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New tests and applications of the worldline path integral in the first order formalism

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We present different nonperturbative calculations within the context of Migdal's representation for the propagator and effective action of quantum particles. We calculate the exact propagators and effective actions for Dirac, scalar, and Proca fields in the presence of constant electromagnetic fields, for an even-dimensional spacetime.

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I. INTRODUCTION

Worldline formulations have been applied since a long time ago [1] to the derivation of many interesting Quantum Field Theory results. More recent applications have emerged as a by-product of the new insights gained by the rederivation of worldline representations by taking the infinite tension limit in (perturbative) string theory amplitudes [2] and by the introduction of new ways to handle the spin degrees of freedom [3–5]. Besides, elegant proposals to treat more general situations, involving internal degrees of freedom and general couplings to higher-spin fields have been advanced [6,7].

In these methods, different sets of variables and alternative constructions have been used in order to "exponentiate" the relevant observables and then perform the path integral. In spite of the *formal* equivalence between the different methods, there are few concrete calculations that may serve as tests to gain a deeper understanding of the method and about the physics involved. Important steps in that direction have already been taken; indeed, some nonperturbative calculations corresponding to external constant electromagnetic fields have been obtained within the worldline representation [8]. Another setting where the worldline approach can be independently tested is in numerical calculations [9].

In this article, we present new tests corresponding to concrete examples, obtained within the worldline path-integral representation for Dirac and other fields in the first-order formalism introduced by Migdal in [10] and further extended in [4,5]. The first-order formalism preserves the geometrical picture and is quite intuitive (for example, it does not involve Grassmann variables). These two features are, we believe, among the main advantages of the worldline method. It is also more adequate for some numerical computations which are specially suited for nonperturbative calculations.

We shall follow our previous work [11], where some features of the method have been discussed in detail, including a proof of the equivalence with standard quantum field theory at the perturbative level.

The structure of the paper is as follows: In Sec. II we briefly review the main properties of the representation introduced in [10], with emphasis in the objects we shall be concerned with in the examples. To elucidate the quite general nature of this approach, we also introduce the worldline representation for the propagator and the effective action corresponding to a Proca field.

In Sec. III we deal with a constant $F_{\mu\nu}$ field in $1+1$ dimensions, for the Dirac, scalar, and Proca cases. In Sec. IV, we generalize the previous cases to $d > 2$ ($d =$ even) dimensions. Finally, in Sec. V we present our conclusions.

II. THE METHOD

Our aim here is to calculate the propagator and effective action corresponding to external electromagnetic fields. We shall be concerned with Dirac, scalar, and Proca models, coupled to Abelian gauge fields.

Let us consider first the case of a massive Dirac field in d Euclidean dimensions, whose action S_f has the following form:

$$S_f(\bar{\psi}, \psi, A) = \int d^d x \bar{\psi}(\not{D} + m)\psi, \tag{1}$$

where

$$\begin{aligned} \not{D} &\equiv \gamma_\mu D_\mu, \qquad D_\mu = \partial_\mu + ieA_\mu, \\ \gamma_\mu^\dagger &= \gamma_\mu, \qquad \mu = 1, \ldots, d. \end{aligned} \tag{2}$$

The γ_μ matrices satisfy the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \qquad \forall \mu = 1, \ldots, d, \tag{3}$$

where A_μ denotes an Abelian gauge field and e is a coupling constant with the dimensions of $[\text{mass}]^{(4-d)/2}$.

In the worldline formulation of [10], the fermion propagator, denoted here by $G_f(x, y)$, is represented by the path integral

$$G_f(x,y) = \int_0^\infty dT\, e^{-mT} \int_{x(0)=y}^{x(T)=x} \mathcal{D}p\mathcal{D}x\, e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \times \mathcal{P}[e^{-i\int_0^T d\tau \not{p}(\tau)}] e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]}, \tag{4}$$

where we have explicitly indicated the boundary conditions for the $x_\mu(\tau)$ paths. The $p_\mu(\tau)$ paths are, on the other hand, unconstrained.

Another object we will be interested in is $\Gamma_f(A)$, the (normalized) contribution of the fermionic determinant to the effective action:

$$\Gamma_f(A) \equiv -\ln\left[\frac{\det(\not{D}+m)}{\det(\not{\partial}+m)}\right] = -\mathrm{Tr}\ln(\not{D}+m) + \mathrm{Tr}\ln(\not{\partial}+m), \tag{5}$$

which (by definition) verifies $\Gamma_f(0) = 0$.

For $\Gamma_f(A)$ we have the worldline representation:

$$\Gamma_f(A) = \int_0^\infty \frac{dT}{T} e^{-mT} \int_{x(0)=x(T)} \mathcal{D}p\mathcal{D}x e^{i\int_0^T d\tau p_\mu(\tau)\dot{x}_\mu(\tau)} \times \mathrm{Tr}[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}] e^{-ie\int_0^T d\tau \dot{x}_\mu(\tau)A_\mu[x(\tau)]}, \tag{6}$$

where the functional integration measure may be formally represented as

$$\mathcal{D}\,p\mathcal{D}x \equiv \prod_{0<\tau\leq T} \frac{d^dx(\tau)d^dp(\tau)}{(2\pi)^d}, \tag{7}$$

and it is (also formally) dimensionless, since there are as many dp's as there are dx's in the integration measure and, in our conventions, $\hbar = 1$. Of course, this formal definition can be made more rigorous by introducing a discrete approximation to it and taking the corresponding limit. This procedure will, indeed, be used later on to deal with some examples.

We also will consider complex scalar fields, their propagators (to be denoted by G_b), and their contribution to the effective action (Γ_b). In the context of the worldline formulation we have explained before, those objects have similar expressions to their Dirac counterparts. Indeed, if the field theory action S_b for φ, $\bar{\varphi}$ is

$$S_b = \int d^dx [\bar{D}_\mu \varphi D_\mu \varphi + m^2 \bar{\varphi}\varphi], \tag{8}$$

then, an entirely analogous definition to the one used for the Dirac field leads to

$$G_b(x,y) = \int_0^\infty dT\; e^{-m^2T} \int_{x(0)=y}^{x(T)=x} \mathcal{D}p\mathcal{D}x\; e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \times e^{-\int_0^T d\tau\, p^2(\tau)} e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]}, \tag{9}$$

and

$$\Gamma_b(A) = -\int_0^\infty \frac{dT}{T} e^{-m^2T} \int_{x(0)=x(T)} \mathcal{D}x\mathcal{D}p\; e^{i\int_0^T d\tau p_\mu(\tau)\dot{x}_\mu(\tau)} \times e^{-\int_0^T d\tau p^2(\tau)} e^{-ie\int_0^T d\tau \dot{x}_\mu(\tau)A_\mu[x(\tau)]}. \tag{10}$$

It is interesting to compare the previous expressions with their Dirac field counterparts: note that the difference amounts to replacing the object

$$\Phi_f(T) \equiv \mathcal{P}[e^{-i\int_0^T d\tau \not{p}(\tau)}], \tag{11}$$

by

$$\Phi_b(T) \equiv e^{-\int_0^T d\tau p^2(\tau)}, \tag{12}$$

in the corresponding fermionic formula. Besides, there is a (-1) factor in Γ_b because of the different statistics; m is replaced by m^2, and the trace of Φ_b is of course absent.

This general structure will reproduce itself with more or less straightforward changes for the next example that we shall consider: the Proca field, for which we use a_μ to denote the field variable in order to avoid confusion with the gauge field A_μ. The Euclidean action, S_P, is defined by

$$S_P = \int d^dx \left(\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \frac{1}{2} m^2 a_\mu a_\mu\right), \tag{13}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. This action corresponds of course to the case of a *real* field, for which it makes sense to define the (free) propagator, but to allow for a coupling to an external gauge field A_μ, we also consider the complex field version:

$$S_P(a^*, a; A) = \int d^dx \left(\frac{1}{2}|D_\mu a_\nu - D_\nu a_\mu|^2 + m^2|a|^2\right). \tag{14}$$

Based on the form of the Euclidean actions, it is rather straightforward to derive the propagator in the presence of an external field A_μ. Indeed, we have

$$G_P(x,y) = \int_0^\infty dT\, e^{-m^2T} \int_{x(0)=y}^{x(T)=x} \mathcal{D}p\mathcal{D}x\, e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \times \Phi_P(T) e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]}, \tag{15}$$

with

$$\Phi_P(T) = \mathcal{P}\exp\left[-\int_0^T d\tau p_\alpha(\tau) p_\beta(\tau) \Gamma^P_{\alpha\beta}\right], \tag{16}$$

and the $\Gamma^P_{\alpha\beta}$ are a set of $d \times d$ matrices whose components are

$$[\Gamma^P_{\alpha\beta}]_{\mu\nu} = \delta_{\alpha\beta}\delta_{\mu\nu} - \tfrac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu}). \tag{17}$$

It should now be clear that, when considering the one-loop effective action one needs to evaluate the expression:

$$\Gamma_P(A) = -\int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(0)=x(T)} \mathcal{D}p\mathcal{D}x\, e^{i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)} \times \mathrm{Tr}[\Phi_P(T)] e^{-ie\int_0^T d\tau \dot{x}(\tau)\cdot A[x(\tau)]}. \tag{18}$$

A quite remarkable property, that we may use to our advantage, is that the functional integral over x_μ, for a given background field A_μ is the same for all the fields. The differences shall of course appear when evaluating the integrals over p_μ, since they are affected by the spin-dependent factor $\Phi(T)$.

We conclude this section by mentioning that the previous representations are not unique (in many ways). One of the reasons is that one may always describe a theory (with any spin) in terms of first-order equations, although for a different set of field variables. Indeed, the equations of motion for a free field φ may always be written as follows [12]:

$$(\Gamma_\mu \partial_\mu + m)\phi(x) = 0, \tag{19}$$

where ϕ is a multicomponent field, defined in terms of φ and its derivatives, while Γ_μ are matrices whose form depends on the spin content of the original field φ. For example, for a massive real scalar field φ, we may write the action:

$$S = \frac{m}{2}\int d^d x \bar{\phi}(\Gamma_\mu \partial_\mu + m)\phi, \tag{20}$$

where the Γ_μ are, in this case, the $(d+1)\times(d+1)$ matrices

$$[\Gamma_\mu]_{ab} = \delta_{a,\mu+1}\delta_{\nu+1,b}, \tag{21}$$

and

$$\phi = \begin{pmatrix} \varphi \\ -\partial_1\varphi/m \\ -\partial_2\varphi/m \\ \dots \\ -\partial_d\varphi/m \end{pmatrix}. \tag{22}$$

The "adjoint" $\bar{\phi}$ is defined as $\bar{\phi} = \phi^T\Gamma_0$, where

$$\Gamma_0 = \begin{pmatrix} 1 & 0_{1\times d} \\ 0_{d\times 1} & I_{d\times d} \end{pmatrix}. \tag{23}$$

If the field is instead complex, and it is coupled to an external gauge field A_μ, we have

$$S = m\int d^d x \bar{\phi}(\Gamma_\mu D_\mu + m)\phi, \tag{24}$$

with the only difference with respect to the previous case is that $\bar{\phi} = \phi^\dagger\Gamma_0$.

The generalization to higher spins J is simple, although care must be taken when considering $J > 1$ [12], due to the existence of nontrivial constraints on the state vectors, depending on representation chosen for the Γ_μ matrices.

Once this first-order formulation is introduced, one may write a worldline representation for the one-loop effective action Γ, which is given by

$$\Gamma(A) = \int_0^\infty \frac{dT}{T} e^{-mT} \int_{x(0)=x(T)} \mathcal{D}p\mathcal{D}x\, e^{i\int_0^T d\tau p_\mu(\tau)\dot{x}_\mu(\tau)} \times \mathrm{Tr}[\mathcal{P}e^{-i\int_0^T d\tau \Gamma_\mu p_\mu(\tau)}] e^{-ie\int_0^T d\tau \dot{x}_\mu(\tau) A_\mu[x(\tau)]}, \tag{25}$$

which has the same structure as the one introduced for the Dirac case.[1]

III. CONSTANT EXTERNAL FIELD IN 1 + 1 DIMENSIONS

A. Dirac field

We shall present here the evaluation of the fermionic determinant and propagator for a massive Dirac field in the presence of a constant external $F_{\mu\nu}$ field, in 1 + 1 dimensions. As usual, rather than working directly with the determinant, we instead use the effective action $\Gamma_f(A)$,

$$\Gamma_f(A) = \int_0^\infty \frac{dT}{T} e^{-mT} \int_{x(0)=x(T)} \mathcal{D}p\mathcal{D}x\, e^{i\int_0^T d\tau p_\mu(\tau)\dot{x}_\mu(\tau)} \times \mathrm{Tr}[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}] e^{-ie\int_0^T d\tau \dot{x}_\mu(\tau) A_\mu[x(\tau)]}, \tag{26}$$

where A_μ is such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = F\varepsilon_{\mu\nu}, \tag{27}$$

with $F \equiv$ constant. For A_μ we adopt a gauge-fixing condition such that

$$A_1(x) = -Fx_2, \qquad A_2 = 0. \tag{28}$$

We then see that

$$\Gamma_f(A) = \int_0^\infty \frac{dT}{T} e^{-mT} \int \mathcal{D}p\, \mathrm{Tr}[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}] \mathcal{Z}(p, F), \tag{29}$$

with

$$\mathcal{Z}(p,F) = \int_{x(0)=x(T)} \mathcal{D}x\, e^{i\int_0^T d\tau p_\mu(\tau)\dot{x}_\mu(\tau)} e^{ieF\int_0^T d\tau\, \dot{x}_1(\tau)x_2(\tau)}. \tag{30}$$

We shall now evaluate $\mathcal{Z}(p, F)$. As it will become clear, the same object appears within the context of the complex scalar field determinant.

To evaluate, we first separate it into two iterated integrals—one for each component:

[1] The structure is more complicated for $J > 1$; see [12].

$$\mathcal{Z}(p,F) = \int_{x_2(0)=x_2(T)} \mathcal{D}x_2 \Big\{ e^{i\int_0^T d\tau \dot{x}_2(\tau) p_2(\tau)} \times \int_{x_1(0)=x_1(T)} \mathcal{D}x_1\, e^{i\int_0^T d\tau \dot{x}_1(\tau)[eFx_2(\tau)+p_1(\tau)]} \Big\}. \tag{31}$$

The two previous integrals are quite simple to evaluate, the result being

$$\mathcal{Z}(p,F) \propto \exp\Big[-\frac{i}{eF}\int_0^T d\tau \dot{p}_1(\tau) p_2(\tau)\Big], \tag{32}$$

an expression which captures the exact dependence on $p_\mu(\tau)$. However, in order to calculate $\Gamma_f(A)$ exactly, we need to know the exact form of $\mathcal{Z}(A)$, including any relevant global factor. One safe way to do that is, as usual, to introduce a discretization of the functional integral. For example, splitting the $[0,T]$ interval into n subintervals, we see that the functional integral over $x_1(\tau)$ is given by the limit:

$$\int_{x_1(0)=x_1(T)} \mathcal{D}x_1\, e^{i\int_0^T d\tau \dot{x}_1(\tau)[eFx_2(\tau)+p_1(\tau)]} = \lim_{n\to\infty}\Big\{\int\Big(\prod_{k=1}^n dx_1^{(k)}\Big) e^{i\sum_{k=1}^n (x_1^{(k+1)}-x_1^{(k)})[eFx_2^{(k)}+p_1^{(k)}]}\Big\},$$

where $x_1^{(k)}$ denotes $x_1(\tau)$ at the discrete time τ_k, with $\tau_k \equiv \frac{kT}{n}$, and a similar convention for x_2 and p_μ. Periodicity requires $x_\mu^{(n+1)} = x_\mu^{(1)}$. It is then immediate to see that

$$\int\Big(\prod_{k=1}^n dx_1^{(k)}\Big) e^{i\sum_{k=1}^n (x_1^{(k+1)}-x_1^{(k)})[eFx_2^{(k)}+p_1^{(k)}]} = L_1 \prod_{l=1}^{n-1} 2\pi\,\delta(eF(x_2^{(l)} - x_2^{(l-1)}) + p_1^{(l)} - p_1^{(l-1)}), \tag{34}$$

where L_1 is the total length of the system along the x_1 coordinate. Discretizing also the $x_2(\tau)$ integral, an analogous calculation yields

$$\begin{aligned}\mathcal{Z}(p,F) &= L_1L_2 \lim_{n\to\infty}\Big\{\Big(\frac{2\pi}{eF}\Big)^{n-1} e^{-[i/(eF)]\sum_{k=1}^n p_2^{(k)}(p_1^{(k+1)}-p_1^{(k)})}\Big\} \\ &= \frac{eFL_1L_2}{2\pi}\lim_{n\to\infty}\Big\{\Big(\frac{2\pi}{eF}\Big)^{n} e^{-[i/(eF)]\sum_{k=1}^n p_2^{(k)}(p_1^{(k+1)}-p_1^{(k)})}\Big\} \\ &= \xi \lim_{n\to\infty}\Big\{\Big(\frac{2\pi}{eF}\Big)^{n} e^{-[i/(eF)]\sum_{k=1}^n p_2^{(k)}(p_1^{(k+1)}-p_1^{(k)})}\Big\},\end{aligned} \tag{35}$$

where L_2 is the system length along the second coordinate. We have factored out the dimensionless quantity $\xi \equiv \frac{eFL_1L_2}{2\pi}$, which measures the "flux" through the system's area L_1L_2, in units of the elementary flux.

Note also that the product of δ functions implies, in particular, that the integral over $p_1(\tau)$ (to be performed next) will be over a space of periodic paths. Namely, the integral over x enforces periodic boundary conditions for the integral over p_1.

Then we insert the previous result for $\mathcal{Z}(p,T)$ into the expression for $\Gamma_f(A)$ and see that

$$\Gamma_f(A) = \xi\int_0^\infty \frac{dT}{T} e^{-mT}\int_{p_1(0)=p_1(T)} \widehat{\mathcal{D}p}\, \mathrm{Tr}[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}] \times e^{-[i/(eF)]\int_0^T d\tau \dot{p}_1(\tau)p_2(\tau)}, \tag{36}$$

where the new integration measure for $p(\tau)$, $\widehat{\mathcal{D}p}$, is defined by

$$\widehat{\mathcal{D}p} = \prod_{0<\tau\leq T}\frac{dp_1(\tau)dp_2(\tau)}{2\pi eF}, \tag{37}$$

[note that one of the two (2π) factors from (7) cancels out]. In the integral over $p(\tau)$, due to the presence of the term $\int d\tau \dot{p}_1(\tau)p_2(\tau)$ in the exponent, the functional integral is equivalent to the operator trace of an evolution operator, with the p_μ's replaced by time-independent, noncommuting operators:

$$\Gamma_f(A) = \xi\int_0^\infty \frac{dT}{T}e^{-mT}\,\mathrm{Tr}(e^{-iT\hat{\not{p}}}), \tag{38}$$

where the $\hat{p}_\mu$'s satisfy the commutation relation:

$$[\hat{p}_1, \hat{p}_2] = -ieF, \tag{39}$$

and the trace is over Hilbert and Dirac spaces.

To evaluate that trace we first write the operator $\hat{\not{p}}$ more explicitly, as follows:

$$\hat{\not{p}} = \sqrt{2eF}\hat{\mathcal{O}}, \tag{40}$$

where

$$\hat{\mathcal{O}} = \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0\end{pmatrix}, \tag{41}$$

and $\hat{a} \equiv \frac{\hat{p}_1 - i\hat{p}_2}{\sqrt{2eF}}$, $\hat{a}^\dagger \equiv \frac{\hat{p}_1 + i\hat{p}_2}{\sqrt{2eF}}$ (we assume that $eF > 0$).

Since the operators $\hat{a}$ and $\hat{a}^\dagger$ verify $[\hat{a}, \hat{a}^\dagger] = 1$, we can calculate the spectrum of the self-adjoint operator $\hat{\mathcal{O}}$ exactly. Indeed, we find the exact eigenvalues and eigenvectors to be the following:

$$\hat{\mathcal{O}}|\varphi_n^{(\pm)}\rangle = \lambda_n^{(\pm)}|\varphi_n^{(\pm)}\rangle, \qquad n\in\mathbb{N}, \qquad \hat{\mathcal{O}}|\varphi_0\rangle = 0, \tag{42}$$

where

$$\lambda_n^{(\pm)} = \pm\sqrt{n}, \qquad n = 1,2,\ldots \tag{43}$$

and

$$|\varphi_n^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\pm|n-1\rangle \\ |n\rangle\end{pmatrix}, \qquad n = 1,2,\ldots, \qquad |\varphi_0\rangle = \begin{pmatrix} 0 \\ |0\rangle\end{pmatrix}. \tag{44}$$

Here, $|n\rangle$ denotes the (normalized) eigenstates of the "number" operator $\hat{a}^\dagger\hat{a}$. Note that the upper element in

$|\varphi_0\rangle$ is 0 (the null vector), while the lower one is the "vacuum" state.

Then the effective action becomes

$$\Gamma_f(A) = \xi \int_0^\infty \frac{dT}{T} e^{-mT}\left[1 + \sum_{n=1}^{\infty}(e^{-iT\sqrt{2eFn}} + e^{iT\sqrt{2eFn}})\right], \tag{45}$$

or, integrating out T:

$$\Gamma_f(A) = \xi\left[\ln m + \sum_{n=1}^{\infty}\ln(m^2 + 2eFn)\right], \tag{46}$$

where we have neglected a constant which is independent of F and m.

Now to sum up the series, we use the representation

$$\ln x = -\lim_{s\to 0}\frac{d}{ds}[x^{-s}], \tag{47}$$

to obtain

$$\Gamma_f(A) = \xi\lim_{s\to 0}\frac{d}{ds}\left\{\sum_{n=1}^{\infty}(m^2 + 2eFn)^{-s} + m^{-s}\right\}, \tag{48}$$

or

$$\Gamma_f(A) = \xi\lim_{s\to 0}\frac{d}{ds}\left\{\sum_{n=1}^{\infty}(2eF)^{-s}\left(n + \frac{m^2}{2eF}\right)^{-s} + m^{-s}\right\}, \tag{49}$$

and

$$\Gamma_f(A) = \xi\lim_{s\to 0}\frac{d}{ds}\left\{(2eF)^{-s}\zeta_H\left(s; 1 + \frac{m^2}{2eF}\right) + m^{-s}\right\}, \tag{50}$$

where ζ_H denotes Hurwitz ζ function. The effective action is then obtained by taking the limit explicitly, and it is coincident with the results of [13], namely,

$$\Gamma_f(A) = L_1L_2\left[\frac{eF + m^2}{4\pi}\ln(2eF) + \frac{eF}{2\pi}\ln\Gamma\left(1 + \frac{m^2}{2eF}\right) - \frac{eF}{4\pi}\ln(2\pi m^2)\right]. \tag{51}$$

The imaginary part of $\tilde{\Gamma}_{(1+1)}(A)$ in Minkowski spacetime may be obtained by Wick rotating: $F \to iF$, so that

$$\Im[\tilde{\Gamma}_{(1+1)}(A)] = \sum_{n=1}^{\infty}\arctan\left[\frac{2eFn}{m^2}\right], \tag{52}$$

which also can be written in terms of the dimensionless vacuum angle [14] θ for the massive Schwinger model

$$\theta = \frac{2\pi F}{e}, \tag{53}$$

as

$$\Im[\tilde{\Gamma}_{(1+1)}(A)] = \sum_{n=1}^{\infty}\arctan\left[\left(\frac{e^2}{m^2}\right)\frac{\theta n}{\pi}\right]. \tag{54}$$

The result for the imaginary part does not exhibit the periodicity in θ of the interacting model, since here the gauge field is not dynamical.

The procedure we have followed for the calculation of the effective action of course also may be applied to the propagator, if one takes into account the main differences, namely, that the integration over x is not over periodic paths and that the spin degrees of freedom are not traced. Thus we are lead to

$$G_f(x, y) = \int_0^\infty dT\, e^{-mT}\langle x|e^{-iT\hat{\not{p}}}|y\rangle, \tag{55}$$

with the same definition for $\hat{p}$ we had in the effective action calculation. In abstract operator form

$$G_f = \int_0^\infty dT\, e^{-mT}e^{-iT\hat{\not{p}}}, \tag{56}$$

and matrix elements may be taken with respect to any convenient basis. Since we already know the eigenvectors of $\hat{p}$, we can use that basis. Integrating out over the "time" T the result is

$$G_f = \frac{1}{m}\mathcal{P}_0 + \sum_{n=1}^{\infty}\left\{\frac{2m}{m^2 + 2eFn}\mathcal{P}_n + \frac{-2im}{m^2 + 2eFn}\mathcal{Q}_n\right\}, \tag{57}$$

where

$$\mathcal{P}_0 = \begin{pmatrix} 0 & 0 \\ 0 & |0\rangle\langle 0| \end{pmatrix}, \tag{58}$$

$$\mathcal{P}_n = \begin{pmatrix} |n-1\rangle\langle n-1| & 0 \\ 0 & |n\rangle\langle n| \end{pmatrix}, \qquad (\forall n > 1), \tag{59}$$

and

$$\mathcal{Q}_n = \begin{pmatrix} 0 & |n-1\rangle\langle n| \\ |n\rangle\langle n-1| & 0 \end{pmatrix}, \qquad (\forall n > 1). \tag{60}$$

B. Complex scalar field

Let us now consider the changes that arise when calculating the effective action Γ_b, for the same gauge field configuration. First, we note that the calculation of $\mathcal{Z}(p, F)$ goes through in the same way as for the Dirac case, and we directly arrive to

$$\Gamma_b(A) = -\xi\int_0^\infty \frac{dT}{T} e^{-m^2T}\,\mathrm{Tr}[e^{-T\hat{p}_\mu\hat{p}_\mu}], \tag{61}$$

where the $\hat{p}_\mu$ operators are the same as the ones from the Dirac field calculation. The trace Tr is now over the Hilbert space only. In terms of the destruction and creation operators $\hat{a}$, $\hat{a}^\dagger$ used in the previous subsection, we see that

$$\Gamma_b(A) = -\xi\int_0^\infty \frac{dT}{T} e^{-m^2T}\,\mathrm{Tr}[e^{-T2eF[\hat{\mathcal{N}}+(1/2)]}], \tag{62}$$

where $\hat{\mathcal{N}}$ is the number operator corresponding to $\hat{a}$ and

$\hat{a}^\dagger$:

$$\hat{\mathcal{N}} = \hat{a}^\dagger \hat{a}. \tag{63}$$

Then we write the trace in terms of the eigenvalues,

$$\Gamma_b(A) = -\xi \int_0^\infty \frac{dT}{T} e^{-m^2 T} \sum_{n=0}^\infty e^{-T2eF[n+(1/2)]}, \tag{64}$$

and integrate over T to obtain

$$\Gamma_b(A) = -\xi \sum_{n=0}^\infty \ln[m^2 + eF(2n+1)]. \tag{65}$$

Of course, this may be evaluated as in the Dirac case in terms of the Hurwitz ζ_H function:

$$\Gamma_b(A) = \xi \frac{d}{ds}\left[(2eF)^{-s}\zeta_H\left(s; \frac{1}{2} + \frac{m^2}{2eF}\right)\right]\bigg|_{s=0}. \tag{66}$$

Again, the result is identical to the one of [13].

The scalar propagator G_b is simpler than its Dirac counterpart. Indeed, a straightforward calculation yields

$$G_b = \int_0^\infty dT\, e^{-m^2 T} e^{-T2eF[\hat{a}^\dagger \hat{a} + (1/2)]}, \tag{67}$$

or

$$G_b = \sum_{n=0}^\infty \frac{1}{m^2 + 2eF(n+\frac{1}{2})} |n\rangle\langle n|. \tag{68}$$

C. Complex Proca field

To calculate the effective action Γ_P (for the same gauge field configuration as before), we make again use of the result for $Z(p, F)$, that in the present case leads to

$$\Gamma_P(A) = -\xi \int_0^\infty \frac{dT}{T} e^{-m^2 T} \operatorname{Tr}[e^{-T\hat{p}_\alpha \hat{p}_\beta \Gamma^P_{\alpha\beta}}], \tag{69}$$

with the same $\hat{p}_\mu$ operators as in the Dirac field case. The trace meant both over Hilbert space and Lorentz indices. In terms of the annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ we have already introduced, we see that

$$\Gamma_P(A) = -\xi \int_0^\infty \frac{dT}{T} e^{-m^2 T} \operatorname{Tr}[e^{-[(eF)/2]T\hat{\mathcal{Q}}}], \tag{70}$$

where $\hat{\mathcal{Q}}$ is the operator

$$\hat{\mathcal{Q}} = \begin{pmatrix} -(\hat{a} - \hat{a}^\dagger)^2 & i(\hat{a}^2 - \hat{a}^{\dagger 2}) \\ i(\hat{a}^2 - \hat{a}^{\dagger 2}) & (\hat{a} + \hat{a}^\dagger)^2 \end{pmatrix}. \tag{71}$$

In order to evaluate the trace, it is convenient to look for the eigenfunctions and eigenvalues of the $\mathcal{Q}$ operator. We first rewrite $\hat{\mathcal{Q}}$ as follows:

$$\hat{\mathcal{Q}} = (2\hat{\mathcal{N}} + 1)I + \hat{a}^2 \eta + (\hat{a}^\dagger)^2 \eta^\dagger, \tag{72}$$

where I is the 2×2 identity matrix, while η denotes the nilpotent matrix

$$\eta = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}. \tag{73}$$

Eigenvalues λ and their corresponding eigenvectors $|\Psi\rangle$ of $\hat{\mathcal{O}}$ may be found, for example, by decomposing (an arbitrary) $|\Psi\rangle$ as follows:

$$|\Psi\rangle = |e_+\rangle \otimes |\chi_+\rangle + |e_-\rangle \otimes |\chi_-\rangle, \tag{74}$$

where $|e_\pm\rangle$ are two-component vectors:

$$|e_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}, \tag{75}$$

which are obviously linearly independent and satisfy

$$\begin{aligned} \eta|e_+\rangle &= -2|e_-\rangle, & \eta^\dagger|e_-\rangle &= -2|e_+\rangle, \\ \eta^\dagger|e_+\rangle &= 0, & \eta|e_-\rangle &= 0, \end{aligned} \tag{76}$$

while $|\chi_\pm\rangle$ are general Hilbert space vectors (scalars with respect to the Lorentz group).

Inserting the general decomposition into the eigensystem equation, we obtain

$$\begin{aligned} (2\hat{\mathcal{N}} + 1 - \lambda)|\chi_+\rangle + 2(\hat{a}^\dagger)^2|\chi_-\rangle &= 0, \\ 2\hat{a}^2|\chi_+\rangle + (2\hat{\mathcal{N}} + 1 - \lambda)|\chi_-\rangle &= 0. \end{aligned} \tag{77}$$

Now it becomes trivial to solve the last system, for example, by using the basis of eigenstates of the number operator for $|\chi_\pm\rangle$:

$$|\chi_\pm\rangle = \sum_{n=0}^\infty C_n^{(\pm)}|n\rangle, \tag{78}$$

where $|n\rangle$ denotes the eigenvalues of the number operator. This yields recurrence relations for the $C_n^{(\pm)}$'s whose solutions are polynomials only if λ equals an odd integer. Otherwise, the resulting eigenfunctions are not regular and must therefore be discarded. For the regular solutions, $\lambda = 2l + 1$, $l = 0, 1, \ldots$, there is no degeneracy. Thus,

$$\Gamma_P(A) = -\xi \int_0^\infty \frac{dT}{T} e^{m^2 T} \sum_{l=0}^\infty e^{-[(eF)/2]T(2l+1)}, \tag{79}$$

which may be integrated over T to obtain:

$$\Gamma_P(A) = -\xi \sum_{l=0}^\infty \ln\left[m^2 + \frac{eF}{2}(2l+1)\right]. \tag{80}$$

Of course, this is equivalent to the massive scalar field, with the trivial replacement $eF \to \frac{eF}{2}$:

$$\Gamma_P(A) = \xi \frac{d}{ds}\left[(eF)^{-s}\zeta_H\left(s; \frac{1}{2} + \frac{m^2}{eF}\right)\right]\bigg|_{s=0}. \tag{81}$$

IV. GENERALIZATION TO $d = 2k$ DIMENSIONS

The calculations of the previous section may be generalized easily to the case of a constant $F_{\mu\nu}$ field configuration in $d = 2k$ dimensions.[2] Indeed, one easily sees that the effective action for the fermionic case shall be given by an expression which is formally identical to (29)

$$\Gamma_f(A) = \int_0^\infty \frac{dT}{T}\, e^{-mT} \int \mathcal{D}p\, \mathrm{Tr}[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}]\mathcal{Z}(p, F), \tag{82}$$

where $\mathcal{Z}(p, F)$ is given by

$$\mathcal{Z}(p, F) = \int_{x(0)=x(T)} \mathcal{D}x\, e^{i\int_0^T d\tau \dot{x}_\mu(\tau) p_\mu(\tau)} \times e^{[(ie)/2]\int_0^T d\tau \dot{x}_\mu(\tau) F_{\mu\nu} x_\nu(\tau)}, \tag{83}$$

as follows from the gauge field configuration

$$A_\mu(x) = -\tfrac{1}{2} F_{\mu\nu} x_\nu, \tag{84}$$

which satisfies the gauge-fixing condition $\partial \cdot A = 0$.

The easiest way to calculate $\mathcal{Z}(p, F)$ is to reduce the problem to a set of decoupled $1 + 1$-dimensional systems, and then to take advantage of the results of the previous section. That may be done by using the fact that $\mathbf{F} \equiv (F_{\mu\nu})$ is a real antisymmetric matrix; hence it may be reduced to a block-diagonal form $\mathbf{f}$ by performing a similarity transformation with an orthogonal matrix $\mathbf{R}$:

$$\mathbf{F} = \mathbf{R}^T \mathbf{f} \mathbf{R}. \tag{85}$$

Each one of the blocks is 2×2 and antisymmetric, so that the reduced matrix has the following structure:

$$\mathbf{f} = \begin{pmatrix} 0 & f^{(1)} & 0 & 0 & 0 & \dots & 0 \\ -f^{(1)} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & f^{(2)} & 0 & \dots & 0 \\ 0 & 0 & -f^{(2)} & 0 & 0 & \dots & 0 \\ \dots & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & f^{(k)} \\ 0 & 0 & 0 & \dots & 0 & -f^{(k)} & 0 \end{pmatrix}, \tag{86}$$

where the $f^{(a)}$, $(a = 1, \dots, k)$ are real numbers, which we assume to be different from zero (although the particular case of one or more of them being equal to zero may of course be dealt with at the end of the calculation). Then we redefine the momenta and coordinates in the path integral, according to the following transformation: $p_\mu \to (\mathbf{R}^{-1})_{\mu\nu} p_\nu$, $x_\mu \to (\mathbf{R}^{-1})_{\mu\nu} x_\nu$. The γ matrices also are redefined with $\mathbf{R}$ and, of course, we arrive to an equivalent representation of the Clifford algebra. We use the same notation for the new γ matrices although we have the new representation in mind.

The general form of the matrix $\mathbf{F}$ can be further simplified in some particular cases, when there are some extra restrictions on the configuration. An interesting example corresponds to $d = 4$, where one has the possibility of considering a self-dual field:

$$\tilde{F}_{\mu\nu} = F_{\mu\nu}, \qquad \tilde{F}_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda} F_{\rho\lambda}. \tag{87}$$

This relation implies that $\mathbf{F}^2 = -f^2\mathbf{I}$, where $\mathbf{I}$ is the unit matrix and $f^2 = \frac{1}{4} F_{\mu\nu} F_{\mu\nu}$. Then the two blocks in the canonical form for $\mathbf{F}$ are degenerate:

$$\mathbf{f} = \begin{pmatrix} 0 & f & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & -f & 0 \end{pmatrix}, \tag{88}$$

which does simplify some calculations.

Rather than using the index μ, we introduce the notation $p_i^{(a)}$ $(a = 1, \dots, k,\ i = 1, 2)$, which distinguishes the components according to the 2×2 block to which they belong. The same convention is adopted for x_μ. Then,

$$\Gamma_f(A) = \int_0^T \frac{dT}{T}\, e^{-mT} \int \mathcal{D}p\, \mathrm{Tr}[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}]\mathcal{Z}(p, F), \tag{89}$$

where

$$\mathcal{Z}(p, F) = \prod_{a=1}^k \int_{x^{(a)}(0)=x^{(a)}(T)} \mathcal{D}x^{(a)} \exp\Big\{ i\int_0^T d\tau [\dot{x}_i^{(a)}(\tau) p_i^{(a)}(\tau) + i\frac{e}{2} f^{(a)} \varepsilon_{ij} \dot{x}_i^{(a)}(\tau) x_j^{(a)}(\tau)]\Big\}. \tag{90}$$

Of course, for each value of a we have an integral which is identical to the one for the $1 + 1$-dimensional case. Thus,

$$\mathcal{Z}(p, F) = \prod_{a=1}^k \Big(\xi^{(a)} \lim_{n\to\infty} \Big(\frac{2\pi}{ef^{(a)}}\Big)^n \times \{e^{-[i/(ef^{(a)})]\sum_{k=1}^n p_2^{(k)}(p_1^{(k+1)} - p_1^{(k)})}\}\Big), \tag{91}$$

a result which we include into (89) to obtain

$$\Gamma_f(A) = \Big[\prod_{a=1}^k \xi^{(a)}\Big] \int_0^\infty \frac{dT}{T}\, e^{-mT} \int_{p_1^{(a)}(0)=p_1^{(a)}(T)} \prod_{a=1}^k \widehat{\mathcal{D}p^{(a)}} \times \mathrm{Tr}\{\mathcal{P}\exp[-i\int_0^T d\tau \gamma^{(a)} p_j^{(a)}(\tau)]\} \times \exp\Big[-i\int_0^T d\tau \sum_{a=1}^k \frac{1}{ef^{(a)}} \dot{p}_1^{(a)}(\tau) p_2^{(a)}(\tau)\Big], \tag{92}$$

where

$$\widehat{\mathcal{D}p^{(a)}} = \prod_{0<\tau\leq T} \frac{dp_1^{(a)}(\tau) dp_2^{(a)}(\tau)}{2\pi e f^{(a)}}. \tag{93}$$

Of course, the expression for $\Gamma_f(A)$ in (92) may be converted to

[2] The essential features of the problem are the same for odd dimensions but they present subtleties which deserve a separate treatment [15].

$$\Gamma_f(A) = \left[\prod_{a=1}^{k} \xi^{(a)}\right] \int_0^\infty \frac{dT}{T}\, e^{-mT}\, \mathrm{Tr}[e^{-iT\sum_{a=1}^{k}\hat{\not{p}}^{(a)}}], \quad (94)$$

where $\hat{\not{p}}^{(a)} \equiv \gamma_1^{(a)}\hat{p}_1^{(a)} + \gamma_2^{(a)}\hat{p}_2^{(a)}$ (a is not summed). The $\hat{p}^{(a)}$ operators verify the commutation relations:

$$[\hat{p}_j^{(a)}, \hat{p}_k^{(b)}] = -if^{(a)}\delta^{ab}\varepsilon_{jk}, \quad (95)$$

and the trace is over Dirac and Hilbert space. Since the γ matrices satisfy the anticommutation relations:

$$\{\gamma_j^{(a)}, \gamma_k^{(b)}\} = 2\delta^{ab}\delta_{jk}, \quad (96)$$

we easily see that

$$\hat{\not{p}}^{(a)}\hat{\not{p}}^{(b)} = 0, \qquad \forall a \neq b. \quad (97)$$

Then

$$\Gamma_f(A) = \left[\prod_{a=1}^{k} \xi^{(a)}\right] \int_0^\infty \frac{dT}{T}\, e^{-mT}\Bigg[1 + \sum_{n_1,\ldots,n_k=1}^{\infty} \times \sum_{a=1}^{k} (e^{-iT\sqrt{2ef^{(a)}n_a}} + e^{+iT\sqrt{2ef^{(a)}n_a}})\Bigg], \quad (98)$$

or, doing the integral

$$\Gamma_f(A) = -\left[\prod_{a=1}^{k} \xi^{(a)}\right]\Bigg[\ln(m) + \sum_{n_1=1,\ldots,n_k=1}^{\infty} \times \ln\left(m^2 + \sum_{a=1}^{k}(ef^{(a)}n_a)\right)\Bigg], \quad (99)$$

which upon regularization leads to the known result ([13]).

V. CONCLUSIONS

We have carried further the first-order spin formalism for the worldline, providing new tests and applications for Migdal's construction. We have thus obtained with this method new expressions for the propagators and effective actions of the Dirac, Proca, and complex scalar fields coupled to Abelian gauge fields. A remarkable result is the universality of the path order spin factor. In fact this can be seen as a natural consequence of the geometric representation and are extendible to higher-spin fields.

For constant electromagnetic fields in two dimensions we have shown that our results agree with the results from the zeta-function renormalization. The results have been then generalized to d-even dimensions. Notice that in the first-order formalism one can incorporate the quantum fluctuations of the gauge field.

This can provide a useful contribution to the progress in nonperturbative quantum field dynamics within the worldline, especially for its numerical implementation, which appears as a powerful new alternative [9]. Work in progress in odd dimensions indicates promising features of transferring internal degrees of freedom to geometrical properties of space time which could hopefully allow the inclusion of non-Abelian fields.

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