to the leading order in g, we have the following contraction of two field strengths:

$$\overbrace{F^a_{MN}(x)F^b_{PQ}}(\tilde{x})\Big|_{x=\tilde{x}} = \frac{2g^2\delta^{ab}}{4\pi^2\epsilon^4}\left(\eta^{(4)}_{MP}\eta_{NQ} - \eta^{(4)}_{MQ}\eta_{NP} - \eta^{(4)}_{NP}\eta_{MQ} + \eta^{(4)}_{NQ}\eta_{MP}\right). \tag{3.3}$$

From these rules the singular part of the $\not{\partial}$-term of (2.19) can be evaluated to the leading order in g as

$$\begin{aligned}\lim_{s_1\to s_2}\left\{i(\overbrace{F_{MN})_{s_1} i(F^M}{}_P)_{s_2} - \frac{1}{2}(\Gamma_N \overbrace{\Psi)_{s_1}(iD_P\overline{\Psi}})_{s_2}\right\} &= \frac{\lambda}{\pi^2}\frac{1}{\epsilon^4}\left(-\left(\eta_{NP} + 2\eta^{(4)}_{NP}\right) + 2\eta^{(4)}_{NP}\right)\\ &= -\frac{\lambda}{\pi^2}\frac{\eta_{NP}}{\epsilon^4},\end{aligned} \tag{3.4}$$

where $\lambda = g^2 N$ is the 't Hooft coupling. It is interesting to observe the following: The contribution to the singular part from the bosonic fields and that from the fermionic fields do not have ten-dimensional covariance separately. However, once they are added using the coefficient $\beta_1 = 1/2$ (2.18), we regain the ten-dimensional covariance as in (3.4). Using (3.4), the most singular part of the $\not{\partial}$-term in (2.19) is given as follows:

$$\not{\partial}\text{-term of (2.19)} = -\frac{\lambda}{\pi^2}\frac{\dot{X}^M(s_1)\dot{X}_M(s_1)}{\epsilon^4}W(C) + O(1/\epsilon^3), \tag{3.5}$$

to the leading order in the coupling constant.

Next let us turn to the linear part in $W(C)$ of the δ-term of (2.19). We already explained that the linear part in $W(C)$ comes from the region $s \sim s_1$ in the s-integration of (2.19). Note that there are two kinds of singularities contained in the last term of (2.19) with $s_1 = s_2$. One is $\delta(s_1 - s_1)$ multiplying (2.19). Besides this, the s-integration around $s = s_1$ is divergent without putting $s_1 = s_2$. We will treat the former singularity in the next subsection. For the latter singularity, we adopt the following regularized four-dimensional delta function:

$$\delta^{(4)}(x) = \frac{2i}{\pi^2}\frac{\epsilon^2}{(x^2+\epsilon^2)^3}. \tag{3.6}$$

This regularization is consistent with the propagators (3.1) in the sense that $\partial_\mu\partial^\mu(x^2+\epsilon^2)^{-1} = i(2\pi)^2\delta^{(4)}(x)$. Using this delta function, we can evaluate the contribution to the s-integration from the region $s \sim s_1$ as follows:

$$\begin{aligned}\delta\text{-term of (2.19)} &\sim -i\delta(s_1 - s_2)\frac{\lambda}{2}\dot{X}_N(s_1)\dot{X}^N(s_1)W(C)\int ds\frac{2i}{\pi^2}\frac{\epsilon^2}{((s-s_1)^2(\dot{x}(s_1))^2+\epsilon^2)^3},\\ &= \frac{3\lambda}{8\pi}\frac{\dot{X}^N(s_1)\dot{X}_N(s_1)}{\epsilon^4}W(C)\frac{\epsilon\delta(s_1-s_1)}{\sqrt{(\dot{x}(s_1))^2}} + O(1/\epsilon^3).\end{aligned} \tag{3.7}$$

Here we have assumed that $\epsilon\delta(s_1 - s_1)/\sqrt{(\dot{x}(s_1))^2}$ is a finite quantity of order ϵ^0 (see the next subsection).

Finally, from (3.5) and (3.7), we obtain the following expression for the loop equation (2.19) with $s_1 = s_2$:

$$\left(-\frac{\delta}{\delta X_M(s_1)}\frac{\delta}{\delta X^M(s_1)} - \frac{1}{2}\frac{\delta}{\delta\eta(s_1)}\frac{\delta}{\delta\bar{\zeta}(s_1)} + \lambda\kappa\frac{\dot{X}^M(s_1)\dot{X}_M(s_1)}{\epsilon^4}\right)W(C)\bigg|_{\zeta=\eta=0} + \cdots = 0, \tag{3.8}$$

where κ is defined by

$$\kappa = \frac{3}{8\pi}\frac{\epsilon\delta(s_1 - s_1)}{\sqrt{(\dot{x}(s_1))^2}} - \frac{1}{\pi^2}. \tag{3.9}$$

In (3.8), the dots ... denote the less singular terms in ϵ, higher order terms in g, and the nonlinear terms in the Wilson loop. In κ of (3.9), the first term is the contribution from the δ-term and the second term is from the $\oint$-term.

B. Hamiltonian constraint on $\mathrm{AdS}_5 \times \mathrm{S}^5$

Let us compare (3.8) with the Hamiltonian constraint of the bosonic string on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The latter can be derived from the Polyakov action:

$$S_{\mathrm{Polyakov}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{ab}G_{MN}\partial_a\mathcal{X}^M(\sigma)\partial_b\mathcal{X}^N(\sigma) = \int d^2\sigma\mathcal{L}_{\mathrm{Polyakov}}, \tag{3.10}$$

where $\mathcal{X}^M(\sigma)$ $(M, N = 0, \ldots, 9)$ is the string coordinate, g_{ab} $(a, b = 0, 1)$ is the world sheet metric, and G_{MN} is the spacetime metric. The Hamiltonian constraint is

$$(2\pi\alpha')^2 G^{MN}\mathcal{P}_M\mathcal{P}_N + G_{MN}\partial_1\mathcal{X}^M\partial_1\mathcal{X}^N = 0. \tag{3.11}$$

We have introduced momentum $\mathcal{P}_M$ conjugate to $\mathcal{X}^M$:

$$\mathcal{P}_M(\sigma) = \frac{\partial\mathcal{L}_{\mathrm{Polyakov}}}{\partial(\partial_0\mathcal{X}^M(\sigma))} = -\frac{1}{2\pi\alpha'}\sqrt{-g}g^{0a}G_{MN}\partial_a\mathcal{X}^N(\sigma). \tag{3.12}$$

In the Poincaré coordinate of $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry with the line element

$$ds^2 = G_{MN}d\mathcal{X}^M d\mathcal{X}^N = \frac{R^2}{Y^2}(dx^\mu dx_\mu + dY^i dY^i), \tag{3.13}$$

(3.11) becomes

$$(2\pi\alpha')^2\frac{Y^2}{R^2}(\mathcal{P}_\mu\mathcal{P}^\mu + \mathcal{P}^Y_i\mathcal{P}^Y_i) + \frac{R^2}{Y^2}(\dot{x}^\mu\dot{x}_\mu + \dot{Y}^i\dot{Y}^i) = 0, \tag{3.14}$$

with $\mathcal{X}^M = (x^\mu, Y^i)$ and $\mathcal{P}_M = (\mathcal{P}_\mu, \mathcal{P}^Y_i)$. In (3.13) and (3.14), the index μ is raised or lowered using the four-dimensional flat metric $\eta^{(4)}_{\mu\nu}$, and the dot denotes the derivative with respect to σ^1. Let us identify x^μ with the four-dimensional coordinate where the SYM lives. The remaining six coordinates Y^i in (3.13), however, do not directly correspond to $y^i = X^{3+i}$ of the Wilson loop operator (2.12). Actually these two sets of coordinates y^i and Y^i should be related through T-duality [8,13]. T-duality on curved backgrounds is a subtle matter, but here we just assume that these two coordinates are related through (here we ignore the ordering problem)

$$\dot{y}^i = 2\pi\alpha' G^{3+i,M}\mathcal{P}_M, \qquad -i\frac{\delta}{\delta y^i} = \frac{1}{2\pi\alpha'}G_{3+i,M}\dot{\mathcal{X}}^M, \tag{3.15}$$

where we identify σ_1 with the parameter s of the Wilson loop. Using the momentum $P_M(s)$ conjugate to the loop coordinate $X^M(s)$ defining the Wilson loop,

$$P_M(s) = -i\frac{\delta}{\delta X^M(s)}, \tag{3.16}$$

the above relation (3.15) can be rewritten as

$$\dot{y}^i = 2\pi\alpha'\frac{Y^2}{R^2}\mathcal{P}^Y_i, \qquad P_{3+i} = \frac{1}{2\pi\alpha'}\frac{R^2}{Y^2}\dot{Y}^i. \tag{3.17}$$

Therefore, the Hamiltonian constraint (3.14) is expressed in terms of the coordinate $X^M = (x^\mu, y^i)$ and its conjugate P_M as

$$(2\pi\alpha')^2\frac{Y^2}{R^2}P_M P^M + \frac{R^2}{Y^2}\dot{X}^M\dot{X}_M = 0, \tag{3.18}$$

where the index M should be raised or lowered using the ten-dimensional flat metric η_{MN}.

Let us compare the loop equation (3.8) with the Hamiltonian constraint (3.18). We find that, if we make the following identifications,

$$\lambda = \frac{R^4}{2\alpha'^2}, \tag{3.19}$$

$$(2\pi^2\kappa)^{-1/4}\epsilon = Y, \tag{3.20}$$

the loop equation (3.8) can be regarded as the Hamiltonian constraint (3.18) acting on the Wilson loop up to the $(\delta/\delta\eta)(\delta/\delta\bar{\zeta})$ term and the omitted ... terms in (3.8). The first identification (3.19) is standard in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [1–5]. On the other hand, the second identification (3.20) is rather problematic and has no justification yet. Roughly, it identifies the UV cutoff ϵ in SYM with the radial coordinate Y of AdS_5. This may look natural if we recall that $Y = 0$ corresponds to the AdS boundary [32], and might imply that we are forced to consider only the strings on the AdS boundary. Another possibility would be that we can treat finite Y through the relation (3.20) in the limit $\epsilon \to 0$ of removing the UV cutoff by fine-tuning κ in such a way that the LHS of (3.20) is finite. For this fine-tuning, the first term $(3/8\pi)\epsilon\delta(s_1 - s_1)/\sqrt{(\dot{x}(s_1))^2}$ in κ (3.9) must be a finite quantity as we mentioned below (3.7), and it should be taken to $1/\pi^2$. This claims that the UV regularization of

$\delta(s_1 - s_1)$, namely, the string world sheet regularization, should be related with the spacetime regularization specified by ϵ. In any case, justification of (3.20) is indispensable for the identification of the Wilson loop with string field mentioned in Sec. I.

IV. LOOP EQUATION IN $\mathcal{N} = 4$ SYM II: LINEAR AND QUADRATIC TERMS IN FERMIONIC COORDINATES

In Sec. II, we have shown that our extended Wilson loop operator (2.12) satisfies the loop equation (2.19) with $\zeta = \eta = 0$. If we do not set $\zeta = \eta = 0$ from outside, the δ-term is no longer proportional to the EOM. For constructing the loop equation valid to higher powers in fermionic coordinates, we must further modify the Wilson loop operator and/or consider other types of functional derivatives. In this section, we will take the former prescription of modifying the Wilson loop and derive the loop equation valid to terms quadratic in fermionic coordinates. In Appendix D, we consider extending the quadratic functional derivative K_{β_1}. We find, however, that this does not change much the results of this section.

Let us take the following Wilson loop which is a generalization of (2.12):

$$W(C) = \mathrm{Tr}\,\mathrm{P}\exp\bigg(i\int_0^l ds(\mathcal{A}_M(x(s), \zeta(s))\dot{X}^M(s) + \overline{\Phi}(x(s), \zeta(s))\dot{\eta}(s) + \dot{\bar{\zeta}}(s)\Omega(x(s), \zeta(s)))\bigg), \tag{4.1}$$

with

$$\mathcal{A}_M = A_M + a_1\delta_\zeta A_M + a_2\delta_\zeta^2 A_M + a_3\delta_\zeta^3 A_M + \cdots, \tag{4.2}$$

$$\Phi_\alpha = \Psi_\alpha + b_1\delta_\zeta\Psi_\alpha + b_2\delta_\zeta^2\Psi_\alpha + b_3\delta_\zeta^3\Psi_\alpha + \cdots, \tag{4.3}$$

$$\Omega_\alpha = i(c_3\delta_\zeta A_M + \cdots)(\Gamma^M\zeta)_\alpha, \tag{4.4}$$

and consider $K_{\beta_1}W(C)$ with K_{β_1} given by (2.13) to quadratic order in ζ and η. Note that we have newly introduced the $\dot{\bar{\zeta}}$-term in (4.1). Operators (4.2), (4.3), and (4.4) are defined by using the SUSY transformation of SYM fields and the parameters a_n, b_n, and c_n.[6] The coefficient β_1 in K_{β_1} helps us to distinguish contributions from $(\delta/\delta X^M)^2$ and $\delta^2/\delta\eta\delta\bar{\zeta}$. Since the calculation of $K_{\beta_1}W(C)$ is lengthy and complicated, we present it in Appendix B. The results are given by (B17) and (B18) using the notations (B9)–(B16), and their explicit forms are found in Appendices B 2–B 4. If we do not put $\zeta = \eta = 0$ from outside, there appears the $\dot{\delta}(s_1 - s_2)$-term as well as the $\delta(s_1 - s_2)$-term in $K_{\beta_1}W(C)$, and we have

$$\begin{aligned} K_{\beta_1}W(C) = &\sum_{\mathcal{G},\tilde{\mathcal{G}}} \mathrm{Tr}\,\mathrm{P}[\mathcal{G}_{s_1}\tilde{\mathcal{G}}_{s_2}\mathcal{W}_0^l] \\ &+ \delta(s_1 - s_2)\,\mathrm{Tr}[\mathcal{O}_{s_1}\mathcal{W}_{s_1}^{s_1+l}] \\ &+ \dot{\delta}(s_1 - s_2)\,\mathrm{Tr}[\mathcal{Q}_{s_1}\mathcal{W}_{s_1}^{s_1+l}], \end{aligned} \tag{4.5}$$

where $\mathcal{G}$, $\tilde{\mathcal{G}}$, $\mathcal{O}$, and $\mathcal{Q}$ are SYM operators. We call the first, the second, and the last term on the RHS of (4.5) $\not{\delta}$-term, δ-term, and $\dot{\delta}$-term, respectively.

The parameters a_n, b_n, and c_n should be determined from the requirement that the operators $\mathcal{O}$ and $\mathcal{Q}$ in the δ- and $\dot{\delta}$-terms in (4.5) be proportional to EOM. In the following, we will summarize these operators and the conditions for them to vanish modulo EOM to terms quadratic in fermionic coordinates. To this order of the fermionic coordinates, it is sufficient to introduce $a_{1,2,3}$, $b_{1,2,3}$, and c_3. Among these, we put $a_1 = 1$ by fixing the normalization of ζ as we have already done in Sec. II. Therefore we have to determine the six remaining parameters.

A. Summary of operators and conditions

In this subsection, we summarize the operators $\mathcal{O}$ and $\mathcal{Q}$ appearing in (4.5) and the corresponding conditions to quadratic order in fermionic coordinates. First, the operator $\mathcal{Q}$ in the $\dot{\delta}$-term is given simply by [see (B20)]

$$\mathcal{Q} = 9\beta_1 b_2\bar{\zeta}\not{D}\Psi. \tag{4.6}$$

This is already proportional to the EOM (2.6) and leads to no conditions on the parameters.

Second, note that the operator $\mathcal{O}$ in the δ-term is given as a sum of operators which are multiplied by one of $(\dot{X}^M, \dot{\zeta}, \dot{\eta})$. We classify these operators by $(\dot{X}^M, \dot{\zeta}, \dot{\eta})$ and the power of ζ [the other fermionic coordinate η appears in $K_{\beta_1}W(C)$ only as $\dot{\eta}$]. In the following we present each operator and the corresponding condition. Detailed calculations are given in Appendix B, and we quote only the results:

(i) $\dot{X}$-term (B35)

$$\text{operator}: \quad i(D^M F_{MN} - 2\beta_1\overline{\Psi}\Gamma_N\Psi)\dot{X}^N, \tag{4.7}$$

$$\text{condition}: \quad \beta_1 = \tfrac{1}{2}. \tag{4.8}$$

(ii) $\dot{\zeta}$-term (B36)

$$\text{operator}: \quad -\dot{\bar{\zeta}}\not{D}\Psi, \tag{4.9}$$

$$\text{condition}: \quad \text{none.} \tag{4.10}$$

[6]The index n denotes the power of ζ in the exponent of (4.1). One might think it natural to introduce $A_M(\Gamma^M\zeta)_\alpha$ as the lowest order term in Ω_α. However, this operator is excluded since it breaks the gauge invariance of the Wilson loop.

(iii) $\dot{\eta}$-term (B37)

$$\text{operator:}\quad i\dot{\bar{\eta}}\not{D}^2\Psi - \frac{1-\beta_1 b_1}{2}[F_{MN}, \bar{\Psi}\Gamma^{MN}\dot{\eta}], \tag{4.11}$$

$$\text{condition:}\quad \beta_1 b_1 = 1. \tag{4.12}$$

(iv) $\zeta\dot{X}$-term (B40)

$$\begin{aligned}\text{operator:}\quad & (\bar{\zeta}\Gamma_M\not{D}^2\Psi - \bar{\zeta}D_M\not{D}\Psi)\dot{X}^M \\ & + \frac{i\beta_1 a_2}{2}[F_{NP}, \bar{\Psi}\{\Gamma_M, \Gamma^{NP}\}\zeta]\dot{X}^M \\ & - \frac{i(1-\beta_1 b_1)}{2}[F_{NP}, \bar{\zeta}\Gamma^{NP}\Gamma_M\Psi]\dot{X}^M,\end{aligned} \tag{4.13}$$

$$\text{conditions:}\quad \beta_1 a_2 = 0, \qquad \beta_1 b_1 = 1. \tag{4.14}$$

(v) $\zeta\dot{\zeta}$-term (B41)

$$\text{operator:}\quad (1 + 3\beta_1 c_3)\dot{\bar{\zeta}}\Gamma^M\zeta\bar{\Psi}\Gamma_M\Psi, \tag{4.15}$$

$$\text{condition:}\quad 1 + 3\beta_1 c_3 = 0. \tag{4.16}$$

(vi) $\zeta\dot{\eta}$-term (B47)

$$\begin{aligned}\text{operator:}\quad & i[\bar{\zeta}\not{D}\Psi, \bar{\Psi}\dot{\eta}] - i\beta_1 b_2(\dot{\bar{\eta}}\zeta\{\bar{\Psi}, \not{D}\Psi\} + [\bar{\Psi}\Gamma_M\zeta, \dot{\bar{\eta}}\Gamma^M\not{D}\Psi] - [\bar{\Psi}\dot{\eta}, \bar{\zeta}\not{D}\Psi]) \\ & + b_1(1-\beta_1 b_1)[F^M{}_N, F_{MP}]\bar{\zeta}\Gamma^{NP}\dot{\eta} - iD_N(b_1 D^M F_{MP} - 2\beta_1 b_2\bar{\Psi}\Gamma_P\Psi)\bar{\zeta}\Gamma^{NP}\dot{\eta} \\ & + 2i(1-\beta_1 b_2)[\bar{\zeta}\Gamma^M\Psi, D_M\bar{\Psi}\dot{\eta}],\end{aligned} \tag{4.17}$$

$$\text{conditions:}\quad b_1(1-\beta_1 b_1) = 0, \qquad b_1 = 2\beta_1 b_2, \qquad \beta_1 b_2 = 1. \tag{4.18}$$

(vii) $\zeta\zeta\dot{X}$-term (B56)

$$\begin{aligned}\text{operator:}\quad & a_2 D_Q(D^M F_{MP} - \bar{\Psi}\Gamma_P\Psi)\bar{\zeta}\Gamma_N\Gamma^{PQ}\zeta\dot{X}^N + [\bar{\zeta}\not{D}\Psi, \bar{\zeta}\Gamma_N\Psi]\dot{X}^N + (a_2 - \beta_1 a_3)([\bar{\zeta}\Gamma_N\Psi, \bar{\zeta}\not{D}\Psi] \\ & + [\bar{\zeta}\Gamma_M\Psi, \bar{\zeta}\Gamma_N\Gamma^M\not{D}\Psi])\dot{X}^N + ia_2(1-2\beta_1 b_1)[F^M{}_P, F_{MQ}]\bar{\zeta}\Gamma_N\Gamma^{PQ}\zeta\dot{X}^N \\ & + ia_2(1-\beta_1 b_1)[F_{PQ}, F_{MN}]\bar{\zeta}\Gamma^M\Gamma^{PQ}\zeta\dot{X}^N + 2(1 - a_2 - \beta_1 b_2)[\bar{\zeta}\Gamma^M\Psi, \bar{\zeta}\Gamma_N D_M\Psi]\dot{X}^N \\ & - (1 - \beta_1(2a_3 + 2b_2 + c_3))[\bar{\zeta}\Gamma^M\Psi, \bar{\zeta}\Gamma_M D_N\Psi]\dot{X}^N \\ & - (a_2 - \beta_1(3a_3 + b_2))[\bar{\zeta}\Gamma_N\Gamma^{MP}\Psi, \bar{\zeta}\Gamma_M D_P\Psi]\dot{X}^N,\end{aligned} \tag{4.19}$$

$$\begin{aligned}\text{conditions:}\quad & a_2(1-2\beta_1 b_1) = 0, \qquad a_2(1-\beta_1 b_1) = 0, \qquad 1 - a_2 - \beta_1 b_2 = 0, \\ & 1 - \beta_1(2a_3 + 2b_2 + c_3) = 0, \qquad a_2 - \beta_1(3a_3 + b_2) = 0.\end{aligned} \tag{4.20}$$

B. Solution to the conditions

In the previous subsection, we obtained 12 conditions on six parameters $a_{2,3}$, $b_{1,2,3}$, and c_3 [(4.8) is merely a repetition of the result (2.18) in Sec. II]. This (apparently overdetermined) set of conditions can in fact be consistently solved to give

$$\begin{aligned}& a_2 = 0, \qquad a_3 = -\tfrac{2}{3}, \qquad b_1 = 2, \qquad b_2 = 2, \\ & b_3 = \text{arbitrary}, \qquad c_3 = -\tfrac{2}{3}.\end{aligned} \tag{4.21}$$

Note that a_n and b_n in (4.21) are different from those for the finite SUSY transformation, $a_n = b_n = 1/n!$.

Using the above results, $K_{\beta_1=1/2}W(C)$ is now given by

$$K_{\beta_1=1/2}W(C) = \not{\partial}\text{-term} + 9\dot{\delta}(s_1 - s_2)\,\mathrm{Tr}[(\bar{\zeta}\not{D}\Psi)_{s_1}\mathcal{W}_{s_1}^{s_1+l}] + \delta(s_1 - s_2)\,\mathrm{Tr}[(i(D^M F_{MN} - \bar{\Psi}\Gamma_N\Psi)\dot{X}^N - \dot{\bar{\zeta}}\not{D}\Psi + i\dot{\bar{\eta}}\not{D}^2\Psi + (\bar{\zeta}\Gamma_M\not{D}^2\Psi - \bar{\zeta}D_M\not{D}\Psi)\dot{X}^M - 2iD_N(D^M F_{MP} - \bar{\Psi}\Gamma_P\Psi)\bar{\zeta}\Gamma^{NP}\dot{\eta} - i\dot{\bar{\eta}}\zeta\{\bar{\Psi}, \not{D}\Psi\} - i[\bar{\Psi}\Gamma_M\zeta, \dot{\bar{\eta}}\Gamma^M\not{D}\Psi] - \tfrac{1}{3}[\bar{\zeta}\Gamma_M\Psi, \bar{\zeta}\Gamma^M\Gamma_N\not{D}\Psi]\dot{X}^N)_{s_1}\mathcal{W}_{s_1}^{s_1+l}], \tag{4.22}$$

up to terms higher than quadratic in fermionic coordinates. The expression of the $\not{\partial}$-term is found in Appendix B 4. In Appendix C, we carry out the analysis of the most singular and linear part of the RHS of (4.22). This is an extension of the analysis presented in Sec. III to the quadratic order in the fermionic coordinates. The contribution from the $\not{\partial}$-term is given by (C12), and that from the δ- and $\dot{\delta}$-terms by (C19). We do not know whether whole of the most singular part, including, in particular, its fermionic coordinate part, has an interpretation as the Hamiltonian constraint of superstring on $AdS_5 \times S^5$. For this, we have to clarify the meaning of our fermionic coordinates ζ and η in the first-quantized superstring theory.

V. CONCLUSION AND DISCUSSIONS

We have investigated the loop equation of the four-dimensional $\mathcal{N} = 4$ SYM. We started with the Wilson loop operator introduced in [6,7], which contains six scalar fields as well as the gauge field, and depends on extra six bosonic coordinates $y^i(s)$ besides four-dimensional space-time coordinate $x^\mu(s)$. We extended this Wilson loop to include fermionic fields by introducing two fermionic coordinates $\zeta(s)$ and $\eta(s)$: $\zeta(s)$ was introduced as the parameter of the SUSY transformation, and $\eta(s)$ was needed by dimension counting arguments of the quadratic functional derivative of the loop equation. In Sec. II, we derived the loop equation by putting these fermionic coordinates equal to zero from outside. We found that a rather simple functional derivative is sufficient to derive the loop equation. In Sec. IV and Appendix B, we extended our loop equation to quadratic order in fermionic coordinates. In deriving this loop equation, we introduced six free parameters in the Wilson loop which should be fixed by requiring that the δ-terms be proportional to the EOM of $\mathcal{N} = 4$ SYM. This requirement leads to 12 conditions on the six parameters, which is apparently overdetermined. However, we can consistently solve this system of equations for the parameters. We expect that there is some cleverer and concise derivation of the loop equation valid to all powers of fermionic coordinates. Understanding the meaning of the fermionic coordinate η would be important for this purpose.

We also extracted the most singular and linear part of our loop equation to the lowest order in the gauge coupling constant. There are two origins of such singular part, which we called $\not{\partial}$-term and δ-term in Sec. III. It is interesting that the singular part from the $\not{\partial}$-term gets ten-dimensionally covariant only after the contributions from both the fermionic and the bosonic quadratic functional derivatives are added.

Our original aim of studying the loop equation is to extract some information about the string field theory on $AdS_5 \times S^5$ or pp-wave background. Although we could not perform such an investigation yet in this paper, we compared the most singular and linear part of our loop equation with the Hamiltonian constraint of bosonic string on $AdS_5 \times S^5$. We found that these two equations take the same form if we identify the UV cutoff ϵ of SYM and the radial coordinate Y of AdS_5.

However, there remain many problems to be clarified in the study of the loop equation in $\mathcal{N} = 4$ SYM: for example, establishing the loop equation valid to higher orders in fermionic coordinates, and more satisfactory analysis of the singular part of the loop equation. The latter problem includes the treatment of the UV cutoff ϵ and $\delta(s_1 - s_1)$, and the analysis beyond the expansion in SYM coupling constant. It is our future subject to resolve these problems to reach complete understanding of the relation between the loop equation in $\mathcal{N} = 4$ SYM and the string field equation on $AdS_5 \times S^5$.

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APPENDIX A: NOTATIONS AND CONVENTIONS

In this appendix, we summarize our notations and conventions. We use the following five sets of indices:

$$M, N = 0, \ldots, 9, \quad (\text{10-dim spacetime index}), \tag{A1}$$

$$\mu, \nu = 0, \ldots, 3, \quad (\text{4-dim spacetime index}), \tag{A2}$$

$$i, j = 1, \ldots, 6, \quad (\text{index for the scalars}), \tag{A3}$$

$$\alpha, \beta = 1, \ldots, 32, \quad (SO(9,1)\ \text{spinor index}), \tag{A4}$$

$$a, b = 1, \ldots, N^2 - 1, \quad (SU(N)\ \text{gauge index}). \tag{A5}$$

Our conventions for the flat metric tensors are

$$\eta_{MN} = \mathrm{diag}(-1, 1, \cdots, 1), \quad \eta^{(4)}_{MN} = \mathrm{diag}(-1, 1, 1, 1, 0, \ldots, 0). \tag{A6}$$

The fields in four-dimensional $\mathcal{N} = 4$ SYM are given as follows:

$$A_\mu = A^a_\mu t^a: \quad \text{gauge field}, \tag{A7}$$

$$A_{3+i} = A^a_{3+i} t^a: \quad \text{six scalar fields}, \tag{A8}$$

$$\Psi_\alpha = \Psi^a_\alpha t^a: \ \text{16 component 10-dim Majorana-Weyl spinor}, \tag{A9}$$

where the generators t^a of $SU(N)$ gauge group are Hermitian matrices normalized by

$$\mathrm{Tr}\,(t^a t^b) = \tfrac{1}{2}\delta^{ab}. \tag{A10}$$

Note that we have gathered four four-dimensional Weyl spinors to define one ten-dimensional Majorana-Weyl spinor Ψ.

The gamma matrices Γ_M and $\Gamma_{MN} = (1/2)[\Gamma_M, \Gamma_N]$ enjoy the following identities:

$$\Gamma_M \Gamma_N = \Gamma_{MN} + \eta_{MN}, \tag{A11}$$

$$\Gamma_M \Gamma^{PQ} = \Gamma^{PQ}\Gamma_M + 2\delta^P_M \Gamma^Q - 2\delta^Q_M \Gamma^P, \tag{A12}$$

$$\Gamma_M \Gamma^{MN} = -\Gamma^{MN}\Gamma_M = 9\Gamma^N, \tag{A13}$$

$$[\Gamma_{MN}, \Gamma_{PQ}] = 2(\eta_{MQ}\Gamma_{NP} - \eta_{MP}\Gamma_{NQ} - \eta_{NQ}\Gamma_{MP} + \eta_{NP}\Gamma_{MQ}). \tag{A14}$$

The Dirac conjugate of Ψ is defined by

$$\overline{\Psi}_\alpha = \Psi_\beta C_{\beta\alpha}, \tag{A15}$$

with the charge conjugation matrix C satisfying

$$(C\Gamma^M C^{-1})_{\alpha\beta} = -(\Gamma^M)_{\beta\alpha}. \tag{A16}$$

The following identity is often used in the calculations in Appendix B and also in showing the invariance of the action (2.1) under the SUSY transformation (2.4):

$$\bar{\xi}_1\Gamma^M\xi_2\bar{\xi}_3\Gamma_M\xi_4 + \bar{\xi}_1\Gamma^M\xi_3\bar{\xi}_4\Gamma_M\xi_2 + \bar{\xi}_1\Gamma^M\xi_4\bar{\xi}_2\Gamma_M\xi_3 = 0, \tag{A17}$$

where ξ_i $(i = 1, \ldots, 4)$ are ten-dimensional Majorana-Weyl spinors with a common chirality.

Finally, the antisymmetrization $A_{[M}B_{N]}$ is defined by

$$A_{[M}B_{N]} = A_M B_N - A_N B_M. \tag{A18}$$

APPENDIX B: CALCULATION OF $K_{\beta_1}W(C)$

In this appendix we present the explicit calculation of $K_{\beta_1}W(C)$ for our Wilson loop operator (4.1). First, in B 1 we give the expressions (B17) and (B18) valid without specifying the form of $\mathcal{A}_M$, Φ, and Ω. Then in B 2–B 4, each term of (B17) and (B18) is evaluated for $\mathcal{A}_M$, Φ, and Ω given by (4.2), (4.3), and (4.4).

1. Functional derivatives of Wilson loop

Let us consider the extended Wilson loop (4.1). The explicit expressions of $\mathcal{A}_M$ (4.2), Φ (4.3), and Ω (4.4) are given by

$$\mathcal{A}_M = A_M - i\bar{\zeta}\Gamma_M\Psi - a_2\frac{i}{2}F_{NP}\bar{\zeta}\Gamma_M\Gamma^{NP}\zeta + a_3\bar{\zeta}\Gamma_N D_P\Psi\bar{\zeta}\Gamma_M\Gamma^{NP}\zeta + \cdots, \tag{B1}$$

$$\Phi_\alpha = \Psi_\alpha + (b_1\tfrac{1}{2}F_{MN} + b_2 i\bar{\zeta}\Gamma_M D_N\Psi + b_3\Upsilon_{MN} + \cdots)(\Gamma^{MN}\zeta)_\alpha, \tag{B2}$$

$$\overline{\Phi}_\alpha = \overline{\Psi}_\alpha - (b_1\tfrac{1}{2}F_{MN} + b_2 i\bar{\zeta}\Gamma_M D_N\Psi + b_3\Upsilon_{MN} + \cdots)(\bar{\zeta}\Gamma^{MN})_\alpha, \tag{B3}$$

$$\Omega_\alpha = c_3\bar{\zeta}\Gamma_M\Psi(\Gamma^M\zeta)_\alpha + \cdots, \tag{B4}$$

where Υ_{MN} is defined by

$$\Upsilon_{MN} = -i[\bar{\zeta}\Gamma_M\Psi, \bar{\zeta}\Gamma_N\Psi] - \frac{i}{2}D_M F_{PQ}\bar{\zeta}\Gamma_N\Gamma^{PQ}\zeta, \tag{B5}$$

and dots denote terms with higher powers of fermionic coordinates. We have put $a_1 = 1$ in (B1). Similarly to the derivation of (1.6), we obtain the following formulas for the functional derivatives of the Wilson line $\mathcal{W}^{u_2}_{u_1}$:

$$\frac{\delta}{\delta X^M(s)}\mathcal{W}^{u_2}_{u_1} = \mathcal{W}^s_{u_1}(\mathcal{O}_M)_s\mathcal{W}^{u_2}_s + \mathcal{W}^{u_2}_{u_1}(i\mathcal{A}_M)_{u_2}\delta(s - u_2) - (i\mathcal{A}_M)_{u_1}\mathcal{W}^{u_2}_{u_1}\delta(s - u_1), \tag{B6}$$

$$\frac{\delta}{\delta \eta_\alpha(s)}\mathcal{W}^{u_2}_{u_1} = \mathcal{W}^s_{u_1}(\mathcal{O}_{\eta_\alpha})_s\mathcal{W}^{u_2}_s - \mathcal{W}^{u_2}_{u_1}(i\overline{\Phi}_\alpha)_{u_2}\delta(s - u_2) + (i\overline{\Phi}_\alpha)_{u_1}\mathcal{W}^{u_2}_{u_1}\delta(s - u_1), \tag{B7}$$

$$\frac{\delta}{\delta \bar{\zeta}_\alpha(s)}\mathcal{W}^{u_2}_{u_1} = \mathcal{W}^s_{u_1}(\mathcal{O}_{\bar{\zeta}_\alpha})_s\mathcal{W}^{u_2}_s + \mathcal{W}^{u_2}_{u_1}(i\Omega_\alpha)_{u_2}\delta(s - u_2) - (i\Omega_\alpha)_{u_1}\mathcal{W}^{u_2}_{u_1}\delta(s - u_1), \tag{B8}$$

with $\mathcal{O}_M$, $\mathcal{O}_{\eta_\alpha}$, and $\mathcal{O}_{\bar{\zeta}_\alpha}$ defined by

$$\mathcal{O}_M = i\mathcal{F}_{MN}\dot{X}^N + i\mathcal{D}_M\overline{\Phi}\,\dot{\eta} + i\dot{\bar{\zeta}}_\alpha\mathcal{F}_{M\alpha}, \tag{B9}$$

$$\mathcal{O}_{\eta_\alpha} = i\mathcal{D}_N\overline{\Phi}_\alpha\dot{X}^N - \{i\overline{\Phi}_\alpha, i\overline{\Phi}_\beta\}\dot{\eta}_\beta + i\dot{\bar{\zeta}}_\beta\mathcal{D}_\beta\overline{\Phi}_\alpha, \tag{B10}$$

$$\mathcal{O}_{\bar{\zeta}_\alpha} = i\mathcal{F}_{\alpha N}\dot{X}^N + i\mathcal{D}_\alpha\overline{\Phi}\,\dot{\eta} - i\dot{\bar{\zeta}}_\beta\mathcal{F}_{\alpha\beta}. \tag{B11}$$

Here we have introduced generalized field strengths $\mathcal{F}$ and covariant derivatives $\mathcal{D}$ as follows:

$$\mathcal{F}_{MN} = \partial_M\mathcal{A}_N - \partial_N\mathcal{A}_M + i[\mathcal{A}_M, \mathcal{A}_N], \tag{B12}$$

$$\mathcal{F}_{M\alpha} = -\mathcal{F}_{\alpha M} = \partial_M\Omega_\alpha - \partial_{\bar{\zeta}_\alpha}\mathcal{A}_M + i[\mathcal{A}_M, \Omega_\alpha], \tag{B13}$$

$$\mathcal{F}_{\alpha\beta} = \partial_{\bar{\zeta}_\alpha}\Omega_\beta + \partial_{\bar{\zeta}_\beta}\Omega_\alpha + i\{\Omega_\alpha, \Omega_\beta\}, \tag{B14}$$

$$\mathcal{D}_M\mathcal{O} = \partial_M\mathcal{O} + i[\mathcal{A}_M, \mathcal{O}], \tag{B15}$$

$$\mathcal{D}_\alpha\mathcal{O} = \partial_{\bar{\zeta}_\alpha}\mathcal{O} + \begin{cases} i[\Omega_\alpha, \mathcal{O}] & \mathcal{O}\text{: bosonic} \\ i\{\Omega_\alpha, \mathcal{O}\} & \mathcal{O}\text{: fermionic.} \end{cases} \tag{B16}$$

Using (B6)–(B8) twice, we get

$$\frac{\delta}{\delta X^M(s_2)}\frac{\delta}{\delta X_M(s_1)}W(C) = \mathrm{Tr}[(\mathcal{O}_M)_{s_1}\mathcal{W}_{s_1}^{s_2}(\mathcal{O}^M)_{s_2}\mathcal{W}_{s_2}^{s_1+l}] + \delta(s_1 - s_2)\,\mathrm{Tr}[(\mathcal{D}_M\mathcal{O}^M)_{s_1}\mathcal{W}_{s_1}^{s_1+l}], \tag{B17}$$

$$\begin{aligned} \frac{\delta}{\delta\eta_\alpha(s_2)}\frac{\delta}{\delta\bar{\zeta}_\alpha(s_1)}W(C) = &-\mathrm{Tr}[(\mathcal{O}_{\bar{\zeta}_\alpha})_{s_1}\mathcal{W}_{s_1}^{s_2}(\mathcal{O}_{\eta_\alpha})_{s_2}\mathcal{W}_{s_2}^{s_1+l}] - \delta(s_1 - s_2)\,\mathrm{Tr}[\{i\overline{\Phi}_\alpha, \mathcal{O}_{\bar{\zeta}_\alpha}\}_{s_1}\mathcal{W}_{s_1}^{s_1+l}] \\ &+ \dot{\delta}(s_1 - s_2)\,\mathrm{Tr}[(i\mathcal{D}_\alpha\overline{\Phi}_\alpha)_{s_1}\mathcal{W}_{s_1}^{s_1+l}]. \end{aligned} \tag{B18}$$

In the rest of this appendix, we will evaluate each term in (B17) and (B18) using the concrete expressions of $\mathcal{A}_M$, Φ, and Ω given by (B1)–(B4).

2. $\dot{\delta}$-term

The $\dot{\delta}$-term appears only in (B18) and is given by

$$\dot{\delta}(s_1 - s_2)\,\mathrm{Tr}[(i\mathcal{D}_\alpha\overline{\Phi}_\alpha)_{s_1}\mathcal{W}_{s_1}^{s_1+l}]. \tag{B19}$$

From (B3) and (B16), we have

$$\begin{aligned} \mathcal{D}_\alpha\overline{\Phi}_\alpha &= \partial_{\bar{\zeta}_\alpha}\overline{\Phi}_\alpha + i\{\Omega_\alpha, \overline{\Phi}_\alpha\} \\ &= -b_2 i(\Gamma_M D_N\Psi)_\alpha(\bar{\zeta}\Gamma^{MN})_\alpha + ib_3([(\Gamma_M\Psi)_\alpha, \bar{\zeta}\Gamma_N\Psi] - (M \leftrightarrow N))(\bar{\zeta}\Gamma^{MN})_\alpha \\ &\quad + b_3\frac{i}{2}D_M F_{PQ}(\{\Gamma_N, \Gamma^{PQ}\}\zeta)_\alpha(\bar{\zeta}\Gamma^{MN})_\alpha + ic_3\{\bar{\zeta}\Gamma_M\Psi(\Gamma^M\zeta)_\alpha, \overline{\Psi}_\alpha\} + O(\zeta^3) \\ &= -9ib_2\bar{\zeta}\not{D}\Psi + O(\zeta^3), \end{aligned} \tag{B20}$$

where we have used (A12) and (A13) and

$$D_M F_{PQ}\bar{\zeta}\Gamma^M\Gamma^P\Gamma^Q\zeta = 0. \tag{B21}$$

The last equation follows from the fact that $\bar{\zeta}\Gamma^M\Gamma^N\Gamma^P\zeta$ is totally antisymmetric and hence is cyclically symmetric with respect to (M, N, P) and that F_{PQ} satisfies the Bianchi identity.

3. δ-term

Let us calculate the δ-terms in (B17) and (B18):

$$\frac{\delta}{\delta X^M(s_2)}\frac{\delta}{\delta X_M(s_1)}W(C) \ni \delta(s_1 - s_2)\,\mathrm{Tr}[\mathcal{D}_M(i\mathcal{F}^M{}_N\dot{X}^N + i\mathcal{D}^M\overline{\Phi}\,\dot{\eta} + i\dot{\bar{\zeta}}_\alpha\mathcal{F}^M{}_\alpha)_{s_1}\mathcal{W}_{s_1}^{s_1+l}], \tag{B22}$$

$$\frac{\delta}{\delta\eta_\alpha(s_2)}\frac{\delta}{\delta\bar{\zeta}_\alpha(s_1)}W(C) \ni -\delta(s_1 - s_2)\,\mathrm{Tr}[\{i\overline{\Phi}_\alpha, i\mathcal{F}_{\alpha N}\dot{X}^N + i\mathcal{D}_\alpha\overline{\Phi}\,\dot{\eta} - i\dot{\bar{\zeta}}_\beta\mathcal{F}_{\alpha\beta}\}_{s_1}\mathcal{W}_{s_1}^{s_1+l}]. \tag{B23}$$

Here we have used the symbol $\ni$ to indicate that the LHS contains the RHS. The field strengths and the covariant derivatives are defined by (B12)–(B16) and their explicit expressions are as follows:

$$\mathcal{F}_{MN} = F_{MN} - i\bar{\zeta}\Gamma_{[N}D_{M]}\Psi - \frac{ia_2}{2}\bar{\zeta}\Gamma_{[N}\Gamma^{PQ}\zeta D_{M]}F_{PQ} - i[\bar{\zeta}\Gamma_M\Psi, \bar{\zeta}\Gamma_N\Psi] + O(\zeta^3), \tag{B24}$$

$$\mathcal{D}_M\bar{\Phi}_\alpha = D_M\Big(\bar{\Psi} - \frac{b_1}{2}F_{NP}\bar{\zeta}\Gamma^{NP}\Big)_\alpha + i[-i\bar{\zeta}\Gamma_M\Psi, \bar{\Psi}_\alpha] + O(\zeta^2), \tag{B25}$$

$$\begin{aligned}\mathcal{F}_{\alpha M} = -\mathcal{F}_{M\alpha} = {}& -i(\Gamma_M\Psi)_\alpha - \frac{ia_2}{2}F_{NP}(\{\Gamma_M, \Gamma^{NP}\}\zeta)_\alpha + a_3(\Gamma_N D_P\Psi)_\alpha\bar{\zeta}\Gamma_M\Gamma^{NP}\zeta + a_3\bar{\zeta}\Gamma_N D_P\Psi(\{\Gamma_M, \Gamma^{NP}\}\zeta)_\alpha \\ & - c_3\bar{\zeta}\Gamma_N D_M\Psi(\Gamma^N\zeta)_\alpha + O(\zeta^3),\end{aligned} \tag{B26}$$

$$\mathcal{D}_\alpha\bar{\Phi}_\beta = -b_2 i(\Gamma_M D_N\Psi)_\alpha(\bar{\zeta}\Gamma^{MN})_\beta - \Big(\frac{b_1}{2}F_{MN} + b_2 i\bar{\zeta}\Gamma_M D_N\Psi\Big)(\Gamma^{MN})_{\alpha\beta} + O(\zeta^2), \tag{B27}$$

$$\mathcal{F}_{\alpha\beta} = c_3(\Gamma_M\Psi)_\alpha(\Gamma^M\zeta)_\beta + c_3\bar{\zeta}\Gamma_M\Psi(-\Gamma^M C^{-1})_{\alpha\beta} + (\alpha \leftrightarrow \beta) + O(\zeta^2). \tag{B28}$$

Using (B24)–(B28), we have the following expressions for the ingredients of (B22) and (B23):

$$\begin{aligned}i\mathcal{D}^M\mathcal{F}_{MN}\dot{X}^N = {}& i\Big(D^M\Big(F_{MN} - i\bar{\zeta}\Gamma_{[N}D_{M]}\Psi - \frac{ia_2}{2}\bar{\zeta}\Gamma_{[N}\Gamma^{PQ}\zeta D_{M]}F_{PQ} - i[\bar{\zeta}\Gamma_M\Psi, \bar{\zeta}\Gamma_N\Psi]\Big) \\ & + [\bar{\zeta}\Gamma^M\Psi, F_{MN} - i\bar{\zeta}\Gamma_{[N}D_{M]}\Psi] + \frac{a_2}{2}[F_{PQ}\bar{\zeta}\Gamma^M\Gamma^{PQ}\zeta, F_{MN}]\Big)\dot{X}^N,\end{aligned} \tag{B29}$$

$$i\mathcal{D}^2\bar{\Phi}\,\dot{\eta} = iD^2\Big(\bar{\Psi}\,\dot{\eta} - \frac{b_1}{2}F_{MN}\bar{\zeta}\Gamma^{MN}\dot{\eta}\Big) + iD^M[\bar{\zeta}\Gamma_M\Psi, \bar{\Psi}\,\dot{\eta}] + i[\bar{\zeta}\Gamma^M\Psi, D_M\bar{\Psi}\,\dot{\eta}], \tag{B30}$$

$$i\dot{\bar{\zeta}}_\alpha\mathcal{D}^M\mathcal{F}_{M\alpha} = -D^M\Big(\dot{\bar{\zeta}}\Gamma_M\Psi + \frac{a_2}{2}F_{NP}\dot{\bar{\zeta}}\{\Gamma_M, \Gamma^{NP}\}\zeta\Big) - [\bar{\zeta}\Gamma^M\Psi, \dot{\bar{\zeta}}\Gamma_M\Psi], \tag{B31}$$

$$\begin{aligned}\{\bar{\Phi}_\alpha, \mathcal{F}_{\alpha M}\dot{X}^M\} = {}& \Big\{\bar{\Psi}_\alpha, -i(\Gamma_M\Psi)_\alpha - \frac{ia_2}{2}F_{NP}(\{\Gamma_M, \Gamma^{NP}\}\zeta)_\alpha + a_3(\Gamma_N D_P\Psi)_\alpha\bar{\zeta}\Gamma_M\Gamma^{NP}\zeta + a_3\bar{\zeta}\Gamma_N D_P\Psi(\{\Gamma_M, \Gamma^{NP}\}\zeta)_\alpha \\ & - c_3\bar{\zeta}\Gamma_N D_M\Psi(\Gamma^N\zeta)_\alpha\Big\}\dot{X}^M - \Big\{\frac{b_1}{2}F_{QR}(\bar{\zeta}\Gamma^{QR})_\alpha, -i(\Gamma_M\Psi)_\alpha - \frac{ia_2}{2}F_{NP}(\{\Gamma_M, \Gamma^{NP}\}\zeta)_\alpha\Big\}\dot{X}^M \\ & - \{b_2 i\bar{\zeta}\Gamma_N D_P\Psi(\bar{\zeta}\Gamma^{NP})_\alpha, -i(\Gamma_M\Psi)_\alpha\}\dot{X}^M,\end{aligned} \tag{B32}$$

$$\begin{aligned}\{\bar{\Phi}_\alpha, \mathcal{D}_\alpha\bar{\Phi}\,\dot{\eta}\} = {}& \Big\{\bar{\Psi}_\alpha, -b_2 i(\Gamma_M D_N\Psi)_\alpha\bar{\zeta}\Gamma^{MN}\dot{\eta} - \Big(\frac{b_1}{2}F_{MN} + b_2 i\bar{\zeta}\Gamma_M D_N\Psi\Big)(\Gamma^{MN}\dot{\eta})_\alpha\Big\} \\ & + \Big\{-\frac{b_1}{2}F_{PQ}(\bar{\zeta}\Gamma^{PQ})_\alpha, -\frac{b_1}{2}F_{MN}(\Gamma^{MN}\dot{\eta})_\alpha\Big\},\end{aligned} \tag{B33}$$

$$-\{\bar{\Phi}_\alpha, \dot{\bar{\zeta}}_\beta\mathcal{F}_{\alpha\beta}\} = -c_3\{\bar{\Psi}_\alpha, -(\Gamma_M\Psi)_\alpha\dot{\bar{\zeta}}\Gamma^M\zeta + 2\bar{\zeta}\Gamma_M\Psi(\Gamma^M\dot{\zeta})_\alpha + \dot{\bar{\zeta}}\Gamma_M\Psi(\Gamma^M\zeta)_\alpha\}. \tag{B34}$$

Here we have kept only terms at most quadratic in fermionic coordinates.

Let us present the explicit form of the operator $\mathcal{O}$ in the δ-term of $K_{\beta_1}W(C)$ [see (4.5)]. As we explained in Sec. IVA, it is given as a sum of terms which are classified by $(\dot{X}^M, \dot{\zeta}, \dot{\eta})$ and the power of ζ multiplying them. In the following, terms without β_1 come from (B22) [and hence from (B29)–(B31)], while those multiplied by β_1 from (B23) [and hence from (B32)–(B34)]:

(i) $\dot{X}$-term

This term comes from (B29) and (B32):

$$iD^M F_{MN}\dot{X}^N + \beta_1\{\overline{\Psi}, -i\Gamma_M\Psi\}\dot{X}^M = i(D^M F_{MN} - 2\beta_1\overline{\Psi}\Gamma_N\Psi)\dot{X}^N. \tag{B35}$$

(ii) $\dot{\zeta}$-term

This term appears only in (B31):

$$-\dot{\overline{\zeta}}\not{D}\Psi. \tag{B36}$$

(iii) $\dot{\eta}$-term

This term comes from (B30) and (B33):

$$iD^2\overline{\Psi}\,\dot{\eta} - \frac{\beta_1 b_1}{2}[\overline{\Psi}\Gamma^{MN}\dot{\eta}, F_{MN}] = i\dot{\overline{\eta}}\not{D}^2\Psi - \frac{1-\beta_1 b_1}{2}[F_{MN}, \overline{\Psi}\Gamma^{MN}\dot{\eta}], \tag{B37}$$

where we have used

$$D^2\Psi = \not{D}^2\Psi - \tfrac{1}{2}\Gamma^{MN}D_{[M}D_{N]}\Psi, \tag{B38}$$

and

$$D_{[M}D_{N]}\mathcal{O} = i[F_{MN}, \mathcal{O}], \tag{B39}$$

for any $\mathcal{O}$.

(iv) $\zeta\dot{X}$-term

The $O(\zeta^1)$ terms in (B29) and (B32) contribute to this term:

$$\begin{aligned} &i(-i\overline{\zeta}D^M\Gamma_{[N}D_{M]}\Psi + [\overline{\zeta}\Gamma^M\Psi, F_{MN}])\dot{X}^N + \frac{\beta_1}{2}([\overline{\Psi}\{\Gamma_M, \Gamma^{NP}\}\zeta, -ia_2F_{NP}] + ib_1[F_{NP}, \overline{\zeta}\Gamma^{NP}\Gamma_M\Psi])\dot{X}^M \\ &\quad = (\overline{\zeta}\Gamma_M\not{D}^2\Psi - \overline{\zeta}D_M\not{D}\Psi)\dot{X}^M + \frac{i\beta_1 a_2}{2}[F_{NP}, \overline{\Psi}\{\Gamma_M, \Gamma^{NP}\}\zeta]\dot{X}^M - \frac{i(1-\beta_1 b_1)}{2}[F_{NP}, \overline{\zeta}\Gamma^{NP}\Gamma_M\Psi]\dot{X}^M, \end{aligned} \tag{B40}$$

where we have used (B38), (B39), and (A12).

(v) $\zeta\dot{\zeta}$-term

This term comes from (B31) and (B34):

$$\begin{aligned} &-\frac{a_2}{2}D_MF_{NP}\dot{\overline{\zeta}}\{\Gamma^M, \Gamma^{NP}\}\zeta - [\overline{\zeta}\Gamma^M\Psi, \dot{\overline{\zeta}}\Gamma_M\Psi] - \beta_1c_3(-\{\overline{\Psi}, \Gamma_M\Psi\}\dot{\overline{\zeta}}\Gamma^M\zeta + 2[\overline{\Psi}\Gamma^M\dot{\zeta}, \overline{\zeta}\Gamma_M\Psi] + [\overline{\Psi}\Gamma^M\zeta, \dot{\overline{\zeta}}\Gamma_M\Psi]) \\ &\quad = (1 + 3\beta_1c_3)\dot{\overline{\zeta}}\Gamma^M\zeta\overline{\Psi}\Gamma_M\Psi. \end{aligned} \tag{B41}$$

Here we have used

$$D_MF_{NP}\dot{\overline{\zeta}}\{\Gamma^M, \Gamma^{NP}\}\zeta = D_MF_{NP}\frac{d}{ds}(\overline{\zeta}\Gamma^M\Gamma^{NP}\zeta) = 0, \tag{B42}$$

where the last equality is due to the same argument as for (B21). We have also used (A17) to rewrite all other terms into the form of the RHS.

(vi) $\zeta\dot{\eta}$-term

This term has contributions from (B30) and (B33):

$$\begin{aligned} &-\frac{ib_1}{2}D^2F_{MN}\overline{\zeta}\Gamma^{MN}\dot{\eta} + iD^M[\overline{\zeta}\Gamma_M\Psi, \overline{\Psi}\,\dot{\eta}] + i[\overline{\zeta}\Gamma^M\Psi, D_M\overline{\Psi}\,\dot{\eta}] + \beta_1\bigg(-ib_2(\{\overline{\Psi}, \Gamma_MD_N\Psi\}\overline{\zeta}\Gamma^{MN}\dot{\eta} \\ &\quad + [\overline{\Psi}\Gamma^{MN}\dot{\eta}, \overline{\zeta}\Gamma_MD_N\Psi]) + \frac{b_1^2}{8}[F_{PQ}, F_{MN}]\overline{\zeta}[\Gamma^{PQ}, \Gamma^{MN}]\dot{\eta}\bigg). \end{aligned} \tag{B43}$$

Let us make the following rewritings of the terms in (B43). The first term is rewritten using

$$D^2F_{MN} = 2i[F^P{}_M, F_{PN}] - D_{[M}D^PF_{N]P}, \tag{B44}$$

which is obtained by covariant differentiating the Bianchi identity and using (B39). For the last term of (B43) we use (A14). The terms multiplied by b_2 are rewritten as follows:

$$\{\overline{\Psi}, \Gamma_M D_N \Psi\}\overline{\zeta}\Gamma^{MN}\dot{\eta} = D_N(\overline{\Psi}\Gamma_M\Psi)\overline{\zeta}\Gamma^{MN}\dot{\eta}, \tag{B45}$$

$$[\overline{\Psi}\Gamma^{MN}\dot{\eta}, \overline{\zeta}\Gamma_M D_N\Psi] = \overline{\zeta}\Gamma^{MN}\dot{\eta}\zeta D_N(\overline{\Psi}\Gamma_M\Psi) + 2[\overline{\zeta}\Gamma^M\Psi, D_M\overline{\Psi}\dot{\eta}] + \dot{\overline{\eta}}\zeta\{\overline{\Psi}, \not{D}\Psi\} + [\overline{\Psi}\Gamma_M\zeta, \dot{\overline{\eta}}\Gamma^M\not{D}\Psi] - [\overline{\Psi}\dot{\eta}, \overline{\zeta}\not{D}\Psi], \tag{B46}$$

where we have used (A17) for the latter. Gathering all the terms, we find that (B43) is rewritten as

$$i[\overline{\zeta}\not{D}\Psi, \overline{\Psi}\dot{\eta}] - i\beta_1 b_2(\dot{\overline{\eta}}\zeta\{\overline{\Psi}, \not{D}\Psi\} + [\overline{\Psi}\Gamma_M\zeta, \dot{\overline{\eta}}\Gamma^M\not{D}\Psi] - [\overline{\Psi}\dot{\eta}, \overline{\zeta}\not{D}\Psi]) + (b_1 - \beta_1 b_1^2)[F^M{}_N, F_{MP}]\overline{\zeta}\Gamma^{NP}\dot{\eta} - iD_N(b_1 D^M F_{MP} - 2\beta_1 b_2\overline{\Psi}\Gamma_P\Psi)\overline{\zeta}\Gamma^{NP}\dot{\eta} + 2i(1 - \beta_1 b_2)[\overline{\zeta}\Gamma^M\Psi, D_M\overline{\Psi}\dot{\eta}]. \tag{B47}$$

(vii) $\zeta\zeta\dot{X}$-term

This term comes from (B29) and (B32):

$$i\Big(-\frac{ia_2}{2}\overline{\zeta}\Gamma_{[N}\Gamma^{PQ}\zeta D^M D_{M]}F_{PQ} + \frac{a_2}{2}[F_{PQ}, F_{MN}]\overline{\zeta}\Gamma^M\Gamma^{PQ}\zeta - iD^M[\overline{\zeta}\Gamma_M\Psi, \overline{\zeta}\Gamma_N\Psi] - i[\overline{\zeta}\Gamma^M\Psi, \overline{\zeta}\Gamma_{[N}D_{M]}\Psi]\Big)\dot{X}^N + \beta_1\Big(a_3\{\overline{\Psi}, \Gamma_N D_P\Psi\}\overline{\zeta}\Gamma_M\Gamma^{NP}\zeta + a_3[\overline{\Psi}\{\Gamma_M, \Gamma^{NP}\}\zeta, \overline{\zeta}\Gamma_N D_P\Psi] - c_3[\overline{\Psi}\Gamma^N\zeta, \overline{\zeta}\Gamma_N D_M\Psi] + \frac{ia_2b_1}{4}[F_{QR}, F_{NP}]\overline{\zeta}\Gamma^{QR}\{\Gamma_M, \Gamma^{NP}\}\zeta - b_2[\overline{\zeta}\Gamma_N D_P\Psi, \overline{\zeta}\Gamma^{NP}\Gamma_M\Psi]\Big)\dot{X}^M. \tag{B48}$$

First we consider the three terms multiplied by a_2. The sum of the first two a_2-terms is rewritten as follows:

$$D^M(D_M F_{PQ}\overline{\zeta}\Gamma_N\Gamma^{PQ}\zeta - D_N F_{PQ}\overline{\zeta}\Gamma_M\Gamma^{PQ}\zeta)\dot{X}^N + i[F_{PQ}, F_{MN}]\overline{\zeta}\Gamma^M\Gamma^{PQ}\zeta\dot{X}^N = 2D_Q(D^M F_{MP} - \overline{\Psi}\Gamma_P\Psi)\overline{\zeta}\Gamma_N\Gamma^{QP}\zeta\dot{X}^N + 2\{\overline{\Psi}, \Gamma_P D_Q\Psi\}\overline{\zeta}\Gamma_N\Gamma^{QP}\zeta\dot{X}^N + 2i[F^M{}_P, F_{MQ}]\overline{\zeta}\Gamma_N\Gamma^{PQ}\zeta\dot{X}^N + 2i[F_{PQ}, F_{MN}]\overline{\zeta}\Gamma^M\Gamma^{PQ}\zeta\dot{X}^N, \tag{B49}$$

where we have used (B39), (B44), and (B21). Here we have added and subtracted the current term:

$$D_Q(\overline{\Psi}\Gamma_P\Psi)\overline{\zeta}\Gamma_N\Gamma^{QP}\zeta\dot{X}^N = \{\overline{\Psi}, \Gamma_P D_Q\Psi\}\overline{\zeta}\Gamma_N\Gamma^{QP}\zeta\dot{X}^N. \tag{B50}$$

The remaining a_2-term which is multiplied by β_1 can be rewritten as follows:

$$[F_{QR}, F_{NP}]\overline{\zeta}\Gamma^{QR}\{\Gamma_M, \Gamma^{NP}\}\zeta\dot{X}^M = -8[F_{QR}, F^Q{}_N]\overline{\zeta}\Gamma_M\Gamma^{RN}\zeta\dot{X}^M - 4[F_{QR}, F_{PM}]\overline{\zeta}\Gamma^P\Gamma^{QR}\zeta\dot{X}^M, \tag{B51}$$

where we have used (A12) and (A14).

Next we proceed to the remaining terms with fermionic fields. We adopt three operators of the following forms as the basis of independent operators:

$$[\overline{\zeta}\Gamma^M\Psi, \overline{\zeta}\Gamma_N D_M\Psi]\dot{X}^N, \qquad [\overline{\zeta}\Gamma^M\Psi, \overline{\zeta}\Gamma_M D_N\Psi]\dot{X}^N, \qquad [\overline{\zeta}\Gamma_N\Gamma^{MP}\Psi, \overline{\zeta}\Gamma_M D_P\Psi]\dot{X}^N, \tag{B52}$$

and rewrite the terms in (B48) in terms of this basis and terms linear in EOM (2.6). Terms with a_3 are rewritten as follows:

$$\{\overline{\Psi}, \Gamma_N D_P\Psi\}\overline{\zeta}\Gamma_M\Gamma^{NP}\zeta\dot{X}^M = (\overline{\zeta}\Gamma_M\Gamma^P\Gamma^N\Psi^a D_P\overline{\Psi}^b\Gamma_N\zeta + \overline{\zeta}\Gamma_M\Gamma^P\Gamma^N D_P\Psi^b\overline{\zeta}\Gamma_N\Psi^a)\dot{X}^M[t^a, t^b] = [\overline{\zeta}\Gamma_M\Gamma^{NP}\Psi, \overline{\zeta}\Gamma_N D_P\Psi]\dot{X}^M - [\overline{\zeta}\Gamma_M\Psi, \overline{\zeta}\not{D}\Psi]\dot{X}^M - [\overline{\zeta}\Gamma_N\Psi, \overline{\zeta}\Gamma_M\Gamma^N\not{D}\Psi]\dot{X}^M + 2[\overline{\zeta}\Gamma^N\Psi, \overline{\zeta}\Gamma_M D_N\Psi]\dot{X}^M, \tag{B53}$$

and

$$[\overline{\Psi}\{\Gamma_N, \Gamma^{MP}\}\zeta, \overline{\zeta}\Gamma_M D_P\Psi]\dot{X}^N = 2([\overline{\zeta}\Gamma_N\Gamma^{MP}\Psi, \overline{\zeta}\Gamma_M D_P\Psi] - [\overline{\zeta}\Gamma^P\Psi, \overline{\zeta}\Gamma_N D_P\Psi] + [\overline{\zeta}\Gamma^M\Psi, \overline{\zeta}\Gamma_M D_N\Psi])\dot{X}^N, \tag{B54}$$

where we have used (A17) for (B53), and (A12) for (B54). Note that (B53) is just the current term (B50) (up to sign) and we also make the rewriting (B53) for the second term on the RHS of (B49). For the b_2-term we have

$$[\bar{\zeta}\Gamma_N D_P\Psi, \bar{\zeta}\Gamma^{NP}\Gamma_M\Psi]\dot{X}^M = (-[\bar{\zeta}\Gamma_M\Gamma^{NP}\Psi, \bar{\zeta}\Gamma_N D_P\Psi] - 2[\bar{\zeta}\Gamma^N\Psi, \bar{\zeta}\Gamma_N D_M\Psi] + 2[\bar{\zeta}\Gamma^P\Psi, \bar{\zeta}\Gamma_M D_P\Psi])\dot{X}^M, \tag{B55}$$

where we have used (A12).

Gathering all the terms we finally obtain the following expression of (B48):

$$\begin{aligned}
&a_2 D_Q(D^M F_{MP} - \bar{\Psi}\Gamma_P\Psi)\bar{\zeta}\Gamma_N\Gamma^{PQ}\zeta\dot{X}^N + [\bar{\zeta}\not{D}\Psi, \bar{\zeta}\Gamma_N\Psi]\dot{X}^N + (a_2 - \beta_1 a_3)([\bar{\zeta}\Gamma_N\Psi, \bar{\zeta}\not{D}\Psi] \\
&\quad + [\bar{\zeta}\Gamma_M\Psi, \bar{\zeta}\Gamma_N\Gamma^M\not{D}\Psi])\dot{X}^N + ia_2(1 - 2\beta_1 b_1)[F^M{}_P, F_{MQ}]\bar{\zeta}\Gamma_N\Gamma^{PQ}\zeta\dot{X}^N \\
&\quad + ia_2(1 - \beta_1 b_1)[F_{PQ}, F_{MN}]\bar{\zeta}\Gamma^M\Gamma^{PQ}\zeta\dot{X}^N + 2(1 - a_2 - \beta_1 b_2)[\bar{\zeta}\Gamma^M\Psi, \bar{\zeta}\Gamma_N D_M\Psi]\dot{X}^N \\
&\quad - (1 - \beta_1(2a_3 + 2b_2 + c_3))[\bar{\zeta}\Gamma^M\Psi, \bar{\zeta}\Gamma_M D_N\Psi]\dot{X}^N - (a_2 - \beta_1(3a_3 + b_2))[\bar{\zeta}\Gamma_N\Gamma^{MP}\Psi, \bar{\zeta}\Gamma_M D_P\Psi]\dot{X}^N.
\end{aligned} \tag{B56}$$

4. $\not{\delta}$-term

The $\not{\delta}$-terms without a delta function are

$$\frac{\delta}{\delta X_M(s_2)}\frac{\delta}{\delta X^M(s_1)}W(C) \ni \mathrm{Tr}[(i\mathcal{F}_{MN}\dot{X}^N + iD_M\bar{\Phi}\,\dot{\eta} + i\dot{\bar{\zeta}}_\alpha\mathcal{F}_{M\alpha})_{s_1}\mathcal{W}_{s_1}^{s_2}(i\mathcal{F}^M{}_P\dot{X}^P + iD^M\bar{\Phi}\,\dot{\eta} + i\dot{\bar{\zeta}}_\beta\mathcal{F}^M{}_\beta)_{s_2}\mathcal{W}_{s_2}^{s_1+l}], \tag{B57}$$

$$\begin{aligned}
\frac{\delta}{\delta\eta_\alpha(s_2)}\frac{\delta}{\delta\bar{\zeta}_\alpha(s_1)}W(C) \ni -\mathrm{Tr}[&(i\mathcal{F}_{\alpha N}\dot{X}^N + i\mathcal{D}_\alpha\bar{\Phi}\,\dot{\eta} - i\dot{\bar{\zeta}}_\beta\mathcal{F}_{\alpha\beta})_{s_1}\mathcal{W}_{s_1}^{s_2}(i\mathcal{D}_N\bar{\Phi}_\alpha\dot{X}^N - \{i\bar{\Phi}_\alpha, i\bar{\Phi}_\beta\}\dot{\eta}_\beta \\
&+ i\dot{\bar{\zeta}}_\beta\mathcal{D}_\beta\bar{\Phi}_\alpha)_{s_2}\mathcal{W}_{s_2}^{s_1+l}],
\end{aligned} \tag{B58}$$

with

$$i\mathcal{F}_{MN}\dot{X}^N = i\left(F_{MN} - i\bar{\zeta}\Gamma_{[N}D_{M]}\Psi - \frac{ia_2}{2}\bar{\zeta}\Gamma_{[N}\Gamma^{PQ}\zeta D_{M]}F_{PQ}\right)\dot{X}^N, \tag{B59}$$

$$i\mathcal{D}_M\bar{\Phi}\,\dot{\eta} = iD_M\left(\bar{\Psi}\,\dot{\eta} - \frac{b_1}{2}F_{NP}\bar{\zeta}\Gamma^{NP}\dot{\eta}\right), \tag{B60}$$

$$i\dot{\bar{\zeta}}_\alpha\mathcal{F}_{M\alpha} = -\dot{\bar{\zeta}}\Gamma_M\Psi - \frac{a_2}{2}F_{NP}\dot{\bar{\zeta}}\{\Gamma_M, \Gamma^{NP}\}\zeta, \tag{B61}$$

$$\begin{aligned}
i\mathcal{F}_{\alpha N}\dot{X}^N = \Big(&(\Gamma_N\Psi)_\alpha + \frac{a_2}{2}F_{MP}(\{\Gamma_N, \Gamma^{MP}\}\zeta)_\alpha + ia_3(\Gamma_M D_P\Psi)_\alpha\bar{\zeta}\Gamma_N\Gamma^{MP}\zeta + ia_3\bar{\zeta}\Gamma_M D_P\Psi(\{\Gamma_N, \Gamma^{MP}\}\zeta)_\alpha \\
&- ic_3\bar{\zeta}\Gamma_M D_N\Psi(\Gamma^M\zeta)_\alpha\Big)\dot{X}^N,
\end{aligned} \tag{B62}$$

$$i\mathcal{D}_\alpha\bar{\Phi}\,\dot{\eta} = -\frac{ib_1}{2}F_{MN}(\Gamma^{MN}\dot{\eta})_\alpha + b_2((\Gamma_M D_N\Psi)_\alpha\bar{\zeta}\Gamma^{MN}\dot{\eta} + \bar{\zeta}\Gamma_M D_N\Psi(\Gamma^{MN}\dot{\eta})_\alpha), \tag{B63}$$

$$-i\dot{\bar{\zeta}}_\beta\mathcal{F}_{\alpha\beta} = ic_3(\Gamma_M\Psi)_\alpha\dot{\bar{\zeta}}\Gamma^M\zeta - 2ic_3\bar{\zeta}\Gamma_M\Psi(\Gamma^M\dot{\zeta})_\alpha - ic_3\dot{\bar{\zeta}}\Gamma_M\Psi(\Gamma^M\zeta)_\alpha, \tag{B64}$$

$$i\mathcal{D}_N\bar{\Phi}_\alpha\dot{X}^N = iD_N\left(\bar{\Psi}_\alpha - \left(\frac{b_1}{2}F_{MP} + b_2 i\bar{\zeta}\Gamma_M D_P\Psi\right)(\bar{\zeta}\Gamma^{MP})_\alpha\right)\dot{X}^N, \tag{B65}$$

$$i\dot{\bar{\zeta}}_\beta \mathcal{D}_\beta \overline{\Phi}_\alpha = b_2 \dot{\bar{\zeta}} \Gamma_M D_N \Psi (\bar{\zeta} \Gamma^{MN})_\alpha + \left(-i\frac{b_1}{2} F_{MN} + b_2 \bar{\zeta} \Gamma_M D_N \Psi\right)(\dot{\bar{\zeta}} \Gamma^{MN})_\alpha. \tag{B66}$$

Here we have kept only terms which are at most quadratic in fermionic coordinates and linear in SYM fields. The latter simplification is because it is sufficient for our use in Appendix C. We do not present the explicit form of $\{\overline{\Phi}, \overline{\Phi}\}$ in (B58) since it is already quadratic in fields.

APPENDIX C: THE MOST SINGULAR PART OF THE LOOP EQUATION

In this appendix we will extract the most singular and linear part of the loop equation with nonzero fermionic coordinates. We will consider the lowest order terms in the coupling constant g. This is an extension of the calculation in Sec. III A. First we consider the $\oint$-term in C 1, and then in C 2 we consider the δ- and the $\dot{\delta}$-terms.

1. $\oint$-term

Let us consider the most singular part of (B57) and (B58) in the limit $s_1 \to s_2$. We use the formulas (3.2), (3.3), and

$$\partial_R \partial_M \overbrace{A_N^a(x) \partial_P A_Q^b}(\tilde{x})\Big|_{x=\tilde{x}} = 0, \tag{C1}$$

$$\begin{aligned}\overbrace{\Psi_\alpha^a(x) \overline{\Psi}_\beta^b}(\tilde{x})\Big|_{x=\tilde{x}} &= \partial_M \overbrace{\Psi_\alpha^a(x) \partial_N \overline{\Psi}_\beta^b}(\tilde{x})\Big|_{x=\tilde{x}} \\ &= \partial_N \partial_M \overbrace{\Psi_\alpha^a(x) \overline{\Psi}_\beta^b}(\tilde{x})\Big|_{x=\tilde{x}} = 0.\end{aligned} \tag{C2}$$

The terms we consider are

$$K_{\beta_1} W(C) \ni \mathrm{Tr}\,\mathrm{P}[((\mathcal{O}_M)_{s_1}(\mathcal{O}^M)_{s_2} - \beta_1 (\mathcal{O}_{\bar{\zeta}_\alpha})_{s_1} \times (\mathcal{O}_{\eta_\alpha})_{s_2}) \mathcal{W}_0^l], \tag{C3}$$

with

$$\begin{aligned}(\mathcal{O}_M)_{s_1}(\mathcal{O}^M)_{s_2} = {}& \left(iF_{MN}\dot{X}^N - \frac{a_2}{2} F_{NP} \dot{\bar{\zeta}} \{\Gamma_M, \Gamma^{NP}\} \zeta\right)_{s_1} \left(iF^M{}_Q \dot{X}^Q - \frac{a_2}{2} F_{QR} \dot{\bar{\zeta}} \{\Gamma^M, \Gamma^{QR}\} \zeta\right)_{s_2} \\ & - (\bar{\zeta} \Gamma_{[N} D_{M]} \Psi \dot{X}^N + iD_M \overline{\Psi}\, \dot{\eta})_{s_1} (\dot{\bar{\zeta}} \Gamma^M \Psi)_{s_2} - (\dot{\bar{\zeta}} \Gamma^M \Psi)_{s_1} (\bar{\zeta} \Gamma_{[N} D_{M]} \Psi \dot{X}^N + iD_M \overline{\Psi}\, \dot{\eta})_{s_2},\end{aligned} \tag{C4}$$

$$\begin{aligned}(\mathcal{O}_{\bar{\zeta}_\alpha})_{s_1}(\mathcal{O}_{\eta_\alpha})_{s_2} = {}& \left(\frac{a_2}{2} F_{MP} (\{\Gamma_N, \Gamma^{MP}\} \zeta)_\alpha \dot{X}^N - \frac{ib_1}{2} F_{MN} (\Gamma^{MN} \dot{\eta})_\alpha\right)_{s_1} \left(-\frac{ib_1}{2} F_{PQ} (\dot{\bar{\zeta}} \Gamma^{PQ})_\alpha\right)_{s_2} \\ & + ((\Gamma_N \Psi)_\alpha \dot{X}^N)_{s_1} (iD_N \overline{\Psi}_\alpha \dot{X}^N + b_2 (\dot{\bar{\zeta}} \Gamma_M D_N \Psi (\bar{\zeta} \Gamma^{MN})_\alpha + \bar{\zeta} \Gamma_M D_N \Psi (\dot{\bar{\zeta}} \Gamma^{MN})_\alpha))_{s_2} \\ & + (ic_3 (\Gamma_M \Psi)_\alpha \dot{\bar{\zeta}} \Gamma^M \zeta - 2ic_3 \bar{\zeta} \Gamma_M \Psi (\Gamma^M \dot{\zeta})_\alpha - ic_3 \dot{\bar{\zeta}} \Gamma_M \Psi (\Gamma^M \zeta)_\alpha)_{s_1} (iD_N \overline{\Psi}_\alpha \dot{X}^N)_{s_2}.\end{aligned} \tag{C5}$$

On the RHS's, we have kept only terms which are relevant to the present analysis to quadratic order in fermionic coordinates. We have also omitted terms like $(F^{MN})_{s_1} \times (D_P F_{QR})_{s_2}$, $(\Psi)_{s_1}(\Psi)_{s_2}$, $(D_M \Psi)_{s_1}(D_N \Psi)_{s_2}$, and $(\Psi)_{s_1} \times (D_N D_P \Psi)_{s_2}$ because these terms do not contribute to the singular part we are considering [see (C1) and (C2)]. The most singular part of each term on the RHS of (C4) and (C5) is given as follows:

The first term of (C4):

$$\frac{\lambda}{4\pi^2} \frac{1}{\epsilon^4} (-4\dot{X}^N \dot{X}_N - 8\dot{x}^\mu \dot{x}_\mu)_{s_1}, \tag{C6}$$

The second and third terms of (C4):

$$\frac{\lambda}{4\pi^2} \frac{1}{\epsilon^4} (-8i\bar{\zeta} \Gamma_N \dot{\zeta} \dot{X}^N - 16i\bar{\zeta} \Gamma_\nu \dot{\zeta} \dot{x}^\nu + 8\dot{\bar{\zeta}}\, \dot{\eta})_{s_1}, \tag{C7}$$

The first term of (C5):

$$\frac{\lambda}{4\pi^2} \frac{1}{\epsilon^4} (16ia_2 b_1 (\dot{\bar{\zeta}} \Gamma_\mu \zeta \dot{x}^\mu - 4\dot{\bar{\zeta}} \Gamma_M \zeta \dot{X}^M) - 36 b_1^2 \dot{\bar{\zeta}}\, \dot{\eta})_{s_1}, \tag{C8}$$

The second term of (C5):

$$\frac{\lambda}{4\pi^2} \frac{1}{\epsilon^4} (-16\dot{x}^\mu \dot{x}_\mu)_{s_1}, \tag{C9}$$

The third term of (C5):

$$\frac{\lambda}{4\pi^2} \frac{1}{\epsilon^4} (-i24c_3 \dot{\bar{\zeta}} \Gamma_\mu \zeta \dot{x}^\mu)_{s_1}. \tag{C10}$$

Then summing all the contributions we get the following expression for the most singular part:

$$(\mathcal{O}_M)_{s_1}(\mathcal{O}^M)_{s_2} - \beta_1(\mathcal{O}_{\bar\zeta_\alpha})_{s_1}(\mathcal{O}_{\eta_\alpha})_{s_2} \underset{s_1\to s_2}{\longrightarrow} \frac{\lambda}{4\pi^2}\frac{1}{\epsilon^4}(-4\dot X^M\dot X_M - (8-16\beta_1)\dot x^\mu \dot x_\mu + (8+36\beta_1 b_1^2)\dot{\bar\zeta}\dot\eta + (8i+64i\beta_1 a_2 b_1)\dot{\bar\zeta}\Gamma_N\zeta\dot X^N + (16i - 16i\beta_1 a_2 b_1 + 24\beta_1 i c_3)\dot{\bar\zeta}\Gamma_\mu\zeta\dot x^\mu)_{s_1}. \tag{C11}$$

Substituting the value of the parameters (4.21), the most singular part in the $\not{\delta}$-term of $K_{\beta_1=1/2}W(C)$ is given by

$$\frac{\lambda}{4\pi^2}\frac{1}{\epsilon^4}(-4\dot X^M\dot X_M + 80\dot{\bar\zeta}\dot\eta + 8i\dot{\bar\zeta}\Gamma_N\zeta\dot X^N + 8i\dot{\bar\zeta}\Gamma_\mu\zeta\dot x^\mu)_{s_1}W(C). \tag{C12}$$

Note that here we have considered only the lowest order terms in g.

2. $\dot\delta$-term and δ-term

Next, let us consider the $\dot\delta$- and δ-terms in (4.22), which consist of terms proportional to EOM. For them we carry out the functional integration by parts with respect to the SYM fields as we did in (1.8) to obtain expressions quadratic in the Wilson loop [the zeroth order in the fermionic coordinates is given by the last term of (2.19)]. Then we extract the most singular part of the s-integration which comes from the region $s\sim s_1$ and is linear in Wilson loop.

First, let us consider the functional integration by parts. Since we are interested in the terms which are at most quadratic in fermionic coordinates in the final expression, we can neglect a number of terms in (4.22). We obtain the following expression for the $\dot\delta$- and δ-terms in (4.22):

$$\begin{aligned}
&9g^2\int_{s_1}^{s_1+l}ds\bar\zeta_{s_1}(\Gamma_M\zeta\dot X^M - i\dot\eta)_s\delta^{(4)}(x_s - x_{s_1})\,\mathrm{Tr}[t^a\mathcal{W}_{s_1}^s t^a\mathcal{W}_s^{s_1+l}]\dot\delta(s_1-s_2)\\
&+ig^2\int_{s_1}^{s_1+l}ds[\{\dot{\bar\zeta}_{s_1}(i\Gamma_N\zeta\dot X^N + \dot\eta)_s - (\dot X^N)_{s_1}(\dot X_N)_s\}\delta^{(4)}(x_s - x_{s_1})\,\mathrm{Tr}[t^a\mathcal{W}_{s_1}^s t^a\mathcal{W}_s^{s_1+l}]\\
&+2(\dot X^N)_{s_1}(\bar\zeta\Gamma^M{}_N\dot\eta)_s\,\mathrm{Tr}[t^a\mathcal{W}_{s_1}^s D_M^s(\delta^{(4)}(x_s - x_{s_1})t^a)\mathcal{W}_s^{s_1+l}] + \{2(\bar\zeta\Gamma^{MN}\dot\eta)_{s_1}(\dot X_N)_s\\
&-(i\dot{\bar\eta}\Gamma^M + \dot X^P\bar\zeta\Gamma_P\Gamma^M - \dot X^M\bar\zeta)_{s_1}(i\Gamma_N\zeta\dot X^N + \dot\eta)_s\}\,\mathrm{Tr}[D_M^{s_1}(\delta^{(4)}(x_s - x_{s_1})t^a)\mathcal{W}_{s_1}^s t^a\mathcal{W}_s^{s_1+l}]]\delta(s_1-s_2),
\end{aligned} \tag{C13}$$

where D_M^s, for example, denotes the covariant derivative with respect to the coordinate $x(s)$. Since we are interested in the lowest order terms in g, we can replace covariant derivatives D_M to partial derivatives ∂_M. After this simplification, (C13) is rewritten as

$$ig^2\int_{s_1}^{s_1+l}ds(f_{(s_1,s_2)}(s)\delta^{(4)}(x(s)-x(s_1)) + f^\mu_{(s_1,s_2)}(s)\partial^s_\mu\delta^{(4)}(x(s)-x(s_1)))\,\mathrm{Tr}[t^a\mathcal{W}_{s_1}^s t^a\mathcal{W}_s^{s_1+l}], \tag{C14}$$

with

$$f_{(s_1,s_2)}(s) = \{-(\dot X^N)_{s_1}(\dot X_N)_s + \dot{\bar\zeta}_{s_1}(i\Gamma_N\zeta\dot X^N + \dot\eta)_s\}\delta(s_1-s_2) - i9\bar\zeta_{s_1}(\Gamma_M\zeta\dot X^M - i\dot\eta)_s\dot\delta(s_1-s_2), \tag{C15}$$

$$f^\mu_{(s_1,s_2)}(s) = \{2(\dot X^N)_{s_1}(\bar\zeta\Gamma^\mu{}_N\dot\eta)_s - 2(\bar\zeta\Gamma^{\mu N}\dot\eta)_{s_1}(\dot X_N)_s + (i\dot{\bar\eta}\Gamma^\mu + \dot X^M\bar\zeta\Gamma_M\Gamma^\mu - \dot X^\mu\bar\zeta)_{s_1}(i\Gamma_N\zeta\dot X^N + \dot\eta)_s\}\delta(s_1-s_2). \tag{C16}$$

Let us evaluate the most singular part of (C14) which arises from the region $s\sim s_1$ and is linear in Wilson loop. Using the regularized delta function (3.6), we obtain

$$\begin{aligned}
\int dsf_{(s_1,s_2)}(s)\delta^{(4)}(x(s)-x(s_1))\,\mathrm{Tr}[t^a\mathcal{W}_{s_1}^s t^a\mathcal{W}_s^{s_1+l}] &\sim \frac{2i}{\pi^2}\int dsf_{(s_1,s_2)}(s_1)\frac{\epsilon^2}{((s-s_1)^2(\dot x(s_1))^2+\epsilon^2)^3}\,\mathrm{Tr}[t^at^a\mathcal{W}_{s_1}^{s_1+l}]\\
&= \frac{3i}{4\pi}\frac{f_{(s_1,s_2)}(s_1)}{\epsilon^4}\frac{\epsilon}{\sqrt{(\dot x(s_1))^2}}\frac{N}{2}W(C),
\end{aligned} \tag{C17}$$

$$\begin{aligned}
&\int dsf^\mu_{(s_1,s_2)}(s)\partial^s_\mu\delta^{(4)}(x(s)-x(s_1))\,\mathrm{Tr}[t^a\mathcal{W}_{s_1}^s t^a\mathcal{W}_s^{s_1+l}]\\
&\sim -\frac{12i}{\pi^2}\int ds\frac{\epsilon^2}{((s-s_1)^2(\dot x(s_1))^2+\epsilon^2)^4}(f^\mu_{(s_1,s_2)}(s_1) + (s-s_1)\partial_s f^\mu_{(s_1,s_2)}(s)|_{s=s_1})\left((s-s_1)\dot x_\mu(s_1) + \frac{1}{2}(s-s_1)^2\ddot x_\mu(s_1)\right)\\
&\times\mathrm{Tr}[t^at^a\mathcal{W}_{s_1}^{s_1+l}] = -\frac{3i}{4\pi}\frac{1}{\epsilon^4}\left\{\frac{1}{2}f^\mu_{(s_1,s_2)}(s_1)\ddot x_\mu(s_1) + \partial_s f^\mu_{(s_1,s_2)}(s)|_{s=s_1}\dot x_\mu(s_1)\right\}\frac{1}{(\dot x(s_1))^2}\frac{\epsilon}{\sqrt{(\dot x(s_1))^2}}\frac{N}{2}W(C).
\end{aligned} \tag{C18}$$

In (C18), we have neglected terms which arise from the Taylor expansion of $\mathrm{Tr}[t^a \mathcal{W}^{s}_{s_1} t^a \mathcal{W}^{s_1+l}_{s}]$. This is because such terms, in general, have additional fields A_M and Ψ and do not contribute in the lowest order in g.

Finally, the most singular and linear part in the $\dot{\delta}$- and δ-terms of $K_{\beta_1=1/2}W(C)$ is given by

$$\frac{3\lambda}{8\pi}\frac{1}{\epsilon^4}\frac{\epsilon}{\sqrt{(\dot{x}(s_1))^2}}\bigg[9(\bar{\zeta}\,\dot{\eta})_{s_1}\dot{\delta}(s_1 - s_1) + \bigg\{\dot{X}^N\dot{X}_N - i\dot{\bar{\zeta}}\Gamma_N\zeta\dot{X}^N - \dot{\bar{\zeta}}\,\dot{\eta} + \frac{1}{(\dot{x})^2}\Big(2\dot{X}^N\dot{x}^\mu\dot{\bar{\zeta}}\Gamma_{\mu N}\dot{\eta} + 2\dot{X}^N\dot{x}^\mu\bar{\zeta}\Gamma_{\mu N}\ddot{\eta}$$
$$- 2\ddot{X}^N\dot{x}^\mu\bar{\zeta}\Gamma_{\mu N}\dot{\eta} - \dot{X}^N\dot{x}^\mu\dot{\bar{\eta}}\Gamma_\mu\Gamma_N\dot{\zeta} - \ddot{X}^N\dot{x}^\mu\dot{\bar{\eta}}\Gamma_\mu\Gamma_N\zeta + i\dot{x}_\mu\dot{\bar{\eta}}\Gamma^\mu\ddot{\eta} + i\dot{X}^M\dot{x}^\mu\dot{X}^N\bar{\zeta}\Gamma_M\Gamma_\mu\Gamma_N\dot{\zeta} + i\dot{X}^M\ddot{X}^N\dot{x}^\mu\bar{\zeta}\Gamma_M\Gamma_\mu\Gamma_N\zeta$$
$$+ \dot{X}^M\dot{x}^\mu\bar{\zeta}\Gamma_M\Gamma_\mu\ddot{\eta} - \frac{1}{2}\dot{x}^\mu\ddot{x}_\mu\bar{\zeta}\,\dot{\eta} - i\dot{x}^\mu\dot{x}_\mu\dot{X}^N\bar{\zeta}\Gamma_N\dot{\zeta} - \dot{x}^\mu\dot{x}_\mu\bar{\zeta}\,\ddot{\eta}\Big)\bigg\}_{s_1}\delta(s_1 - s_1)\bigg]W(C). \tag{C19}$$

This reduces to our previous (3.7) if we put $\zeta(s) = \eta(s) = 0$. The total of the most singular and linear part in $K_{\beta_1=1/2}W(C)$ is the sum of (C12) and (C19).

APPENDIX D: MORE GENERAL QUADRATIC FUNCTIONAL DERIVATIVES

In Secs. II, III, and IV, we considered only the simplest version of the quadratic functional derivative, K_{β_1} (2.13). In this section, we will consider more general quadratic functional derivatives and discuss their influence on the analysis given in Sec. IV. The new functional derivatives we will consider are the following five which are classified into type 1 and type 2:

$$\text{type 1:}\quad \dot{X}^M\frac{\delta}{\delta\eta}\Gamma_M\frac{\delta}{\delta\dot{\bar{\eta}}},\qquad \bar{\zeta}\Gamma^M\dot{\zeta}\frac{\delta}{\delta\eta}\Gamma_M\frac{\delta}{\delta\dot{\bar{\eta}}},\qquad \frac{\delta}{\delta\eta}\Gamma_M\zeta\dot{\bar{\zeta}}\Gamma^M\frac{\delta}{\delta\dot{\bar{\eta}}}, \tag{D1}$$

$$\text{type 2:}\quad \frac{\delta}{\delta\eta}\Gamma_M\zeta\frac{\delta}{\delta X^M},\qquad \Big(\frac{\delta}{\delta\eta}\Gamma_M\zeta\Big)^2. \tag{D2}$$

All of these operators are of mass-dimension 2. Note that the type 1 operators contain $\delta/\delta\dot{\bar{\eta}}$. As in K_{β_1} (2.13), one of the two functional derivatives in each of the five operators in (D1) and (D2) is at s_1 and the other at s_2, and we take the limit $s_1 \to s_2$ in the end. As we will explain below, the type 2 operators (D2) have ambiguity related to the choice of argument s of $\zeta(s)$ multiplying them even after taking the limit $s_1 \to s_2$. For this reason, we will not consider the type 2 operators in detail. On the other hand, the type 1 operators (D1) are free from such ambiguity.

1. Type 1 operators

Let us generalize K_{β_1} to the following $K_{\{\beta\}}$ obtained by adding the type 1 operators:

$$K_{\{\beta\}} = \frac{\delta}{\delta X^M(s_2)}\frac{\delta}{\delta X_M(s_1)} + \beta_1\frac{\delta}{\delta\eta(s_2)}\frac{\delta}{\delta\bar{\zeta}(s_1)} + \beta_2\dot{X}^M(s_2)\frac{\delta}{\delta\eta(s_2)}\Gamma_M\frac{\delta}{\delta\dot{\bar{\eta}}(s_1)} + \beta_3\bar{\zeta}(s_2)\Gamma^M\dot{\zeta}(s_1)\frac{\delta}{\delta\eta(s_2)}\Gamma_M\frac{\delta}{\delta\dot{\bar{\eta}}(s_1)} + \beta_4\frac{\delta}{\delta\eta(s_2)}\Gamma^M\zeta(s_2)\dot{\bar{\zeta}}(s_1)\Gamma_M\frac{\delta}{\delta\dot{\bar{\eta}}(s_1)}. \tag{D3}$$

In the above type 1 operators, we have chosen deliberately the argument s of $\dot{X}^M(s)$ and $\zeta(s)$ multiplying them. However, the choice of these arguments does not affect the following discussion. We will repeat the analysis of Sec. IV by taking the quadratic functional derivative $K_{\{\beta\}}$ (D3) and the Wilson loop $W(C)$ (4.1). Namely, we will determine the coefficients $\beta_{1,2,3,4}$ in $K_{\{\beta\}}$ and $a_{2,3}$, $b_{1,2,3}$, and c_3 in $W(C)$ from the requirement that the δ-term in $K_{\{\beta\}}W(C)$ be proportional to EOM.

The type 1 operators (D1) contain differentiation with respect to $\dot{\bar{\eta}}$:

$$\frac{\delta}{\delta\dot{\bar{\eta}}(s)}W(C) = \mathrm{Tr}[i\Phi_s\mathcal{W}^{s+l}_s]. \tag{D4}$$

Using this we have

$$\dot{X}^M(s_2)\frac{\delta}{\delta\eta(s_2)}\Gamma_M\frac{\delta}{\delta\dot{\bar{\eta}}(s_1)}W(C) = -\mathrm{Tr}[(i(\Gamma_M\Phi)_\alpha)_{s_1}\mathcal{W}^{s_2}_{s_1}(\mathcal{O}_{\eta_\alpha})_{s_2}\mathcal{W}^{s_1+l}_{s_2}]\dot{X}^M(s_2) - \delta(s_1 - s_2)\,\mathrm{Tr}[\{i\bar{\Phi}, i\Gamma_M\Phi\}_{s_1}\mathcal{W}^{s_1+l}_{s_1}]\dot{X}^M(s_2), \tag{D5}$$

$$\bar{\zeta}(s_2)\Gamma^M\dot{\zeta}(s_1)\frac{\delta}{\delta\eta(s_2)}\Gamma_M\frac{\delta}{\delta\dot{\bar{\eta}}(s_1)}W(C) = -\mathrm{Tr}[(i(\Gamma_M\Phi)_\alpha)_{s_1}\mathcal{W}^{s_2}_{s_1}(\mathcal{O}_{\eta_\alpha})_{s_2}\mathcal{W}^{s_1+l}_{s_2}]\bar{\zeta}(s_2)\Gamma^M\dot{\zeta}(s_1) - \delta(s_1 - s_2)\,\mathrm{Tr}[\{i\bar{\Phi}, i\Gamma_M\Phi\}_{s_1}\mathcal{W}^{s_1+l}_{s_1}]\bar{\zeta}(s_2)\Gamma^M\dot{\zeta}(s_1), \tag{D6}$$

$$\frac{\delta}{\delta\eta(s_2)}\Gamma^M\zeta(s_2)\dot{\bar{\zeta}}(s_1)\Gamma_M\frac{\delta}{\delta\dot{\bar{\eta}}(s_1)}W(C) = \mathrm{Tr}[(i\dot{\bar{\zeta}}\Gamma_M\Phi)_{s_1}\mathcal{W}_{s_1}^{s_2}(\mathcal{O}_\eta\Gamma^M\zeta)_{s_2}\mathcal{W}_{s_2}^{s_1+l}] + \delta(s_1-s_2)\,\mathrm{Tr}[[i\dot{\bar{\zeta}}\Gamma_M\Phi, i\bar{\Phi}\Gamma^M\zeta]_{s_1}\mathcal{W}_{s_1}^{s_1+l}], \quad \text{(D7)}$$

where $\mathcal{O}_\eta\Gamma^M\zeta$ is the abbreviation of $\mathcal{O}_{\eta_\alpha}(\Gamma^M\zeta)_\alpha$. Following the same steps as in Sec. IV, we obtain the following conditions on the parameters:

(i) $\dot{X}$-term

$$2(\beta_1 + i\beta_2) = 1, \quad \text{(D8)}$$

(ii) $\dot{\zeta}$-term

$$\text{none}, \quad \text{(D9)}$$

(iii) $\dot{\eta}$-term

$$\beta_1 b_1 = 1, \quad \text{(D10)}$$

(iv) $\zeta\dot{X}$-term

$$\beta_1 a_2 = 0, \qquad 1 - \beta_1 b_1 - 2i\beta_2 b_1 = 0, \quad \text{(D11)}$$

(v) $\zeta\dot{\zeta}$-term

$$1 + 3\beta_1 c_3 - 2\beta_3 + \beta_4 = 0, \quad \text{(D12)}$$

(vi) $\zeta\dot{\eta}$-term

$$b_1(1-\beta_1 b_1) = 0, \qquad b_1 = 2\beta_1 b_2, \qquad \beta_1 b_2 = 1, \quad \text{(D13)}$$

(vii) $\zeta\zeta\dot{X}$-term

$$\begin{aligned} ia_2 - 2i\beta_1 a_2 b_1 + \beta_2 b_1^2 &= 0,\\ ia_2 - i\beta_1 a_2 b_1 + \beta_2 b_1^2 &= 0,\\ 1 - a_2 - \beta_1 b_2 - 2i\beta_2 b_2 &= 0,\\ 1 - \beta_1(2a_3 + 2b_2 + c_3) - 4i\beta_2 b_2 &= 0,\\ a_2 - \beta_1(3a_3 + b_2) - 2i\beta_2 b_2 &= 0. \end{aligned} \quad \text{(D14)}$$

The set of Eqs. (D8)–(D14) can again be consistently solved to give

$$\beta_1 = \tfrac{1}{2}, \qquad \beta_2 = 0, \qquad \beta_3 = \tfrac{1}{2}\beta_4, \quad \text{(D15)}$$

and

$$a_2 = 0, \qquad a_3 = -\tfrac{2}{3}, \qquad b_1 = 2, \qquad b_2 = 2, \qquad b_3 = \text{arbitrary}, \qquad c_3 = -\tfrac{2}{3}. \quad \text{(D16)}$$

Namely, the values of the "old parameters" remain the same as before: (2.18) and (4.21). Thus we have found that our new quadratic functional derivative $K_{\{\beta\}}$ does not change essentially the results of Sec. IV.

2. Type 2 operators

Next, let us consider the type 2 operators (D2). Their action on the Wilson loop (4.1) is given by

$$\frac{\delta}{\delta\eta(s_2)}\Gamma^M\zeta(s_2)\frac{\delta}{\delta X^M(s_1)}W(C) = \mathrm{Tr}[(\mathcal{O}_M)_{s_1}\mathcal{W}_{s_1}^{s_2}(\mathcal{O}_\eta\Gamma^M\zeta)_{s_2}\mathcal{W}_{s_2}^{s_1+l}] + \delta(s_1-s_2)\,\mathrm{Tr}[[\mathcal{O}_M, i\bar{\Phi}\Gamma^M\zeta]_{s_1}\mathcal{W}_{s_1}^{s_1+l}] - i\dot{\delta}(s_1-s_2)\,\mathrm{Tr}[(\mathcal{D}_M\bar{\Phi})_{s_1}\Gamma^M\zeta_{s_2}\mathcal{W}_{s_1}^{s_1+l}], \quad \text{(D17)}$$

$$\frac{\delta}{\delta\eta(s_2)}\Gamma_M\zeta(s_2)\frac{\delta}{\delta\eta(s_1)}\Gamma^M\zeta(s_1)W(C) = \mathrm{Tr}[(\mathcal{O}_\eta\Gamma^M\zeta)_{s_1}\mathcal{W}_{s_1}^{s_2}(\mathcal{O}_\eta\Gamma_M\zeta)_{s_2}\mathcal{W}_{s_2}^{s_1+l}] + \delta(s_1-s_2)\,\mathrm{Tr}[[\mathcal{O}_\eta\Gamma^M\zeta, i\bar{\Phi}\Gamma_M\zeta]_{s_1}\mathcal{W}_{s_1}^{s_1+l}] + \dot{\delta}(s_1-s_2)\,\mathrm{Tr}[[i(\bar{\Phi}\Gamma^M\zeta)_{s_1}, i\bar{\Phi}_{s_1}\Gamma_M\zeta_{s_2}]\mathcal{W}_{s_1}^{s_1+l}]. \quad \text{(D18)}$$

In the above type 2 operators, we have taken particular choices of the arguments s of $\zeta(s)$ multiplying them. Let us consider the Taylor expansion of the $\dot{\delta}$-terms in (D17) and (D18) with respect to s_2 around s_1 by using the formula:

$$\begin{aligned} f(s_1,s_2)\dot{\delta}(s_1-s_2) &= (f(s_1,s_1) + (s_2-s_1)\partial_s f(s_1,s)|_{s=s_1} + \cdots)\dot{\delta}(s_1-s_2)\\ &= f(s_1,s_1)\dot{\delta}(s_1-s_2) + \partial_s f(s_1,s)|_{s=s_1}\delta(s_1-s_2), \end{aligned} \quad \text{(D19)}$$

which follows from

$$s\dot{\delta}(s) = -\delta(s), \qquad s^n\dot{\delta}(s) = 0 \quad (n \geq 2). \quad \text{(D20)}$$

We have

$$(\mathcal{D}_M\overline{\Phi})_{s_1}\Gamma^M\zeta_{s_2}\dot{\delta}(s_1 - s_2) = (D_M\overline{\Psi}\Gamma^M\zeta)_{s_1}\dot{\delta}(s_1 - s_2) + \left(D_M\overline{\Psi}\Gamma^M\dot{\zeta} - \frac{b_1}{2}D_M F_{NP}\bar{\zeta}\Gamma^{NP}\Gamma^M\dot{\zeta} + [\bar{\zeta}\Gamma_M\Psi, \overline{\Psi}\Gamma^M\dot{\zeta}]\right)_{s_1}\delta(s_1 - s_2), \qquad \text{(D21)}$$

$$[i(\overline{\Phi}\Gamma^M\zeta)_{s_1}, i\overline{\Phi}_{s_1}\Gamma_M\zeta_{s_2}]\dot{\delta}(s_1 - s_2) = 0 \times \dot{\delta}(s_1 - s_2) + [i\overline{\Psi}\Gamma^M\zeta, i\overline{\Psi}\Gamma_M\dot{\zeta}]_{s_1}\delta(s_1 - s_2), \qquad \text{(D22)}$$

where we have kept only terms at most quadratic in fermionic coordinates. From (D21) and (D22), we find the following. First, as in Sec. IV, the $\dot{\delta}$-terms on the RHS of (D21) and (D22) are already proportional to the EOM (or equal to zero). Second, the Taylor expansion of the $\dot{\delta}$-terms in (D17) and (D18) give additional contributions to the δ-terms. However, these new δ-terms have ambiguities depending on the choice of the arguments s of $\zeta(s)$ multiplying the type 2 operators. For example, if we had adopted s_1 as the arguments of all $\zeta(s)$ differently from (D17) and (D18), we would have obtained no additional δ-terms at all. Therefore, the conditions determining the parameters depend on the choice of the arguments of $\zeta(s)$. For this reason, we do not consider the type 2 operators seriously in this paper. Note, however, that this kind of ambiguity does not appear in the case of the type 1 operators since there are no $\dot{\delta}$-terms in (D5)–(D7).

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