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Out of equilibrium quantum field dynamics of an initial thermal state after a change in the external field

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The effects of the initial temperature in the out of equilibrium quantum field dynamics in the presence of a homogeneous external field are investigated. We consider an initial thermal state of temperature T for a constant external field $\vec{J}$. A subsequent sign flip of the external field, $\vec{J} \to -\vec{J}$, gives rise to an out of equilibrium nonperturbative quantum field dynamics. The dynamics is studied here for the symmetry broken $\lambda(\vec{\Phi}^2)^2$ scalar N component field theory in the large N limit. We find a dynamical effective potential for the expectation value that helps us to understand the dynamics. The dynamics presents two regimes defined by the presence or absence of a temporal trapping close to the metastable equilibrium position of the potential. The two regimes are separated by a critical value of the external field that depends on the initial temperature. The temporal trapping is shorter for larger initial temperatures or larger external fields. Parametric resonances and spinodal instabilities amplify the quantum fluctuations in the field components transverse to the external field. When there is a temporal trapping, this is the main mechanism that allows the system to escape from the metastable state for large N. Subsequently, backreaction stops the growth of the quantum fluctuations and the system enters a quasiperiodic regime.

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I. INTRODUCTION

Out of equilibrium dense concentrations of particles are present in several important physical systems, such as the ultrarelativistic heavy ion collisions [1] and the early universe [2]. These out of equilibrium dense concentrations of particles imply the need for out of equilibrium nonperturbative quantum field theory methods, such as the large N limit.

We study here the effects of the initial temperature in the out of equilibrium dynamics induced by a change in the external field. We consider the $O(N)$ $\lambda\vec{\Phi}^4$ theory with spontaneously broken symmetry. Initially, a homogeneous external field $\vec{\mathcal{J}}$ breaks the vacuum degeneracy and the system is in a thermal equilibrium state at a given temperature T. Subsequently, the external field $\vec{J}$ flips the sign inducing an out of equilibrium dynamics in the system. We study this out of equilibrium dynamics using the large N limit method. We pay particular attention to the effects of the initial temperature in the dynamics of the system. This system for the particular case of zero temperature has been studied in Ref. [3]. We show here how those results are extended or modified for nonzero initial temperatures. On the other hand, the effects of uniform external fields in $\lambda\vec{\Phi}^4$ theory with broken symmetry have been studied in a different framework through classical evolution with random initial conditions [4]. Here we study the out of equilibrium quantum field dynamics of an initial thermal state after a change in the external field.

In Sec. II we present the $\lambda(\vec{\Phi}^2)^2$ model and its out of equilibrium evolution equations in the large N limit. The evolution equation for the expectation value can be restated in terms of an effective dynamical potential as it is shown in Sec. III. This effective dynamical potential and its maxima and minima help us to understand the dynamics. The early time dynamics is presented in Sec. IV, where we show that there are two dynamical regimes defined by whether the system is temporally trapped close to a metastable state or not. The dynamical regime is determined by the values of the initial temperature and the external field, because the two regimes are separated by a critical value of the external field, $J_c(T)$, that depends on the initial temperature. In this section we also compute the trapping time, finding that it is shorter for larger initial temperatures or larger external fields. In the next section, Sec. V, we study the intermediate time dynamics and find a quasiperiodic evolution with a clear separation of slow and fast variables. Finally, we present the conclusions and an appendix where the analytic expression for the spinodal time (the trapping time) is derived as a function of the external field and the initial temperature.

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II. THE MODEL

We consider N scalar fields, $\vec{\Phi}$, with a $\lambda(\vec{\Phi}^2)^2$ self-interaction in the presence of an external field $\vec{\mathcal{J}}$. The action and the Lagrangian density are given by

$$S = \int d^4x \mathcal{L}, \tag{1}$$

$$\mathcal{L} = \frac{1}{2}[\partial_\mu \vec{\Phi}(x)]^2 - \frac{1}{2}m^2\vec{\Phi}^2 - \frac{\lambda}{8N}(\vec{\Phi}^2)^2 - \frac{m^4 N}{2\lambda} + \vec{\mathcal{J}}\cdot\vec{\Phi}. \tag{2}$$

We restrict ourselves to the case where the symmetry is spontaneously broken, i.e., $m^2 < 0$; we mainly consider small coupling constants λ, because this slows the dynamics and allows a better study of its different parts.

We consider here the evolution of an initial thermal state of temperature T after a flip in the homogeneous external field $\vec{\mathcal{J}} \to -\vec{\mathcal{J}}$. A thermal state of temperature T has translational invariance. This translational invariance of the initial state is preserved by the evolution equations when the external field $\vec{\mathcal{J}}$ is homogeneous, as in our case. Thus, the expectation values of the fields, in particular $\langle\vec{\Phi}\rangle$ and $\langle\vec{\Phi}^2\rangle$, are independent of the spatial coordinates and only depend on time. We want to study the dynamics after the change $\vec{\mathcal{J}} \to -\vec{\mathcal{J}}$; this implies that the direction of the external field is fixed, and we can choose the axes in the N-dimensional internal space such that

$$\vec{\mathcal{J}} = \begin{cases} (\sqrt{N}J, 0, \ldots, 0) & \text{for } t \leq 0, \\ (-\sqrt{N}J, 0, \ldots, 0) & \text{for } t > 0. \end{cases} \tag{3}$$

For an initial thermal state we have an expectation value parallel to the external field. The $O(N)$ invariance of the Lagrangian (2) for $\vec{\mathcal{J}} = 0$, together with the fixed direction of $\vec{\mathcal{J}}$, guarantees that the expectation value remains parallel to the external field during the evolution. Therefore, the following decomposition can be done:

$$\vec{\Phi}(x) = (\sigma(x), \vec{\pi}(x)) = (\sqrt{N}\phi(t) + \chi(x), \vec{\pi}(x)), \tag{4}$$

with $\sqrt{N}\phi(t) = \langle\sigma(x)\rangle$; thus, $\langle\chi(x)\rangle = 0$. While in the remaining $N-1$ directions transversal to the expectation value, $\langle\vec{\pi}(x)\rangle = 0$.

A. Evolution equations in the large N limit

We present here the main concepts and results that lead to the derivation of the evolution equations in the large N limit. More details can be found in Refs. [3,5].

As we have one direction parallel to the expectation value and $N-1$ transversal, the fluctuations in the transverse directions dominate in the large N limit, while those in the longitudinal direction only contribute to the evolution equations as corrections of order $1/N$.

In the Heisenberg picture we have

$$\vec{\pi}(\vec{x}, t) = \int \frac{d^3\vec{k}}{\sqrt{2}(2\pi)^3}[\vec{a}_{\vec{k}}\varphi_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} + \vec{a}^\dagger_{\vec{k}}\varphi^*_{\vec{k}}(t)e^{-i\vec{k}\cdot\vec{x}}], \tag{5}$$

where $\vec{a}_{\vec{k}}$, $\vec{a}^\dagger_{\vec{k}}$ are the annihilation and creation operators, respectively, that satisfy the usual canonical commutation rules. The functions $\varphi_{\vec{k}}(t)$ are the mode functions of the $\vec{\pi}$ field.

The evolution equations for $t > 0$ are

$$\ddot{\phi}(t) + \mathcal{M}^2_d(t)\phi(t) = -J, \tag{6}$$

$$\ddot{\varphi}_{\vec{k}}(t) + \omega^2_{\vec{k}}(t)\varphi_{\vec{k}}(t) = 0, \tag{7}$$

where we define the effective frequency $\omega_{\vec{k}}$ as

$$\omega^2_{\vec{k}}(t) \equiv k^2 + \mathcal{M}^2_d(t), \tag{8}$$

$$\mathcal{M}^2_d \equiv m^2 + \frac{\lambda}{2}\left[\phi^2(t) + \frac{\langle\vec{\pi}^2\rangle(t)}{N}\right]. \tag{9}$$

The last term in the effective mass squared, $\mathcal{M}^2_d$, gives the quantum and thermal contributions. It can be interpreted as the mean effect due to the $\vec{\pi}$ particles, and it is given by

$$\frac{\langle\vec{\pi}^2\rangle}{N} = \frac{1}{2}\int\frac{d^3\vec{k}}{(2\pi)^3}\left[|\varphi_{\vec{k}}(t)|^2\coth\left(\frac{\beta_d\omega_{\vec{k}}(0)}{2}\right) - \mathcal{S}_d\right], \tag{10}$$

$$\mathcal{S}_d \equiv \frac{1}{k} - \frac{\theta(k-\kappa)}{2k^3}\mathcal{M}^2_d(t), \tag{11}$$

with $\beta_d^{-1} = k_B T$, and κ an arbitrary renormalization scale that we choose, for simplicity, equal to the renormalized mass, $\kappa = |m_R|$. (For details on the renormalization procedure that leads to the subtraction $\mathcal{S}_d$, see Ref. [6].) The previous equations are already written for the renormalized magnitudes. It is important to stress that the renormalization is independent of the temperature [7]. The hyperbolic cotangent factor in Eq. (10) is due to the initial thermal state (see, for example, Ref. [8]). In addition, the fact that the initial state is a thermal state implies the following initial conditions for the expectation values and the modes:

$$\phi(0) = \phi_0, \qquad \dot{\phi}(0) = 0, \tag{12}$$

$$\varphi_{\vec{k}}(0) = \frac{1}{\sqrt{\omega_k(0)}}, \qquad \dot{\varphi}_{\vec{k}}(0) = -i\sqrt{\omega_k(0)}, \tag{13}$$

where ϕ_0 is the stationary solution for $t \leq 0$ with minimal total energy. The evolution equation for $t \leq 0$ is Eq. (6) changing J for $-J$, therefore the stationary condition for $t \leq 0$ is $\mathcal{M}^2_d\phi_0 = J$. Note that the initial conditions for the modes are spherically symmetric in the momentum space and the evolution keeps this symmetry (because the evolution equations are also spherically symmetric).

We introduce the following adimensional variables:

$$\tau \equiv |m|t, \qquad \eta(\tau) \equiv \sqrt{\frac{\lambda}{2}}\frac{\phi(t)}{|m|}, \qquad \vec{j} \equiv \sqrt{\frac{\lambda}{2N}}\frac{\vec{\mathcal{J}}}{|m|^3}, \tag{14}$$

$$q \equiv \frac{k}{|m|}, \quad g \equiv \frac{\lambda}{8\pi^2}, \quad \mathcal{M}(\tau) \equiv \frac{\mathcal{M}_d(t)}{|m|}, \quad \beta \equiv \frac{\beta_d}{|m|}, \tag{15}$$

$$\varphi_q(\tau) \equiv \sqrt{|m|}\varphi_k(t), \qquad g\Sigma(\tau) \equiv \frac{\lambda}{2|m|^2}\frac{\langle\vec{\pi}^2\rangle(t)}{N}, \qquad \mathcal{S} \equiv \frac{\mathcal{S}_d}{|m|}. \tag{16}$$

In terms of these adimensional variables the evolution equations for $\tau > 0$ are

$$\ddot{\eta}(\tau) + \mathcal{M}^2(\tau)\eta(\tau) = -j, \qquad \ddot{\varphi}_q(\tau) + \omega_q^2(\tau)\varphi_q(\tau) = 0, \tag{17}$$

where

$$\mathcal{M}^2(\tau) = -1 + \eta^2(\tau) + g\Sigma(\tau), \qquad \omega_q^2 = q^2 + \mathcal{M}^2(\tau), \tag{18}$$

$$g\Sigma(\tau) = g\int q^2 dq\left[|\varphi_q(\tau)|^2 \coth\left(\frac{\beta\omega_q(0)}{2}\right) - \mathcal{S}\right], \tag{19}$$

$$\mathcal{S}(\tau) \equiv \frac{1}{q} - \frac{\theta(q-1)}{2q^3}\mathcal{M}^2(\tau). \tag{20}$$

The initial conditions (12) and (13) are expressed as

$$\eta(0) = \eta_0, \qquad \dot{\eta}(0) = 0, \tag{21}$$

$$\varphi_q(0) = \frac{1}{\sqrt{\omega_q(0)}}, \qquad \dot{\varphi}_q(0) = -i\sqrt{\omega_q(0)}. \tag{22}$$

III. EFFECTIVE DYNAMICAL POTENTIAL FOR THE EXPECTATION VALUE

We can define an (adimensionalized) energy for positive times as the (adimensionalized) expectation value of the T^{00} component of the energy-momentum tensor.

$$\begin{aligned}\epsilon(\tau) &\equiv \frac{\lambda}{2N|m|^4}\langle T^{00}\rangle \\ &= \frac{\eta^4}{4} - \frac{\eta^2}{2} + \frac{1}{2}\eta^2 g\Sigma - \frac{g\Sigma}{2} + \frac{(g\Sigma)^2}{4} + \frac{1}{4} + j\eta \\ &\quad + \frac{\dot{\eta}^2}{2} + \frac{g}{2}\int q^2 dq\left[|\dot{\varphi}_q|^2\coth\left(\frac{\beta\omega_q(0)}{2}\right) - \mathcal{S}_1\right] \\ &\quad + \frac{g}{2}\int q^2 dq q^2\left[|\varphi_q|^2\coth\left(\frac{\beta\omega_q(0)}{2}\right) - \mathcal{S}_2\right],\end{aligned} \tag{23}$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are two subtractions due to the renormalization

$$\begin{aligned}\mathcal{S}_1 &\equiv q + \frac{\mathcal{M}^2}{2q} - \frac{\theta(q-1)}{8q^3}\left[(\mathcal{M}^2)^2 + \frac{d^2\mathcal{M}^2}{dt^2}\right], \\ \mathcal{S}_2 &\equiv \frac{1}{q} - \frac{\mathcal{M}^2}{2q^3} + \frac{\theta(q-1)}{8q^5}\left[3(\mathcal{M}^2)^2 + \frac{d^2\mathcal{M}^2}{dt^2}\right].\end{aligned} \tag{24}$$

For a time independent external field j, as it is the case for $\tau > 0$, the energy is conserved. From the expression (23) we have in the $(\eta, g\Sigma)$ plane the restriction $\epsilon \geq V_{\mathrm{de};\tau>0}(\eta, \Sigma)$, where

$$V_{\mathrm{de};\tau>0}(\eta, \Sigma) \equiv \frac{\eta^4}{4} - \frac{\eta^2}{2} + \frac{1}{2}\eta^2 g\Sigma - \frac{g\Sigma}{2} + \frac{(g\Sigma)^2}{4} + \frac{1}{4} + j\eta. \tag{25}$$

$V_{\mathrm{de};\tau>0}(\eta, \Sigma)$ can be interpreted as a dynamical effective potential because the evolution equation (17) for η can be written as

$$\ddot{\eta}(\tau) = -\frac{\partial}{\partial\eta}V_{\mathrm{de};\tau>0}(\eta, \Sigma). \tag{26}$$

It must be stressed that $V_{\mathrm{de};\tau>0}$ is an effective potential *only* for η (and *not* for the modes).

A. Equilibrium states for the effective dynamical potential

The dynamical effective potential for $\tau \leq 0$ can be defined as $V_{\mathrm{de};\tau\leq 0}(\eta, \Sigma) = V_{\mathrm{de};\tau>0}(\eta, \Sigma) - 2j\eta$. Therefore, the stationary states for the initial dynamics for times $\tau \leq 0$ (before the external field has been flipped) are the solutions of $V'_{\mathrm{de};\tau\leq 0}(\eta) = 0$ (the apostrophe means η derivative). Thus, they verify

$$\eta^3 + (-1 + g\Sigma)\eta - j = 0. \tag{27}$$

For small external fields,

$$j < j_d \equiv 2\sqrt{\frac{(1-g\Sigma)}{27}}, \tag{28}$$

we have three roots. There is a global minimum that corresponds to a stable equilibrium state, at the value of η,

$$\begin{aligned}\eta_0 &\equiv 2\sqrt{\frac{1-g\Sigma}{3}}\cos\left\{\frac{1}{3}\arccos\left[\frac{j}{2}\sqrt{\frac{27}{(1-g\Sigma)^3}}\right]\right\} \\ &= (1-g\Sigma)^{1/2} + \frac{1}{2}(1-g\Sigma)^{-1}j - \frac{3}{8}(1-g\Sigma)^{-5/2}j^2 \\ &\quad + \frac{1}{2}(1-g\Sigma)^{-4}j^3 - \frac{105}{128}(1-g\Sigma)^{-11/2}j^4 + \mathcal{O}(j^5),\end{aligned} \tag{29}$$

a local minimum (metastable equilibrium state) at

$$\eta_1 \equiv 2\sqrt{\frac{1-g\Sigma}{3}}\cos\left\{\frac{1}{3}\arccos\left[\frac{j}{2}\sqrt{\frac{27}{(1-g\Sigma)^3}}\right]+\frac{2\pi}{3}\right\}$$
$$= -(1-g\Sigma)^{1/2}+\frac{1}{2}(1-g\Sigma)^{-1}j+\frac{3}{8}(1-g\Sigma)^{-5/2}j^2 + \frac{1}{2}(1-g\Sigma)^{-4}j^3+\frac{105}{128}(1-g\Sigma)^{-11/2}j^4+\mathcal{O}(j^5), \quad (30)$$

and a local maximum (unstable equilibrium) at

$$\eta_2 \equiv 2\sqrt{\frac{1-g\Sigma}{3}}\cos\left\{\frac{1}{3}\arccos\left[\frac{j}{2}\sqrt{\frac{27}{(1-g\Sigma)^3}}\right]+\frac{4\pi}{3}\right\}$$
$$= -(1-g\Sigma)^{-1}j-(1-g\Sigma)^{-4}j^3+\mathcal{O}(j^5). \quad (31)$$

On the other hand, for large external fields ($j > j_d$) there is a single extreme that corresponds to a global minimum (stable equilibrium), at the value

$$\eta_0 \equiv \left(\frac{j}{2}\right)^{1/3}\left\{\left[1+\sqrt{1-\frac{4(1-g\Sigma)^3}{27j^2}}\right]^{1/3} + \left[1-\sqrt{1-\frac{4(1-g\Sigma)^3}{27j^2}}\right]^{1/3}\right\}$$
$$= j^{1/3}+\frac{1}{3}(1-g\Sigma)j^{-1/3}-\frac{1}{81}(1-g\Sigma)^3j^{-5/3} + \mathcal{O}(j^{-7/3}). \quad (32)$$

We consider here the more interesting case, $j < j_d$, where a potential barrier is present. The initial conditions are those of the stable equilibrium state of the dynamics for times $\tau \leq 0$. In Fig. 1 we show the location of the minima and the maximum of $V_{\text{de};\tau\leq 0}(\eta)$ for $g\Sigma(0) = 0.05$ and an external field $j = 0.2 < j_d$.

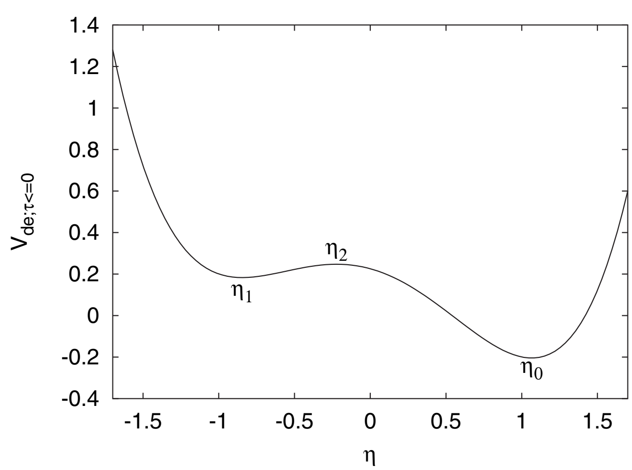

FIG. 1. Dynamical effective potential $V_{\text{de};\tau\leq 0}$ as a function of η for $g\Sigma(0) = 0.05$. The value of the external field is $j = 0.2$. The positions of the global minimum η_0, the local minimum η_1 (metastable state), and the relative maximum η_2 are shown.

For zero initial temperature, $g\Sigma(0) \sim g \ll 1$. Thus, the zero temperature results of Ref. [3] are recovered neglecting $g\Sigma$ in Eqs. (27)–(32).

IV. EARLY TIME DYNAMICS AND DYNAMICAL REGIMES

The fact that the initial state is not the ground state but a thermal state increases the initial value of $g\Sigma$. It can be shown that $\Sigma(0)$ is approximately given by

$$\Sigma(0) = \Sigma^{T=0}(0) + \frac{\pi^2}{3}\beta^{-2}. \quad (33)$$

$\Sigma^{T=0}(0)$ is the zero temperature value that for $g \ll 1$ only depends on j (see Table I). The second term on the right-hand side of Eq. (33) is the thermal contribution computed in the hard thermal loop approximation [9], i.e., assuming the main contribution comes from the modes with $q \sim \beta^{-1}$ and we are in the case $\beta^{-1} \gg \mathcal{M}^2(0)$. [On the other hand, for $\beta^{-1} \ll \mathcal{M}^2(0)$ thermal effects are negligible, because $\beta^{-1} \ll \omega_q$ and $\coth(\beta\omega_q/2) = 1 + \mathcal{O}(e^{-\beta\omega_q})$.]

After the initial flip of the external field sign at $\tau = 0$ the positions of the absolute minimum and the relative minimum of the potential are interchanged, and the state of the system becomes a metastable state (Figs. 2 and 3). The height of the potential barrier is $V_{\text{de};\tau>0}(\eta_2')$, where $\eta_2' = -\eta_2$ is the maximum of $V_{\text{de};\tau>0}(\eta)$. The existence of this potential barrier gives rise to two different dynamical regimes. In the first one, the system can directly overcome the barrier $V_{\text{de};\tau>0}(\eta_0) > V_{\text{de};\tau>0}(\eta_2')$, and rapidly reaches the neighborhoods of the global minimum, while in the second regime the system cannot overcome the barrier directly, $V_{\text{de};\tau>0}(\eta_0) < V_{\text{de};\tau>0}(\eta_2')$, and it gets temporally trapped close to the metastable state. Solving the equation

$$V_{\text{de};\tau>0}(\eta_0) = V_{\text{de};\tau>0}(\eta_2') \quad (34)$$

the external field critical value j_c that separates the two regimes, untrapped $|j| > j_c$ or trapped $|j| < j_c$, is obtained:

$$j_c = \frac{j_c^{T=0}}{(1-g\Sigma(0))^{3/2}} \quad \text{with}$$
$$j_c^{T=0} = \sqrt{2\frac{(13^2+15\sqrt{5})}{19^3}} = 0.243\,019\ldots, \quad (35)$$

where we have used that for very early times we can make the approximation $g\Sigma(\tau) \approx g\Sigma(0)$. An explicit analytic expression for the critical external field as a function of the initial temperature is obtained using Eq. (33),

TABLE I. $\Sigma^{T=0}(0)$ values obtained for various values of the external field j ($g \ll 1$).

j	0.01	0.05	0.10	0.20	0.25
$-10^2\Sigma^{T=0}(0)$	1.2	4.2	6.5	9.6	11

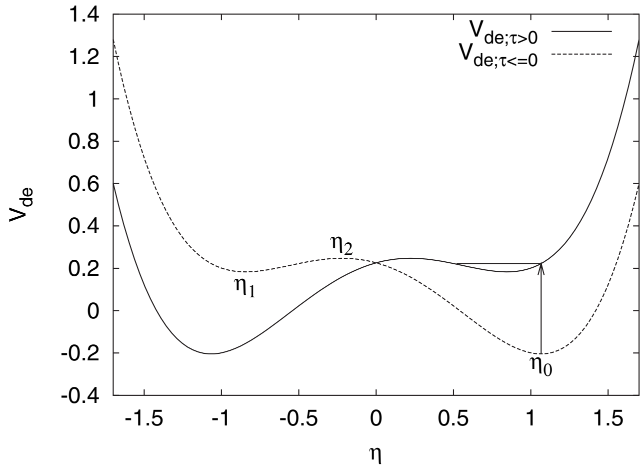

FIG. 2. Dynamical effective potentials $V_{\mathrm{de};\tau\leq 0}$ and $V_{\mathrm{de};\tau>0}$ as a function of η for $g\Sigma(0) = 0.05$. The value of the external field is $j = 0.2$. The positions of the global minimum η_0, the local minimum η_1, and the local maximum η_2 are shown. There is a potential barrier ($j < j_d$), and the system gets temporally trapped ($j < j_c$).

$$j_c(\beta) = \frac{j_c^{T=0}}{[1 - (\frac{\beta_c^{J=0}}{\beta})^2]^{3/2}} \quad \text{with} \quad \beta_c^{J=0} = \pi\sqrt{\frac{g}{3}}. \tag{36}$$

This analytical formula is compared with the numerical results for $g = 10^{-2}$ and $g = 10^{-3}$ in Fig. 4 finding a good fit. The two dynamical regimes (untrapped $|j| > j_c$ and trapped $|j| < j_c$) correspond to the two regions in the (j, β^{-1}) plane separated by the $j_c(\beta)$ curve. This curve can be better understood recalling the effects of the temperature and of the external field on the potential. As the

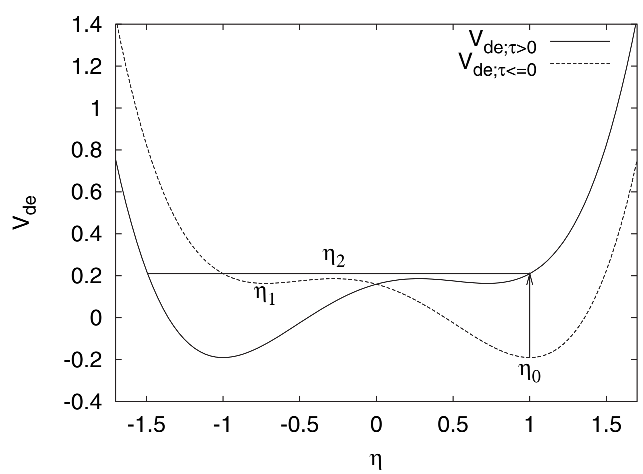

FIG. 3. Dynamical effective potentials $V_{\mathrm{de};\tau\leq 0}$ and $V_{\mathrm{de};\tau>0}$ as a function of η for $g\Sigma(0) = 0.2$. The value of the external field is $j = 0.2$. The positions of the global minimum η_0, the local minimum η_1, and the local maximum η_2 are shown. The potential barrier is not high enough ($j < j_c$), and the system rapidly evolves towards the absolute minimum.

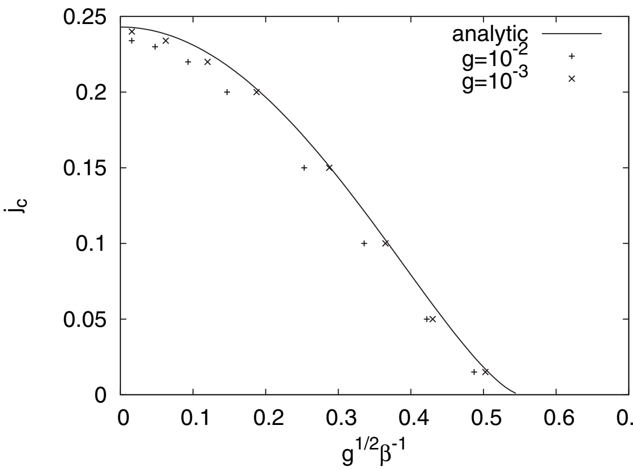

FIG. 4. Critical external field j_c as a function of β^{-1} for $g = 10^{-2}$ and $g = 10^{-3}$ obtained from numerical simulations (dots) and from the analytical formula in Eq. (36) (solid line). There are two regions in the (j, β^{-1}) plane corresponding to the two dynamical regimes: for $j < j_c$ the system is temporally trapped in the metastable state; for $j > j_c$ the system rapidly reaches the neighborhood of the global minimum of the potential.

temperature increases, $g\Sigma$ grows, and the barrier in the η direction between the two minima of the potential diminishes until it disappears. The barrier disappears even for $j = 0$ when $\beta^{-1} > (\beta_c^{J=0})^{-1}$. On the other hand, the effect of the external field j is to tilt the potential; for $j > j_c^{T=0}$ the potential is so tilted that there is no temporal trapping even for $\beta^{-1} = 0$. Therefore, Eq. (36) states how the trapping can disappear due to a combination of both effects.

We now analyze the two dynamical regimes.

A. $j < j_c(\beta)$

In this regime (Fig. 2) the system does not have enough energy to jump over the potential barrier and it remains oscillating around the metastable minimum for some time (Fig. 5). The system is only *temporally trapped* because the spinodal instability of the modes [due to the fact that $\mathcal{M}^2(\tau) < 0$] makes the amplitude of the modes grow. This growth allows the field to surround the maximum of the potential in the $N - 1$ directions transversal to the external field, and finally the system reaches the neighborhood of the stable minimum. (Another effect that leads the system to the global minimum is tunneling. However, our computation neglects this effect because it only takes place in one of the N internal directions and thereby it is subleading in the large N limit.)

The classical evolution equations (recovered for $\Sigma = 0$) predict that the expectation value η oscillates forever between the value η_0 and the turning point value η_r given by

$$V_{\mathrm{de};\tau>0}(\eta_r, \Sigma = 0) = V_{\mathrm{de};\tau>0}(\eta_0, \Sigma = 0). \tag{37}$$

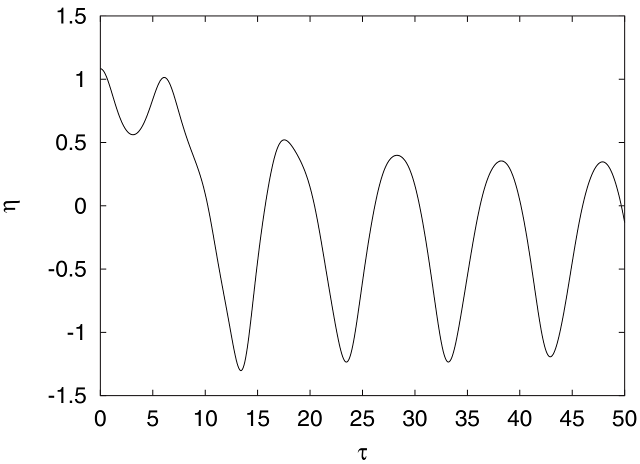

FIG. 5. η as a function of time for $j < j_c(\beta)$. It corresponds to the values $g = 10^{-3}$, $j = 0.2$, $\beta^{-1} = 2$.

This is also the case when $g\Sigma(0) \ll 1$ (otherwise the trapping is very short). Solving Eq. (37) order by order gives

$$\eta_r = 1 - \frac{3}{2}j - \frac{11}{8}j^2 - \frac{7}{2}j^3 - \frac{1049}{128}j^4 + \mathcal{O}(j^5). \qquad (38)$$

In Fig. 6 we plot the behavior of the squared effective mass. For early times, $\mathcal{M}^2$ oscillates with a negative average value. This makes the low momentum modes spinodally unstable and they grow exponentially. This implies a quasiexponential growth of $g\Sigma$ for early times, and finally $g\Sigma$ becomes of order one (see Fig. 7). This makes it so η is no longer trapped close to the metastable minimum (see Fig. 5).

Parametric resonances are also present on the evolution, due to the oscillations of $\mathcal{M}^2$, implying the transfer of energy to certain momenta that grow in amplitude. These two mechanisms, spinodal instability and parametric resonance, transfer energy from the expectation value η to the

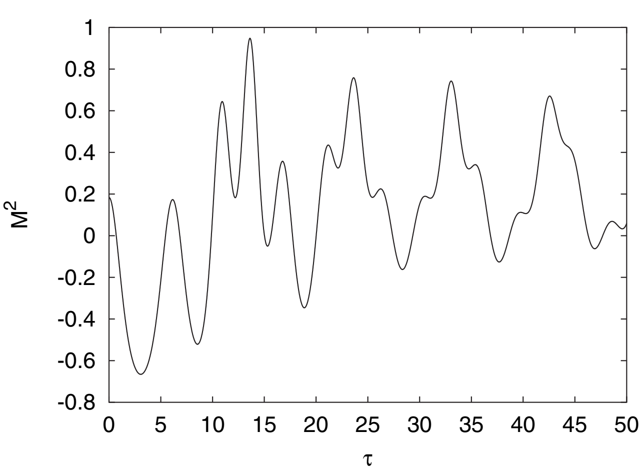

FIG. 6. $\mathcal{M}^2$ as a function of time for $j < j_c(\beta)$. It corresponds to the values $g = 10^{-3}$, $j = 0.2$, $\beta^{-1} = 2$.

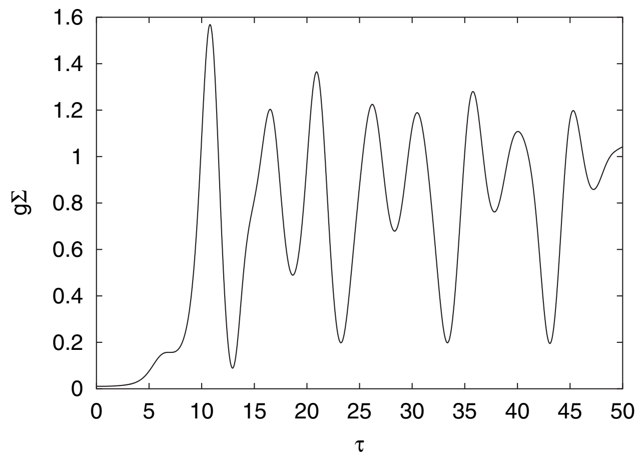

FIG. 7. $g\Sigma$ as a function of time for $j < j_c(\beta)$. It corresponds to the values $g = 10^{-3}$, $j = 0.2$, $\beta^{-1} = 2$.

modes φ_q increasing $g\Sigma$. However in this case, $|j| < j_c$, the main contribution is the spinodal instability.

A detailed analysis of the spinodal instability allows us to obtain the trapping time, or spinodal time τ_s. (Here we summarize the results; the details can be found in the Appendix.) At early times ($\tau < \tau_s$), the effective squared mass, $\mathcal{M}^2$, oscillate with a negative average. Thus, we can define an effective average squared mass $-\mu^2$, and in the modes evolution equation (17) replace $\mathcal{M}^2(\tau)$ by $-\mu^2$ to obtain an estimate for $|\varphi_q|$. This is a good approximation after several complete oscillations of $\mathcal{M}^2(\tau)$ (i.e., $\tau > 1/\mu$), and when the time dependence of $\mathcal{M}^2(\tau)$ is slow (adiabatic approximation). The effective average squared mass is given by [Eq. (A4)]

$$-\mu^2 = -j - \frac{1}{2}j^2 - \frac{9}{8}j^3 - \frac{5}{2}j^4 + \mathcal{O}(j^5). \qquad (39)$$

The quasiexponential growth of $g\Sigma(\tau)$ for $\tau < \tau_s$ is

$$g\Sigma_s(\tau) \approx \begin{cases} \frac{2}{\beta\mu} g\Sigma_s^{T=0}(\tau) & \text{for } \beta^{-1} \gg \mu, \\ g\Sigma_s^{T=0}(\tau) & \text{for } \beta^{-1} \ll \mu, \end{cases} \qquad (40)$$

with

$$g\Sigma_s^{T=0}(\tau) \approx \frac{g\sqrt{\pi\mu}e^{2\tau\mu}}{8\tau^{3/2}} \qquad (41)$$

the value for zero temperature.

After a certain time, the spinodal time τ_s, the quantum and thermal effects start to be important in the dynamics, $g\Sigma_s(\tau_s)$ compensates $-\mu^2$, and the exponential growth of the mode functions stops; then the mode functions start to have an oscillatory behavior. Thus, the spinodal time τ_s is defined as

$$g\Sigma_s(\tau_s) = \mu^2. \qquad (42)$$

A good approximate expression for the spinodal time is

$$\tau_s = \begin{cases} \frac{1}{2\mu}\left[\log\left(\frac{8}{g\sqrt{\pi}}\right) - \log\left(\frac{2}{\beta\mu}\right) + \frac{3}{2}\log(\mu\tau_s)\right] & \text{for } \beta^{-1} \gg \mu, \\ \frac{1}{2\mu}\left[\log\left(\frac{8}{g\sqrt{\pi}}\right) + \frac{3}{2}\log(\mu\tau_s)\right] & \text{for } \beta^{-1} \ll \mu. \end{cases} \quad (43)$$

We see that the spinodal time for $\beta^{-1} \ll \mu$ is the same as that for zero temperature $\tau_s^{T=0}$ (see Ref. [3]). On the other hand, for $\beta^{-1} \gg \mu$ we see that by increasing the initial temperature, the spinodal time decreases. Therefore, a higher initial temperature implies a shorter trapping period; that is consistent with the fact that for greater temperatures the initial transversal fluctuations are larger and the system is closer to the end of the trapping period. The contribution of the transversal fields grows rapidly following Eqs. (40) and (41), turning around the maximum of the potential and reaching the neighborhood of the absolute minimum at earlier times. Increasing the external field j also shortens the trapping period, as can be shown in Eq. (43) ($\mu \sim \sqrt{j}$).

In Table II we compare the numerical results obtained solving the evolution equations (6)–(13) and applying the τ_s definition in Eq. (42) with the values predicted by the analytical expression in Eq. (43). We see a good agreement (with discrepancies of 10% or smaller) except for low $\frac{\beta}{g}$ values (that correspond to high temperatures). This later discrepancy is not troublesome because the expression (43) has been obtained for times $\tau \gtrsim \frac{1}{\mu}$ and this condition does not hold for high temperatures, as the spinodal time decreases for decreasing $\frac{\beta}{g}$. In fact, when $j = 0.2$ we have $\mu^2 \approx 0.23$ [Eq. (39)] and therefore $\frac{1}{\mu} \approx 2.1$; as for $\frac{\beta}{g} = 2 \times 10^2$ we have a spinodal time $\tau_s^{\rm sim} \approx 4.8 \sim 2.1$ (Table II), and Eq. (43) can no longer be applied. However, for greater spinodal times (that corresponds to $\frac{\beta}{g} \geq 2 \times 10^3$) the behavior predicted by Eq. (43) is reproduced with good agreement, as we show in Table II.

B. $j > j_c(\beta)$

The main characteristic of this regime (Fig. 3) is that the system has enough energy to overcome the potential barrier directly. Therefore *no* trapping is present in this case.

Also in this case there is energy transfer between the expectation value and the modes of the field. The low

TABLE II. Spinodal time values for different values of $\frac{\beta}{g}$ with fixed $\beta = 0.2$ and $j = 0.2$. The values have been obtained from the numerical simulation of the complete evolution equations, and from the analytical approximation equation (43).

$\frac{\beta}{g}$	2×10^2	2×10^3	2×10^4	2×10^5	2×10^6	2×10^7
τ_s (numerical)	4.8	9.6	11.5	15.1	16.7	20.1
τ_s (analytical)	7.1	10.1	12.8	15.6	18.2	20.8
Error (%)	33.1	4.1	10.8	3.0	8.4	3.6

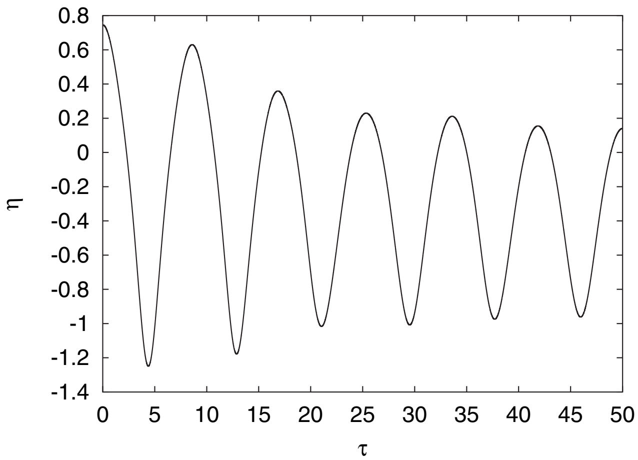

FIG. 8. η as a function of time for $j > j_c(\beta)$. It corresponds to the values $g = 10^{-3}$, $j = 0.2$, $\beta^{-1} = 15$.

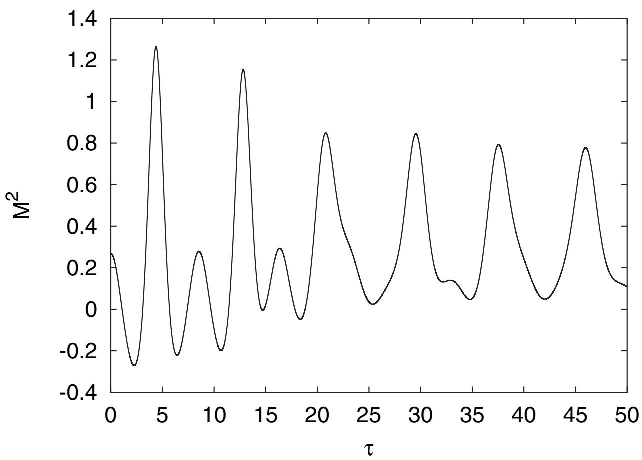

FIG. 9. η as a function of time for $j > j_c(\beta)$. It corresponds to the values $g = 10^{-3}$, $j = 0.2$, $\beta^{-1} = 15$.

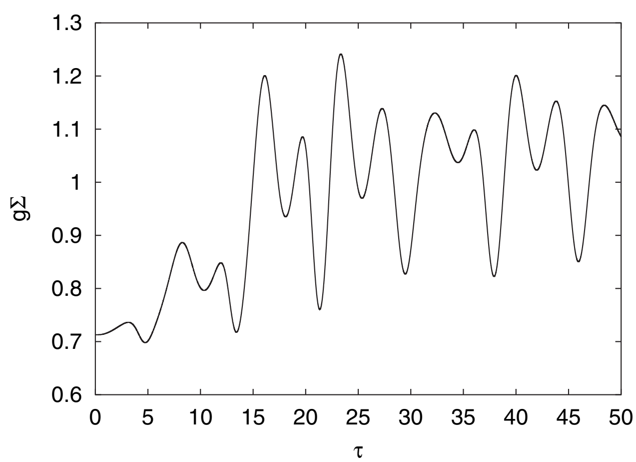

FIG. 10. $g\Sigma$ as a function of time for $j > j_c(\beta)$. It corresponds to the values $g = 10^{-3}$, $j = 0.2$, $\beta^{-1} = 15$.

momenta modes are spinodally unstable and grow exponentially only during the time intervals when $\mathcal{M}^2(\tau) < 0$ (Fig. 9). Parametric resonances are also present due to the oscillations of the effective squared mass. Figures 8–10 reflect the system dynamics in this regime. Both mechanisms transfer energy from the expectation value η to the quantum fluctuation φ_k. Therefore, $g\Sigma(\tau)$ increases while the amplitude of the oscillations of η decreases.

V. INTERMEDIATE TIME DYNAMICS

After the early period described in the previous section, the system enters a quasiperiodic regime. This behavior has already been observed for the particular case $T = 0$ in Ref. [3], while for different initial conditions the oscillations damped much faster [5,10]

This quasiperiodic behavior found for $\eta(\tau)$ and $g\Sigma(\tau)$ suggests that these quantities are approximately governed by an effective Hamiltonian with a few degrees of freedom. Actually, we found (as in Ref. [3]) that the full numerical solution of Eqs. (17)–(20) gives that $\eta(\tau)$ and $g\Sigma(\tau)$ are approximately related as

$$g\Sigma(\eta) = 1 + c_0 j - (1 - c_1 j)(\eta + j)^2, \tag{44}$$

where c_0 and c_1 are positive numbers of the order j^0 and g^0 for small g that are obtained fitting the numerical solution. See Fig. 11, and Tables III and IV.

Using this approximate expression, the effective squared mass can be expressed as

$$\begin{aligned}\mathcal{M}^2(\eta) &= -1 + \eta^2 + g\Sigma(\eta) \\ &= j[c_0 - j + c_1 j^2 - 2(1 - c_1 j)\eta + c_1\eta^2]. \end{aligned}\tag{45}$$

The evolution equations (17)–(20) in this approximation take the form

$$\ddot{\eta} + \mathcal{M}^2(\eta)\eta = -j. \tag{46}$$

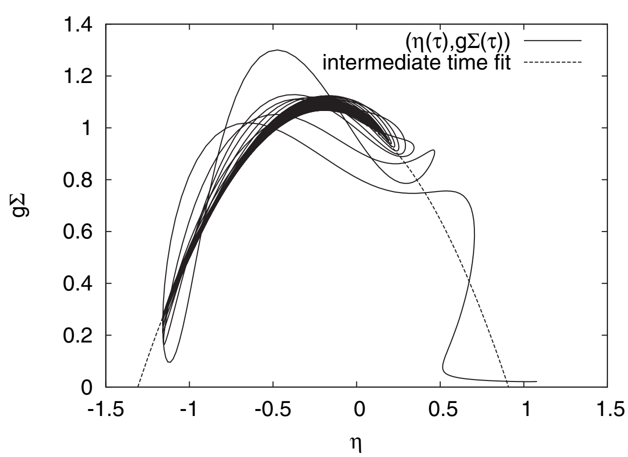

FIG. 11. Trajectory in the $(\eta, g\Sigma)$ plane for $\tau \leq 1000$ and intermediate time ($\tau \in [800, 1000]$) fit to Eq. (44), corresponding to the values $g = 10^{-2}$, $j = 0.20$, $\beta^{-1} = 1$.

TABLE III. c_0 values obtained fitting $g\Sigma(\eta)$ to Eq. (44) in the time interval $\tau \in [800, 1000]$ for $g = 10^{-2}$ for various values of the external field j and the initial temperature.

j $\sqrt{g}\beta^{-1}$	0.05	0.10	0.20
0	0.56	0.54	0.40
0.1	0.48	0.48	0.37
0.2	0.42	0.43	0.20
0.3	0.40	0.40	0.11

We find, integrating on η,

$$\tfrac{1}{2}\dot{\eta}^2 + jV_{\rm int}(\eta) = jE_{\rm int}, \tag{47}$$

where

$$\begin{aligned}V_{\rm int}(\eta) = \eta + \frac{1}{2}(c_0 - j + c_1 j^2)\eta^2 - \frac{2}{3}(1 - c_1 j)\eta^3 \\ + \frac{c_1}{4}\eta^4,\end{aligned}\tag{48}$$

$$E_{\rm int} = V_{\rm int}(\eta_1). \tag{49}$$

Notice that $E_{\rm int}$ depends on the initial conditions, and that η_1 is a turning point of the motion. Equation (47) can be integrated as follows,

$$\sqrt{2j}(\tau - \tau_1) = \int_{\eta_1}^{\eta} \frac{d\eta}{\sqrt{E_{\rm int} - V_{\rm int}(\eta)}} \tag{50}$$

with $\eta_1 = \eta(\tau_1)$.

The fourth order polynomial $E_{\rm int} - V_{\rm int}(\eta)$ always has two real roots $\eta_1 < \eta_2$ corresponding to the turning points. Depending on the value of j, the two other roots are a complex conjugated pair or two more real roots. Depending on the value of j, the two other roots can be

(i) a pair of complex conjugate roots $\eta_R \pm i\eta_I$, for which we define

$$\begin{aligned} a &\equiv \frac{\eta_R - \eta_1}{(\eta_R - \eta_1)^2 + \eta_I^2}, \\ b &\equiv \frac{\eta_I}{(\eta_R - \eta_1)^2 + \eta_I^2}, \end{aligned}\tag{51}$$

(ii) or a pair of real roots $\eta_1 < \eta_2 < \eta_3 < \eta_4$; we then define

TABLE IV. c_1 values obtained fitting $g\Sigma(\eta)$ to Eq. (44) in the time interval $\tau \in [800, 1000]$ for $g = 10^{-2}$ for various values of the external field j and the initial temperature.

j $\sqrt{g}\beta^{-1}$	0.10	0.20
0	1.02	0.55
0.1	1.12	0.63
0.2	1.42	0.98
0.3	2.04	1.49

$$a \equiv \frac{2}{\eta_3 - \eta_1} - \frac{1}{\eta_4 - \eta_1}, \qquad b^2 \equiv \frac{4(\eta_3 - \eta_2)(\eta_4 - \eta_3)}{(\eta_2 - \eta_1)(\eta_4 - \eta_1)(\eta_3 - \eta_1)^2}. \tag{52}$$

We also introduce two other quantities to simplify the formulas:

$$d \equiv \frac{1}{\eta_2 - \eta_1}, \qquad X \equiv [(a-d)^2 + b^2]^{1/4}, \qquad C \equiv X\sqrt{2|\mathcal{M}^2(\eta_1)\eta_1 + j|}. \tag{53}$$

It is convenient to use $u \equiv \frac{1}{\eta - \eta_1}$ as an integration variable in Eq. (50). The solution of Eq. (50) can be expressed after calculation as

$$\eta(\tau) = \eta_1 + \frac{(\eta_2 - \eta_1)[1 - \mathrm{cn}(C(\tau - \tau_1), k)]}{1 + (\eta_2 - \eta_1)X^2 - [1 - (\eta_2 - \eta_1)X^2]\mathrm{cn}(C(\tau - \tau_1), k)}, \tag{54}$$

where $\mathrm{cn}(z, k)$ is the Jacobi cosine function, and

$$k = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{a - d}{X^2}} \tag{55}$$

the elliptic modulus.

The solution (54) oscillates between η_1 and η_2 with the period

$$\mathcal{T} = \frac{4}{C}K(k), \tag{56}$$

where $K(k)$ stands for the complete elliptic integral of the first kind and C is given by Eq. (53). The analytical and the numerical solutions are compared in Fig. 12.

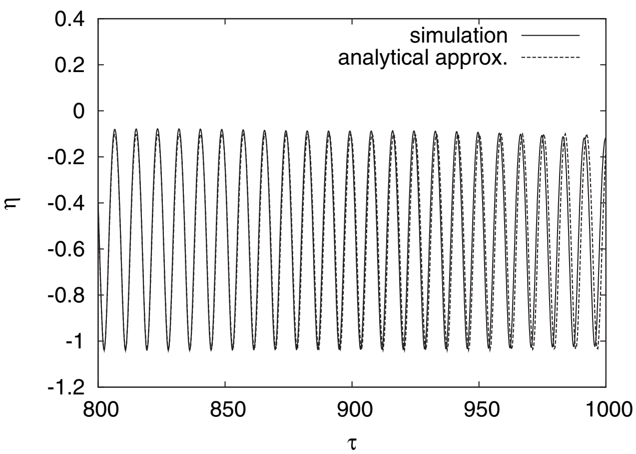

FIG. 12. Quantum evolution for the expectation value $\eta(\tau)$ (solid line) compared to the analytical intermediate time approximation of Eq. (54) (dashed line), corresponding to $g = 10^{-2}$, $j = 0.20$, and $\beta^{-1} = 1$ [for this case $j < j_c(\beta)$].

TABLE V. E_{int} [Eq. (47)] for $g = 10^{-2}$ and various values of the external field j and the initial temperature.

τ	400	700	1000
$jE_{\mathrm{int}}(j = 0.05, \sqrt{g}\beta^{-1} = 0.0)$	0.022	0.016	0.015
$jE_{\mathrm{int}}(j = 0.05, \sqrt{g}\beta^{-1} = 0.1)$	0.019	0.013	0.009
$jE_{\mathrm{int}}(j = 0.05, \sqrt{g}\beta^{-1} = 0.2)$	0.015	0.011	0.006
$jE_{\mathrm{int}}(j = 0.05, \sqrt{g}\beta^{-1} = 0.3)$	0.012	0.007	0.006
$jE_{\mathrm{int}}(j = 0.20, \sqrt{g}\beta^{-1} = 0.0)$	0.011	0.004	−0.011
$jE_{\mathrm{int}}(j = 0.20, \sqrt{g}\beta^{-1} = 0.1)$	−0.004	−0.014	−0.024
$jE_{\mathrm{int}}(j = 0.20, \sqrt{g}\beta^{-1} = 0.2)$	−0.027	−0.035	−0.046
$jE_{\mathrm{int}}(j = 0.20, \sqrt{g}\beta^{-1} = 0.3)$	−0.050	−0.058	−0.064

As in the zero temperature case (Ref. [3]), the numerical solution of the large N limit evolution equations (17)–(20) would be periodic if E_{int} were exactly conserved; however it is slowly decreasing, implying a slow damping of the oscillations. This damping is smaller the smaller j is. See Table V.

Therefore, we have seen that the system at intermediate times presents a clear separation between fast variables and slow variables. $\eta(\tau)$, $g\Sigma(\tau)$, and $\mathcal{M}^2(\tau)$ oscillate fast, while E_{int} slowly decreases. We have found the fast time dependence of the order parameter and the quantum fluctuations in terms of rational functions of Jacobi cosines.

The analytic expression for $g\Sigma(\tau)$, and $\mathcal{M}^2(\tau)$ are obtained from the relations $g\Sigma(\eta)$ [Eq. (44)] and $\mathcal{M}^2(\eta)$ [Eq. (45)], just using the analytical solution for $\eta(\tau)$. The relation $g\Sigma(\eta)$ indicates that $g\Sigma$ and η oscillate with opposite phase, up to order j terms; $g\Sigma(\eta) = 1 - \eta^2 + \mathcal{O}(j)$. These $\mathcal{O}(j)$ terms are very important because they imply $\mathcal{M}^2(\tau) = -1 + \eta^2 + g\Sigma = \mathcal{O}(j)$. Therefore the time averaged squared mass is positive, and tends asymptotically to a positive value. This can also be seen from Eq. (45). Thus, in the limit of a zero external field we recover a zero effective mass squared consistently with the presence of Goldstone bosons for a zero external field. [Recall that for $j = 0$ in the broken symmetry case the squared mass goes to zero for initial energies below the potential at the local maximum ($\eta = 0$) [5,10].]

VI. CONCLUSIONS

We have studied here the effects of the initial temperature in the out of equilibrium dynamics induced by a flip in the sign of the external field. The study has been done in the leading order of large N for the $\lambda(\vec{\Phi}^2)^2$ theory.

Before the flip of the external field, the system is in a thermal state, with the expectation value of the field in the minimum of its effective potential. After the flip, the system evolves in either of the two dynamical regimes. The difference between the two regimes is the presence or absence of a temporal trapping of the system in the neighborhood of a metastable state. The presence of this trapping close to a metastable state is reflected in the fact that the expectation value oscillates without changing sign (despite of the sign change of the external field).

From the evolution equation in the large N limit, we have found a dynamical effective potential for the expectation value in the direction of the external field that helps us to understand the dynamics. The trapped regime corresponds to the case where the system cannot overcome the barrier appearing in the dynamical effective potential. We have shown that the dynamical regime is determined by the initial temperature and the external field. There is a temperature dependent critical value of the external field that separates both regimes; see Eq. (36). For external fields smaller than the critical one, the system is temporally trapped close to the metastable equilibrium position of the effective potential. For external fields greater than the critical one, the neighborhood of the stable equilibrium is rapidly reached and no trapping is observed.

Spinodal growth of the quantum fluctuations is the main mechanism that makes the trapping temporal. The increase of the quantum fluctuations diminishes the potential barrier between the two minima of the dynamical effective potential for the expectation value. We have found an approximate analytic expression for the trapping time (spinodal time); see Eq. (43). It shows that the trapping time becomes shorter for larger initial temperatures or larger external fields.

After the trapping ends, the transfer of energy from the expectation value to the quantum fluctuations takes place through two mechanisms: spinodal instabilities and parametric resonances. The spinodal instabilities appear during the time intervals when the effective mass is negative, while the parametric resonances are due to the oscillations of the expectation value.

We have found that these processes lead, for intermediate times, to a quasiperiodic regime with a clear separation of slow and fast variables. We have explicitly solved the time evolution for the fast variables that oscillate periodically, while the other variables change slowly due to a tiny damping of the oscillations. The effective squared mass oscillates around a positive value of the order of magnitude of the external field. Thus, in the limit of a zero external field we recover a zero effective mass squared consistently with the presence of Goldstone bosons for a zero external field [5].

An interesting question that remains still open is which of these results would be modified by the next to leading order terms of the large N approximation [11–13] and how much. The inclusion of next to leading order terms allows us to include the effects of longitudinal fluctuations. (The longitudinal fluctuations have quantum and thermal origin, and later they reflect the effects of thermal jumping over the barrier and of quantum tunneling through the potential barrier.) This inclusion will allow one to compute the probability of finding the system close to the stable minimum before the spinodal time, and it will also give corrections to the spinodal time; these corrections are expected to be small, as the spinodal time will be mainly determined by the quasiexponential growth of the transversal fluctuations. Also, the inclusion of next to leading order terms has been shown to damp faster the oscillations on the magnitudes during evolution [13]. Therefore, how much the next to leading order terms will increase the damping rate of the intermediate time quasiperiodic behavior is still an open question.

ACKNOWLEDGMENTS

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APPENDIX: SPINODAL TIME

We present here the early time solution for the modes in the spinodally resonant band, and an estimate for the spinodal time τ_s in the case where the external field is smaller than the critical value, $j < j_c(\beta)$. In this case the spinodal time corresponds to the time that the system is trapped close to the metastable state.

At early times (before the spinodal time τ_s), the effective squared mass $\mathcal{M}^2(\tau)$ oscillates with a negative average. Thus, we can define an effective average squared mass $-\mu^2$ and replace $\mathcal{M}^2(\tau)$ by $-\mu^2$ in the modes evolution equation (17) to obtain an estimate for $|\varphi_q|$. This is a good approximation when the time dependence of $\mathcal{M}^2(\tau)$ is slow (adiabatic approximation), and after several complete oscillations of $\mathcal{M}^2(\tau)$. Assuming $g\Sigma(\tau) \ll 1$ (otherwise τ_s is very short), we have $\mathcal{M}^2(\tau) \approx -1 + \eta^2(\tau)$. As η evolves according to Eq. (17) with a frequency $\mathcal{M}(\tau)$, both η and $\mathcal{M}^2(\tau)$ oscillate with a period of the same order $\frac{1}{\mu}$. Therefore, as the effective average squared mass approximation is valid when we average over several periods, this requires

$$\tau \gtrsim \frac{1}{\mu}. \tag{A1}$$

(We will show at the end of this appendix that requiring $\tau_s > 1/\mu$ implies $g\Sigma(\tau) \ll 1$ during most of the time interval $\tau \in [0, \tau_s]$.)

The average squared mass $-\mu^2$ is estimated as the average value of $\mathcal{M}^2$ between its maximum $\mathcal{M}^2_{\rm max}$ and minimum $\mathcal{M}^2_{\rm min}$ values. These values correspond to the η field oscillations between η_0 [Eq. (29)] and the turning point value η_r [Eq. (38)]. Let us call τ_P the time over which the average is done; it results in

$$-\mu^2 = \frac{1}{\tau_P}\int_0^{\tau_P} \mathcal{M}^2(\tau)d\tau \simeq \frac{1}{2}(\mathcal{M}^2_{\min} + \mathcal{M}^2_{\max}). \quad \text{(A2)}$$

Neglecting $g\Sigma$ for these times $\tau < \tau_P$,

$$\mathcal{M}^2_{\max} \approx -1 + \eta_0^2 \approx j - \tfrac{1}{2}j^2 + \tfrac{5}{8}j^3 - j^4 + \mathcal{O}(j^5),$$
$$\mathcal{M}^2_{\min} \approx -1 + \eta_r^2 \approx -3j - \tfrac{1}{2}j^2 - \tfrac{23}{8}j^3 - 4j^4 + \mathcal{O}(j^5). \quad \text{(A3)}$$

Using these expansions on j we get from Eq. (A2)

$$-\mu^2 = -j - \tfrac{1}{2}j^2 - \tfrac{9}{8}j^3 - \tfrac{5}{2}j^4 + \mathcal{O}(j^5). \quad \text{(A4)}$$

The evolution equations for the modes are then

$$\left(\frac{d^2}{d\tau^2} + q^2 - \mu^2\right)\varphi_q(\tau) = 0 \quad \text{(A5)}$$

with the initial conditions

$$\varphi_q(0) = (q^2 + \mathcal{M}^2(0))^{-1/4},$$
$$\dot\varphi_q(0) = -i(q^2 + \mathcal{M}^2(0))^{1/4}. \quad \text{(A6)}$$

For low momenta (those in the band $0 \le q \le \mu$) the effective squared frequency $q^2 - \mu^2$ is negative (spinodal instability) and the $\varphi_q(\tau)$ have an exponential behavior.

Solving this second order ordinary differential equation,

$$\varphi_q(\tau) = \frac{1}{2\sqrt{\mu^2 - q^2}(q^2 + \mathcal{M}^2(0))^{1/4}} \times\left[\left(\sqrt{\mu^2 - q^2} - i\sqrt{q^2 + \mathcal{M}^2(0)}\right)e^{\tau\sqrt{\mu^2-q^2}} + \left(\sqrt{\mu^2 - q^2} + i\sqrt{q^2 + \mathcal{M}^2(0)}\right)e^{-\tau\sqrt{\mu^2-q^2}}\right]. \quad \text{(A7)}$$

The contribution of the spinodal band to $g\Sigma$ is

$$g\Sigma_s(\tau) = g\int_0^{\mu} dq q^2|\varphi_q(\tau)|^2 \coth\left(\frac{\beta\omega_q(0)}{2}\right). \quad \text{(A8)}$$

This spinodal band gives the dominant contribution to $g\Sigma_s(\tau)$. Because of the exponential growth of these low momenta modes, after a short time they give the main contribution.

Inserting Eq. (A7) in Eq. (A8) we have

$$g\Sigma_s(\tau) = g\int_0^{\mu} dq\left\{q^2\frac{\mu^2 + \mathcal{M}(0)^2}{4\mu^2(1 - \frac{q^2}{\mu^2})\sqrt{q^2 + \mathcal{M}(0)^2}} \times e^{2\tau\mu\sqrt{1-(q^2/\mu^2)}}\coth\left[\frac{\beta\sqrt{q^2 + \mathcal{M}(0)^2}}{2}\right]\right\}, \quad \text{(A9)}$$

where we have neglected the exponentially decreasing term of φ_q. This is justified by the condition (A1).

It is convenient to separate three dynamical regimes:

(i) High temperatures, $\beta^{-1} \gg \mu/g$; this implies $g\Sigma(0) \gtrsim 1$, i.e., $j \gtrsim j_c(\beta)$, and there is no trapping.

(ii) Medium temperatures, $\mu/g \gg \beta^{-1} \gg \mu$; this implies $g\Sigma_s(\tau) \approx \frac{2}{\beta\mu}g\Sigma_s^{T=0}(\tau)$, as we show later.

(iii) Low temperatures, $\beta^{-1} \ll \mu$; this implies $g\Sigma_s(\tau) \approx g\Sigma_s^{T=0}(\tau)$, as we show later.

Therefore, here we are only interested in the medium and low temperature cases where the system can be trapped close to the metastable state giving a nonzero spinodal time. We devote the following subsections to these cases.

1. Medium temperatures ($\mu/g \gg \beta^{-1} \gg \mu$)

We call medium temperatures those that verify $\beta^{-1} \gg \omega_q(0)$ for all the modes in the spinodal band, i.e.,

$$\beta^{-1} \gg \mu, \quad \text{(A10)}$$

and also give $g\Sigma(0) \ll 1$, which implies

$$\beta^{-1} \ll \mu/g. \quad \text{(A11)}$$

This allows the following simplifications:

$$\mathcal{M}^2(0) = -1 + \eta^2(0) + g\Sigma(0) \approx -1 + \eta^2(0), \quad \text{(A12)}$$

and, also in this regime ($\beta^{-1} \gg \mu$),

$$\coth\left(\frac{\beta\omega_q(0)}{2}\right) = \coth\left(\frac{\beta\sqrt{q^2 + \mathcal{M}^2(0)}}{2}\right) \approx \frac{2}{\beta\sqrt{q^2 + \mathcal{M}^2(0)}}. \quad \text{(A13)}$$

Therefore,

$$g\Sigma_s(\tau) = g\int_0^{\mu} dq\left\{q^2\frac{\mu^2 + \mathcal{M}(0)^2}{4\mu^2(1 - \frac{q^2}{\mu^2})\sqrt{q^2 + \mathcal{M}(0)^2}} \times e^{2\tau\mu\sqrt{1-(q^2/\mu^2)}}\frac{2}{\beta\sqrt{k + \mathcal{M}^2(0)}}\right\}. \quad \text{(A14)}$$

In addition, $\mathcal{M}^2(0)$ is of the same order of magnitude as μ^2, and we make the approximation $\mathcal{M}^2(0) \approx \mu^2$. We get

$$g\Sigma_s(\tau) = \frac{g}{\beta}\int_0^{\mu} dq q^2\frac{1}{\mu^2(1 - \frac{q^2}{\mu^2})(1 + \frac{q^2}{\mu^2})} \times e^{2\tau\mu\sqrt{1-(q^2/\mu^2)}}. \quad \text{(A15)}$$

It can be shown that the greatest contributions to the integral come from the integrand values that have $q = \mathcal{O}(0.1\mu)$ (the exponential enhances low momenta while other factors suppress very low momenta). This justifies making a Taylor expansion to second order in $\frac{q}{\mu}$, both in the exponential and in the previous factor. The result obtained is

$$g\Sigma_s(\tau) = \frac{g}{\beta}\int_0^{\mu} dq\frac{q^2}{\mu^2}e^{2\tau\mu(1-(q^2/2\mu^2))}, \quad \text{(A16)}$$

and making the change of variable $\xi = q\sqrt{\frac{\tau}{\mu}}$, the previous expression becomes

$$g\Sigma_s(\tau) = \frac{g}{\beta}\frac{e^{2\tau\mu}}{\sqrt{\mu}\tau^{3/2}}\int_0^{\sqrt{\mu\tau}} d\xi\xi^2 e^{-\xi^2}. \quad \text{(A17)}$$

For $\mu\tau \to \infty$ the integral gives $\sqrt{\pi}/4$, which is a good approximation for $\mu\tau \gg 1$ [condition (A1)]. Therefore, a good approximate expression for the growth of the quantum and thermal contributions in the spinodal band is

$$g\Sigma(\tau) \approx \frac{g}{4\beta}\sqrt{\frac{\pi}{\mu}}\frac{e^{2\tau\mu}}{\tau^{3/2}}, \quad \text{(A18)}$$

which is valid for times $\frac{1}{\mu} \lesssim \tau \lesssim \tau_s$ and temperatures $\mu \ll \beta^{-1} \ll \frac{\mu}{g}$.

In the case of zero temperature we have $g\Sigma_s^{T=0}(\tau) \approx (g\sqrt{\pi\mu}e^{2\tau\mu})/(8\tau^{3/2})$ (see Ref. [3]) and we can express the result obtained as

$$g\Sigma_s(\tau) \approx \frac{2}{\beta\mu} g\Sigma_s^{T=0}(\tau). \quad \text{(A19)}$$

After a certain time, the spinodal time τ_s, the quantum and thermal effects start to be important in the dynamics, $g\Sigma_s(\tau_s)$ compensates $-\mu^2$, and the exponential growth of the mode functions stops, and they start to all have an oscillatory behavior. Thus, the spinodal time τ_s is defined as

$$g\Sigma_s(\tau_s) = \mu^2. \quad \text{(A20)}$$

Using Eqs. (A18) and (A20) we have

$$\frac{g}{4\beta}\sqrt{\frac{\pi}{\mu}}\frac{e^{2\tau_s\mu}}{\tau_s^{3/2}} = \mu^2, \quad \text{(A21)}$$

and taking logarithms we obtain

$$\tau_s = \frac{1}{2\mu}\left[\log\left(\frac{4\beta\mu}{g\sqrt{\pi}}\right) + \frac{3}{2}\log\left(\mu\tau_s\right)\right], \quad \text{(A22)}$$

implying

$$\tau_s = \frac{1}{2\mu}\log\left(\frac{4\beta\mu}{g\sqrt{\pi}}\right) + \frac{3}{4\mu}\log\left[\frac{1}{2}\log\left(\frac{4\beta\mu}{g\sqrt{\pi}}\right)\right] + \mathcal{O}\left(\frac{\log\log\frac{1}{g}}{\log\frac{1}{g}}\right); \quad \text{(A23)}$$

this expression is related to the spinodal time for zero temperature, $\tau_s^{T=0}$ (see Ref. [3]), through the equation

$$\tau_s = \tau_s^{T=0} - \log\left(\frac{2}{\beta\mu}\right) + \mathcal{O}\left(\log\log\frac{1}{g}\right). \quad \text{(A24)}$$

One direct conclusion from the expression (A24) is that an increase of the initial temperature diminishes the spinodal time.

2. Low temperatures ($\beta^{-1} \ll \mu$)

For temperatures that satisfy

$$\beta^{-1} \ll \mu \quad \text{(A25)}$$

we can make the following development for the hyperbolic cotangent,

$$\coth\left(\frac{\beta\omega_q(0)}{2}\right) = \coth\left(\frac{\beta\sqrt{q + \mathcal{M}^2(0)}}{2}\right) = 1 + \mathcal{O}(e^{-\beta\mu}). \quad \text{(A26)}$$

Thus, for these low temperatures we recover as a good approximation the zero temperature expressions, and we obtain for the spinodal time a value very close to $\tau_s^{T=0}$.

In summary, a good approximate expression for the spinodal time to order $\mathcal{O}[(\log\log\frac{1}{g})/\log\frac{1}{g}]$ is

$$\tau_s = \begin{cases} \frac{1}{2\mu}\log\left(\frac{4\beta\mu}{g\sqrt{\pi}}\right) + \frac{3}{4\mu}\log\left[\frac{1}{2}\log\left(\frac{4\beta\mu}{g\sqrt{\pi}}\right)\right] & \text{for } \beta^{-1} \gg \mu, \\ \frac{1}{2\mu}\log\left(\frac{8}{g\sqrt{\pi}}\right) + \frac{3}{4\mu}\log\left[\frac{1}{2}\log\left(\frac{8}{g\sqrt{\pi}}\right)\right] & \text{for } \beta^{-1} \ll \mu. \end{cases} \quad \text{(A27)}$$

This analytic expression is in good agreement with the numerical results, as can be seen in Table II.

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