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Failure of microcausality in quantum field theory on noncommutative spacetime

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The commutator of $:\phi(x) \star \phi(x):$ with $\partial^y_\mu :\phi(y) \star \phi(y):$ fails to vanish at equal times and thus also fails to obey microcausality at spacelike separation even for the case in which $\theta^{0i} = 0$. The failure to obey microcausality for these sample observables implies that this form of noncommutative field theory fails to obey microcausality in general. This result holds generally when there are time derivatives in the observables. We discuss possible responses to this problem.

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I. INTRODUCTION

There is broad agreement that new possibilities, beyond the standard model, must be explored to understand how to reconcile relativistic quantum theory with the theory of gravity provided by Einstein's general theory of relativity, as well as to reduce the number of parameters that must be found empirically in order to make the standard model precise. String theory is the most far-reaching of the extensions of the standard model. Quantum field theory on noncommutative spacetime stands as an intermediate framework between string theory and the usual quantum theory of fields. Noncommutative spacetime was considered as long ago as 1947 [1]. Because this intermediate theory is more manageable than string theory, quantum field theory on noncommutative spacetime has aroused a good deal of interest following the work by Doplicher, *et al.* [2]. Reviews appear in [3,4]. The specific type of theory on noncommutative spacetime that has been studied the most is the one in which the noncommutativity takes the form

$$[\hat{x}^\mu, \hat{x}^\nu]_- = i\theta^{\mu\nu}, \tag{1}$$

with $\theta^{\mu\nu}$ chosen to be a constant matrix. Most authors make this choice only for the case Eq. (1). Some authors also assume

$$[\hat{x}^\mu, \hat{y}^\nu]_- = i\theta^{\mu\nu}, \qquad y \neq x. \tag{2}$$

The argument for Eq. (1) is that it follows both from string theory in a background "magnetic" field [5,6] and from the equation for the motion of an electron in a magnetic field [7]. The argument that one should also adopt Eq. (2) because otherwise there would be a discontinuity for $\hat{x} \to \hat{y}$ seems to be based on too naïve an interpretation of the symbol $\hat{x}$. We will not adopt Eq. (2); rather we will assume

$$[\hat{x}^\mu, \hat{y}^\nu]_- = 0, \qquad y \neq x \tag{3}$$

for most of our discussion. Since some authors do use the star product that follows for $x \neq y$ both for field products and in between the terms of a commutator [8–10], we will discuss this case in a later section. With the assumption of Eq. (1) we replace the field $\phi(\hat{x})$ by $\phi(x)$ and use the star product [3,4] for the product of fields at the *same* spacetime point. This means that field theory on noncommutative spacetime becomes a particular form of nonlocal field theory, with the nonlocality expressed in terms of the Moyal phases that occur in the star product.

One of the major problems with this case of constant $\theta^{\mu\nu}$ is that it breaks the Lorentz group $SO(1,3)$ to $SO(1,1) \times SO(2)$ which is Abelian and thus has only one-dimensional irreducible representations. Because of this, no spinor, vector, etc. fields would exist. M. Chaichian, *et al.* [11,12], J. Wess [13], and P. Aschieri, *et al.* [8], have shown that the theory has a twisted Lorentz (and also Poincaré) symmetry in which the full $SO(1,3)$ symmetry remains, and thus the spinor, vector, etc. representations do occur. To date the full significance of this twisted symmetry is unclear.

In this paper we consider the question of microcausality of observables; i.e., of vanishing of the commutator of observables at spacelike separation. This condition is often called locality, but since locality can have several meanings, we will use "microcausality" for this requirement.

Chaichian, *et al.* [14], studied microcausality for the choice of $\mathcal{O}(x) \equiv :\phi(x) \star \phi(x):$ as a sample observable and found that it obeys microcausality provided that $\theta^{0i} = 0$. We will take $\theta^{0i} = 0$ throughout this paper [15]. Since this condition is required for unitarity, this is not a further restriction on the theory. These authors stated that microcausality would hold generally for observables, but we show below that this is not the case. Because microcausality should hold for all observables, we also want $\partial_\mu \mathcal{O}(x)$ to obey microcausality relative to $\mathcal{O}(y)$ as well as relative to $\partial_\nu \mathcal{O}(y)$. We find that microcausality fails for some of these cases.

In the discussion of microcausality, to prove a positive result one must show that all matrix elements of the commutator obey microcausality. To show a negative result, that the commutator violates microcausality, one need only show that any single matrix element of the commutator violates microcausality.

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Although we give detailed calculations for the sample observable considered by Chaichian, *et al.*, our results are valid for any fields or observables, as we discuss later.

II. CALCULATION OF $[\mathcal{O}(x), \partial_\nu \mathcal{O}(y)]_-$

Here are our normalization and other conventions which differ from those of Chaichian, *et al.*,

$$\phi(x) = (2\pi)^{-D/2} \int \tilde{\phi}(k) e^{-ik\cdot x} d^D k, \tag{4}$$

$$\langle 0|\tilde{\phi}(k)\tilde{\phi}(l)|0\rangle = 2\pi\theta(k^0)\delta(k^2 - m^2)\delta(k + l), \tag{5}$$

$$\langle 0|\tilde{\phi}(k)|p\rangle = \delta(k - p), \qquad E_k = \sqrt{\mathbf{k}^2 + m^2}. \tag{6}$$

We find

$$\langle 0|[: \phi(x) \star \phi(x) :, : \phi(y) \star \phi(y) :]_- |p, p'\rangle = (e^{-ip\cdot x - ip'\cdot y} + e^{-ip'\cdot x - ip\cdot y}) \frac{4}{(2\pi)^{2D-1}} \int d^D k \epsilon(k^0)\delta(k^2 - m^2) e^{-ik\cdot(x-y)} \times \cos\left(\frac{1}{2}\theta^{\mu\nu} k_\mu p_\nu\right) \cos\left(\frac{1}{2}\theta^{\mu\nu} k_\mu p'_\nu\right), \tag{7}$$

which agrees, up to irrelevant numerical factors, with the calculation of Chaichian, *et al.* [14]. We also calculated the anticommutator, obtained by replacing $\epsilon(k^0)$ by 1 in Eq. (7), and also checked by direct calculation

$$\langle 0|[: \phi(x) \star \phi(x) :, : \phi(y) \star \phi(y) :]_+ |p, p'\rangle = (e^{-ip\cdot x - ip'\cdot y} + e^{-ip'\cdot x - ip\cdot y}) \frac{4}{(2\pi)^{2D-1}} \int d^D k \delta(k^2 - m^2) e^{-ik\cdot(x-y)} \times \cos\left(\frac{1}{2}\theta^{\mu\nu} k_\mu p_\nu\right) \cos\left(\frac{1}{2}\theta^{\mu\nu} k_\mu p'_\nu\right). \tag{8}$$

From Eq. (7),

$$[\mathcal{O}(x), \partial_\nu \mathcal{O}(y)]_- = -i(p'_\nu e^{-ip\cdot x - ip'\cdot y} + p_\nu e^{-ip'\cdot x - ip\cdot y}) \frac{4}{(2\pi)^{2D-1}} \int d^D k \epsilon(k^0)\delta(k^2 - m^2) e^{-ik\cdot(x-y)} \cos\left(\frac{1}{2}\theta^{\mu\nu} k_\mu p_\nu\right) \times \cos\left(\frac{1}{2}\theta^{\mu\nu} k_\mu p'_\nu\right) + (e^{-ip\cdot x - ip'\cdot y} + e^{-ip'\cdot x - ip\cdot y}) \frac{4}{(2\pi)^{2D-1}} \int d^D k (ik_\nu)\epsilon(k^0)\delta(k^2 - m^2) e^{-ik\cdot(x-y)} \times \cos\left(\frac{1}{2}\theta^{\mu\nu} k_\mu p_\nu\right) \cos\left(\frac{1}{2}\theta^{\mu\nu} k_\mu p'_\nu\right). \tag{9}$$

At $x^o = y^0$, the $\nu = 0$ term is

$$(e^{-ip\cdot x - ip'\cdot y} + e^{-ip'\cdot x - ip\cdot y}) \frac{4i}{(2\pi)^{2D-1}} \int d^{D-1} k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \cos\left(\frac{1}{2}\theta^{ij} k_i p_j\right) \cos\left(\frac{1}{2}\theta^{ij} k_i p'_j\right). \tag{10}$$

In order for this to vanish for $\mathbf{x} - \mathbf{y} \neq 0$, the Fourier transform of Eq. (10) must be a *polynomial* in k. Since this Fourier transform is $\cos(\frac{1}{2}\theta^{ij} k_i p_j)\cos(\frac{1}{2}\theta^{ij} k_i p'_j)$, it is not a polynomial in k, and thus this commutator violates microcausality. If we carry out the $\int d^{D-1}k$ we get a sum of delta functions that exhibits the violation of microcausality explicitly. To be explicit, let $\theta^{12} = -\theta^{12} = \theta$, other values of $\theta = 0$, then, up to irrelevant factors, the nonlocality is

$$\sum_{s=\pm1, t=\pm1} \delta(x^1 - y^1 - s\theta(p_2 + tp'_2)) \times \delta(x^2 - y^2 - s\theta(p_1 + tp'_1))\delta(x^3 - y^3). \tag{11}$$

The nonlocality increases with the sum or difference of the momenta of the particles.

We expect that the commutator of any observable which is a polynomial in free fields with odd numbers of time derivatives will fail to commute at spacelike separation, just as in the case calculated above. A relevant case of such an observable is the current of a charged scalar field. The basic reason for these violations of spacelike commutativity is that the space averaging of zero, which occurs for $\Delta(x - y)$ at $x^0 = y^0$, is still zero. By contrast, the space averaging of $\delta(\mathbf{x} - \mathbf{y})$, which occurs for $\partial_{x^0}\Delta(x - y)$ at $x^0 = y^0$, is not zero.

III. CALCULATION OF A MATRIX ELEMENT OF THE STAR COMMUTATOR

In the study of $[\mathcal{O}(x), \mathcal{O}(y)]_-$ Chaichian, *et al.* [14] considered the ordinary commutator rather than the star commutator,

$$[\mathcal{O}(x), \mathcal{O}(y)]_{\star -} = [: \phi(x) \star \phi(x) :, : \phi(y) \star \phi(y) :]_{\star -}$$
$$\equiv : \phi(x) \star \phi(x) : \star : \phi(y) \star \phi(y) : - : \phi(y) \star \phi(y) : \star : \phi(x) \star \phi(x) :. \quad (12)$$

We have calculated the star commutator for this sample observable. We anticipate that the star commutator will give a qualitatively different result than the ordinary one, because the Moyal phases in the star commutator will be sensitive to both coordinates x and y and thus to the separation of x and y, while the star product in the observable itself is not aware of this separation. From a more calculational point of view, the new Moyal phases in the terms of the star commutator will have opposite signs in the two terms. Thus if the Moyal phase in one term of the star commutator is $e^{i\Theta}$ the phase in the other term will be $e^{-i\Theta}$ and the star commutator will have the form

$$[\mathcal{O}(x), \mathcal{O}(y)]_{\star -} = \cos\Theta[\mathcal{O}(x), \mathcal{O}(y)]_{-} + i\sin\Theta[\mathcal{O}(x), \mathcal{O}(y)]_{+}, \quad (13)$$

where Θ is the differential operator,

$$\Theta = \frac{i}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y. \quad (14)$$

The anticommutator term will not vanish at spacelike separation. We only have to convert the differential operator, Θ, defined in Eq. (14), to momentum space and insert it in Eq. (13) to find

$$[\mathcal{O}(x), \mathcal{O}(y)]_{\star -} = (e^{-ip\cdot x - ip'\cdot y} + e^{-ip'\cdot x - ip\cdot y}) \quad (15)$$

$$\times \frac{4}{(2\pi)^{2D-1}}\int d^D k(\epsilon(k)\cos\tilde{\Theta} + i\sin\tilde{\Theta})\delta(k^2 - m^2)e^{-ik\cdot(x-y)}\cos\left(\frac{1}{2}\theta^{\mu\nu}k_\mu p_\nu\right)\cos\left(\frac{1}{2}\theta^{\mu\nu}k_\mu p'_\nu\right), \quad (16)$$

where now

$$\tilde{\Theta} = -\tfrac{1}{2}\theta^{ij}(k_i(p + p')_j + p_i p'_j). \quad (17)$$

At equal times,

$$[\mathcal{O}(x), \mathcal{O}(y)]_{\star - ET} = (e^{-ip\cdot x - ip'\cdot y} + e^{-ip'\cdot x - ip\cdot y}) \quad (18)$$

$$\times \frac{4}{(2\pi)^{2D-1}}\int \frac{d^{D-1}k}{2E_k}[\cos\tilde{\Theta}(e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} - e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}) + i\sin\tilde{\Theta}(e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} + e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})})]$$
$$\times\cos\left(\frac{1}{2}\theta^{ij}k_i p_j\right)\cos\left(\frac{1}{2}\theta^{ij}k_i p'_j\right), \quad (19)$$

where we have exhibited explicitly the contributions from the two mass shells. Obviously the coefficient of the $\cos\tilde{\Theta}$ term vanishes. The final result is

$$[\mathcal{O}(x), \mathcal{O}(y)]_{\star - ET} = (e^{-ip\cdot x - ip'\cdot y} + e^{-ip'\cdot x - ip\cdot y}) \quad (20)$$

$$\times \frac{8i}{(2\pi)^{2D-1}}\int \frac{d^{D-1}k}{2E_k}\sin\left(-\frac{1}{2}\theta^{ij}(k_i(p + p')_j + p_i p'_j)\right)(e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})})\cos\left(\frac{1}{2}\theta^{ij}k_i p_j\right)\cos\left(\frac{1}{2}\theta^{ij}k_i p'_j\right). \quad (21)$$

As in the previous section, the Fourier transform of this is not a polynomial in k, so this quantity does not vanish at spacelike separation. Clearly if we drop the $\tilde{\Theta}$ term we recover the result for the ordinary commutator which does obey microcausality. (We can equip the $\tilde{\Theta}$ parameter with a factor λ if we want to go continuously between the ordinary and the star commutator.) This completes the demonstration that the star commutator of this sample observable does not vanish at spacelike separation.

IV. STAR COMMUTATOR AND ANTICOMMUTATOR OF GENERAL FIELDS AND OBSERVABLES

The analog of the result we gave in Eq. (13) holds for any fields and observables. Although our discussion above concerned neutral scalar fields, our conclusions hold for fields, neutral or charged, of any spin provided the usual connection of spin and type of commutation relation (using a commutator or an anticommutator) is used. For fields or observables whose commutators vanish at spacelike separation in ordinary field theory, the star commutators of the fields or observables on noncommutative spacetime fail to vanish. Correspondingly, for fields or observables whose anticommutators vanish at spacelike separation in ordinary field theory, the star anticommutators of the fields or observables on noncommutative spacetime fail to vanish. Thus even the free field star commutator (anticommutator) does not vanish at spacelike separation. Because of the anticommutator (commutator) term in Eq. (13) the free field commutator (anticommutator) on noncommutative spacetime is neither translation invariant nor a c-number. On the other hand, the vacuum matrix element, and thus also the propagator of the free field on noncommutative spacetime, is the usual one, because for the vacuum matrix element the derivatives in Eq. (14) or the momenta in Eq. (17) are linearly dependent so that the Moyal phase vanishes.

V. RELATED WORK

The increase of nonlocality with momentum that we found in Eq. (11) is similar to that found by [16]. H. Bozkaya, *et al.*, [17] studied microcausality in noncommutative field theory in the context of perturbation theory using different definitions of time-ordering and concluded that microcausality and unitarity are in conflict. Our simpler calculation was done in the context of free field observables rather than in perturbation theory. L. Alvarez-Gaume’, *et al.*, [18] also studied microcausality in perturbation theory and found that $SO(1,3)$ microcausality is violated but that $SO(1,1)$ microcausality, i.e. microcausality in the light wedge, holds if and only if perturbative unitarity holds.

VI. COMMENTS ABOUT THE FAILURE OF MICROCAUSALITY

Since the light cone has no status in a theory with constant $\theta^{\mu\nu}$ it is surprising that microcausality can hold in some special cases, such as the case in which the observables are constructed from scalar fields with no time derivatives [14]. What one should expect is that only the light wedge, $x^{02} - x^{32} \leq 0$, has meaning as discussed by [18]. Assuming $\theta^{12} = \theta$, with the other elements of the θ matrix equal to zero, both the ordinary and the star commutator of observables vanish trivially in the light wedge. Very likely the choice of constant θ should be abandoned in favor of a θ that transforms under the Lorentz group. This was the point of view of Snyder [1] in his early work and recently has been suggested by Doplicher, *et al.* [2,19]. Other responses to the failure of microcausality that we demonstrated in the previous sections include: (a) that massive string states cannot be neglected in quantum field theory on noncommutative spacetime, at least in the version in which the noncommutativity occurs as a constant matrix $\theta^{\mu\nu}$ as in Eq. (1), and the noncommutativity is implemented via the star product as described above. Gomis and Mehen [20] have shown that theories with electric ($\theta^{0i} \neq 0$) noncommutativity violate unitarity, except for the case of lightlike noncommutativity [21], and do not represent a low-energy limit of string theory, while theories, at least in perturbation theory to one loop, with magnetic ($\theta^{0i} = 0$, $\theta^{ij} \neq 0$) noncommutativity obey unitarity and can serve as a low-energy limit of string theory. The situation here differs from the case considered by Gomis and Mehen, not only because microcausality is at stake instead of unitarity, but also because the problem occurs even when $\theta^{0i} = 0$. Nonetheless, the results of Gomis and Mehen may give a hint that the problem arises because of the neglect of massive string states. If this is the correct way to understand the failure of microcausality, we should ask if there is some way to amend the usual space-space noncommutativity so that the massive string modes can be incorporated and microcausality can be restored; or (b) to drop the requirement of microcausality. We do not speculate on the implications of this last response in this paper.

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