

**$R$ -charged  $\text{AdS}_5$  black holes and large  $N$  unitary matrix models**

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Using the anti-de Sitter (AdS)/conformal field theories correspondence, we establish a correspondence between the intricate thermal phases of  $R$ -charged  $\text{AdS}_5$  black holes and the  $R$ -charge sector of the  $N = 4$  gauge theory, in the large  $N$  limit. Integrating out all fields in the gauge theory, except the thermal Polyakov line, leads to an effective unitary matrix model. In the canonical ensemble, a logarithmic term is generated in the nonzero charge sector of the matrix model. This term is important for discussing various supergravity properties like (i) the nonexistence of a thermal AdS phase as a solution, (ii) the existence of a point of cusp catastrophe in the phase diagram, and (iii) the matching of saddle points and the critical exponents of supergravity and those of the effective matrix model. Our study is restricted to a unitary matrix model where the action depends only on  $\text{Tr}U \text{Tr}U^\dagger$ .

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**I. INTRODUCTION**

The anti-de Sitter (AdS)/conformal field theories (CFT) correspondence [1] implies that the phases of string theory can be studied by studying those of the dual gauge theory [2–4]. In the case of type IIB string theory in  $\text{AdS}^5 \times S^5$  the phases can be studied using the large  $N$  limit of a unitary matrix model [5–8].<sup>1</sup> The unitary matrix is the finite temperature Polyakov loop which does not depend on the points of  $S^3$ . This fortunate circumstance is due to the fact that in the Hamiltonian formulation,  $N = 4$  SYM theory at a given time slice, is defined on the compact space  $S^3$  and the  $SO(6)$  scalars are massive because of their coupling to the curvature of  $S^3$ . These facts imply that, in principle, one can integrate out almost all the fields and obtain an effective theory of the zero mode of the gauge potential  $A_0$ .

Using this method, a detailed correspondence of the critical points of the gauge theory effective Lagrangian and the critical points of supergravity (discussed by Hawking and Page [11]) can be constructed at the leading order of the  $1/N$  expansion. These are  $\text{AdS}_5$  and the small and big black holes (as in [12], we refer to these as SBH and BBH). It turns out that in the gauge theory these critical points are in the gapped phase, where the density of eigenvalues vanishes in a finite arc of the circle around which the eigenvalues are distributed. The closing of the gap, corresponds to the Gross-Witten (GW) phase transition [13–15]. In a window around this transition, the supergravity description of string theory is likely to smoothly cross over into a description in terms of heavy string modes.<sup>2</sup>

In this paper we extend the discussion of the correspondence between  $R$ -charged  $\text{AdS}_5$  blackholes [16–20], and the effective unitary matrix model [13–15,21].  $R$ -charged black holes are known to have a rich phase structure in the canonical and grand canonical ensembles. In the canonical ensemble, the fixed charge constraint contributes an additional logarithmic term  $\log(\text{Tr}U \text{Tr}U^\dagger)$ , involving the order parameter, to the gauge theory effective action. This term is crucial for matching with supergravity. We analyze the implications of this term in the large  $N$  limit and compare with the various supergravity properties like the existence of only black hole solutions in the canonical ensemble and also the existence of a point of cusp catastrophe in the phase diagram.

The plan of this paper is as follows. In Sec. II we give a brief review of charged  $\text{AdS}_5$  black holes. In Secs. III and IV we discuss the effective action of the gauge theory at zero and small coupling, in the fixed charged sector. At zero coupling there is exactly one saddle point and the value of  $\text{Tr}U \text{Tr}U^{-1}$  at the saddle point is always nonzero. For a small positive coupling there are two stable and one unstable saddle points, all with a nonzero value of  $\text{Tr}U \text{Tr}U^{-1}$ . They merge at the GW point. In Sec. V we discuss the model effective action at strong coupling. Here, too, there are three saddle points, two stable (I, III) and one unstable (II). In the region  $\rho > \frac{1}{2}$ , I and III can be identified with a stable small black hole and stable big black hole, respectively. Saddle point II is identified with the small unstable black hole. The merging of saddle points leads to critical phenomenon whose exponents can be calculated and shown to agree with supergravity. This is discussed in Sec. VI. We have also calculated the  $o(1)$  part of the partition function near the critical point.

**II.  $R$ -CHARGED BLACK HOLES IN  $\text{AdS}_5$  AND CRITICAL PHENOMENA**

The  $R$ -charged  $\text{AdS}_5$  black hole and relevant phase structure were discussed by Chamblin, Emparan,

\*Electronic address: [pallab@theory.tifr.res.in](mailto:pallab@theory.tifr.res.in)†Electronic address: [wadia@theory.tifr.res.in](mailto:wadia@theory.tifr.res.in)<sup>1</sup>Phases of large  $N$  gauge theory are also discussed in [9,10]<sup>2</sup>In [12], besides the vicinity of the Gross-Witten transition, authors also studied  $1/N$  corrections and presented formulas for the partition function in the vicinity of black hole nucleation and the Hagedorn transition.

Johnson, and Myers [16,17]. Here we review their result. The Einstein-Maxwell-anti-de Sitter (EMAdS $_{n+1}$ ) action may be written as

$$I = -\frac{1}{16\pi G} \int_M d^{n+1}x \sqrt{-g} \left[ \tilde{R} - F^2 + \frac{n(n-1)}{R^2} \right], \quad (1)$$

where  $\tilde{R}$  is the Ricci scalar and  $R$  is the characteristic length scale of the AdS geometry. The metric of the Reissner-Nordström-anti-de Sitter solution is given in static coordinates by

$$ds^2 = -\left(1 - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2n-4}} + \frac{r^2}{R^2}\right) dt^2 + \frac{dr^2}{1 - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2n-4}} + \frac{r^2}{R^2}} + r^2 d\Omega_{n-1}^2. \quad (2)$$

The parameter  $q$  is proportional to the charge

$$Q = \sqrt{2(n-1)(n-2)} \left( \frac{\omega_{n-1}}{8\pi G} \right) q \quad (3)$$

and  $m$  is proportional to the ADM mass  $M$  of the black hole,

$$M = \frac{(n-1)\omega_{n-1}}{16\pi G} m. \quad (4)$$

$\omega_{n-1}$  is the volume of the unit  $(n-1)$  sphere, and the gauge potential is given by

$$A_0 = \left( -\frac{1}{\sqrt{\frac{2(n-2)}{n-1}}} \frac{q}{r^{n-2}} + \Phi \right) \quad (5)$$

where  $\Phi$  is the electrostatic potential difference between the black hole horizon and infinity.

For  $n=4$ , the solution (2) can be considered as a rotating black hole in AdS $_5 \times S^5$  with angular momentum in the internal space  $S^5$ .<sup>3</sup> The symmetry group of  $S^5$  is  $SO(6)$  and the black hole we are discussing has equal  $U(1)$  charges for all three commuting  $U(1)$  subgroups of  $SO(6)$ , and the  $R$ -symmetry group of the  $N=4$  SYM. Hence we are dealing with a system which has the same chemical potential  $\mu$  for all three  $U(1)$  charges in the grand canonical ensemble or, equivalently, three fixed equal  $U(1)$  charges in the canonical ensemble.

### A. Equation of state

In order to discuss the thermodynamics, we consider the Euclidean continuation ( $t \rightarrow i\tau$ ) of the solution, and identify the imaginary time period  $\beta$  with the inverse temperature. Using the formula for the period,  $\beta = 4\pi/V'(r_+)$  (for

<sup>3</sup>The EMAdS system described here may be thought of as the dimensional reduction of AdS $_5 \times S^5$  to AdS $_5$ . Generally, introducing angular momentum in  $S^5$  will distort  $S^5$ . This distortion is not taken into account in the EMAdS reduction [22,23]. We thank S. Trivedi for pointing this out to us. However these details do not change our main consideration.

a review, see [24]), we get

$$\beta = \frac{4\pi l^2 r_+^{2n-3}}{nr_+^{2n-2} + (n-2)l^2 r_+^{2n-4} - (n-2)q^2 l^2}. \quad (6)$$

This may be rewritten in terms of the potential as

$$\beta = \frac{4\pi l^2 r_+}{(n-2)l^2 \left(1 - \frac{2(n-2)}{n-1} \Phi^2\right) + nr_+^2}. \quad (7)$$

The condition for Euclidean regularity used to derive (6) is equivalent to the condition that the black hole is in thermodynamical equilibrium. Equation (6) may therefore be written as an equation of state  $T = T(\Phi, Q)$ . From this equation of state we see that for fixed  $\Phi$  we get two branches, one for each sign, when the discriminant under the square root is positive [17]. For fixed  $Q$ ,  $T(\Phi)$  has three branches for  $Q < Q_{\text{crit}}$  (let us call them I, II, III) and one for  $Q > Q_{\text{crit}}$ . The critical charge is determined at the ‘‘point of inflection’’ by  $(\partial Q/\partial \Phi)_T = (\partial^2 Q/\partial \Phi^2)_T = 0$ .

The qualitative features of  $\beta(r_+)$  for varying  $q$  are shown in Fig. 1. There is a critical charge,  $q_{\text{crit}}$ , below which there are three solutions for  $r_+$  for a range of values of  $T$ , corresponding to small (I), unstable (II), and large (III) black holes. For fixed  $q < q_{\text{crit}}$ , only branch I is available at low temperatures. At  $T = T_{01}(q)$ , there is a nucleation of two new solutions—II, the unstable small black hole solution, and III, the stable big black hole solution. As the temperature is increased further, black hole II approaches black hole I and at  $T = T_{02}(q)$  the two solutions merge.

As  $q$  is increased further,  $T_{02}$  decreases, whereas  $T_{01}$  increases. At  $q = q_{\text{crit}}$  we have  $T_{\text{crit}} = T_{01} = T_{02}$ . At  $q_{\text{crit}}$  and  $T_{\text{crit}}$  all three solutions merge. In the language of catastrophe theory this is a cusp catastrophe. As we increase  $q$  beyond  $q_{\text{crit}}$  there will be just one solution for all temperatures  $T$ .

We wish to take note of some properties of the phase diagram.

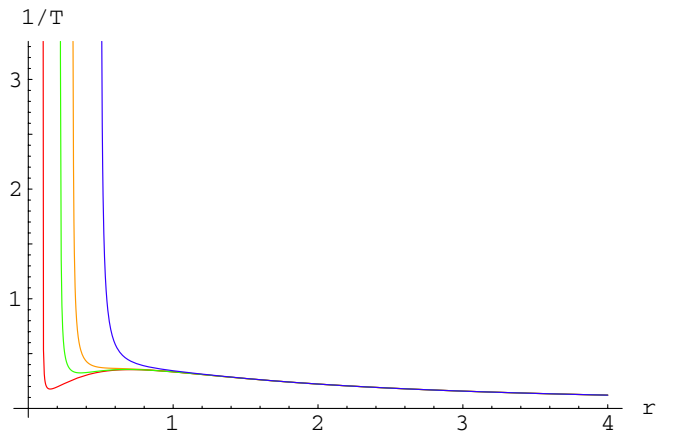


FIG. 1 (color online). Plot of  $\beta(r_+)$  for  $q$  increasing from the left. The third line from the left is for  $q_{\text{crit}}$ .

- (1) The thermal AdS phase is not a solution and all three branches of the solution represent black holes.
- (2) There exists a critical point where the three solutions of the system merge. It is a point of fold catastrophe.

### B. Critical phenomena

The critical point  $(Q_{\text{crit}}, T_{\text{crit}})$  may be approached from various directions in the parameter space. If we set  $T = T_{\text{crit}}$ , then the equation determining  $(r - r_{\text{crit}})$  takes the form ( $r_{\text{crit}}$  is the radius of the critical black hole)

$$(r - r_{\text{crit}})^3 = C(Q - Q_{\text{crit}}). \quad (8)$$

$C$  is a numerical constant. The critical exponent here is  $\frac{1}{3}$ , since

$$(r - r_{\text{crit}}) \propto (Q - Q_{\text{crit}})^{1/3}. \quad (9)$$

As discussed in [17] (Fig. 16), the critical point may also be approached through the coexistence line in the parameter space. The coexistence line is the line with the property

$$S_{\text{I}} = S_{\text{III}} \quad (10)$$

for the parametric range  $q < q_{\text{crit}}$ . It is the line where the Hawking-Page (first order) transition from the small black hole to the big black hole takes place. As we approach the critical point through this line, we have the relation

$$(r_{\text{I}} - r_{\text{II}}) \propto (T - T_{\text{crit}})^{1/2}. \quad (11)$$

In the following we will present an understanding of these properties. Before we do the matching with supergravity we would like to present a discussion of the gauge theory in the limit when  $\lambda = g^2 N = 0$  and also when  $\lambda \ll 1$ . In these cases we of course cannot compare with supergravity which requires  $\lambda \gg 1$ .

## III. FREE YM THEORY

### A. Effective action with chemical potential

In this section we briefly review (see [7,8]) the effective action for a free  $SU(N)$  Yang-Mills (YM) theory (with adjoint matter) on a compact manifold  $\text{Im}$  in the large  $N$  limit. The basic idea is to integrate out all fields in the theory except for the zero mode of the Polyakov line. The partition function is then reduced to a single unitary matrix integral.

Expanding all fields in the gauge theory in terms of harmonics on  $\text{Im}$ , the theory reduces to a zero dimensional problem of free  $N \times N$  Hermitian matrices,

$$\mathcal{L} = \frac{1}{2} \sum_n \text{Tr}[(D_t M_n)^2 - \omega_n^2 M_n^2]. \quad (12)$$

The sum in (12) is over all field types and their Kaluza-Klein modes on  $\text{Im}$ .  $\omega_n$  is the frequency of each mode. The covariant derivative in (12) is

$$D_t M_n = \partial_t M_n - i[A_0, M_n]. \quad (13)$$

$A_0$  comes from the zero mode (i.e. the mode independent of coordinates on  $\text{Im}$ ) of the time component of the gauge field and is not dynamical. The partition function of the theory at finite temperature can be written as a unitary matrix integral by integrating out all fields in (12) except for  $A_0$ . Hence we have

$$Z = \int DU \prod_n (\det_{\text{adj}}(1 - \epsilon_n e^{-\beta \omega_n U}))^{-\epsilon_n} \quad (14)$$

where  $U = \exp(i\beta A_0)$  is a  $U(N)$ .  $\det_{\text{adj}}$  denotes the determinant in the adjoint representation and  $\epsilon_n = 1$  ( $-1$ ) for bosonic (fermionic)  $M_n$ . The above equation can be expressed as

$$Z = \int DU \exp\left(\sum_{n=1}^{\infty} \frac{z_n(\beta)}{n}\right) \text{Tr} U^n \text{Tr} U^{-n} \quad (15)$$

where

$$z_n(\beta) = z_B(n\beta) + (-1)^{n+1} z_F(n\beta). \quad (16)$$

Here  $z_B(\beta)$ ,  $z_F(\beta)$  are the single particle partition functions of the bosonic and fermionic sectors, respectively (see [7] for the explicit formulas for  $z(\beta)$  in various gauge theories).

If we introduce a chemical potential  $\mu = (\log m)$ , the formula for the partition function changes to

$$\mathcal{Z}[\beta, \mu] = \int DU e^{i \sum_{n=1}^{\infty} [z_n(\beta, \mu)/n] \text{Tr} U^n \text{Tr} U^{-n}} \quad (17)$$

where

$$z_n(\beta, \mu) = z_B(n\beta, n\mu) + (-1)^{n+1} z_F(n\beta, n\mu). \quad (18)$$

$z_B(\beta, \mu)$ ,  $z_F(\beta, \mu)$  are now the single particle partition functions with chemical potential  $\mu$ . Hence

$$z_B(\beta, \mu) = \sum_{\text{bosons}} \exp(-E_i \beta + Q_i \mu), \quad (19)$$

$$z_F(\beta, \mu) = \sum_{\text{fermions}} \exp(-E_i \beta + Q_i \mu). \quad (20)$$

$Q_i$  is the charge of the state whose energy is  $E_i$  (see [7]).

As we have already mentioned, we wish to describe a system which has the same chemical potential  $\mu$  for all three  $U(1)$  charges of the  $R$ -symmetry group  $SO(6)$  in the grand canonical ensemble. Equivalently, we can work with fixed and equal values of the  $U(1)$  charges in the canonical ensemble.

Let us for simplicity confine ourselves to the bosonic sector of the  $N = 4$  SYM theory. The gauge fields have no  $R$  charge. The six scalars  $\phi_i$  ( $i = 1, \dots, 6$ ) can be grouped in pairs of two. We define

$$\phi^+ = \phi^1 + i\phi^2, \quad \phi^- = \phi^1 - i\phi^2. \quad (21)$$

(We can similarly define complex fields for the other two

pairs.)  $\phi^\pm$  have charge  $\pm 1$  for each of the three commuting  $U(1)$ s of  $SO(6)$ . Hence, if we consider the single particle partition function for these fields, it will be

$$\begin{aligned} z[x, \mu] &= (\exp(+\mu) + \exp(-\mu))z[x, 0]/2 \\ &= \cosh(\mu)z[x, 0] \end{aligned} \quad (22)$$

where  $z[x, 0]$  is the single particle partition function without any chemical potential.

### B. Canonical ensemble

We will now discuss the free gauge theory partition function for a canonical ensemble with constant charge  $Q_0$ , by introducing a delta function  $\delta(\hat{Q} - Q_0)$  in the functional integration of the gauge theory.  $\hat{Q} = Q[\phi]$  is the corresponding functional for the charge which we want to keep fixed. In our case,  $\hat{Q}$  is just the functional for  $R$  charge in gauge theory.

The fixed charge partition function is defined by

$$Z(\beta, Q_0) = \int DX e^{\int_0^\beta S[X]} \delta(\hat{Q} - Q_0) \quad (23)$$

$$= \int DX e^{\int_0^\beta S[X]} \int \exp(i\mu \hat{Q}) \exp(-i\mu Q_0) d\mu \quad (24)$$

$$= \int d\mu \exp(-i\mu Q_0) \left( \int DX e^{\int_0^\beta S[X]} e^{i\mu \hat{Q}} \right) \quad (25)$$

$$\begin{aligned} &= \int d\mu \exp(-i\mu Q_0) \\ &\quad \times \int DU \exp\left(\sum \frac{z_n[\beta, i\mu]}{n} \text{Tr} U^n \text{Tr} U^{-n}\right) \end{aligned} \quad (26)$$

where  $z_n[\beta, i\mu] = z_n^V[\beta, 0] + \cos(\mu)z_n^S[\beta, 0] + \cos(\frac{\mu}{2})z_n^F[\beta, 0]$ .

We can now make the approximation<sup>4</sup> that  $|z_n[x, i\mu]|$  for  $n > 1$  is negligible in comparison to  $|z_1[x, i\mu]|$ . Neglecting the contribution from the  $n > 1$  modes we arrive at a model which contains only  $\text{Tr} U \text{Tr} U^{-1}$ . Using the specific formula for  $z[x, \mu]$ , of the bosonic<sup>5</sup> sector,

<sup>4</sup>This approximation can be thought of as a low temperature approximation. This is because at low temperatures  $z_n^S[\beta, 0]$  approaches zero as  $e^{-\beta n}$ . Hence the higher  $z_n$  are suppressed relative to  $z_1$ . It is also true that for all temperatures  $z_n < z_1$ , and for very high temperatures we have  $z_n \sim \frac{z_1}{n}$ . As an example, the total contribution for all other  $z_n$ , even near the Hagedorn transition in free  $N = 4$  SYM theory, is only about 7% of  $z_1$  [7]. Unitary matrix models involving  $\text{Tr} U^n$ ,  $n > 1$  have been discussed in [25,26].

<sup>5</sup>The effect of the fermions is discussed in Appendix A.

$$\begin{aligned} Z(\beta, Q_0) &= \int d\mu \exp(-i\mu Q_0) \int DU \exp((a + c \cos(\mu)) \\ &\quad \times \text{Tr} U \text{Tr} U^{-1}) \\ &= \int DU \exp(a \text{Tr} U \text{Tr} U^{-1}) \int d\mu \exp(-i\mu Q_0) \\ &\quad \times \exp(c \cos(\mu) \text{Tr} U \text{Tr} U^{-1}) \\ &= \int DU \exp(a \text{Tr} U \text{Tr} U^{-1}) I_{Q_0}(c \text{Tr} U \text{Tr} U^{-1}). \end{aligned} \quad (27)$$

Here  $a(\beta) = z_n^V[\beta, 0]$ ,  $c(\beta) = z_n^S[\beta, 0]$ , and for convenience we did not show the explicit  $\beta$  dependence in the equations.  $I_n(x)$  is the Bessel function.

Hence we end up with a matrix model with an effective potential

$$S_{\text{eff}} = a(\text{Tr} U \text{Tr} U^{-1}) + \log[I_{Q_0}(c \text{Tr} U \text{Tr} U^{-1})] \quad (28)$$

where  $a > 0$ ,  $c > 0$ . We define  $\rho^2 = \text{Tr} U \text{Tr} U^{-1}/N^2$  to get

$$S_{\text{eff}}(\rho) = N^2(a\rho^2 + (1/N^2) \log[I_{Q_0}(cN^2\rho^2)]). \quad (29)$$

It may seem that the logarithmic term is suppressed by a factor of  $1/N^2$  and hence negligible for large  $N$ . But this need not be the case because, in the semiclassical large  $N$  limit, we must deal with a system with a charge of order  $N^2$ . Hence we define  $Q_0 = N^2 q$  ( $q \sim o(1)$ ). Using the asymptotic expansion of  $I_n(nx)$  for large  $n$ , the effective action<sup>6</sup> becomes

$$\begin{aligned} S_{\text{eff}}(\rho) &= N^2 \left( a\rho^2 + q \left( \sqrt{1 + \frac{c^2}{q^2} \rho^4} \right) \right. \\ &\quad \left. + \log \left( \frac{\frac{c}{q} \rho^2}{1 + \sqrt{1 + \frac{c^2}{q^2} \rho^4}} \right) \right) + O(1). \end{aligned} \quad (30)$$

### C. Phase structure

To understand the phase diagram of this model at large  $N$ , we have to locate the saddle points of (30) after including the relevant contribution from the path integral measure depending on whether  $\rho < \frac{1}{2}$  or  $\rho > \frac{1}{2}$ .<sup>7</sup>

<sup>6</sup>It should be noted that when  $Q = 0$  we should use the asymptotic expansion of  $I_0(x)$  for large  $x$ . Then we get the expected answer  $S_{\text{eff}} = (a + c) \text{Tr} U \text{Tr} U^{-1}$  which is the same as a model without any constraint on charge.

<sup>7</sup>The term in the right-hand side of Eq. (32) originates from the path integral measure over a unitary matrix. (See [27], and the Appendix of [12].)

Differentiating  $S_{\text{eff}}(\rho)$  we get

$$\begin{aligned} \frac{1}{N^2} \frac{\partial}{\partial \rho^2} S_{\text{eff}}(\rho) &= a + \frac{\partial}{\partial \rho^2} \left( \frac{1}{N^2} \log \left[ I_Q \left( Q \frac{cN^2 \rho^2}{Q} \right) \right] \right) \\ &= a + b \frac{I'_Q \left( Q \frac{c\rho^2}{q} \right)}{I_Q \left( Q \frac{c\rho^2}{q} \right)} \\ &= a + \frac{q}{\rho^2} \left( 1 + \frac{c^2}{q^2} \rho^4 \right)^{1/2} + O(1/Q). \end{aligned} \quad (31)$$

Hence the equations to solve are

$$\begin{aligned} a\rho + \frac{q}{\rho} \left( 1 + \frac{c^2}{q^2} \rho^4 \right)^{1/2} &= \rho, \quad \rho < \frac{1}{2}, \\ a\rho + \frac{q}{\rho} \left( 1 + \frac{c^2}{q^2} \rho^4 \right)^{1/2} &= \frac{1}{4(1-\rho)}, \quad \rho > \frac{1}{2}. \end{aligned} \quad (32)$$

The left side in (32) can be written as

$$\begin{aligned} a\rho + \frac{q}{\rho} \left( 1 + \frac{c^2}{q^2} \rho^4 \right)^{1/2} &= a\rho + c\rho + \frac{q}{\rho} \\ &\times \frac{1}{\left( 1 + \frac{c^2}{q^2} \rho^4 \right)^{1/2} + \frac{c}{q} \rho^2}. \end{aligned} \quad (33)$$

So fixing the charge gives rise to a term of type  $\frac{q}{\rho} \times \{1/[ (1 + \frac{c^2}{q^2} \rho^4)^{1/2} + \frac{c}{q} \rho^2 ]\}$  which has some important properties.

- (1) For all values of  $q > 0, c > 0$ , this term is a decreasing positive function of  $\rho$  and it diverges as  $\rho \rightarrow 0$ .
- (2) For all values of  $c > 0$  it is a monotonically increasing function of  $q$ .

We can now discuss the solution of this model at  $N = \infty$ . Let us assume that we are discussing the phase where  $a(T) + c(T) < 1$ . This condition is valid for low temperatures since  $a(T), c(T) \rightarrow 0$  as  $T \rightarrow 0$ . It should also be recalled that without any charge fixing, the Hagedorn transition occurs when  $a(T) + c(T) = 1$ . Unlike the situation with no charge, here we have a function, on the left-hand side of (32), which diverges as  $\rho \rightarrow 0$ . Hence  $\rho = 0$  cannot be a solution. We get only one solution at a finite value of  $\rho$  which we will describe in the next paragraph.

Equation (32) is solved in the region  $\rho < \frac{1}{2}$  with solution

$$\rho^4 = \frac{q^2}{(1-a)^2 - c^2} = \frac{q^2}{(1-a-c)(1-a+c)}. \quad (34)$$

The self consistency condition for a solution in the region  $\rho < \frac{1}{2}$  is

$$\rho^4 = \frac{q^2}{(1-a)^2 - c^2} < \frac{1}{16}. \quad (35)$$

At low temperatures the condition is naturally satisfied for a small enough value of  $q$ . If we gradually increase the temperature [i.e. the value of  $a(T)$  and  $c(T)$ ] while keeping

the value of the  $q$  fixed, then the value of  $\rho$  at this saddle point will increase. At some temperature  $T_3(q)$ ,  $\rho$  will become equal to  $\frac{1}{2}$ . Since the measure part [i.e. right-hand side of (32)] has a third order discontinuity at  $\rho = \frac{1}{2}$ , we will get a third order phase transition at the temperature  $T_3$ . From (35) we have the following condition at  $T_3$ ,

$$\frac{q^2}{(1-a)^2 - c^2} = \frac{1}{16}. \quad (36)$$

If the temperature is increased beyond  $T_3$  then we have to solve (32) in the region  $\rho > \frac{1}{2}$ .

If we increase  $q$ , then the value of  $\rho$  at the saddle point for a fixed temperature will increase. At some  $q_3$  we get a third order phase transition satisfying

$$16q_3(T)^2 = (a-1)^2 - c^2. \quad (37)$$

Since the minimum value of  $a(T)$  and  $c(T)$  is zero, the maximum possible value of  $q_3^2(T)$  is  $q_{\text{crit}}^2 = \frac{1}{16}$ . If we increase the  $q$  beyond  $q_{\text{crit}}$ , the saddle point will always be confined in the parameter range  $\rho > \frac{1}{2}$ . Consequently, as we increase the temperature from zero we will not get a third order phase transition for  $q > q_{\text{crit}} = \frac{1}{4}$ .

This free model, at zero gauge coupling ( $l_s \gg R$  in bulk), has some similarities with AdS<sub>5</sub> black holes in a fixed charge ensemble. However, unlike the three black hole branches in AdS<sub>5</sub>, we get only one branch in the free theory. But, most importantly, the solution always has a nonzero value of  $\rho$ .

It should be recalled that before the Hagedorn transition, a free gauge theory with zero charge has the solution  $\rho = 0$  [5,7].

Some properties of the free theory will be important when we analyze the situation for the weakly coupled gauge theory. Just above the temperature  $T_3(q)$ , the difference of the two sides of (32) can be expanded in the region  $\rho > \frac{1}{2}$ . Defining  $\rho = \frac{1}{2} + x$  ( $x > 0$ ), the difference is

$$-\epsilon(q)x - C_1 x^2. \quad (38)$$

Here  $\epsilon(q) > 0$  and  $\epsilon(q) \rightarrow 0$  as  $q \rightarrow 0$ . It is important to note that  $C_1 > 2$  because the measure function [i.e. right-hand side of (32)] has a third order discontinuity at the point  $\rho = \frac{1}{2}$ . We will also discuss the significance of this in what follows.

#### IV. SMALL COUPLING MODEL

We will now discuss the problem with a small nonzero gauge coupling  $\lambda = g_{\text{YM}}^2 N$ . By AdS/CFT correspondence, it corresponds to a finite string length in the AdS phase. It has been shown in [12] that by considering a phenomenological model of type

$$\begin{aligned} S[\text{Tr}U \text{Tr}U^{-1}] &= a(\lambda, T)(\text{Tr}U \text{Tr}U^{-1}) \\ &+ \frac{b(\lambda, T)}{N^2} (\text{Tr}U \text{Tr}U^{-1})^2, \quad b > 0 \end{aligned} \quad (39)$$

we can map out the possible phase diagram of type IIB string theory in AdS<sub>5</sub>.<sup>8</sup> Even though the model in (39) can be derived from a weak coupling analysis of the gauge theory, it can be thought of as a phenomenological model describing supergravity in AdS<sub>5</sub>.

We are motivated to discuss the fixed charge ensemble in the same spirit. Let us add a small interaction term to (29). The effective action is then given by

$$S_{\text{eff}}(\rho) = N^2 \left( a\rho^2 + b\rho^4 + q \left( \sqrt{1 + \frac{c^2}{q^2} \rho^4} + \log \left( \frac{\frac{c}{q} \rho^2}{1 + \sqrt{1 + \frac{c^2}{q^2} \rho^4}} \right) \right) \right) + O(1). \quad (40)$$

Here  $b$  is proportional to  $\lambda$  and is also a function of charge. Depending on the theory considered, the sign of  $b$  can be either positive or negative. It has been shown in [28] that  $b$  is positive for a pure YM theory. In the following discussion we will assume that this is the case in order to motivate a similarity with the supergravity picture. The equations determining the saddle points of (40), including the contribution from the path integral measure, are

$$(a + c)\rho^2 + 2b\rho^4 + \frac{q}{(1 + \frac{c^2}{q^2} \rho^4)^{1/2} + \frac{c}{q} \rho^2} = \rho^2, \quad \rho < \frac{1}{2},$$

$$(a + c)\rho^2 + 2b\rho^4 + \frac{q}{(1 + \frac{c^2}{q^2} \rho^4)^{1/2} + \frac{c}{q} \rho^2} = \frac{\rho}{4(1 - \rho)}, \quad \rho > \frac{1}{2}. \quad (41)$$

In what follows it will be useful to introduce a function

$$M(\rho) = \rho^2, \quad \rho < \frac{1}{2} \quad = \quad \frac{\rho}{4(1 - \rho)}, \quad \rho > \frac{1}{2}. \quad (42)$$

$M(\rho)$  is an increasing convex function and is the right-hand side of (41). It is also useful to introduce  $D(x) = S'_{\text{eff}}(x) - M(x)$ . Equation (41) is then equivalent to  $D(\rho) = 0$ .

It has been shown in [12] that for an interacting model with zero charge, we get nucleation of black holes along the curve given by  $a = (1 - 2w)/[(1 - w)^2(1 + w)]$  and  $b = 2w/[(1 - w)^2(1 + w)^3]$ ,  $1 > w > 0$ . Here we want to analyze a similar type of phenomenon. Let us consider the different cases.

### A. $T$ is varying and $q$ is small

Let us start with a value of charge which is small (i.e.  $q \ll 1$ ) and let us increase the temperature from zero. At low temperatures all the parameters  $a, c$  will be small (for small  $T$  these parameters have a dependence like  $e^{-c\beta}$ ;  $c$  is a constant). Hence we get just one solution for  $\rho < \frac{1}{2}$  which

<sup>8</sup>In fact, in [12] an arbitrary convex function is considered, and shown to map out the phase diagram of IIB theory in AdS<sub>5</sub>. The simplified model (39) leads to a similar qualitative result.

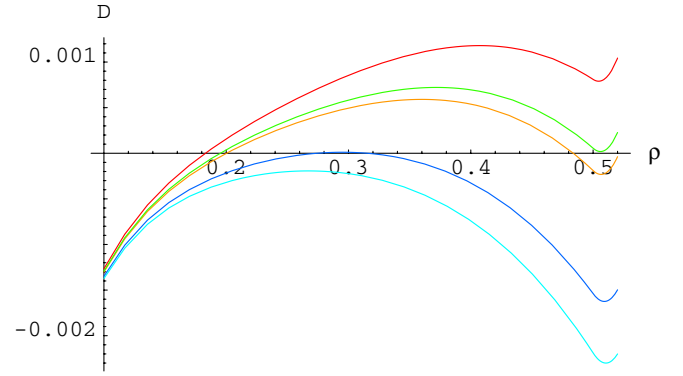


FIG. 2 (color online). Plots of  $D(\rho)$  with increasing  $a = c$  from the top and with fixed  $b$  and  $q < q_{\text{crit}}$ .

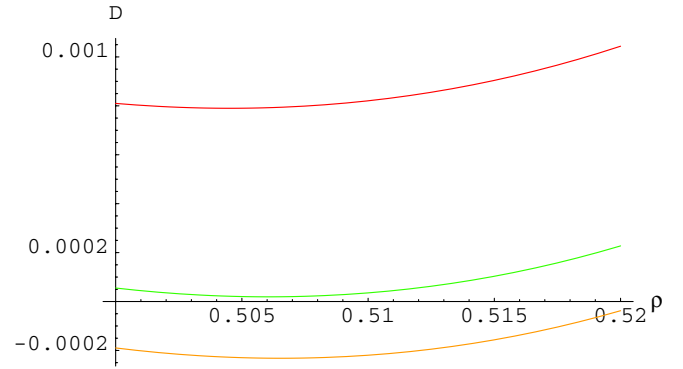


FIG. 3 (color online). Top three lines of Fig. 2 showing emergence of two saddle points in the region  $\rho > \frac{1}{2}$ .

we call I. There is no solution for  $\rho > \frac{1}{2}$  (see the topmost curve of Fig. 2) because the left-hand side of (41) is less than the right-hand side.

The situation is quite similar to supergravity where for small charge and low enough temperatures we get a stable small black hole solution.<sup>9</sup>

The function  $M(\rho)$  [i.e. right-hand side of (41)] is a convex increasing function. Hence we will generate new solutions of (41) in the region  $\rho > \frac{1}{2}$  as we increase the temperature (i.e.  $a(T), c(T)$  as discussed in Appendix B) keeping  $q$  fixed (Figs. 2 and 3). The new solutions will always come in pairs (Fig. 2). Let us call the solution nucleation temperature  $T_{01}(q)$ . At  $T = T_{01}$  we have

$$D(\rho) = S'_{\text{eff}}(\rho) - M(\rho) = 0, \quad D'(\rho) = 0. \quad (43)$$

As we increase the temperature further [i.e.  $a(T), c(T)$ ] the function on the left-hand side of (41) will also increase. Hence the solutions will start to separate. Let us call them II and III. Here  $\rho_{\text{III}} > \rho_{\text{II}}$ . Also, III is a stable saddle point whereas II is an unstable one. They are similar to stable big

<sup>9</sup>We should keep in mind that  $\rho < \frac{1}{2}$  is not the supergravity regime. The solutions of gauge theory effective action there should be represented as excited string states.

and unstable small black holes in supergravity. As the temperature is increased beyond  $T_{01}$  the value  $\rho_{II}$  decreases whereas  $\rho_{III}$  increases. At some temperature  $T_H(q)$ , we will have  $S(III) < S(I)$  and consequently we expect a first order phase transition. At a temperature  $T_{HP}$  (for a similar discussion on the no-charge case, see [11]), the dominant saddle point of the system changes from I to III. As the temperature increases, the saddle point II goes through a third order transition when  $\rho(II)$  crosses the  $\rho = \frac{1}{2}$  point. We call this temperature  $T_3$  which is determined by the following relation between the parameters,

$$\left(a + \frac{b}{2}\right) + 4q\left(1 + \frac{c^2}{16q^2}\right)^{1/2} = 1. \quad (44)$$

Increasing the temperature further, the saddle point II approaches the saddle point I and they merge at a temperature  $T_{02}$  (4th line from the top in Fig. 2). In the language of catastrophe theory this is a fold catastrophe. For  $T > T_{02}$ , the only saddle point is III. This then is the thermal history as we increase the temperature for a small  $q$ .

In summary, at low temperatures we have only one saddle point, I, and then two new saddle points, II, III, are created at  $T_{01}$ . As the temperature increases the saddle points I, II merge at a temperature  $T_{02}$ . Beyond that we have only one saddle point, III. In the next paragraph we will discuss what happens when we increase the value of the  $q$ .

### B. Varying $q$

Let us discuss how the various temperatures discussed above change as we increase the value of  $q$ .

- (1) The first is  $T_{01}(q)$ , the nucleation temperature for saddle points II and III.  $T_{01}$  will decrease as we increase the value of  $q$ . This is so because all three terms in the right-hand side of (41) are positive and increasing functions of  $\rho$  and the left-hand side is a positive convex function. Consequently the saddle point value of  $\rho$  at  $T_{01}$  will also decrease.
- (2) The temperature  $T_{02}$ , at which the stable saddle point I and unstable saddle point II merge, and the value of  $\rho$  at  $T_{02}$  will increase as we increase  $q$ . The reason is that all the coefficients in (41) are positive and hence increasing the value of  $q$  increases the function on the left-hand side.

As we increase the value of  $q$  further,  $T_{01}$  and  $T_{02}$  will become equal for some value of  $q = q_{\text{crit}}$ . In the language of catastrophe theory this is a cusp catastrophe. Corresponding to  $q_{\text{crit}}$  there will be a  $T_{\text{crit}} = T_{01} = T_{02}$  and a value of  $\rho_{\text{crit}}$  (saddle point  $\rho$  at  $q_{\text{crit}}$ ,  $T_{\text{crit}}$ ). As we increase the  $q$  beyond  $q_{\text{crit}}$ , we do not get any new saddle point and consequently there is only one saddle point for all temperatures. We will discuss the physics near this phase transition in detail in what follows.

The above discussion is true for any increasing positive function  $a(T)$ ,  $c(T)$  with the constraint  $a + c < 1$ . We have also assumed that  $b(T)$  is small and positive.

### C. Value of $\rho_{\text{crit}}$

Let us determine the value of  $\rho_{\text{crit}}$ . As already discussed at the end of the previous section, in (38)  $C$  is always a finite quantity. Hence, (41) cannot have three solutions in the region  $\rho < \frac{1}{2}$  for small  $b$ . Therefore the saddle point I is always in the region  $\rho < \frac{1}{2}$ . Whereas the saddle point III will be in the region  $\rho > \frac{1}{2}$ . Hence the only place where these three saddle points can meet is  $\rho_{\text{crit}} = \frac{1}{2}$  which is also the point of the third order phase transition. Here we have analyzed an effective action which consists only of  $\text{Tr}U \text{Tr}U^\dagger$ . Inclusion of the higher modes like  $\text{Tr}U^2$  etc. will definitely change characteristics of the transition.

### D. Physical interpretation

Before proceeding further we will briefly discuss the bulk interpretation of saddle points of weakly coupled gauge theory. Weak coupling in gauge theory means  $l_s \gg R_{\text{AdS}_5}$ . Hence the supergravity picture is not valid in the bulk. However, at large  $N$ , the string coupling (i.e.  $\frac{1}{N}$ ) will be small and we may conclude that the saddle points discussed above can be described by exact (in all orders in  $l_s$ ) conformal field theories. These CFTs are characterized by the values of  $q$  and  $\rho$  at the saddle points.

We would like to end this section by emphasizing that the coincidence of the three saddle points at  $\rho = \frac{1}{2}$  is a property of the weak coupling ( $\lambda \ll 1$ ) limit of the gauge theory. In what follows we will see that this fact is not necessarily true at strong coupling. We will show that there the coincidence happens in the gapped phase where  $\rho > \frac{1}{2}$ .

## V. EFFECTIVE ACTION AND PHASE DIAGRAM AT STRONG COUPLING

In this section we will discuss the effective action and the phase diagram in the strongly coupled gauge theory which is dual to the supergravity (discussed in Sec. II) regime of IIB string theory.

### A. Finite temperature effective actions in the gauge theory

Let us first summarize the situation in the zero charge sector. The propagator of adjoint fields in the free gauge theory, on a compact manifold  $S^3$ , coupled to a space independent  $A_0$  is given by (see [29])

$$G_{U \ kl}^{ij}(x, t, y, 0) = \sum_{n=-\infty}^{n=\infty} (U^n)_j^i G_0(x, t + n\beta, y, 0) (U^{-n})_k^j \quad (45)$$

where  $G_0(x, t, y, 0)$  is the zero temperature Green's function and  $U$  is the constant Polyakov line. We know that at

any temperature<sup>10</sup>  $G_0(x, t + n\beta, y, 0) > G_0(x, t, y, 0)$  and also at low temperatures

$$G_0(y, t + n\beta, x, 0) \sim e^{-n\beta}. \quad (46)$$

Using the above Green's function one can develop the large  $N$  diagrammatics to arrive at an effective action involving  $Z_N$  invariant terms built out of products of  $\text{Tr}U^n$ . In the weak coupling example (small  $\lambda$ ) one can envisage that  $\text{Tr}U^n (n > 1)$  can be solved in terms of  $\text{Tr}U$  and the effective action will be of the form

$$S_{\text{eff}} = N^2 \sum_{n=1}^{n=\infty} a_n(\beta, \lambda) \left( \frac{\text{Tr}U \text{Tr}U^\dagger}{N^2} \right)^n. \quad (47)$$

We will assume that the form of the effective action remains the same as we increase the coupling; however, the dependence of the parameters on the temperature and the 't Hooft coupling would change. We will also assume that the above reasoning is valid for the  $q \neq 0$  case.

### B. Nonzero charge sector

If we include the fixed charge constraint, as in (26), then we get the following expression for the fixed charge path integral,

$$Z_q = \int d\mu e^{i\mu Q} \int DU e^{N^2 (\sum_{n=0}^{n=\infty} S_n(\rho, \lambda, \beta) \cos(n\mu))} \quad (48)$$

where  $S_n(\rho, \lambda, \beta)$  is the coefficient of  $\cos(n\mu)$  in the effective action. For large  $N$  we can do the  $\mu$  integral by the saddle point method. The saddle point of  $\mu$  is on the imaginary axis. Hence we set  $\mu = im$ , to get the saddle point equation

$$q = \sum_{n=1}^{n=\infty} n S_n(\rho, \lambda, \beta) \sinh(nm), \quad q = Q/N^2. \quad (49)$$

At small values of  $\rho$ ,  $S_n(\rho)$  goes as  $\rho^{n+1}$  and hence in the  $\rho \rightarrow 0$  limit we can approximate Eq. (49) as

$$q \approx C\rho \sinh(m) \quad (50)$$

where  $C$  is a constant independent of  $\rho$ . The solution is

$$m \approx q \log(\rho) + C, \quad \rho \rightarrow 0. \quad (51)$$

<sup>10</sup>Temperature here is measured in units of  $1/R_{S^3}$ .

<sup>11</sup>Introduction of a chemical potential changes the formula (45) as

$$G_U^{\mu ij}(x, t, y, 0) = e^{i\mu t/\beta} \sum_{n=-\infty}^{n=\infty} (U^n)_j^i G_0^\mu(x, t + n\beta, y, 0) (U^{-n})_j^k$$

where

$$G_0^\mu(x, t + n\beta, y, 0) = e^{in\mu} G_0(x, t + n\beta, y, 0).$$

Hence in each order in perturbation theory the terms containing  $\cos(n\mu)$  also get multiplied by  $(\text{Tr}U \text{Tr}U^{-1})^m$ ,  $m > n$  or the higher operators like  $\text{Tr}U^n$  which can be integrated out to give again a term like  $(\text{Tr}U \text{Tr}U^{-1})^m$ .

Substituting  $m$  in (48) we get a logarithmic term for  $\rho$ . We conclude that the logarithmic term is a general feature and it implies, among other things, that  $\text{Tr}U = 0$  is never a solution in the nonzero charge sector.

### C. Model effective action at strong coupling

Following our previous discussion, we will include the generic logarithmic term in the effective potential for a fixed nonzero charge, and proceed to analyze the saddle point structure following [12].

Our proposal for the gauge theory effective action is

$$S_q = S(a(T), b(T), \dots, \rho) + q \log(\rho) \quad (52)$$

and the saddle point equations are

$$\begin{aligned} \rho F(a(T), b(T), c(T), \dots, \rho^2) + q &= \rho^2, & \rho < \frac{1}{2}, \\ \rho F(a(T), b(T), c(T), \dots, \rho^2) + q &= \frac{\rho}{4(1-\rho)}, & \rho > \frac{1}{2}, \end{aligned} \quad (53)$$

where  $F(\rho) = S'_{\text{eff}}(\rho)$ . We assume that

- (1)  $F(x, T)$  is a monotonically increasing function of  $x$  and  $F(0, T) = 0$ .
- (2) The value of  $F(x, T)$  increases for fixed  $x$  as we increase the temperature and  $F(x, 0) = 0$ .

These global properties of  $F(x)$  reproduce the phase diagram of supergravity.

### D. Analysis of solution structure

Let us consider the function  $D(T, \rho) = \rho F(\rho, T) - M(\rho)$  [where  $M(\rho)$  is the contribution from the measure appearing in the right-hand side of (53)]. At  $T = 0$ ,  $F(\rho, T)$  is zero. Hence  $D(T, \rho)$  is a monotonically decreasing function of  $\rho$  at  $T = 0$ .

We know that at  $T = T_{01}$  a pair of two new saddle points appears at  $\rho_{01} > \frac{1}{2}$ . Hence at  $T = T_{01}$  we have  $D(T_{01}, \rho_{01}) = 0$  and  $D'(T_{01}, \rho_{01}) = 0$ .  $D(T_{01}, \rho)$  has a zero for  $\rho = 0$  and it is a decreasing function in the neighborhood of  $\rho = 0$ . It again increases and becomes zero at  $\rho_{01}$  and then the function again decreases as  $\rho \rightarrow 1$ . This implies that the function has a local maximum and local minimum.

In summary,

- (1)  $D(0, \rho)$  is a monotonically decreasing function of  $\rho$ .
- (2)  $D(T_{01}, \rho)$  has a maximum and minimum.

There is a temperature  $T_{\text{crit}}$  at which the local maximum and local minimum appear (Fig. 4). Let us call this temperature  $T_{\text{crit}}$ . At  $T_{\text{crit}}$  the curve  $D(T_{\text{crit}}, \rho)$  will have a point of inflection at  $\rho = \rho_{\text{crit}}$ , say. Let the value of  $D(T_{\text{crit}}, \rho_{\text{crit}}) = q_{\text{crit}}$ .

Increasing the value of  $q$  from zero we need to solve the equation  $D(T, \rho) = q$ . We will get a solution for a nonzero value of  $\rho$ . Denote this solution as I. As the temperature increases, two new solutions appear at  $T = T_{01}$ . We call



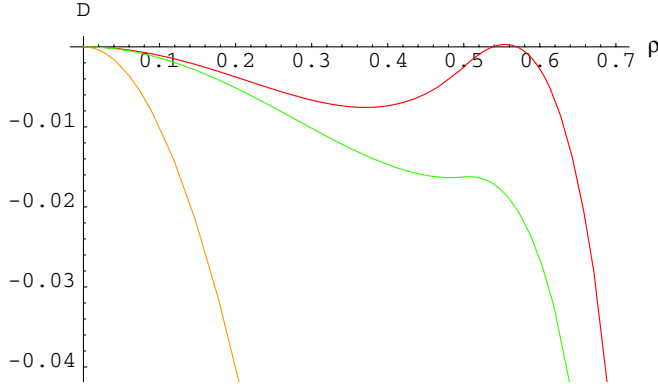


FIG. 4 (color online). Plots of  $D(T, \rho)$  for  $T = 0$ ,  $T = T_{\text{crit}}$ ,  $T = T_{01}$  from the bottom.

the stable solution III, and the unstable solution II. As the temperature is further increased to  $T_{02}$ , the unstable solution II and the stable solution I merge. For  $T > T_{02}$ , the only solution is III.

As  $q$  approaches  $q_{\text{crit}}$  from below, the two temperatures  $T_{01}$  and  $T_{02}$  approach each other. At  $q = q_{\text{crit}}$ , we have  $T_{01} = T_{02} = T_{\text{crit}}$ . If we increase  $q$  beyond  $q_{\text{crit}}$  only one solution appears for all temperatures. These facts are consistent with supergravity solutions (Sec. II).

With a sufficiently sharp rising function  $F(T, \rho)$  in (53) we can obtain this critical point in the region  $\rho > \frac{1}{2}$ .<sup>12</sup> As the function  $D(\rho, T)$  is smooth in the region  $\rho > \frac{1}{2}$ , the second derivative of the function  $D(T, \rho)$  will vanish at the inflection point, and we will get a third order phase transition. We can calculate the partition function in a suitable double scaling limit near the critical point. This is discussed in Sec. VI.

### E. A specific example

We will now illustrate the above phenomenon in a simple model defined by

$$F(\rho) = a\rho + b\rho^3 \quad (54)$$

where  $a, b > 0$ .

We will determine the parameter ranges of  $a, b$  for which all three saddle points of (54) are in the range  $\rho > \frac{1}{2}$ .

At  $\rho = \frac{1}{2}$  we have the constraints

$$\begin{aligned} \partial_\rho \left( \rho F(\rho) - \frac{\rho}{4(1-\rho)} \right) &< 0, \\ \partial_\rho^2 \left( \rho F(\rho) - \frac{\rho}{4(1-\rho)} \right) &> 0. \end{aligned} \quad (55)$$

Putting the value of  $\rho = \frac{1}{2}$  in the above inequality we get the following constraints on the parameters,  $a + b < 1$  and  $a + 3b > 2$ . Simplifying we have  $b > \frac{1}{2}$  and  $a < \frac{1}{2}$ .

<sup>12</sup>which describes supergravity in the bulk [12].

As the coupling becomes stronger, we expect that  $b$  is not necessarily small and will be of  $o(1)$  or greater. All the saddle points of (53) are then naturally shifted to the region  $\rho > \frac{1}{2}$ . Here, as was discussed in [12], we can expect to match the solutions of the gauge theory with those of supergravity. The stable saddle point I corresponds to the stable black hole branch I of supergravity. And the unstable saddle point II is matched with the unstable black hole branch II. The stable saddle point III is matched with the big stable black hole in supergravity. With this identification, the thermal history and critical behavior of the gauge theory, discussed earlier in this section, match with the thermal history and critical behavior of supergravity (discussed in Sec. II and [17]).

## VI. UNIVERSAL NEIGHBORHOOD OF THE CRITICAL POINT AND THE CRITICAL EXPONENTS

Let us consider the effective action  $S_{\text{tot}}(\rho, T, q)$  which includes the contribution from the path integral measure over a unitary matrix. The derivative of  $S_{\text{tot}}$  with respect to  $\rho$ , say  $G(\rho, T, q)$ , gives the saddle point equations (53). We have already discussed that by a suitable choice of parameters the critical point appears in the region  $\rho > \frac{1}{2}$ . This critical point is a third order critical point because three saddle points of the system merge here. Hence the first and second derivatives of  $G_{\text{tot}}(\rho, T, q)$  with respect to  $\rho$  vanish at  $\rho = \rho_{\text{crit}}$ ,  $q = q_{\text{crit}}$ ,  $T = T_{\text{crit}}$ . Expanding  $G(\rho, T, q)$  around the critical point, we get

$$\begin{aligned} G(\rho, T, q) &= (\delta\rho)^3 \frac{\partial_\rho^3 G}{3!} + (\delta T)\partial_T G + (\delta q)\partial_q G + (\delta\rho) \\ &\quad \times (\delta T)\partial_\rho \partial_T G + (\delta q)(\delta T)\partial_\rho \partial_q G. \end{aligned} \quad (56)$$

Let us fix  $T = T_{\text{crit}}$  or  $\delta T = 0$ . Then Eq. (56) has one solution. In order to know how the saddle point value of  $\rho$  approaches  $\rho_{\text{crit}}$  ( $\delta\rho \rightarrow 0$ ) as  $\delta q \rightarrow 0$ , we equate the leading part of (56) to zero,

$$(\delta\rho)^3 \frac{\partial_\rho^3 G}{3!} + (\delta q)\partial_q G = 0. \quad (57)$$

Hence  $\delta\rho \propto \delta q^{1/3}$  and we get the same universal exponent  $\frac{1}{3}$  as in supergravity [17].

### A. Partition function near the critical point

Near the critical point we can write  $S_{\text{tot}}$  as

$$\begin{aligned} S_{\text{tot}} &= S_{\text{tot}}(\rho_{\text{crit}}, T_{\text{crit}}, q_{\text{crit}}) + (\delta\rho)^4 \frac{\partial_\rho^4 S}{4!} + (\delta q)\partial_q S \\ &\quad + (\delta q)(\delta\rho)\partial_\rho \partial_q S + O(\delta\rho^5). \end{aligned} \quad (58)$$

If we define a double scaling limit  $N^{1/2}\rho = x$ ,  $N^{3/2}q = z$ , we can write the  $o(1)$  part of the partition function, after suitable rescaling of the variables, as

$$Z_2 \propto \int dx e^{-(x^4 - zx)}. \quad (59)$$

This can be calculated in a power series

$$Z_2 \propto \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \Gamma\left(\frac{n}{2} + \frac{1}{4}\right). \quad (60)$$

### B. Approaching the critical point through a line of first order transitions

Another type of double scaling limit is possible in this problem. We can set<sup>13</sup>

$$(\delta T)\partial_T G + (\delta q)\partial_q G = 0 \quad (61)$$

by choosing a suitable relation between  $\delta T$  and  $\delta q$ . Using (61) in (56) we get

$$(\delta\rho)^3 \frac{\partial_\rho^3 G}{3!} + (\delta\rho)((\delta T)\partial_\rho \partial_T G + (\delta q)\partial_\rho \partial_q G) = 0 \quad (62)$$

with the solutions

$$\delta\rho = 0, \quad \delta\rho \propto \pm(\delta T)^{1/2}. \quad (63)$$

We can expand  $S_{\text{eff}}$  as

$$S_{\text{eff}} \approx S_{\text{crit}} + (\delta\rho)^4 \frac{\partial_\rho^4 S}{4!} + C_1(\delta T)(\delta\rho)^2 + OT \quad (64)$$

where  $OT$  are terms independent of  $\delta\rho$ . Defining a suitable double scaling limit,  $N^{3/2}\delta T = z$ ,  $N^{1/2}\delta\rho = x$ , and a suitable rescaling of the parameters, we can evaluate the  $o(1)$  factors in the partition function as

$$\begin{aligned} Z_2 &\propto \int dx e^{-(x^4 + 2zx^2)} \propto \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} \Gamma\left(\frac{n}{2} + \frac{1}{4}\right) \\ &\propto \sqrt{z} e^{z^2/z} K_{1/4}\left(\frac{z^2}{2}\right) \end{aligned} \quad (65)$$

where  $z < 0$ .

## VII. CONCLUSION

In this paper we have studied the logarithmic matrix model generated by fixing the  $R$  charge in the gauge theory partition function. In the free gauge theory it has been shown that there is no solution with  $\rho = 0$  (AdS type solution). We then studied the effect of adding an interaction term in our model and discussed the generic nature of the logarithmic term even at an arbitrary value of the coupling. We identified the supergravity saddle points and their critical behavior, which were discussed in [17].

Our main aim was to give another example of the utility of unitary matrix methods in providing a nonperturbative dual description of black holes in the AdS phase and to

<sup>13</sup>It is the same as following the Hawking Page (HP) (first order) transition line in parameter space.

understand the relation between matrix models and string theory, in general. It would be interesting to consider an effective unitary matrix model to describe phases of Kerr-AdS black holes.

## ACKNOWLEDGMENTS

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## APPENDIX A: INCLUSION OF FERMIONS

Including the contributions from the fermions of  $N = 4$  SYM theory changes (27) to

$$\begin{aligned} Z(\beta, Q_0) &= \int DU \int d\mu \exp\left(N^2\left(a + c \cos(\mu)\right.\right. \\ &\quad \left.\left.+ d \cos\left(\frac{\mu}{2}\right)\right)\rho^2 - i\mu Q_0\right), \end{aligned} \quad (A1)$$

where  $d(\beta)$  is the single particle partition function for the fermions.

At large  $N$ , the integral in (A1) could be evaluated by the saddle point method. The equations determining the saddle points of  $\mu = im$  and  $\rho$  are

$$\left(c \sinh(m) + \frac{d}{2} \sinh\left(\frac{m}{2}\right)\right) = \frac{q}{\rho^2} \quad (A2)$$

and

$$\rho\left(a + \cosh(m) + d \cosh\left(\frac{m}{2}\right)\right) = \rho. \quad (A3)$$

We would like to see whether there is a solution with  $\rho = 0$ . As the right-hand side of (A2) becomes large in the limit  $\rho \rightarrow 0$ , we can self-consistently approximate  $\cosh(m)$  and  $\sinh(m)$  as  $e^m$  and we get

$$m \approx \log \frac{q}{c\rho^2}. \quad (A4)$$

Hence a logarithmic potential for  $\rho$  is once again generated. One can also confirm this by putting (A4) in (A3).

## APPENDIX B: POSITIVITY OF THE COEFFICIENT OF THE QUADRATIC TERM IN THE EFFECTIVE ACTION

Let us consider the partition function of YM theory on a compact manifold written as an integral over the effective action of  $\rho = \text{Tr}U \text{Tr}U^{-1}$ .

$$Z(\beta) = \int DU e^{N^2(S_{\text{eff}}(\rho))} \quad (\text{B1})$$

$$= \int d\rho e^{N^2(S_{\text{eff}}(\rho) - S_M(\rho))}, \quad (\text{B2})$$

where  $S_M(\rho)$  is the contribution from the measure part<sup>14</sup> of the path integral and

$$S_{\text{eff}}(\rho) = a(\beta)\rho^2 + \sum_{n=4} a_n(\beta)\rho^n \quad (\text{B3})$$

i.e. a polynomial in  $\rho$ . As  $\beta \rightarrow \infty$  we have  $S_{\text{eff}}(\rho) \rightarrow 0$ .

<sup>14</sup>See discussions before (31).

The contribution from the measure part  $S_M(0)$  has only one minimum at  $\rho = 0$ . Hence at low temperatures the system will have a saddle point at  $\rho = 0$ . Expanding  $\rho$  around this saddle point as  $\rho = 0 + \frac{\delta\rho}{N}$  we get

$$Z(\beta) = \int_{-\infty}^{\infty} d(\delta\rho) e^{-(1-a(\beta))(\delta\rho)^2 + o(1/N^2)}. \quad (\text{B4})$$

Like any thermal partition function, (B2) or (B4) is a decreasing function of  $\beta$ . Hence  $a(\beta)$  should also be a decreasing function of  $\beta$ . Since  $a(\infty) = 0$ , for any finite  $\beta$ ,  $a(\beta)$  is a positive decreasing function. The Hagedorn transition happens when  $a(\beta_H) = 1$ , but whether  $a(\beta)$  will reach 1 or not depends on the model.

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