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Response of strongly interacting matter to a magnetic field: Some exact results

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We derive some exact results concerning the response of strongly interacting matter to external magnetic fields. Our results come from consideration of triangle anomalies in medium. First, we define an ''axial magnetic susceptibility,'' then we examine its behavior in two-flavor QCD via response theory. In the chirally restored phase, this quantity is proportional to the fermion chemical potential, while in the phase of broken chiral symmetry it can be related, through triangle anomalies, to an in-medium amplitude for $\pi^0 \to 2\gamma$. We confirm the latter result by calculation in a linear sigma model, where this amplitude is already known in the literature.

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I. INTRODUCTION

Much attention has been focused recently on properties of matter at very high temperatures or baryon density [1]. The interest is driven by the physics of heavy-ion collisions and of the core of neutron stars. Most of the discussion is focused on properties of the system at finite temperature T and chemical potential μ. In this paper, we are interested in the properties of hot and dense matter under external magnetic field B. This question is of potential interest for the physics of compact objects.

In contrast to the temperature and chemical potential, the magnetic field that can be achieved in nature seems to be always small compared to the strong scale (perhaps as large as $\sim 10^{18}$ G in magnetar [2,3]). This means that the effect of magnetic field on the medium, in most cases, can be treated as a small perturbation, and the linear response theory is appropriate. In this paper we derive two results related to the response of a strongly interacting medium to the external magnetic field. We are interested in the axial current created by a uniform magnetic field $\mathbf{B}$. For small magnetic fields the axial current is linear in $\mathbf{B}$, with a proportionality coefficient which we call the axial magnetic susceptibility χ. We show that if chiral symmetry is unbroken, then the value of χ is equal to the baryon chemical potential, with a known numerical coefficient. This result is universal and receives no correction due to strong coupling. We also find that, when chiral symmetry is unbroken, this universal result no longer holds. However, in this case, we derive a relation between the axial magnetic susceptibility and the in-medium $\pi^0 \to 2\gamma$ amplitude in spacelike domain (a more precise definition is given below). In both cases the results come from the consideration of triangle anomalies, and hence are reminiscent to those of Refs. [4,5]. The difference between this work and Refs. [4,5] is that here we are interested in the properties of the ambient matter, not of topological defects as in Refs. [4,5].

The paper is organized as follows. In Sec. II we define the quantities of interest in a simplified version of the real world. In Sec. III we derive the exact relations. We explicitly check this relation in Sec. IV in the linear sigma model. We extend the results to the real world in Sec. V, and conclude with Sec. VI.

II. BASIC DEFINITIONS

For simplicity, we consider QCD with massless u and d quarks and neglect other quarks. We shall first assume nonzero baryon chemical potential, but zero isospin chemical potential. Moreover, let us assume that electromagnetism couples to the third component of the isospin current $\frac{1}{2}\bar{q}\gamma^\mu\tau^3 q$, but not a linear combination of isospin and baryon current as in the real world. We shall modify our result to the real world later. We will assume no superconductivity, so the magnetic field can penetrate the matter without being confined in magnetic flux tubes (this may require that we work at sufficiently high temperatures).

Let us first define the axial magnetic susceptibility. Assume we have a medium at temperature T and baryon chemical potential μ in a uniform magnetic field $\mathbf{B}$. The magnetic field induces an axial current in the medium,

$$\langle \mathbf{j}^5 \rangle = \chi(T, \mu)\mathbf{B}, \tag{1}$$

where, in terms of quark fields,

$$j^{5\mu} = \tfrac{1}{2}(\bar{u}\gamma^\mu\gamma^5 u - \bar{d}\gamma^\mu\gamma^5 d). \tag{2}$$

That χ is nonzero is permitted by symmetries. With respect to parity, both $\mathbf{j}^5$ and $\mathbf{B}$ are axial vectors. Note that B and A_i have different C parity, so χ must be an odd function of the baryon chemical potential μ. In particular, $\chi = 0$ at zero chemical potential. In the model described in Sec. IV, χ is proportional to μ at small μ.

If the medium consists of nonrelativistic nucleons, then the axial current is proportional to the nucleon spin. The coefficient χ therefore has a simple physical interpretation as the spin polarizability of the medium (more precisely, the difference between proton and neutron spin polarizability). When nucleons are relativistic, it becomes impossible to separate nucleon spin from nucleon total angular momentum. However, even in this case, the axial magnetic susceptibility is still a well-defined concept. In fact, this axial current interacts, through Z^0 exchange, with neutrinos and modifies their dispersion relations.

We also define the in-medium coupling of the neutral pion to two photons, $g_{\pi^0\gamma\gamma}$, in the chirally broken phase. In vacuum, the anomalous coupling of a pion to external gauge fields is given by a term in the chiral Lagrangian,

$$-\frac{1}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu\phi A^B_\nu F_{\alpha\beta}, \qquad \phi = \frac{\pi^0}{f_\pi}, \tag{3}$$

where A^B_μ is the gauge potential coupled to the baryon current (note that A_μ couples to the isospin current). The dimensionless field ϕ is normalized to have periodicity 2π. At finite temperature, the coupling is more subtle. As noted in Ref. [6], there is an ambiguity with the zero momentum limit. We choose the following definition. Let us look at the free energy of a *static* field configuration where the π^0 field changes slowly in space, in the presence of a background static magnetic field **B** and static baryon scalar potential A^B_0. The free energy (density) has the form

$$F = \frac{f_s^2}{2}(\partial_i\phi)^2 - g_{\pi^0\gamma\gamma}\frac{1}{8\pi^2 f_s}\epsilon^{ijk}\partial_i\phi A^B_0 F_{jk}. \tag{4}$$

Here f_s is the spatial pion decay constant [7], and $g_{\pi^0\gamma\gamma}$ will be called the $\pi^0 \to 2\gamma$ amplitude. The free energy (4) can be thought of as arising from integrating out all degrees of freedom of QCD except the Goldstone boson, and restricting to the lowest Matsubara frequency $\omega = 0$. At zero temperature $f_s = f_\pi$ and $g_{\pi^0\gamma\gamma} = 1$, but in general both f_s and $g_{\pi^0\gamma\gamma}$ are functions of temperature and baryon chemical potential. We also know that $f_s \to 0$ at the second-order chiral phase transition, with the critical exponent of Josephson's scaling [8]. The way we define $g_{\pi^0\gamma\gamma}$ corresponds to the $\pi^0 \to 2\gamma$ amplitude in the spacelike region of Ref. [6].

To summarize, our results are

(i) When the point (T, μ) lies in the chirally restored phase, χ is directly proportional to the chemical potential,

$$\chi = \frac{1}{4\pi^2}\mu. \tag{5}$$

The numerical coefficient $1/(4\pi^2)$ is exact and is related to triangle anomaly.

(ii) When chiral symmetry is spontaneous broken, the relation between χ and the anomaly is lost. However, there is an exact equation relating the susceptibility and the in-medium amplitude of $\pi^0 \to 2\gamma$. Namely,

$$4\pi^2\frac{d\chi}{d\mu} + g_{\pi^0\gamma\gamma}(T, \mu) = 1. \tag{6}$$

Here $g_{\pi^0\gamma\gamma}(T)$ is the $\pi^0 \to 2\gamma$ amplitude defined above.

We note that the first part of our results, which concerns the chirally restored phase, has been observed in Ref. [9]. In addition, it is only a slight variation on the result that is found in Ref. [10], where it was determined that the magnetic susceptibility of the electric current is proportional to a chemical potential for fermion chirality. This relation can also be checked explicitly in models of strongly interacting field theory with gravity dual description [11]. However, to our knowledge, the case with spontaneous breaking of chiral symmetry has never been considered before; hence the second part of our results is new. We first show the validity of these relations in a general setting. Then we shall verify them explicitly in a model with anomaly, namely, the linear sigma model.

III. EXACT RELATIONS

To derive the exact relations, we consider a three-point correlation function of the axial current j^5_μ, the isospin current j^μ, and the baryon current B^μ,

$$i\Gamma^{\mu\nu\lambda}(p, q) = \int d^4x d^4y e^{ip\cdot x + iq\cdot y}\langle j^{5\mu}(x) j^\nu(y) B^\lambda(0)\rangle, \tag{7}$$

where $j^{5\mu}$ is defined in Eq. (2) and other currents are defined as follows:

$$j^\mu = \tfrac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d), \qquad B^\mu = \tfrac{1}{3}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d). \tag{8}$$

We can always define the correlator so that the triangle anomaly resides entirely in the derivative of the axial current:

$$p_\mu\Gamma^{\mu\nu\lambda}(p, q) = -\frac{1}{4\pi^2}\epsilon^{\nu\lambda\alpha\beta}p_\alpha q_\beta, \tag{9}$$

$$q_\nu\Gamma^{\mu\nu\lambda}(p, q) = (p_\lambda + q_\lambda)\Gamma^{\mu\nu\lambda}(p, q) = 0. \tag{10}$$

In this paper we will be interested only in static (time-independent) problems, therefore we shall set $p_0 = q_0 = 0$. Moreover, the baryon chemical potential couples to the zeroth component of B_λ. Thus the quantity of interest for us will be

$$i\Gamma^{ij0}(\mathbf{p}, \mathbf{q}) = \int d^4x d^4y e^{-i\mathbf{p}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{y}}\langle A^i(x) V^j(y) B^0(0)\rangle. \tag{11}$$

On the other hand, the axial magnetic susceptibility χ is related to the low-momentum behavior of a two-point function. Indeed, the axial current created by a background electromagnetic field is

$$\langle j^{5\mu}(x)\rangle = -i\int d^4y G^{\mu\nu}_{5I}(x-y)A_\nu(y), \tag{12}$$

where

$$G^{\mu\nu}_{5I}(x-y) = \langle j^{5\mu}(x) j^\nu(y)\rangle. \tag{13}$$

In order to reproduce Eq. (1), the infrared behavior of the G_{5I} correlator must be as follows:

$$\int d^4 x e^{-i\mathbf{p}\cdot\mathbf{x}} \langle j^{5i}(x) j^k(0)\rangle_\mu = \chi \epsilon^{ijk} p_j + O(p^2). \qquad (14)$$

In Eq. (14) we emphasize that the average is taken at nonzero chemical potential μ. Differentiating Eq. (14) with respect to μ we get

$$\Gamma^{ik0}(\mathbf{p}, -\mathbf{p}) = -\frac{\partial\chi(\mu)}{\partial\mu}\epsilon^{ijk} p^j + O(p^2). \qquad (15)$$

Let us now look at the structure of the correlator $\Gamma_{ij0}(\mathbf{p}, \mathbf{q})$ in the regime of small $\mathbf{p}$ and $\mathbf{q}$. It changes sign under parity. This static correlator is not singular in the chirally restored phase, and may have a pion pole in the chirally broken phase. In light of this, the general form of the three-point function is

$$\Gamma^{ij0}(\mathbf{p}, \mathbf{q}) = C_1 \epsilon^{ijk} q^k + C_2 \frac{p^i}{p^2}\epsilon^{jkl} q^k p^l. \qquad (16)$$

Since Γ is even under C, the dimensionless constants C_1 and C_2 must both be even under C parity, with C_2 vanishing in the chirally restored phase.

From the relation of triangle anomaly (9) we find

$$C_1 + C_2 = \frac{1}{4\pi^2}. \qquad (17)$$

Consider first the case when chiral symmetry is restored. Then, as the singular term in Eq. (16) is absent, $C_1 = 1/(4\pi^2)$. But by comparing Eq. (16) with Eq. (15), we find

$$\frac{\partial\chi(\mu)}{\partial\mu} = C_1 = \frac{1}{4\pi^2}. \qquad (18)$$

Requiring χ to be an odd function of μ, we determine

$$\chi(\mu, T) = \frac{\mu}{4\pi^2}, \qquad (19)$$

which is the first part of our result.

Now we turn to the chirally broken phase. The singular term in Eq. (16) comes from the Feynman diagram with an intermediate pion line, which can be computed using the free energy (4) as an effective Lagrangian. This leads us to

$$C_2 = \frac{g_{\pi^0\gamma\gamma}}{4\pi^2}. \qquad (20)$$

Eq. (17) then implies

$$4\pi^2 \frac{\partial\chi}{\partial\mu} + g_{\pi^0\gamma\gamma} = 1, \qquad (21)$$

which is the second part of our result.

IV. EXAMPLE: LINEAR SIGMA MODEL

Since QCD is strongly coupled for temperatures below the chiral phase transition, we cannot directly check the formula (6) there (although it should be possible to verify it in the high-density phase of three-flavor QCD, where chiral symmetry is broken at weak coupling). We shall instead verify this formula in a weakly coupled field theory with anomaly, namely, the linear sigma model. This model was employed before to understand the effects of temperature on anomaly [12]. Furthermore, the finite-temperature $\pi^0 \to 2\gamma$ amplitude has been computed in this model in various kinematic limits, including the limit where the outgoing photons are at zero frequency [6,13]. Thus, we can confirm our result (6) by calculating the axial magnetic susceptibility in this model.

The model is given by the Lagrangian

$$\mathcal{L} = \bar{Q}(i\gamma^\mu D_\mu - g\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma^5)Q + \frac{1}{2}(D_\mu\sigma D^\mu\sigma + D_\mu\boldsymbol{\pi}\cdot D^\mu\boldsymbol{\pi}) + \frac{\mu^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2. \qquad (22)$$

The couplings g and λ are small. The expectation value of σ is v, which is temperature dependent (being equal to $\sqrt{\mu^2/\lambda}$ in vacuum.) The covariant derivative, $D_\mu = \partial_\mu + iqA_\mu$, is that of $\mathrm{U}(1)_{\mathrm{EM}}$. In the phase with chiral symmetry breaking, fermions (“constituent quarks”) have mass $m = gv$. We shall calculate the axial current induced by a baryon chemical potential μ_B on the background of a constant magnetic field, using the single-particle Hamiltonian in the regime where $T \gg \mu, m, \sqrt{eB}$.

Placing a static, homogeneous magnetic field pointing in the $\hat{z}$ direction can be accomplished by means of the vector potential, $A^\mu = (0, 0, Bx, 0)$. The fermion spectrum can be found by solving the Dirac equation $i\boldsymbol{\gamma}\cdot\mathbf{D}Q = EQ$ with $D^\mu = \partial^\mu - iq\mathcal{A}^\mu$, where q is the electric charge of the quark $Q = (u, d)$, $q_u = -q_d = \frac{1}{2}$. The axial-vector current can then be directly calculated as a thermal expectation value of (2).

Within this setup it is convenient to parametrize the quark wave functions as

$$Q(x, y, z) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \phi_1(x) \\ \phi_2(x) \end{pmatrix} e^{i(p_y y + p_z z)}. \qquad (23)$$

Because of our choice of coordinate system, “p_y” will always appear in the same component of the Dirac equation as “eBx” in the linear combination $p_y + eBx$. Thus, by making a convenient coordinate shift, $x \to x - \frac{p_y}{qB}$, we can eliminate the explicit appearance of “p_y” in the Dirac equation. We have four first order coupled differential equations, from which we can obtain two second order equations for ψ_2 and ϕ_2 separately:

$$[\nabla^2 - (qBx)^2 + E^2 - p_z^2 - m^2 + qB]\psi_2(x) = 0, \qquad (24)$$

$$[\nabla^2 - (qBx)^2 + E^2 - p_z^2 - m^2 + qB]\phi_2(x) = 0. \qquad (25)$$

The other components can be expressed via ψ_2 and ϕ_2, as

$$\psi_1 = i[\nabla + qBx]\frac{(E - p_z)\psi_2 - m\phi_2}{E^2 - p_z^2 - m^2}, \tag{26}$$

$$\phi_1 = -i[\nabla + qBx]\frac{(E + p_z)\phi_2 - m\psi_2}{E^2 - p_z^2 - m^2}. \tag{27}$$

The components $\psi_2(x)$ and $\phi_2(x)$ are clearly the eigenfunctions of a harmonic oscillator, while the other components are obtained by acting a lowering operator on linear combinations of ψ_2 and ϕ_2. The energy eigenvalues are thus the familiar Landau levels, $E = \pm\sqrt{p_z^2 + m^2 + 2nqB}$. The constituent quark wave function, properly normalized, is

$$Q(x, y, z) = \frac{1}{\sqrt{2^{n+2}n!\sqrt{\pi}}} \times \begin{pmatrix} i\frac{\alpha(E+p_z)-\beta m}{\sqrt{qB}}\mathcal{H}_{n-1}(\sqrt{qB}x) \\ \alpha\mathcal{H}_n(\sqrt{qB}x) \\ i\frac{\alpha m-\beta(E-p_z)}{\sqrt{qB}}\mathcal{H}_{n-1}(\sqrt{qB}x) \\ \beta\mathcal{H}_n(\sqrt{qB}x) \end{pmatrix} \times e^{-qBx/2}e^{i(p_y y+p_z z)}. \tag{28}$$

The Landau levels, n, replace the p_x quantum number.

The $\mathcal{H}$ are the Hermite polynomials, with the addition that, for $n = 0$, we define $\mathcal{H}_{-1} \equiv 0$. For each $n > 0$ there are two eigenstates with each choice of the sign of E:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\frac{qB}{16\pi}\right)^{1/4}\begin{pmatrix} \frac{m - \sqrt{2nqB}}{\sqrt{E(E - p_z)}} \\ \sqrt{\frac{E - p_z}{E}} \end{pmatrix}, \quad \times\left(\frac{qB}{16\pi}\right)^{1/4}\begin{pmatrix} \sqrt{\frac{E + p_z}{E}} \\ \frac{m + \sqrt{2nqB}}{\sqrt{E(E + p_z)}} \end{pmatrix}. \tag{29}$$

Meanwhile, for $n = 0$ there is only one positive energy eigenstate and one negative energy eigenstate,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\frac{qB}{4\pi}\right)^{1/4}\begin{pmatrix} \sqrt{\frac{E + p_z}{E}} \\ \sqrt{\frac{E - p_z}{E}} \end{pmatrix}. \tag{30}$$

Now we calculate,

$$\langle A_z^3 \rangle \equiv \langle \bar{Q}\gamma_z\gamma^5 Q\rangle_{\mu,T} - \langle \bar{Q}\gamma^z\gamma^5 Q\rangle_{0,T}, \tag{31}$$

by filling up the fermion energy levels with the Fermi-Dirac distribution function,

$$\langle \mathbf{j}^5 \rangle = \frac{q\mathbf{B}}{2\pi}\sum_{n,\lambda}\sum_{sgn(E)}\int\frac{dp_z}{2\pi}[|\psi_1|^2 - |\psi_2|^2 + |\phi_1|^2 - |\phi_2|^2] \times\left[\frac{1}{e^{\beta(E-\mu_q)} + 1} - \frac{1}{e^{\beta E} + 1}\right]\hat{z}. \tag{32}$$

Here $\mu_q = \frac{1}{3}\mu$ is the quark chemical potential, and the prefactor $(qB)/(2\pi)$ arises from the degeneracy of the energy eigenstates in p_y. The sum is taken over all Landau levels n, both sign of energy E and all polarization λ. For regularization we also subtracted the value of the axial current at $\mu = 0$, which is zero by C parity. For $n > 0$ the sum over polarizations reads

$$\sum_\lambda[|\psi_1|^2 - |\psi_2|^2 + |\phi_1|^2 - |\phi_2|^2] = \frac{1}{4nqB}\left\{2m^2 - 4\frac{p_z}{E}\left(m^2 + 2nqB - 4\frac{m^2\sqrt{2nqB}}{m^2 + 2nqB}\right) - 2m^2\right\}, \tag{33}$$

which is an odd function of p_z and contributes nothing to the axial current after integration over dp_z. It is very easy to see that the $n > 0$ Landau levels do not contribute to the axial current in two limits, when the fermions are massless and when the fermions are nonrelativistic (very massive), so it is not entirely surprising that they do not contribute for any value of m.

The contribution from the lowest Landau level is considerably simpler, since here

$$[|\psi_1|^2 - |\psi_2|^2 + |\phi_1|^2 - |\phi_2|^2] = \frac{|\alpha|^2 + |\beta|^2}{\sqrt{eB}} = 1. \tag{34}$$

All that remains is to calculate the integral over the statistical factor. This can be done analytically at large temperatures by expanding to first order in the small quantities, μ/T and m^2/T^2. For the sake of brevity we use $n(\xi) \equiv (e^\xi + 1)^{-1}$, and $n'(\xi) = \partial_\xi n(\xi)$, with $\xi \equiv p_z/T$. In particular,

$$\frac{1}{e^{\beta(E-\mu_q)} + 1} - \frac{1}{e^{\beta E} + 1} \approx -\beta\mu_q\frac{d}{d(\beta E)}n(\beta E) = -\beta\mu_q\left[n'(\xi) + \frac{m^2}{T^2}\frac{1}{2\xi}n''(\xi)\right]. \tag{35}$$

The sum over flavors yields $q_u - q_d = 1$ in place of "q" in the prefactor $qB/(2\pi)$. Also, a factor of $N_c = 3$ is gained from the sum over colors. Thus, we find

$$\langle \mathbf{j}^5 \rangle = \frac{\mu\mathbf{B}}{4\pi^2}\left[1 - \frac{m^2}{T^2}\int_0^\infty d\xi\frac{n''(\xi)}{2\xi} + \mathcal{O}\left(\frac{m^4}{T^4}\right) + \mathcal{O}\left(\frac{\mu}{T}\right)\right]. \tag{36}$$

Performing the integral as in Ref. [6], we obtain

$$\frac{\partial \chi}{\partial \mu} = \frac{1}{4\pi^2}\left[1 - \frac{7\zeta(3)}{4\pi^2}\frac{m^2}{T^2} + \mathcal{O}\left(\frac{m^4}{T^4}\right) + \mathcal{O}\left(\frac{\mu}{T}\right)\right]. \quad (37)$$

Note that we recover Eq. (5) for $m = 0$.

On the other hand, the result of Refs. [6,13], in our language, corresponds to

$$g_{\pi^0\gamma\gamma} = \frac{7\zeta(3)m^2}{4\pi^2 T^2}. \quad (38)$$

We see that

$$4\pi^2 \frac{\partial \chi}{\partial \mu} + g_{\pi^0\gamma\gamma} = 1, \quad (39)$$

in accordance with Eq. (6).

V. ISOSPIN CHEMICAL POTENTIAL, REAL WORLD EM COUPLING

Realistic dense matter can have an isospin chemical potential beside the baryon chemical potential. In addition, in the real world the electromagnetic field is coupled to a linear combination of the third component of the isospin current and the baryon current. In this section we extend the results of the last two sections to two-flavor QCD with nonzero μ_I and μ_B in an applied magnetic field coupled to the electromagnetic current $j^\mu_{\rm Q} \equiv j^\mu_{\rm EM} = e(\frac{1}{2} j^\mu_{\rm B} + j^\mu_{\rm I})$.

Based on our previous experience, it is clear that we should expect

$$\langle \mathbf{j}^5 \rangle = \chi(T, \mu_{\rm I}, \mu_{\rm B})\mathbf{B}_{\rm Q}. \quad (40)$$

In the chirally symmetric phase, we follow Sec. III but replace the field A^ν coupled to isospin with one coupled to electric charge, $A^\nu_{\rm Q}$. Then the response of the chiral current can be expressed in terms of the correlator G_{5Q} as

$$\langle j^{5\mu}(x)\rangle = -i \int d^4y G^{\mu\nu}_{5Q}(x-y)A_{\nu Q}(y). \quad (41)$$

For this system, we must have

$$\int d^4x e^{ip\cdot x} G^{ik}_{5Q}(x) = \chi \epsilon^{ijk} p_j + O(p^2). \quad (42)$$

Now, two different anomaly relations—one for $\langle j^{5i} j^j_Q B^0\rangle$ and one for $\langle j^{5i} j^j_Q V^0\rangle$—give, respectively,

$$\frac{\partial \chi}{\partial \mu_B} = \frac{1}{4\pi^2}, \qquad \frac{\partial \chi}{\partial \mu_I} = \frac{1}{8\pi^2}. \quad (43)$$

which means that χ is a simple linear combination of μ_B and μ_I in the chirally unbroken phase:

$$\chi = \frac{\mu_B}{4\pi^2} + \frac{\mu_I}{8\pi^2}. \quad (44)$$

For the phase with broken chiral symmetry, the analysis of Sec. III still holds as well, but with the distinction that, instead of one $g_{\pi^0\gamma\gamma}$ coupling defined in Sec. II, one needs to introduce two coupling constants $g_{\pi^0\gamma I}$ and $g_{\pi^0\gamma B}$. Specifically, we write down the free energy of a slowly varying pion field configuration ϕ in the background static baryon and isospin scalar potentials A^B_0 and A^I_0 and a background magnetic field F_{ij},

$$F = \frac{f_s^2}{2}(\partial_i\phi)^2 - g_{\pi^0\gamma B}\frac{1}{8\pi^2 f_s}\epsilon^{ijk}\partial_i\phi A^B_0 F_{jk} + g_{\pi^0\gamma I}\frac{1}{8\pi^2 f_s}\epsilon^{ijk}\partial_i\phi A^I_0 F_{jk}. \quad (45)$$

Again, in complete analogy with our derivation of Eq. (39), we obtains two exact relations,

$$8\pi^2\frac{\partial\chi}{\partial\mu_I} + g_{\pi^0\gamma I} = 1, \qquad 4\pi^2\frac{\partial\chi}{\partial\mu_B} + g_{\pi^0\gamma B} = 1. \quad (46)$$

One may be interested in the $\pi^0 \to \gamma\gamma$ amplitude, defined through the interaction between the axial current and two EM currents. This amplitude is a linear combination of $g_{\pi^0\gamma I}$ and $g_{\pi^0\gamma B}$:

$$\frac{g_{\pi^0\gamma\gamma}}{4\pi^2} = \frac{1}{2}\frac{g_{\pi^0\gamma B}}{4\pi^2} + \frac{g_{\pi^0\gamma I}}{8\pi^2}. \quad (47)$$

As before, $g_{\pi^0\gamma\gamma}$ is normalized to 1 at zero temperature. From Eqs. (46) we find

$$4\pi^2\left(\frac{\partial\chi}{\partial\mu_I} + \frac{1}{2}\frac{\partial\chi}{\partial\mu_B}\right) + g_{\pi^0\gamma\gamma} = 1. \quad (48)$$

This result could again be checked explicitly in the sigma model by noting that the sum over flavors should be performed with $\mu_u \neq \mu_d$, resulting in "q" in the prefactor being replaced by $q_u\mu_u - q_d\mu_d$. Invoking $\mu_B = \frac{1}{3}(\mu_u + \mu_d)$ and $\mu_I = \frac{1}{2}(\mu_u - \mu_d)$, we find

$$q_u\mu_u - q_d\mu_d = \frac{e}{3}\left(\mu_B + \frac{1}{2}\mu_I\right). \quad (49)$$

Thus,

$$\chi(T, \mu_{\rm I}, \mu_{\rm B}) = \frac{e}{4\pi^2}\left(\mu_B + \frac{1}{2}\mu_I\right)\left[1 - \frac{7\zeta(3)}{4\pi^2}\frac{m^2}{T^2} + \mathcal{O}\left(\frac{m^4}{T^4}\right) + \mathcal{O}\left(\frac{\mu_B}{T}\right) + \mathcal{O}\left(\frac{\mu_I}{T}\right)\right]. \quad (50)$$

This result should be contrasted with that of Refs. [6,13], where it is found that $g_{\pi^0\gamma\gamma} = 7\zeta(3)m^2/(4\pi^2T^2)$. Equation (48) is obviously valid in this model.

VI. CONCLUSIONS

We have shown that there is a close connection between the response of strongly interacting matter on external magnetic field and the axial anomaly. By considering the properties of the three-point function of isovector, axial isovector, and baryon currents in the presence of nonvanishing baryon chemical potential and QED magnetic field, we were able to show that the axial magnetic susceptibility $\chi(\mu, T)$ is directly proportional to the baryon chemical potential in the absence of Goldstone modes carrying the axial current. Alekseev *et al.* previously found the same to be true for massless QED, using quite general methods [10]. This result follows from the fact that the anomaly coefficient, which determines the coefficient of proportionality, is not renormalized, and receives no finite-temperature contribution. In the presence of massless pions, this direct relation no longer holds, but one still can relate $\chi(\mu, T)$ to the anomaly coefficient through the amplitude for $\pi^0 \to \gamma\gamma$. We confirmed the second relation for a particular case of weakly coupled linear sigma model.

These results may be applicable to the study of compact objects such as neutron stars, where both baryonic chemical potential and magnetic field may be large. Specifically, the self-energy of neutrinos is affected by interaction with the axial isovector current through Z^0 exchange [14,15]. Calculation of $g_{\pi^0\gamma\gamma}(\mu, T)$ in nuclear matter is complicated by the need to deal with singularities arising from particle-hole interactions, but our results could be easily employed to find this neutrino self-energy contribution in deep cores of neutron stars, if chiral symmetry is restored there. In a strong magnetic field, the momentum distribution of neutrinos is asymmetric already in equilibrium, so they will stream out in asymmetric fashion, giving rise to a small contribution to pulsar velocities. The presence of an axial current in matter will not affect oscillations between active neutrinos, but does change the oscillations between an active neutrino and a sterile one.

We end the paper by noting that currently we lack an understanding of the critical behavior of $g_{\pi^0\gamma\gamma}$ near the second-order chiral phase transition. While it is natural that this coefficient goes to zero smoothly at the phase transition, the question about the critical exponent remains open.

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