Construction of the second-order gravitational perturbations produced by a compact object

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Accurate calculation of the gradual inspiral motion in an extreme mass-ratio binary system, in which a compact object inspiral towards a supermassive black hole requires calculation of the interaction between the compact object and the gravitational perturbations that it induces. These metric perturbations satisfy linear partial differential equations on a curved background space-time induced by the supermassive black hole. At the point-particle limit the second-order perturbations equations have source terms that diverge as r^{-4}, where r is the distance from the particle. This singular behavior renders the standard retarded solutions of these equations ill defined. Here we resolve this problem and construct well-defined and physically meaningful solutions to these equations. We recently presented an outline of this resolution [E. Rosenthal, Phys. Rev. D **72**, 121503 (2005).]. Here we provide the full details of this analysis. These second-order solutions are important for practical calculations: the planned gravitational-wave detector LISA requires preparation of waveform templates for the potential gravitational waves. Construction of templates with desired accuracy for extreme mass-ratio binaries requires accurate calculation of the inspiral motion including the interaction with the second-order gravitational perturbations.

I. INTRODUCTION

Consider a binary system composed of a small compact object with mass μ (e.g. a neutron star or a stellar-mass black hole) that inspirals towards a supermassive black hole with a mass M. Such extreme mass-ratio binaries (e.g., $M/\mu = 10^5$) are valuable sources for gravitational waves (GW) that could be detected by the planned laser interferometer space antenna (LISA) [1]. To detect these binaries and determine their parameters using matched-filtering data-analysis techniques one has to prepare gravitational waveform templates for the expected GW. An important part in the calculation of the templates is keeping track of the GW phase. Successful determination of the binary parameters using matched-filtering techniques often requires one to prepare templates with a phase error of less than one cycle over a year of inspiral [2]. Calculating waveform templates to this accuracy is a challenging task since a waveform from a year of inspiral may contain 10^5 wave cycles [3].

To carry out this calculation one is required to calculate the compact-object inspiral trajectory. By virtue of the smallness of the mass ratio μ/M one may use perturbations analysis to simplify this calculation. In this analysis the full space-time metric is represented as a sum of a background metric—induced by the supermassive black hole, and a sequence of perturbations—induced by the compact object. The object's trajectory is also treated with perturbations techniques. At leading order of this approximation the object's trajectory was found to be a geodesic in the background geometry (see e.g., [4]). At higher orders, the interaction between the object and its own gravitational field gives rise to a gravitational self-force that acts on the object. The leading-order effect of the self-force originates from interaction between the object

and its own first-order gravitational perturbations that are linear in μ. This first-order self-force (that scales like μ^2) induces an acceleration of order μ for the object's trajectory. In the case of a vacuum background geometry, formal and general expression for this first-order gravitational self-force was derived by Mino, Sasaki, and Tanaka [5], and independently by Quinn and Wald [6] using a different method. Later, practical methods to calculate this self-force were developed by several authors [7–15], see also [16,17] for a different approach to this problem. The next order corrections to the object's trajectory originate from the interaction between the object and its own second-order gravitational perturbations (that are quadratic in μ). Higher-order corrections will not be considered in this article.

We shall now estimate the effect of the gravitational self-force (more accurately the dissipative part of the gravitational self-force, see below) on the accumulated phase of the emitted GW. This will allow us to determine how many terms should be retained in the perturbations expansion (see also [18–20]). For simplicity consider a compact object which inspirals between two circular orbits in a strong field region of a Schwarzschild black hole. As the object inspirals towards the black hole its orbital frequency slowly changes from its value at an initial time. This shift in the orbital frequency is approximated by $\dot{\omega}t$, where $\dot{\omega} \equiv \frac{d\omega}{dt}$, and t denotes the elapsed time (from initial time) in Schwarzschild coordinates. Let $\Delta\phi$ denote the part of the phase shift of the GW (between two fixed times) which is induced by the shift in the orbital frequency. Since the GW frequency is proportional to the orbital frequency we find that after an inspiral time Δt_{ins}, the phase shift $\Delta\phi$ is approximately proportional to $\Delta t_{\text{ins}}^2 \dot{\omega}$. Let us find how the quantities in this expression scale with μ. The inspiral time Δt_{ins} scales like $\Delta E \dot{E}^{-1}$, here E denotes the particle's

energy per unit mass, ΔE is the energy difference between initial and final circular orbits, and $\dot{E} \equiv \frac{dE}{dt}$. The first-order gravitational self-force produces the leading term in an expansion of $\dot{E}$, which we denote $\dot{E}_1$. Since $\dot{E}_1$ scales like μ we find that Δt_{ins} scales like $M^2 \mu^{-1}$. Turning now to $\dot{\omega}$, we write this quantity as $\dot{\omega} = \frac{d\omega}{dE} \dot{E}$. At the leading order $\frac{d\omega}{dE}$ is independent of μ—it is obtained from the equations describing a circular geodesic world line. At higher orders the conservative part of the self-force will produce a correction to this geodesic orbit. Since we focus on the contribution to the phase coming from the dissipative part of the self-force (i.e. the part of the self-force responsible for a nonzero $\dot{E}$) we ignore the conservative corrections. Denoting the leading term in an expansion of $\dot{\omega}$ with $\dot{\omega}_1$, and recalling that $\dot{E}_1$ scales like μ we find that $\dot{\omega}_1$ is of $O(\mu M^{-3})$. Combining the expressions for Δt_{ins} and $\dot{\omega}_1$ we find that the first-order self-force produces a phase shift $\Delta \phi$ of order $\Delta t_{\text{ins}}^2 \dot{\omega}_1 = O(M/\mu)$. The second-order self-force gives rise to second-order terms in the expansions of $\dot{E}$ and $\dot{\omega}$. These terms are denoted here $\dot{E}_2$ and $\dot{\omega}_2$, respectively. Since $\dot{E}_2$ scales like μ^2 we find that $\dot{\omega}_2$ is $O(\mu^2 M^{-4})$. After Δt_{ins} the term $\dot{\omega}_2$ will produce a phase shift of order $\Delta t_{\text{ins}}^2 \dot{\omega}_2 = O[(M/\mu)^0]$. Therefore, a calculation of $\Delta \phi$ to the desired accuracy of order $(M/\mu)^0$ (needed for LISA data analysis) requires the calculation of the compact-object interaction with its own second-order metric perturbations.

The goal of constructing long waveform templates (e.g. for 1 yr of inspiral) which do not deviate by more than one cycle from the true GW provides practical motivation for the study of the second-order metric perturbations in this article. Moreover, construction of second-order metric perturbations allows one to extend the applicability of the perturbation analysis to binary systems with smaller M/μ mass ratios. This study can also shed light on the problem of waveform construction. Suppose that we attempt to construct a waveform for an inspiraling compact object (including the correct leading term for $\Delta \phi$) by using the following procedure. First, we calculate a corrected world line by including the contributions coming from the first-order self-force. Then we substitute this world line into the expression of the source term in the first-order perturbations wave equation, and finally we construct a waveform by solving this equation. Here there is a subtlety, since the first-order gravitational self-force is a gauge dependent quantity [21] (e.g. one may set it to zero by an appropriate choice of gauge, see below). Therefore, by invoking a first-order gauge transformation we can change the path of the corrected world line. A waveform constructed from this new corrected world line may not encode the correct gauge invariant information. This argument reveals that for some gauge choices the above procedure does not provide us with a waveform that includes the correct leading term for $\Delta \phi$. For these gauge choices it is reasonable to expect that the correct waveform could be obtained by including the

second-order gravitational perturbations in the waveform calculation. In this way all the contributions to the waveform which scale like μ^2 are being included in the calculation.

To construct the metric perturbations produced by a compact object it is useful to consider the point-particle limit—where the dimensions of the compact object approach zero (below we give more precise definitions of the compact object and the limiting process that we use). In this limit the first-order metric perturbations in the Lorenz gauge satisfy a wave equation with a delta function source term (see e.g. [22]). It is well known that (certain components of) the retarded solution of this wave equation diverge as r^{-1} as the world line of the object is approached, where r denotes the spatial distance from the object. The construction of the second-order perturbations is more difficult. In the limit the second-order metric perturbation equation away from the compact object take the following schematic form

$$D[h^{(2)}] = \nabla h^{(1)} \nabla h^{(1)} \& h^{(1)} \nabla \nabla h^{(1)}. \qquad (1)$$

Here $\mu h^{(1)}$ and $\mu^2 h^{(2)}$ denote the first-order metric perturbations and the second-order metric perturbations, respectively. D denotes a certain linear partial differential operator, ∇ schematically denotes the covariant derivative with respect to the background metric, and & denotes "and terms of the form...". Since $h^{(1)}$ diverge as r^{-1} we find that the source term of Eq. (1) diverges as r^{-4}. One might naively attempt to construct a standard retarded solution to Eq. (1), by imposing Lorenz-gauge conditions on $h^{(2)}$ and then formally integrating the singular source term with the corresponding retarded Green's function. However, the resultant integral turns out to be ill defined, in fact it diverges at every point in space-time. To see this notice that the invariant four-dimensional volume element scales like r^2 while the source term which is being integrated diverges as r^{-4}.

In this article we develop a regularization method for the construction of well-defined and physically meaningful solutions to Eq. (1). A similar problem in a scalar toy model was recently studied [23]. An outline of the resolution presented in this article was recently published [24]. Here we provide the complete details of this analysis, including derivations of results which were mentioned without derivations in [24]. In addition we provide a prescription for the construction of Fermi gauge which was only briefly mentioned in [24].

For simplicity suppose that the small compact object is a Schwarzschild black hole. We consider the black hole to be "small" compared to the length scales that characterize Riemann curvature tensor of the vacuum background geometry (e.g. a stellar-mass Schwarzschild black hole in a strong field region of a background geometry induced by a supermassive Kerr black hole). Denoting these length scales with $\{\mathcal{R}_i\}$ we express our restriction as $\mu \ll \mathcal{R}$,

where $\mathcal{R} = \min\{\mathcal{R}_i\}$. The presence of length scales with different orders of magnitude allows one to analyze this problem using the method of matched asymptotic expansions (see e.g. [22,25]). In this method one employs different approximation methods to calculate the metric in different overlapping regions of space-time, where each approximation method is adapted to a particular subset of space-time. Later, one matches the various metrics in these overlapping regions, and thereby obtain a complete approximate solution to Einstein's field equations. In this article we shall consider the following decomposition of space-time into two overlapping regions. Let r be a meaningful notion of spatial distance, we define the internal zone to lie within a world tube which surrounds the black hole and extends out to $r = r_I(\mathcal{R})$ such that $r_I \ll \mathcal{R}$, and define the external zone to lie outside another world tube $r = r_E(\mu)$, such that $\mu \ll r_E$. We denote the interior of this inner world tube with S. Since $\mu \ll \mathcal{R}$ we may choose r_E to be smaller than r_I such that there is an overlap between the above mentioned regions in $r_E < r < r_I$. We shall refer to this overlap region as the buffer-zone (in the buffer-zone r can be of order $\sqrt{\mu \mathcal{R}}$).

This article is organized as follows: First in Sec. II we discuss the perturbative approximation method to Einstein's field equations in the external zone; in Sec. III we employ this approximation and construct the well-known first-order metric perturbations; in Sec. IV we discuss the construction of the second-order metric perturbations in the external zone; in Sec. V we complete the construction of the physical second-order perturbations by matching the second-order external-zone solution to a solution in the internal zone; finally Sec. VI provides conclusions.

II. APPROXIMATION IN THE EXTERNAL ZONE

In the external zone the space-time geometry is dominated by the background geometry. Therefore, it is convenient to decompose the full space-time metric $g_{\mu\nu}^{\mathrm{full}}$ into a background metric $g_{\mu\nu}$, and perturbations $\delta g_{\mu\nu}$ that are induced by the small black hole, reading

$$g_{\mu\nu}^{\mathrm{full}}(x) = g_{\mu\nu}(x) + \delta g_{\mu\nu}(x). \tag{2}$$

Throughout this paper we use the background metric $g_{\mu\nu}$ to raise and lower tensor indices and to evaluate covariant derivatives. We expand $\delta g_{\mu\nu}$ in an asymptotic series, reading

$$\delta g_{\mu\nu}(x) = \mu h_{\mu\nu}^{(1)}(x) + \mu^2 h_{\mu\nu}^{(2)}(x) + O(\mu^3). \tag{3}$$

Here the perturbations $\{h_{\mu\nu}^{(i)}\}$ are independent of μ.

We shall now substitute the asymptotic expansion of $g_{\mu\nu}^{\mathrm{full}}$ into Einstein's field equations and obtain linear partial differential equations for the first-order and second-order gravitational perturbations $h_{\mu\nu}^{(1)}$ and $h_{\mu\nu}^{(2)}$, respectively. We assume that full space-time metric satisfies Einstein's field

equations in vacuum, reading

$$R_{\mu\nu}^{\mathrm{full}} = 0. \tag{4}$$

Here $R_{\mu\nu}^{\mathrm{full}}$ is Ricci tensor of the full space-time. Substituting decomposition (2) into Ricci tensor we obtain the following formal expansion[1]

$$R_{\mu\nu}^{\mathrm{full}} = R_{\mu\nu}^{(0)} + R_{\mu\nu}^{(L)}[\delta g] + R_{\mu\nu}^{(Q)}[\delta g] + O(\delta g^3). \tag{5}$$

Here the superscripts denote the type of dependence on $\delta g_{\mu\nu}$: (0)—no dependence on $\delta g_{\mu\nu}$, (L)—linear dependence on $\delta g_{\mu\nu}$, and (Q)—quadratic dependence on $\delta g_{\mu\nu}$. Substituting Eq. (5) into Einstein Eqs. (4) and using expansion (3) we obtain

$$R_{\mu\nu}^{(0)} = 0, \qquad x \notin S, \qquad \mu^0, \tag{6}$$

$$R_{\mu\nu}^{(L)}[h^{(1)}] = 0, \qquad x \notin S, \qquad \mu^1, \tag{7}$$

$$R_{\mu\nu}^{(L)}[h^{(2)}] = -R_{\mu\nu}^{(Q)}[h^{(1)}], \qquad x \notin S, \qquad \mu^2. \tag{8}$$

Note that Eq. (6) is an equation for the background metric $g_{\mu\nu}$. This metric satisfies Einstein's field equations in the absence of the small black hole. We can therefore omit the restriction $x \notin S$ from Eq. (6).

To further simplify the calculation it is useful to consider the limit $\mu \to 0$ of the series (3). Notice that by definition $h_{\mu\nu}^{(1)}$ and $h_{\mu\nu}^{(2)}$ do not depend on μ and therefore the form Eqs. (7) and (8) are not affected by this limit. However, the domain of validity of these equations is in fact expanded as $\mu \to 0$. At this limit we let $r_E(\mu)$ approach zero and Eqs. (7) and (8) become valid throughout the entire background space-time excluding a timelike world line $z(\tau)$, where τ denotes proper time with respect to the background metric. At this limit Eqs. (7) and (8) take the form of

$$R_{\mu\nu}^{(L)}[h^{(1)}] = 0, \qquad x \notin z(\tau), \tag{9}$$

$$R_{\mu\nu}^{(L)}[h^{(2)}] = -R_{\mu\nu}^{(Q)}[h^{(1)}], \qquad x \notin z(\tau). \tag{10}$$

As they stand Eqs. (9) and (10) contain insufficient information about the physical properties of the sources that induce the perturbations. To obtain unique solutions to these equations, we must provide additional information about these sources. The gravitational perturbations $h_{\mu\nu}^{(1)}$ and $h_{\mu\nu}^{(2)}$ are induced by a Schwarzschild black hole, and therefore their properties on the world line are determined from the physical properties of this source. As we show below these properties can be communicated to the external zone by specifying a set of divergent boundary conditions as $x \to z(\tau)$. Once these divergent boundary conditions are specified, a physical solution (defined be-

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[1]For brevity we shall omit the tensors indices inside the squared brackets of the differential operators.

low) to the perturbation Eqs. (9) and (10) is uniquely determined. In Sec. V below we obtain the desired divergent boundary conditions for Eq. (10) from the corresponding internal-zone solution. For Eq. (9) D'Eath has shown [4] that at the limit $\mu \rightarrow 0$ the retarded perturbations $h^{(1)}_{\mu\nu}$ are identical to the retarded first-order perturbations that are induced by a unit-mass point particle tracing the same world line $z(\tau)$.

Before tackling Eqs. (9) and (10) we must provide additional information about $z(\tau)$. Recall that as $\mu \rightarrow 0$ the world tube S collapses to the world line $z(\tau)$. Roughly speaking one may choose S to follow the motion of the black hole keeping it "centered" at all times with respect to S in some well-defined manner.[2] This point of view allows one to identify $z(\tau)$ as a representative world line of the black hole in the background space-time. At the leading order of approximation the representative world line of the black hole was found to be a geodesic in the background space-time (see e.g., [4,5]). Alternatively one may let the black hole drift with respect to the center of S. In this article we find that this alternative point of view is more suitable for our purposes of studying the second-order perturbations. The main reason is that it allows us to choose $z(\tau)$ to be exactly a geodesic in the background space-time which we denote with $z_G(\tau)$. Setting $z(\tau) = z_G(\tau)$ guarantees that Eq. (9) has an exact retarded solution [given by Eq. (13) below]. Notice that if we had chosen a point of view where $z(\tau)$ represents the (generically accelerated) motion of the black hole we would have found that Eq. (9) does not have any exact solution. This difficulty originates from the fact that application of the divergence operator to the left-hand side of Eq. (9) gives $\nabla^\nu R^{(L)}_{\mu\nu}[h^{(1)}] \equiv 0$, which restricts the possible sources allowed on the right-hand side. As we already mentioned, matching with the internal-zone solution implies that one may replace the right-hand side of Eq. (9) with a point-particle (delta-function) source at $z(\tau)$. Here we find that the above mentioned restriction on the source implies that $z(\tau)$ must be a geodesic world line. However, the motion of the black hole in the background space-time is generically an accelerated motion. This acceleration originates from the gravitational self-force acting on the small black hole. Therefore, had we chosen $z(\tau)$ to represent the (accelerated) motion of the black hole we would have found that Eq. (9) does not have any exact solution. Our point of view is different, since we set $z(\tau) = z_G(\tau)$ and therefore Eq. (9) has a well-defined exact solution. The black-hole acceleration which will gradually shift the black hole from the center of the world tube S will show up as a term in the

boundary conditions for the external-zone solution as $x \rightarrow z_G(\tau)$. Notice that since the leading-order acceleration scales like μ the difference in the boundary conditions induced by this leading-order acceleration will affect only $h^{(2)}_{\mu\nu}$, thus placing the effect of the first-order self-force at the boundary conditions for the second-order Eq. (10).

With the above choice of world line the equations for $h^{(1)}_{\mu\nu}$ and $h^{(2)}_{\mu\nu}$ now read

$$R^{(L)}_{\mu\nu}[h^{(1)}] = 0, \qquad x \notin z_G(\tau), \qquad (11)$$

$$R^{(L)}_{\mu\nu}[h^{(2)}] = -R^{(Q)}_{\mu\nu}[h^{(1)}], \qquad x \notin z_G(\tau). \qquad (12)$$

III. FIRST-ORDER METRIC PERTURBATIONS

First, we consider the construction of the first-order metric perturbations $h^{(1)}_{\mu\nu}$, which satisfy Eq. (11). As was previously mentioned, at the limit $\mu \rightarrow 0$ the perturbations $h^{(1)}_{\mu\nu}$ are identical to the first-order metric perturbations induced by a unit-mass point particle which traces a geodesic $z_G(\tau)$ on the vacuum background metric. We impose the Lorenz-gauge conditions on these first-order perturbations, reading

$$\bar{h}^{(1)\mu\nu}{}_{;\nu} = 0,$$

where overbar denotes the trace-reversal operator defined by $\bar{h}^{(1)}_{\mu\nu} \equiv h^{(1)}_{\mu\nu} - 1/2 g_{\mu\nu} h^{(1)\alpha}_\alpha$. In this gauge the first-order perturbations $h^{(1)}_{\mu\nu}$ read (see e.g. [22])

$$\bar{h}^{(1)}_{\mu\nu}(x) = 4 \int_{-\infty}^{\infty} G^{\text{ret}}_{\mu\nu\alpha\beta}[x|z_G(\tau)] u^\alpha(\tau) u^\beta(\tau) d\tau. \quad (13)$$

Here $u^\alpha \equiv \frac{dz_G^\alpha}{d\tau}$, and $G^{\text{ret}}_{\mu\nu\alpha\beta}[x|z_G(\tau)]$ is the gravitational retarded Green's function which is a bitensor, where the indices α, β refer to $z(\tau)$, and the indices μ, ν refer to x. This Green's function satisfies

$$\Box G^{\mu\nu}{}_{\alpha'\beta'}[x|x'] + 2R_\eta{}^\mu{}_\rho{}^\nu(x) G^{\eta\rho}{}_{\alpha'\beta'}[x|x']$$
$$= -4\pi \bar{g}^{(\mu}_{\alpha'}(x, x') \bar{g}^{\nu)}_{\beta'}(x, x') [-g]^{-1/2} \delta^4(x - x').$$

Here $\Box \equiv g^{\rho\sigma} \nabla_\rho \nabla_\sigma$ is a differential operator at x, $\bar{g}^\mu_{\alpha'}(x, x')$ denotes the bivector of a geodesic parallel transport with respect to the background metric (for the properties of this bivector see e.g. [22,26]), $R_{\eta\mu\rho\nu}$ denotes Riemann tensor of the background geometry with the sign convention of Reference [27], g denotes the determinant of the background metric, and $\delta^4(x - x')$ denotes the four-dimensional (coordinate) Dirac delta-function. Throughout this article we use the signature $(-, +, +, +)$ and geometric units $G = c = 1$.

IV. SECOND-ORDER METRIC PERTURBATIONS

We now focus our attention on the construction of the second-order perturbations $h^{(2)}_{\mu\nu}$ which satisfy Eq. (12).

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[2]Within the internal zone the full space-time metric is approximated by the metric of a perturbed Schwarzschild black hole. Here fixing the center of the black hole amounts to fixing the internal-zone dipole perturbations (these dipole perturbations are purely gauge and one can therefore set them to zero) [5].

Here it will be useful to apply the trace reverse operator to Eq. (12) which gives $\bar{R}^{(L)}_{\mu\nu}[h^{(2)}] = -\bar{R}^{(Q)}_{\mu\nu}[h^{(1)}]$. To simplify the notation we rewrite this equation as

$$D_{\mu\nu}[\bar{h}^{(2)}] = S_{\mu\nu}[\bar{h}^{(1)}], \qquad x \notin z_G(\tau). \qquad (14)$$

Here we substituted $h^{(2)}_{\mu\nu} = \bar{h}^{(2)}_{\mu\nu} - (1/2)g_{\mu\nu}\bar{h}^{(2)\alpha}{}_{\alpha}$ into $\bar{R}^{(L)}_{\mu\nu}[h^{(2)}]$ and denoted the resultant expression with $D_{\mu\nu}[\bar{h}^{(2)}]$ (notice that $\bar{R}^{(L)}_{\mu\nu}[h^{(2)}] \neq R^{(L)}_{\mu\nu}[\bar{h}^{(2)}]$). Similarly the source term $S_{\mu\nu}[\bar{h}^{(1)}]$ is defined by $S_{\mu\nu}[\bar{h}^{(1)}] \equiv -\bar{R}^{(Q)}_{\mu\nu}[h^{(1)}]$; and for abbreviation we shall often simply write $S_{\mu\nu}$. The explicit form of these terms is provided in Appendix A. For the moment we do not impose any second-order gauge conditions and Eq. (14) is in a general second-order gauge.

Before constructing a solution to Eq. (14) let us first study the singular properties of $S_{\mu\nu}$ near $z_G(\tau)$. In what follows we shall expand $\bar{h}^{(1)}_{\mu\nu}$ and $S_{\mu\nu}$ in the vicinity of $z_G(\tau)$. Throughout this paper such tensor expansions are considered on a family of hypersurfaces $\tau = $ const that are generated by geodesics which are normal to world line. Each point x on such a hypersurface is associated with the same point on the world line $z_G(\tau_x)$. On each of these hypersurfaces the expansions are valid only in a local neighborhood of $z_G(\tau_x)$ excluding a sphere of arbitrarily small volume which surrounds $z_G(\tau_x)$. Here it should be noticed that since the dynamical Eqs. (11) and (12) are not valid on the world line we do not have to keep track of the singularities on the world line, for example, distributions of the form $\delta^3(x - z_G)$ which may arise in the tensor expansions below due to application of a Laplacian operator on terms of the form $1/r$ may be completely discarded. Throughout this paper (unless we explicitly indicate otherwise) we shall represent the expansions of tensor fields using Fermi normal coordinates based on $z_G(\tau)$. These expansions take a particularly simple form in these coordinates, and often many of the leading terms are found to be identical to the corresponding terms in an expansion over a flat background space-time using Lorentz coordinates. We shall use the symbol $\overset{*}{=}$ to denote equality in a particular coordinate system. Note that the covariant nature of Eqs. (11) and (12) implies that working in a particular (background) coordinate system does not reduce the generality of our analysis, since once a solution is constructed in one particular coordinate system it can be transformed to any other coordinate system by a coordinate transformation. Moreover, our results which provide a prescription for constructing the second-order gravitational perturbations are stated in a covariant manner, and therefore can be implemented in any coordinate system.

Using Eq. (13) we expand $\bar{h}^{(1)}_{\mu\nu}$ in the vicinity of $z_G(\tau)$, which gives

$$\bar{h}^{(1)}_{\mu\nu}(x) \overset{*}{=} 4u_\mu u_\nu r^{-1} + O(r^0). \qquad (15)$$

In the external zone r denotes the invariant spatial distance along a geodesic connecting $z(\tau_x)$ and x, $r = \sqrt{\delta_{ab}x^a x^b}$, where x^a are the spatial Fermi coordinates; $u^\mu \overset{*}{=} \delta^\mu_0$ is a vector field which coincide with four-velocity on the world line. By substituting Eq. (15) into $S_{\mu\nu}$ we obtain the following expansion

$$S_{\mu\nu}(x) \overset{*}{=} [4u_\mu u_\nu + 7\eta_{\mu\nu} - 14\Omega_\mu\Omega_\nu]r^{-4} + O(r^{-3}). \qquad (16)$$

Here $\eta_{\mu\nu}$ denotes Minkowski metric, and we defined $\Omega^a \overset{*}{=} x^a/r$, $\Omega^0 \overset{*}{=} 0$, and $\Omega_\mu \overset{*}{=} g_{\mu\nu}\Omega^\nu$. Naively one may try to construct the standard retarded solution to Eq. (14), say by imposing Lorenz-gauge conditions on $h^{(2)}_{\mu\nu}$ and then formally integrating $S_{\mu\nu}$ with the retarded Green's function which gives

$$\bar{h}^{(2)}_{\mu\nu}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G^{\text{ret}}_{\mu\nu\alpha'\beta'}[x|x']S^{\alpha'\beta'}(x')\sqrt{-g(x')}d^4x'. \qquad (17)$$

Here there is a problem, examining Eq. (16) reveals that $S_{\mu\nu}$ diverges like r^{-4} in the vicinity of the world line $z_G(\tau)$. Recalling that $\sqrt{-g(x')}d^4x'$ scales like r^2, we find that the integral in Eq. (17) diverges at every point in space-time. Furthermore, the next order term in Eq. (16) that diverges like r^{-3} also gives rise to a divergent integral.

We will now develop a method to obtain well-defined solutions for Eq. (14). This method is based on consecutive steps, where in each step we reduce the degree of singularity of the field equation at hand. Eventually we end up with a field equation for a certain residual potential which has a source term which diverges like r^{-2}, this equation has well-defined retarded solutions. At this point we shall construct a retarded solution to this equation and discuss the matching to the internal-zone solution.

We should mention here another difficulty in calculating the integral in Eq. (17). Asymptotically $h^{(1)}$ has a form of a gravitational wave. Therefore, the leading asymptotic behavior of $\nabla h^{(1)}$ is $O(R^{-1})$, where R is the area coordinate—in this paragraph, for simplicity, we specialize to a background metric of a Schwarzschild black hole. This implies that for an infinitely long world line the source term $S_{\alpha\beta}$ decays asymptotically as R^{-2}. This term has a static $O(R^{-2})$ part which does not vanish after time averaging. An attempt to calculate the integral (17) over this static $O(R^{-2})$ part produces a divergent integral (even if we resolve the difficulty with the singularity near the world line). The regularization of this divergency lies outside the scope of this article (Ori has recently suggested a resolution to this problem [28]). In what follows we shall assume that such a regularization at infinity has been carried out.

A. r^{-4} singularity

First we tackle the strongest (r^{-4}) singularity in the source term of Eq. (14). For this purpose let us decompose $\bar{h}^{(2)}_{\mu\nu}$ into two tensor potentials, reading

$$\bar{h}^{(2)}_{\mu\nu} = \bar{\psi}_{\mu\nu} + \delta\bar{h}^{(2)}_{\mu\nu}, \qquad (18)$$

where $\bar{\psi}_{\mu\nu}$ satisfies

$$D_{\mu\nu}[\bar{\psi}] \overset{*}{=} [4u_\mu u_\nu + 7\eta_{\mu\nu} - 14\Omega_\mu\Omega_\nu]r^{-4} + O(r^{-3}). \qquad (19)$$

Notice that the r^{-4} singular term in Eq. (19) is the same as the r^{-4} singular term in the expansion of $S_{\mu\nu}$ [see Eq. (16)], but the lower-order terms of Eqs. (14) and (19) are in general different. In fact, we do not impose any restrictions on the lower-order terms in Eq. (19). Suppose that we construct a solution to Eq. (19), then by subtracting $D_{\mu\nu}[\bar{\psi}]$ from both sides of Eq. (14) we obtain the following equation for $\delta\bar{h}^{(2)}_{\mu\nu}$

$$D_{\mu\nu}[\delta\bar{h}^{(2)}] = S_{\mu\nu} - D_{\mu\nu}[\bar{\psi}], \qquad x \notin z_G(\tau). \qquad (20)$$

By construction the source term in this equation diverges only like r^{-3}, while the original field equation [Eq. (14)] has a source term which diverges like r^{-4}. In this sense Eq. (20) is simpler then Eq. (14).

We now face the problem of solving Eq. (19). To construct a particular solution we use a linear combination of terms which are quadratic in $\bar{h}^{(1)}_{\mu\nu}$. Since $\bar{h}^{(1)}_{\mu\nu}$ diverges like r^{-1} we find that by applying the differential operator $D_{\mu\nu}$ to terms which are quadratic in $\bar{h}^{(1)}_{\mu\nu}$, we can obtain terms which diverge like r^{-4}. First we construct four independent quadratic tensor fields reading

$$\varphi^A_{\mu\nu} = \bar{h}^{(1)\rho}{}_\mu \bar{h}^{(1)}_{\rho\nu}, \qquad \varphi^B_{\mu\nu} = \bar{h}^{(1)\rho}{}_\rho \bar{h}^{(1)}_{\mu\nu},$$
$$\varphi^C_{\mu\nu} = \bar{h}^{(1)\eta\rho} \bar{h}^{(1)}{}_{\eta\rho} g_{\mu\nu}, \qquad \varphi^D_{\mu\nu} = (\bar{h}^{(1)\rho}{}_\rho)^2 g_{\mu\nu}. \qquad (21)$$

These terms can be combined to form a solution to Eq. (19) which reads

$$\bar{\psi}_{\mu\nu} = \tfrac{1}{64}[2(c_A\varphi^A_{\mu\nu} + c_B\varphi^B_{\mu\nu}) - 7(c_C\varphi^C_{\mu\nu} + c_D\varphi^D_{\mu\nu})]. \qquad (22)$$

Here the constants c_A, c_B, c_C, c_D must satisfy

$$c_A + c_B = 1, \qquad c_C + c_D = 1, \qquad (23)$$

but are otherwise arbitrary. One may directly substitute Eq. (22) into Eq. (19) and verify that $\bar{\psi}_{\mu\nu}$ satisfies this equation.

Equation (22) can be derived as follows. First notice that the coefficient in front of the r^{-4} term in both Eq. (16) and Eq. (19) does not depend on the curvature of the background space-time. In fact the form of this expression would not change if we would replace the curved background space-time with a flat space-time. Therefore in deriving $\bar{\psi}_{\mu\nu}$ we may consider a simple case of flat back-

ground space-time. In this case, the exact nonlinear solution to Einstein's field equations in our problem is simply the Schwarzschild solution. Far from the black hole this Schwarzschild solution may be approximated by an expansion which schematically reads

$$g^{\text{Sch}}_{\mu\nu} = \eta_{\mu\nu} + \mu H^{(1)}_{\mu\nu} + \mu^2 H^{(2)}_{\mu\nu} + O(\mu^3). \qquad (24)$$

In the appropriate coordinates the second-order term $\bar{H}^{(2)}_{\mu\nu}$ satisfies Eqs. (14) and (19) in the flat background case. To be consistent with our first-order (Lorenz) gauge conditions we have to make sure that the term $H^{(1)}_{\mu\nu}$ satisfies the Lorenz-gauge conditions for a flat space-time metric. As we will immediately show this condition is satisfied if we express the Schwarzschild solution in isotropic Cartesian coordinates. The Schwarzschild metric in the isotropic Cartesian coordinates takes the form of

$$ds^2 = -\left(\frac{2\tilde{r} - \mu}{2\tilde{r} + \mu}\right)^2 dt^2 + \left(1 + \frac{\mu}{2\tilde{r}}\right)^4 (dx^2 + dy^2 + dz^2). \qquad (25)$$

Here $\tilde{r}^2 = x^2 + y^2 + z^2$. Expanding this metric in powers of $\mu/\tilde{r}$ gives the following expressions for the trace-reversed first-order and second-order perturbations

$$\bar{H}^{(1)}_{\mu\nu} \overset{*}{=} \frac{4}{\tilde{r}}\tilde{u}_\mu\tilde{u}_\nu, \qquad \bar{H}^{(2)}_{\mu\nu} \overset{*}{=} -\frac{1}{4\tilde{r}^2}[2\tilde{u}_\mu\tilde{u}_\nu + 7\eta_{\mu\nu}]. \qquad (26)$$

Here $\tilde{u}^\mu \overset{*}{=} \delta^\mu_0$ is a vector field. Notice that the Lorenz-gauge conditions $\bar{H}^{(1)\mu\nu}{}_{,\nu} \overset{*}{=} 0$ are satisfied. We may replace $\bar{h}^{(1)}_{\mu\nu}$ with $\bar{H}^{(1)}_{\mu\nu}$ in the quadratic terms (21) and employ Eqs. (26) to express $\bar{H}^{(2)}_{\mu\nu}$ as a linear combination of these quadratic terms. The coefficients of this linear combination are the desired coefficients in Eqs. (22) and (23). Notice that for the case of a flat background space-time $\bar{\psi}_{\mu\nu}$ is an *exact solution* to Eq. (14).

B. r^{-3} singularity

We now consider the construction of $\delta\bar{h}^{(2)}_{\mu\nu}$ which satisfies Eq. (20). This equation has a source term which diverges like r^{-3}, and therefore its standard retarded solution diverges. Let us examine the terms that give rise to this r^{-3} singularity. It is convenient to express the source term of Eq. (20) schematically (and without indices) as

$$S - D[\bar{\psi}] = \nabla\bar{h}^{(1)}\nabla\bar{h}^{(1)} \& \bar{h}^{(1)}\nabla\nabla\bar{h}^{(1)}. \qquad (27)$$

Notice that here only sum over all the terms diverges as r^{-3} (since the individual terms which diverge as r^{-4} cancel each other). Using a decomposition devised by Detweiler and Whiting [29] we decompose $\bar{h}^{(1)}_{\mu\nu}$ as follows[3]

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[3] Here we define $\bar{h}^{(1)R}_{\mu\nu}$ and $\bar{h}^{(1)S}_{\mu\nu}$ to be independent of μ, as opposed to [29].

$$\bar{h}^{(1)}_{\mu\nu} = \bar{h}^{(1)S}_{\mu\nu} + \bar{h}^{(1)R}_{\mu\nu}. \qquad (28)$$

Here $\bar{h}^{(1)S}_{\mu\nu}$ is a certain singular potential which diverges as r^{-1} as $r \to 0$, and $\bar{h}^{(1)R}_{\mu\nu}$ is a certain regular potential which satisfies the following homogeneous wave equation

$$\Box \bar{h}^{(1)R}_{\mu\nu} + 2R^{\eta}{}_{\mu}{}^{\rho}{}_{\nu}\bar{h}^{(1)R}_{\eta\rho} = 0. \qquad (29)$$

Decomposition (28) is particularly useful for expressing the first-order gravitational self-force, since the general expression of this self-force is completely determined from $\bar{h}^{(1)R}_{\mu\nu}$ [see [29] and also Eq. (35) below]. Generically $\bar{h}^{(1)R}_{\mu\nu}$ is a smooth field of $O(r^0)$ on $z_G(\tau)$. Expanding $\bar{h}^{(1)S}_{\mu\nu}$ and its covariant derivatives in the vicinity of the world line $z_G(\tau)$ gives (a method of constructing these expressions is described in detail in [22])

$$\bar{h}^{(1)S}_{\mu\nu} \stackrel{*}{=} \frac{4}{r}u_\mu u_\nu + O(r^1), \qquad (30)$$

$$\nabla_\rho \bar{h}^{(1)S}_{\mu\nu} \stackrel{*}{=} -\frac{4}{r^2}u_\mu u_\nu \Omega_\rho + O(r^0), \qquad (31)$$

$$\nabla_\eta \nabla_\rho \bar{h}^{(1)S}_{\mu\nu} \stackrel{*}{=} \frac{4u_\mu u_\nu}{r^3}[3\Omega_\eta \Omega_\rho - u_\eta u_\rho - \eta_{\eta\rho}] + O(r^{-1}). \qquad (32)$$

Notice that the orders r^0, r^{-1}, and r^{-2} are missing from the expansions of $\bar{h}^{(1)S}_{\mu\nu}$, $\nabla_\rho \bar{h}^{(1)S}_{\mu\nu}$, and $\nabla_\eta \nabla_\rho \bar{h}^{(1)S}_{\mu\nu}$, respectively. The absence of these terms can be traced to the vanishing acceleration of $z_G(\tau)$. We now substitute Eq. (28) into Eq. (27) and examine the various terms that give rise to the problematic r^{-3} singularity in the source term $S - D[\bar{\psi}]$. Expansions (30)–(32) imply that the only combination that produces this r^{-3} singularity is of the form $\bar{h}^{(1)R}\nabla\nabla\bar{h}^{(1)S}$.

To eliminate this problematic r^{-3} singularity we utilize gauge freedom and employ a first-order (regular) gauge transformation $x^\nu \to x^\nu - \mu\xi^\nu$, which gives

$$h^{(1)}_{\mu\nu} \to h^{(1)S}_{\mu\nu} + h^{(1)R(\text{new})}_{\mu\nu}, \quad h^{(1)R(\text{new})}_{\mu\nu} \equiv h^{(1)R}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}. \qquad (33)$$

Here ξ_ν does not depend on μ. Notice that we have included the entire gauge transformation in the definition of the new regular potential $h^{(1)R(\text{new})}_{\mu\nu}$. This is a natural identification since the gravitational self-force in the new gauge is obtained by replacing $h^{(1)R}_{\mu\nu}$ with $h^{(1)R(\text{new})}_{\mu\nu}$ in the expression of the self-force [30]. We now impose the following gauge conditions

$$[h^{(1)R(\text{new})}_{\mu\nu}]_{z_G(\tau)} \equiv [h^{(1)R}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}]_{z_G(\tau)} = 0. \qquad (34)$$

Expanding $h^{(1)R(\text{new})}_{\mu\nu}$ in the vicinity of the world line $z_G(\tau)$ gives

$$h^{(1)R(\text{new})}_{\mu\nu} = O(r). $$

Most beneficially in this new gauge the previously mentioned problematic terms $\bar{h}^{(1)R(\text{new})}\nabla\nabla\bar{h}^{(1)S}$ in the source term $S - D[\bar{\psi}]$ diverge only like r^{-2}. This property will allow us to construct well-defined retarded solution to Eq. (20). Notice that we invoked a regular gauge transformation in the sense that it did not change the singular properties of $\bar{h}^{(1)}_{\mu\nu}$ near the world line, meaning that Eq. (15) is unchanged by this gauge transformation. Therefore, the coefficient in front of the r^{-4} term in Eqs. (14) and (19) is not affected by the gauge transformation. This implies that even though the numerical values of $\bar{\psi}_{\mu\nu}$ are changed by the gauge transformation, the general form of $\bar{\psi}_{\mu\nu}$ given by Eqs. (22) and (23) is invariant to any such regular gauge transformations [e.g., a transformation satisfying Eq. (34)].

C. Construction of the first-order gauge

As was previously mentioned the first-order gravitational self-force must be accounted for when imposing boundary conditions as $x \to z_G(\tau)$ for Eq. (12). To simplify the calculation of these boundary conditions we once more use the gauge freedom. The first-order gravitational self-force is a gauge dependent quantity [21], and in fact one can always choose a convenient first-order gauge in which the first-order gravitational self-force vanishes (see below). In this gauge the geodesic world line $z_G(\tau)$ represents the black hole's world line accurately up to errors of order μ^2. In this case the contributions to the boundary conditions of Eq. (12) that originate from the black hole's acceleration due to the first-order self-force simply vanish.

To spell out the desired gauge conditions let us examine the expression for the $O(\mu)$ acceleration which is induced by the first-order self-force [29]

$$a^\mu = -\mu(g^{\mu\nu} + u^\mu u^\nu)u^\rho u^\eta(\nabla_\rho h^{(1)R}_{\eta\nu} - \tfrac{1}{2}\nabla_\nu h^{(1)R}_{\rho\eta}). \qquad (35)$$

Here all quantities are evaluated on the world line. Originally this expression was derived for $h^{(1)R}_{\mu\nu}$ which satisfies the Lorenz-gauge conditions. But following the analysis in [21] we find that this expression is also valid in any new gauge provided that the gauge transformation from Lorenz gauge to this new gauge is sufficiently smooth. To obtain a gauge with a vanishing first-order gravitational self-force we should require that $a^\mu = 0$. This requirement conforms with many gauge choices. For example it is satisfied if all the first covariant derivatives of the regular field in the new gauge vanish. Putting this gauge condition together with our previous gauge condition (34) yields

$$[h^{(1)R(\text{Fermi})}_{\mu\nu}]_{z_G(\tau)} \equiv [h^{(1)R}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}]_{z_G(\tau)} = 0, \qquad (36)$$

$$[\nabla_\rho h^{(1)R(\text{Fermi})}_{\mu\nu}]_{z_G(\tau)} \equiv [\nabla_\rho(h^{(1)R}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu})]_{z_G(\tau)} = 0. \qquad (37)$$

We shall refer to this new gauge as Fermi gauge.

To construct Fermi gauge consider contracting Eq. (37) with u^ρ. The resultant equation states that $h_{\mu\nu}^{(1)R(\text{Fermi})}$ is constant along $z_G(\tau)$, which is consistent with Eq. (36). But more importantly it implies that once Eq. (36) is satisfied at an initial point $z_G(\tau_0)$ Eq. (37) will guarantee its validity everywhere along $z_G(\tau)$. We now choose an arbitrary gauge vector $\xi_{(0)\mu}$ at some initial point $z_G(\tau_0)$, and construct its first covariant derivatives at this point such that Eq. (36) is satisfied. For example we may choose

$$\xi_{(0)\mu;\nu} = \xi_{(0)\nu;\mu} = -\tfrac{1}{2}h_{\mu\nu}^{(1)R}(\tau_0). \qquad (38)$$

To transport ξ_μ and $\xi_{\mu;\nu}$ along $z_G(\tau)$ we derive transport equations as follows. We treat Eq. (37) and the commutation relation $2\xi_{\mu;[\nu\alpha]} = R^\epsilon{}_{\mu\nu\alpha}\xi_\epsilon$ as a set of algebraic equations for $\xi_{\alpha;\beta\gamma}$ and use the identities of Riemann tensor to obtain the following relation

$$\xi_{\nu;\mu\alpha} = R^\epsilon{}_{\alpha\mu\nu}\xi_\epsilon - \tfrac{1}{2}(h_{\mu\nu;\alpha}^{(1)R} + h_{\nu\alpha;\mu}^{(1)R} - h_{\mu\alpha;\nu}^{(1)R}). \qquad (39)$$

Here all quantities are evaluated on the world line. One may substitute Eq. (39) into Eq. (37), and into the above mentioned commutation relation; and thereby verify that these two equations are identically satisfied by Eq. (39). We now construct a second-order transport equation for $\xi_\mu(\tau)$ by contracting Eq. (39) with $u^\alpha u^\mu$ which gives

$$\frac{D^2}{D\tau^2}\xi_\nu = R^\epsilon{}_{\alpha\mu\nu}\xi_\epsilon u^\alpha u^\mu - \frac{1}{2}u^\alpha u^\mu(h_{\mu\nu;\alpha}^{(1)R} + h_{\nu\alpha;\mu}^{(1)R} - h_{\mu\alpha;\nu}^{(1)R}).$$
$$(40)$$

Solving this equation with the above mentioned initial conditions provides us with the gauge vector $\xi_\mu(\tau)$ along the world line. Similarly we can construct a first-order transport equation for $\xi_{\mu;\nu}$ by contracting Eq. (39) with u^α. Using the solution $\xi_\mu(\tau)$ of Eq. (40) together with the initial conditions (38) this first-order transport equation can be integrated to give $\xi_{\mu;\nu}(\tau)$. $\xi_{\mu;\nu\alpha}(\tau)$ is then obtained by substituting $\xi_\mu(\tau)$ into Eq. (39). Once $\xi_\mu(\tau)$, $\xi_{\mu;\nu}(\tau)$, and $\xi_{\mu;\nu\alpha}(\tau)$ are obtained, one can use these quantities to construct a local expansion of the gauge vector field $\xi^\mu(x)$ in a local neighborhood of the world line. Notice that Eq. (39) implies that Fermi gauge satisfies the Lorenz-gauge conditions along the world line.

The above construction only provides leading terms in an expansion of ξ_μ in a local neighborhood of the world line. One may continue ξ_μ globally to the entire spacetime. Since gauge freedom is associated with nonphysical degrees of freedom we are allowed to introduce a gauge continuation which depends on arbitrary parameters. Nevertheless, it is sometimes helpful to work in a gauge which is manifestly causal, such a gauge may be constructed by continuing ξ_μ along future null cones based on $z_G(\tau)$ [31]. In the analysis below we assume that such a global gauge continuation has been performed and that the first-order gauge is fixed globally.

D. Particular solution $\delta\bar{h}^{(2)}$

We shall now construct a particular retarded solution to Eq. (20). Here it is useful to remove the restriction $x \notin z_G(\tau)$, and continue the source of Eq. (20) to the world line. Clearly a particular solution to Eq. (20) with the world line included also satisfies the original equation (i.e. with the world line excluded). However, not every continuation of the source of Eq. (20) to the world line produces an equation which is self-consistent. This is easily demonstrated by taking the covariant divergence of both sides of Eq. (20). In Appendix B we show that $\nabla^\mu D_{\mu\nu}[\delta\bar{h}^{(2)}]$ vanishes identically, and furthermore the covariant divergence of the source term of Eq. (20) in $x \notin z_G(\tau)$ vanishes as well. These facts constrain the permitted continuations of the source of Eq. (20) to the world line. If one naively chooses a continuation to the world line which has a non-vanishing covariant divergence, the resulted equation will not be self-consistent. Here we choose the simplest possible continuation by requiring that no additional singularities are introduced on the world line. Meaning that the singularities of the continued source term on the world line are completely specified by its expansion in $x \notin z_G(\tau)$, and no additional singularities (e.g. delta functions) are introduced on the world line. Equation (20) now takes the form

$$D_{\mu\nu}[\delta\bar{h}^{(2)}] = \delta S_{\mu\nu}^F. \qquad (41)$$

Here $\delta S_{\mu\nu}^F \equiv S_{\mu\nu}^F - D_{\mu\nu}[\bar{\psi}^F]$, where the superscript F indicates that source terms are evaluated in Fermi gauge. Below we show that Eq. (41) is self-consistent by constructing a solution to this equation.

Recall that we have fixed the first-order gauge, but we still have the freedom to invoke a purely second-order gauge transformation of the form

$$x^\mu \rightarrow x^\mu - \mu^2 \xi_{(2)}^\mu.$$

Here the gauge vector $\xi_{(2)}^\mu$ is independent of the mass μ. Similar to the first-order case, one may choose the gauge vector $\xi_{(2)}^\mu$ such that $\delta\bar{h}_{\mu\nu}^{(2)}$ satisfies the Lorenz-gauge conditions, reading

$$\delta\bar{h}^{(2)\mu\nu;}{}_\nu = 0. \qquad (42)$$

Equation (41) now takes the form of

$$\Box\delta\bar{h}_{\mu\nu}^{(2)} + 2R^\eta{}_\mu{}^\rho{}_\nu\delta\bar{h}_{\eta\rho}^{(2)} = -2\delta S_{\mu\nu}^F. \qquad (43)$$

We define $\delta\bar{h}_{\mu\nu}^{(2)}$ to be the retarded solution of Eq. (43) reading

$$\delta\bar{h}_{\mu\nu}^{(2)}(x) = \frac{1}{2\pi}\int G_{\mu\nu}{}^{\alpha'\beta'\text{ret}}[x|x']\delta S_{\alpha'\beta'}^F(x')\sqrt{-g(x')}d^4x'. \qquad (44)$$

Since the source term of Eq. (43) diverges only like r^{-2} the integral in Eq. (44) has a finite contribution originating from the vicinity of $z_G(\tau)$. Notice that even though the

expression in Eq. (44) satisfies Eq. (43) it is not *a priori* guaranteed that it also satisfies Eq. (41). This equation will be satisfied only if the retarded solution (44) satisfies the Lorenz-gauge conditions (42), this is shown in Appendix B.

E. General second-order solution

So far we have constructed a particular solution to Eq. (14), reading

$$\bar{h}^{(2)}_{\mu\nu} = \bar{\psi}^F_{\mu\nu} + \delta\bar{h}^{(2)}_{\mu\nu}. \tag{45}$$

Having found one particular solution does not complete the construction, since we need to make sure that the constructed solution satisfies several required physical properties (e.g. it has to match the internal-zone solution). To find the desired physical solution we first construct the general solution to Eq. (14), and then impose a set of additional requirements on this solution. In this way we obtain a particular solution which is physically meaningful.

Since Eq. (14) is valid for $x \notin z_G(\tau)$, we find that we can construct a new solution by adding to $\bar{h}^{(2)}_{\mu\nu}$ a potential that satisfies a *semihomogeneous* equation i.e., a homogeneous equation for $x \notin z_G(\tau)$, reading

$$D_{\mu\nu}[\bar{h}^{(2)SH}] = 0, \qquad x \notin z_G(\tau). \tag{46}$$

The general solution to Eq. (14) is given by

$$\bar{h}^{(2)G}_{\mu\nu} \equiv \bar{h}^{(2)SH}_{\mu\nu} + \bar{h}^{(2)}_{\mu\nu}, \tag{47}$$

where $\bar{h}^{(2)SH}_{\mu\nu}$ is the general solution to Eq. (46).

F. Physical second-order solution

To find a (particular) physically meaningful solution to Eq. (14) we need to impose additional requirements on $\bar{h}^{(2)G}_{\mu\nu}$. These requirements can also be expressed as requirements imposed on the general semihomogeneous solution $\bar{h}^{(2)SH}_{\mu\nu}$, thus obtaining a particular semihomogeneous solution. For abbreviation we denote this particular semihomogeneous solution with $\bar{\gamma}_{\mu\nu}$. Using Eqs. (45) and (47) the desired physical solution $\bar{h}^{(2)P}_{\mu\nu}$ is expressed as

$$\bar{h}^{(2)P}_{\mu\nu} = \bar{h}^{(2)}_{\mu\nu} + \bar{\gamma}_{\mu\nu} = \bar{\psi}^F_{\mu\nu} + \delta\bar{h}^{(2)}_{\mu\nu} + \bar{\gamma}_{\mu\nu}. \tag{48}$$

We group the additional requirements into four groups (i) gauge conditions, (ii) causality requirements, (iii) global boundary conditions, and (iv) boundary conditions at the world line.

(i) First we impose gauge conditions. To obtain a simple representation for $\bar{\gamma}_{\mu\nu}$ we impose the Lorenz-gauge conditions on $\bar{\gamma}_{\mu\nu}$. In this gauge Eq. (46) takes the form of

$$\Box\bar{\gamma}_{\mu\nu} + 2R^{\eta}{}_{\mu}{}^{\rho}{}_{\nu}\bar{\gamma}_{\eta\rho} = 0, \qquad x \notin z_G(\tau). \tag{49}$$

Notice that by construction both $\delta\bar{h}^{(2)}_{\mu\nu}$ and $\bar{\gamma}_{\mu\nu}$ satisfy the

Lorenz-gauge conditions. However, $\bar{\psi}^F_{\mu\nu}$ does not satisfy these conditions, and therefore our particular second-order solution $\bar{h}^{(2)P}_{\mu\nu}$ is not in the (second-order) Lorenz gauge.

(ii) Next we discuss causality. Causality is more easily discussed in terms of an initial value formulation of the problem. Therefore, in this paragraph only we consider such an initial value formulation. Suppose that we prescribe initial data for the first-order and second-order metric perturbations on some initial spacelike hypersurface Σ_0, such that the corresponding constraint equations are satisfied on this hypersurface. Here we consider a standard extension of the retarded solutions $\bar{h}^{(1)}_{\mu\nu}$, $\delta\bar{h}^{(2)}_{\mu\nu}$ [given by Eq. (13), Eq. (44)] to include additional terms which depend on the initial data. Let x be a point within the causal future $J^+(\Sigma_0)$. Then by construction the retarded solution $\bar{h}^{(1)}_{\mu\nu}(x)$ is unaffected by an arbitrary modification of the initial data on outside $J^-(x) \cap \Sigma_0$. At second order we define $\bar{\gamma}_{\mu\nu}$ to be the retarded solution of Eq. (49). Recall that $\delta\bar{h}^{(2)}_{\mu\nu}$ is the retarded solution of Eq. (43), and $\bar{\psi}^F_{\mu\nu}$ is completely determined from the first-order metric perturbations. Equation (48) now implies that the particular solution $\bar{h}^{(2)P}_{\mu\nu}$ is unaffected by an arbitrary modification of the initial data outside $J^-(x) \cap \Sigma_0$. In this sense the constructed second-order solution $\bar{h}^{(2)P}_{\mu\nu}(x)$ is manifestly causal.

(iii) In addition we require that the only source for $\bar{\gamma}_{\mu\nu}$ is the small black hole. Since $\delta\bar{h}^{(2)}_{\mu\nu}$ satisfies an inhomogeneous equation [Eq. (43)] it may contain waves that are not sourced by the world line. However $\bar{\gamma}_{\mu\nu}$ satisfies a semihomogeneous Eq. (49), and therefore the waves within $\bar{\gamma}_{\mu\nu}$ can originate either from the world line or from global boundary conditions (e.g. prescribed boundary conditions at I^-). Since we are interested only in perturbations that are induced by the black hole, we exclude any supplementary perturbations coming from these global boundary conditions.

(iv) We now turn to discuss the boundary conditions as $x \rightarrow z_G(\tau)$. For this purpose let us consider once more a small but finite value of μ. In this case Eq. (49) is valid only for $x \notin S$. Following the D'Eath analysis of first-order metric perturbations [4] we express a solution to this equation using a Kirchhoff representation. In this way $\bar{\gamma}_{\mu\nu}$ is expressed as a certain integral over a surface of a world tube. Recall that the external zone lies outside a world tube with radius $r_E(\mu)$. Denoting the surface of this world tube with Σ_E, we express $\bar{\gamma}_{\mu\nu}$ as

$$\bar{\gamma}_{\mu\nu}(x) = -\frac{1}{4\pi} \int_{\Sigma_E(\mu)} (G^{\text{ret}}_{\mu\nu\alpha'\beta'}[x|x']\nabla^{\epsilon'}\bar{\gamma}^{\alpha'\beta'}(x')$$
$$- \bar{\gamma}^{\alpha'\beta'}(x')\nabla^{\epsilon'}G^{\text{ret}}_{\mu\nu\alpha'\beta'}[x|x'])d\Sigma_{\epsilon'}. \tag{50}$$

Here $d\Sigma_{\epsilon'}$ denotes an outward directed three-surface element on Σ_E. In the derivation of Eq. (50) we assumed that $\bar{\gamma}_{\mu\nu}$ decays sufficiently fast at spatial infinity. Furthermore,

we assumed that the retarded Green's function falls sufficiently fast into the past. Consider substituting a given expansion of $\bar{\gamma}_{\mu\nu}$ (in powers of r) into Eq. (50) and then taking the limit $\mu \to 0$. Recall that at this limit $r_E \to 0$ and notice that $d\Sigma_\epsilon$ scales like r_E^2. Therefore, only the diverging terms (as $r \to 0$) in this expansion give rise to a nonvanishing contribution to $\bar{\gamma}_{\mu\nu}$ at the limit $\mu \to 0$. We conclude that at the limit, it is sufficient to specify *divergent boundary conditions* to obtain a unique physical solution to Eq. (49). To obtain these boundary conditions we examine the divergent behavior of $\bar{\gamma}_{\mu\nu}(x)$ as $x \to z_G(\tau)$. For this we use Eq. (48) together with an analysis of the behavior of $\bar{\psi}_{\mu\nu}^F$, $\delta\bar{h}_{\mu\nu}^{(2)}$, and $\bar{h}_{\mu\nu}^{(2)P}$ near $r = 0$.

First let us consider the divergent behavior of $\bar{\psi}_{\mu\nu}^F$. Using Eqs. (28), (30), and (36) together with Eqs. (21) and (22) we obtain the following expansion for $\bar{\psi}_{\mu\nu}^F$ in Fermi normal coordinates

$$\bar{\psi}_{\mu\nu}^F \overset{*}{=} -\frac{1}{4r^2}[2u_\mu u_\nu + 7\eta_{\mu\nu}] + O(r^0). \quad (51)$$

Next we examine the behavior of $\delta\bar{h}_{\mu\nu}^{(2)}$ near $r = 0$. Solving Eq. (43) iteratively (see Appendix C) shows that $\delta\bar{h}_{\mu\nu}^{(2)}$ is bounded as $r \to 0$. Finally, we have to determine the divergent behavior of $\bar{h}_{\mu\nu}^{(2)P}$ in the vicinity of $r = 0$. This requires an analysis of the internal-zone solution, which is discussed in the next section.

V. APPROXIMATION IN THE INTERNAL ZONE

To simplify the discussion we assume that all the length scales characterizing the background space-time $\{\mathcal{R}_i\}$ are of the same order of magnitude $\mathcal{R}$. Recall that $\mu \ll \mathcal{R}$, and furthermore in the internal zone we have $r \ll \mathcal{R}$. By virtue of the smallness of $r/\mathcal{R}$ and $\mu/\mathcal{R}$ we may expand the full space-time metric in the internal zone as follows

$$g_{\mu\nu} = g_{\mu\nu}^{\text{Sch}} + \mathcal{R}^{-1}g_{\mu\nu}^{(1)} + \mathcal{R}^{-2}g_{\mu\nu}^{(2)} + O(\mathcal{R}^{-3}). \quad (52)$$

Here $g_{\mu\nu}^{\text{Sch}}$ is the metric of the small Schwarzschild black hole. Recall that in the buffer zone both expansions (3) and (52) are valid, and therefore we can formally expand these two equations simultaneously using the fact that in the buffer zone both μr^{-1} and $r\mathcal{R}^{-1}$ are small. Following Thorne and Hartle [25] the various dimensional combinations involved in these expressions can be summarized in a table (see Table I).

The top row in Table I gives the external-zone expansion, and the left most column in this table gives the internal-zone expansion. In the buffer zone where both expansions are valid one can find appropriate coordinates in which these two expansions coincide. Each entry in this table schematically represents the combinations of the dimensional quantities (μ, $\mathcal{R}$, and r) obtained from simultaneous expansions of the top row and left most column. Note that this table provides only the powers of

TABLE I. Schematic representation of dimensional quantities combinations in expansions of the metric in the buffer zone. The top row gives the metric's external-zone expansion (3), and the left column gives the metric's internal-zone expansion (52).

	$g_{\mu\nu}$	$\mu h_{\mu\nu}^{(1)}$	$\mu^2 h_{\mu\nu}^{(2)}$	$\cdots$
$g_{\mu\nu}^{\text{Sch}}$	η	μr^{-1}	$\mu^2 r^{-2}$	$\cdots$
$\mathcal{R}^{-1}g_{\mu\nu}^{(1)}$	$r\mathcal{R}^{-1}$	$\mu\mathcal{R}^{-1}$	$\mu^2(r\mathcal{R})^{-1}$	$\cdots$
$\mathcal{R}^{-2}g_{\mu\nu}^{(2)}$	$r^2\mathcal{R}^{-2}$	$\mu r\mathcal{R}^{-2}$	$\mu^2\mathcal{R}^{-2}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

the relevant dimensional combinations and does not give the exact expressions.

As before the background metric $g_{\mu\nu}$ is described in Fermi normal coordinates based on $z_G(\tau)$. The Schwarzschild metric $g_{\mu\nu}^{\text{Sch}}$ is described in the Schwarzschild isotropic coordinates (25). η (schematically) denotes the Minkowski metric, which is the leading order is both the expansions of $g_{\mu\nu}$ and $g_{\mu\nu}^{\text{Sch}}$ in the buffer zone.

We number the rows from top to bottom starting at the row of $g_{\mu\nu}^{\text{Sch}}$ (row 0), and number the columns from left to right starting from the column of $g_{\mu\nu}$ (column 0). Each entry in the table is associated with an ordered pair of numbers: (row, column). The divergent behavior of $\bar{h}_{\mu\nu}^{(2)P}$ near the world line follows from column 2. In this column only terms (0, 2) and (1, 2) need to be considered since only these terms have the potential of producing divergent terms as $r \to 0$. These terms follow from an expansion for $g_{\mu\nu}^{\text{Sch}}$ and $\mathcal{R}^{-1}g_{\mu\nu}^{(1)}$ that we discuss next.

Let us consider first row 0. In the Schwarzschild isotropic coordinates (25) the entries (0, 1) and (0, 2) are nothing but the terms $\mu H_{\mu\nu}^{(1)}$ and $\mu^2 H_{\mu\nu}^{(2)}$ [see Eq. (24)], respectively; and their explicit form can be easily obtained from Eq. (26). We identify $\tilde{r}$ with r, with this we find that the term (0, 1) coincide with the corresponding terms of order r^{-1} in the expansion of $\mu h_{\mu\nu}^{(1)}$.

We now discuss row 1. Expanding the background metric in the vicinity of the world line gives $g_{\mu\nu} = \eta_{\mu\nu} + O(r^2\mathcal{R}^{-2})$, which implies that the term (1, 0) vanishes. This vanishing term serves as a boundary condition for the perturbations equations for $\mathcal{R}^{-1}g^{(1)}$ as $\tilde{r} \to \infty$. The term $\mathcal{R}^{-1}g^{(1)}$ is obtained by solving the gravitational vacuum perturbations equation in a Schwarzschild background, with these boundary conditions. However, it is well known that these $O(\mathcal{R}^{-1})$ perturbations can be eliminated by a gauge choice and mass redefinition. Therefore, we may always choose a gauge in which row 1 vanishes identically (see also [25]). Notice that the vanishing of the (1, 1) term conforms with the fact that the $O(r^0)$ terms are absent from the expansion of $\mu h_{\mu\nu}^{(1)}$ in Fermi gauge. Since the (1, 2) term vanishes we conclude that the only divergent term in the expansion of $\bar{h}_{\mu\nu}^{(2)P}$ near the world line, is a term which

diverges like r^{-2}. The form of this term is provided by $\bar{H}^{(2)}_{\mu\nu}$ in Eq. (26). Notice that $\bar{H}^{(2)}_{\mu\nu}$ is identical to the divergent expression of $\bar{\psi}^F_{\mu\nu}$ given by Eq. (51).

VI. CONCLUSIONS

We have found that the divergent terms in the expansion for $\bar{h}^{(2)P}_{\mu\nu}$ coincides with the divergent terms in the expansion for $\bar{\psi}^F_{\mu\nu}$ given by Eq. (51). Rewriting Eq. (48) as

$$\bar{\gamma}_{\mu\nu} = \bar{h}^{(2)P}_{\mu\nu} - \bar{\psi}^F_{\mu\nu} - \delta\bar{h}^{(2)}_{\mu\nu},$$

and recalling that $\delta\bar{h}^{(2)}_{\mu\nu}$ is bounded as $r \rightarrow 0$, we find that at this limit $\bar{\gamma}_{\mu\nu}$ is bounded as well. Recall that only divergent boundary conditions as $r \rightarrow 0$ can produce a nonvanishing semihomogeneous retarded solution $\bar{\gamma}_{\mu\nu}$. Since the divergent boundary conditions of Eq. (49) vanish, we find that $\bar{\gamma}_{\mu\nu}$ vanishes identically.

From Eq. (48) we finally conclude that the physical second-order gravitational perturbations in the external zone are given by

$$\bar{h}^{(2)P}_{\mu\nu} = \bar{\psi}^F_{\mu\nu} + \delta\bar{h}^{(2)}_{\mu\nu}. \tag{53}$$

Here $\bar{\psi}^F_{\mu\nu}$ is given by Eq. (22), where the perturbations $\bar{h}^{(1)}_{\mu\nu}$ are in Fermi gauge, and $\delta\bar{h}^{(2)}_{\mu\nu}$ is given by Eq. (44). Equation (53) provides a simple covariant prescription for the construction of the second-order metric perturbations without any reference to a particular (background) coordinate system.

ACKNOWLEDGMENTS

APPENDIX A: EXPANSION OF RICCI TENSOR

The linear and quadratic terms in the expansion of Ricci tensor [see Eq. (5)] are given by (see e.g. [5])

$$\bar{R}^{(L)}_{\mu\nu}[h] \equiv D_{\mu\nu}[\bar{h}]$$
$$\equiv \tfrac{1}{2}[-\bar{h}_{\mu\nu;\alpha}{}^{\alpha} + \bar{h}_{\alpha\mu;\nu}{}^{\alpha} + \bar{h}_{\alpha\nu;\mu}{}^{\alpha} - g_{\mu\nu}\bar{h}_{\beta\alpha;}{}^{\beta\alpha}], \tag{A1}$$

$$R^{(Q)}_{\mu\nu}[h] \equiv \tfrac{1}{2}[\tfrac{1}{2}h_{\alpha\beta;\mu}h^{\alpha\beta}{}_{;\nu} + h^{\alpha\beta}(h_{\alpha\beta;\mu\nu} + h_{\mu\nu;\alpha\beta}$$
$$- 2h_{\alpha(\mu;\nu)\beta}) + 2h_{\nu}{}^{\alpha;\beta}h_{\mu[\alpha;\beta]}$$
$$- (h^{\alpha\beta}{}_{;\beta} - \tfrac{1}{2}h^{;\alpha})(2h_{\alpha(\mu;\nu)} - h_{\mu\nu;\alpha})]. \tag{A2}$$

We also used the notation $S_{\mu\nu}[\bar{h}] \equiv -\bar{R}^{(Q)}_{\mu\nu}[h]$, where on the right-hand side $h_{\mu\nu}$ is expressed with $\bar{h}_{\mu\nu}$.

APPENDIX B: $\delta\bar{h}^{(2)}_{\mu\nu}$ SATISFIES LORENZ-GAUGE CONDITIONS

Here we show that the Lorenz-gauge conditions are indeed satisfied by the retarded solution (44). For this purpose we follow a standard method of deriving differential equations for $\nabla^\nu \delta\bar{h}^{(2)}_{\mu\nu}$. By applying divergence operator to Eq. (43) and using a contraction of Bianchi identities together with the fact that background geometry is a vacuum space-time, we obtain

$$\Box(\nabla^\nu \delta\bar{h}^{(2)}_{\mu\nu}) = -2\nabla^\nu \delta S^F_{\mu\nu}. \tag{B1}$$

We assume that Lorenz-gauge conditions (42) are satisfied on an initial spacelike hypersurface Σ_I and moreover that $[(n^\alpha \nabla_\alpha)\nabla^\nu \delta\bar{h}^{(2)}_{\mu\nu}]_{\Sigma_I} = 0$ where n^α is normal to Σ_I. We shall now show that the retarded solution of Eq. (B1) vanishes, and therefore $\delta\bar{h}^{(2)}_{\mu\nu}$ satisfies the Lorenz-gauge conditions as required.

First, we will show that the source of Eq. (B1) vanishes for $x \notin z_G(\tau)$. Consider a metric $\hat{g}_{\mu\nu}$ that depends on a small parameter μ, and may be expanded as follows

$$\hat{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \mu g^{(1)}_{\mu\nu}(x) + \mu^2 g^{(2)}_{\mu\nu}(x) + O(\mu^3). \tag{B2}$$

Here $\hat{g}_{\mu\nu}(x)$ is not necessarily a solution of Einstein's field equations in vacuum, whereas $g_{\mu\nu}$ maintains its definition as a vacuum solution to Einstein's field equations. We now employ Bianchi identities reading

$$\hat{g}^{\alpha\mu}\hat{\nabla}_\alpha \hat{G}_{\mu\nu} \equiv 0. \tag{B3}$$

Here the contravariant metric satisfies $\hat{g}^{\alpha\mu}\hat{g}_{\mu\beta} = \delta^\alpha_\beta$, $\hat{\nabla}_\mu$ denotes the covariant derivative with respect to $\hat{g}_{\mu\nu}$, and $\hat{G}_{\mu\nu}$ is Einstein tensor evaluated with this metric. We now employ decomposition (B2) to formally expand Einstein tensor, and the covariant derivative (for rank-2 tensors), giving

$$\hat{G}_{\mu\nu} = G^{(0)}_{\mu\nu} + \mu G^{(1)}_{\mu\nu} + \mu^2 G^{(2)}_{\mu\nu} + O(\mu^3), \tag{B4}$$

$$\hat{g}^{\alpha\mu}\hat{\nabla}_\alpha = \nabla^\mu + \mu\Gamma^\mu_1 + \mu^2\Gamma^\mu_2 + O(\mu^3). \tag{B5}$$

In these expansions the dependence on μ is only through the explicit powers μ^i, Γ^μ_1, and Γ^μ_2 denote linear operators (defined on rank-2 tensors), whose explicit form is not required here. We now substitute Eqs. (B4) and (B5) into (B3) and obtain a perturbative expansion of Bianchi identities. The Bianchi identities are valid for any value of μ and therefore the individual terms in their expansion in powers of μ vanish identically, yielding the following set of identities for arbitrary tensor fields $g^{(1)}_{\mu\nu}$ and $g^{(2)}_{\mu\nu}$

$$\nabla^\mu D_{\mu\nu}[\bar{g}^{(1)}] \equiv 0, \tag{B6}$$

$$\nabla^\mu G^{(2)}_{\mu\nu} + \Gamma^\mu_1[D_{\alpha\beta}[\bar{g}^{(1)}]] \equiv 0. \tag{B7}$$

Here we denoted

$$G^{(2)}_{\mu\nu} \equiv D_{\mu\nu}[\bar{g}_2] - S_{\mu\nu}[\bar{g}_1] + \tfrac{1}{2}R^{(L)}_{\alpha\beta}[g_1](g^{(1)\alpha\beta}g_{\mu\nu}$$
$$- g^{(1)}_{\mu\nu}g^{\alpha\beta}). \tag{B8}$$

Employing Eqs. (11) and (B6)–(B8) we find that the source term of Eq. (B1) vanishes for $x \notin z_G(\tau)$.

Next, we show that the source of Eq. (B1) on the world line is too weak to produce a nonvanishing $\nabla^\nu \delta\bar{h}^{(2)}_{\mu\nu}$. We shall now estimate the strength of the source Eq. (B1) on the world line. Consider a hypersurface of constant time, generated by spacelike geodesics which are normal to $z_G(\tau)$. In this hypersurface we consider a small sphere $D(\epsilon)$ of radius ϵ, centered at $r = 0$; and calculate the following three-dimensional volume integral over $\nabla^\nu \delta S_{\mu\nu}$ inside this sphere, reading

$$\int_{D(\epsilon)} \bar{g}^{\mu'}_\mu(z_G, x')\nabla^{\nu'}\delta S^F_{\mu'\nu'}dV'. \tag{B9}$$

If this integral vanishes then the strength of the source term on the world line is weaker than a delta-function source term, and it is too weak to produce a nonvanishing $\nabla^\nu \delta\bar{h}^{(2)}_{\mu\nu}$. Recall that $\nabla^\nu \delta S^F_{\mu\nu}$ vanishes for $r \neq 0$. Therefore, we may take the limit $\epsilon \to 0$ without changing the value of this integral. Explicit expression of $\nabla_\nu \delta S^{F\nu}_\mu$ reads

$$\nabla_\nu \delta S^{F\nu}_\mu = (-g)^{-1/2}\frac{\partial}{\partial x^a}(\delta S^{Fa}_\mu \sqrt{-g}$$
$$+ (-g)^{-1/2}\frac{\partial}{\partial x^0}(\delta S^{F0}_\mu \sqrt{-g}) - \frac{1}{2}g_{\nu\rho,\mu}\delta S^{F\nu\rho}. \tag{B10}$$

We now substitute Eq. (B10) into Eq. (B9), and evaluate this integral using Fermi normal coordinates, based on the world line. Notice that in these coordinates $\bar{g}^{\mu'}_\mu(z_G, x') = \delta^{\mu'}_\mu + O(r^2)$, dV scales like r^2, while the second and third terms in Eq. (B10) scale like r^{-2} and r^{-1}, respectively. We therefore find that at the limit $\epsilon \to 0$ the integral (B9) over the second and third terms in Eq. (B10) vanishes. Substituting the first term in Eq. (B10) into integral (B9) and using Gauss theorem we find that at for small values of ϵ the integral (B9) is approximated by

$$\oint_{\partial D(\epsilon)} \delta S^{Fa'}_\mu d\Sigma_{a'}. \tag{B11}$$

Here $\partial D(\epsilon)$ is the surface of the sphere. Consider an expansion of $\delta S^{Fa'}_\mu$ in powers of r in the vicinity of $r = 0$. Here only terms which scale like r^{-2} have the potential of producing a nonvanishing integral at the limit $\epsilon \to 0$. Using the schematic form (27) one finds that only terms of the form $\bar{h}^{(1)S}\nabla\nabla\bar{h}^{(1)S}$ and $\nabla\bar{h}^{(1)S}\nabla\bar{h}^{(1)S}$ produce terms in

$\delta S^{Fa'}_\mu$ which scale like r^{-2}. We now consider an expansion of $\bar{h}^{(1)S}$ and its derivatives, see Eqs. (30)–(32) for the leading terms in these expansions. Examining these equations we see that these expansions depend on dimensionless quantities of the form u^μ, Ω^μ, $\eta_{\mu\nu}$, higher-order terms in these expansions also include the Riemann tensor and its derivatives. Dimensional analysis implies that the terms in $\delta S^{Fa'}_\mu$ which scale like r^{-2} must be linear in Riemann tensor. The integral in Eq. (B9) provides us with a vector at $r = 0$. When integrating over the above mentioned local expansions we find that this vector must be composed of Riemann tensor u^μ and the background metric. However, in a vacuum background space-time one cannot construct from these quantities a nonvanishing vector. Therefore the integral in Eq. (B9) vanishes.

In the above calculations we showed that the source of Eq. (B1) vanishes for $x \notin z_G(\tau)$, and furthermore that a volume integral over this source, which includes the world line vanishes as well. We therefore find (with the above mentioned initial conditions) that the retarded solution of Eq. (B1) vanishes, and therefore $\delta\bar{h}^{(2)}_{\mu\nu}$ satisfies the Lorenz-gauge conditions.

APPENDIX C: $\delta\bar{h}^{(2)}_{\mu\nu}$ IS BOUNDED AS $r \to 0$

We show that the retarded potential $\delta\bar{h}^{(2)}_{\mu\nu}$ given by Eq. (44) is bounded as $r \to 0$. Recall that the source term in Eq. (43) diverges like r^{-2} as $r \to 0$. Therefore, the integral in Eq. (44) converges for $x \notin z_G(\tau)$. In particular $\delta\bar{h}^{(2)}_{\mu\nu}$ is finite on the surface of a world tube which surrounds the world line at a fixed spatial distance $r = r_B$, where $r_B \ll \mathcal{R}$. We now consider the solution of Eq. (43) within this world tube. By virtue of the smallness of $r\mathcal{R}^{-1}$ within this world tube, Eq. (43) can be solved iteratively using the following expansions of $\delta\bar{h}^{(2)}_{\mu\nu}$ and $g_{\mu\nu}$

$$\delta\bar{h}^{(2)}_{\mu\nu} = \delta\bar{h}^{(2)}_{(0)\mu\nu} + \mathcal{R}^{-1}\delta\bar{h}^{(2)}_{(1)\mu\nu} + \mathcal{R}^{-2}\delta\bar{h}^{(2)}_{(2)\mu\nu} + O(\mathcal{R}^{-3}), \tag{C1}$$

$$g_{\mu\nu} \overset{*}{=} \eta_{\mu\nu} + O(\mathcal{R}^{-2}). \tag{C2}$$

Here again we employ Fermi normal coordinates based on $z_G(\tau)$. Note that the time scale in which the source term of Eq. (43) changes is of $O(\mathcal{R})$ and therefore the leading term $\delta\bar{h}^{(2)}_{(0)\mu\nu}$ satisfies the following equation

$$(\delta^{ab}\partial_a\partial_b)\delta\bar{h}^{(2)}_{(0)\mu\nu} \overset{*}{=} -2\delta S^F_{\mu\nu}. \tag{C3}$$

Here x^a, x^b denote the spatial Fermi normal coordinates. Equation (C3) is a set of Poisson's equations for each tensorial component of $\delta\bar{h}^{(2)}_{(0)\mu\nu}$. To solve these equations we decompose into spherical harmonics centered at $r = 0$, reading

$$\delta \bar{h}^{(2)}_{(0)\mu\nu} \stackrel{*}{=} \sum_{lm} Y^{lm} \phi^{lm}_{\mu\nu},$$

$$-2\delta S^F_{\mu\nu} \stackrel{*}{=} \sum_{lm} Y^{lm} \rho^{lm}_{\mu\nu}.$$

The solution for each spherical harmonics component is given by

$$\phi^{lm}_{\mu\nu}(r) \stackrel{*}{=} -\frac{1}{2l+1} \int_0^{r_B} \frac{r_<^l}{r_>^{l+1}} r'^2 \rho^{lm}_{\mu\nu}(r') dr' + \text{B.T.} \quad \text{(C4)}$$

Here $r_>$ and $r_<$ are the larger and smaller terms from the pair $\{r, r'\}$, respectively; B.T. denotes finite boundary terms coming from the contribution of the surface of the world tube. Expanding $\rho^{lm}_{\mu\nu}$ in a power series gives

$$\rho^{lm}_{\mu\nu}(r) \stackrel{*}{=} a^{lm}_{\mu\nu(-2)} r^{-2} + a^{lm}_{\mu\nu(-1)} r^{-1} + O(r^0).$$

Equation (C4) implies that in the expansion of $\rho^{lm}_{\mu\nu}$ only the term $a^{00}_{\mu\nu(-2)} r^{-2}$ gives rise to a (logarithmic) divergency in $\phi^{lm}_{\mu\nu}$, while all the other terms produce a bounded potential at $r = 0$. Using the schematic expression (27) one finds that the terms in the source of Eq. (43) that can possibly contribute to the problematic term $a^{00}_{\mu\nu(-2)} r^{-2}$, are of the form $\bar{h}^{(1)S} \nabla\nabla \bar{h}^{(1)S}$ and $\nabla\bar{h}^{(1)S} \nabla\bar{h}^{(1)S}$. These terms can be expanded in the vicinity $z_G(\tau)$ using expansions (30)–(32) evaluated to a higher accuracy (see discussion at the end of Appendix B). Dimensional analysis implies that $a^{00}_{\mu\nu(-2)}$ has to be proportional to a component of Riemann tensor. Since we assumed a vacuum background space-time the only possible nonvanishing candidate for $a^{00}_{\mu\nu(-2)}$ is $R_{\mu\alpha\nu\beta} u^\alpha u^\beta$ times a constant. Explicit calculation (using MATHEMATICA software) of the constant coefficient of this $l = 0$ term shows that it vanishes, which implies that $\delta \bar{h}^{(2)}_{(0)\mu\nu}$ is bounded at $r = 0$. The higher-order corrections to $\delta \bar{h}^{(2)}_{(0)\mu\nu}$ given by Eq. (C1) are smaller than the leading term by at least a factor of $r\mathcal{R}^{-1}$, and are therefore bounded as well. We conclude that $\delta \bar{h}^{(2)}_{\mu\nu}$ is bounded as $r \to 0$.

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[31] To construct this continuation consider the future null cones Σ_τ emanating from the world line $z_G(\tau)$. Here we focus on a local neighborhood of the world line in which these null cones do not intersect each other. For an arbitrary point $z_G(\tau^-)$ one may choose an arbitrary continuation of ξ^μ on Σ_{τ^-}, such that ξ^μ decays to zero away from the world line. In this way the constructed gauge preserve causality in the following sense: The perturbations on the null cones Σ_τ for $\tau \le \tau^-$ will remain unchanged if one modifies the world line for $\tau > \tau^-$. Such a modification of the world line is possible by introducing additional GW that interact with the world line for $\tau > \tau^-$.