PHYSICAL REVIEW D **73,** 044029 (2006)

Quantum motion of a neutron in a waveguide in the gravitational field

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(Received 15 December 2005; published 23 February 2006)

We study theoretically the quantum motion of a neutron in a horizontal waveguide in the gravitational field of the Earth. The waveguide in question is equipped with a mirror below and a rough surface absorber above. We show that such a system acts as a quantum filter, i.e. it effectively absorbs quantum states with sufficiently high transversal energy but transmits low-energy states. The states transmitted are determined mainly by the potential well formed by the gravitational field of the Earth and the mirror. The formalism developed for quantum motion in an absorbing waveguide is applied to the description of the recent experiment on the observation of the quantum states of neutrons in the Earth's gravitational field.

DOI: 10.1103/PhysRevD.73.044029 PACS numbers: 04.80.Cc, 04.25.Nx

I. INTRODUCTION

Although the solution of the problem of the quantization of particle motion in a well, formed by a linear potential and ideal mirror, has been known for a long time [1–6] the experimental observation of such a phenomenon in the case of a gravitational field is an extremely challenging task.

The electric neutrality of neutrons [7–11] is an advantage for this kind of research. Thus, in earlier experiments the use of cold neutrons has allowed the gravitationally induced phase-shift of neutrons to be measured [12–16].

The direct observation of the lowest quantum states of neutrons in the Earth's gravitational field above a mirror has become possible recently. The experiment consists of the measurement of the neutron flux through a slit between a mirror and an absorber (scatterer) as a function of the slit size. Slit size could be finely adjusted and precisely measured. The neutron flux in front of the experimental installation (in Fig. 1 on the left) is uniform over height and isotropic over angle. A low-background detector measures the neutron flux at the exit (in Fig. 1 on the right). The main aim of this experiment was to demonstrate, for the first time, the existence of the quantum states of matter in a gravitational field. The detailed description of the experiment and a discussion of its reliability and precision can be found in Refs. [17–27].

The gravitationally bound quantum states of neutrons and the related experimental techniques provide a unique tool for a broad range of investigations in the fundamental physics of particles and fields. These include the equivalence principle tests in the quantum domain as well as short-range fundamental forces studies [21,28–35] and the study of the foundations of quantum mechanics [36,37]. The experiment on neutron gravitational quantum states stimulated progress in surface studies (see, for in-

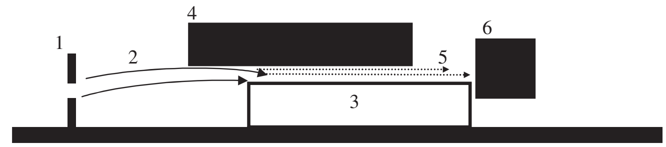

FIG. 1. Schematic view of the experiment. From left to the right: the vertical bold lines indicate the upper and lower plates of the input collimator (1); the solid arrows correspond to classical neutron trajectories (2) between the input collimator and the entry slit between a mirror [(3), empty rectangle below] and a scatterer [(4), black rectangle above]. The dotted horizontal arrows illustrate the quantum motion of neutrons above a mirror (5), and the black box represents the neutron detector (6).

stance, [38,39]). A short overview of the applications can be found in [40].

These studies require clear understanding of the quantum-mechanical problem of neutron passage through an absorbing waveguide in the presence of gravitational field. Here we develop a theoretical model of neutron quantum motion in such a waveguide.

In Sec. II we summarize the main known facts about a solution of the quantum-mechanical problem for a particle in the potential well formed by a linear potential and an ideal horizontal mirror. In Sec. III we discuss the main principles of observation of neutron quantum states using the absorbing waveguide. We show that such a waveguide turns out to be a quantum filter, which absorbs states with high transversal energy and transmits low-energy states. These transmitted states are mainly determined by the potential well formed by the gravitational field and the mirror.

The latter condition is a specific feature of our problem, which, to our knowledge, has not been explicitly considered in the literature (see, for instance, Refs. [41–45] and the references therein, devoted to the theory of the interaction of waves with rough surfaces). Section IV is devoted to the passage of neutrons through the waveguide with a flat neutron absorber, as proposed in [7,8] and Sec. V to their passage with a rough absorber, as proposed in [17]. We examine several models for the mechanism of neutron loss as a result of their interaction with an absorber and discuss the limits of their validity.

The final chapter summarizes the conclusions. The results obtained are rather general in character and can be applied to different physical problems, involving the transmission of quantum particles through absorbing waveguides.

II. QUANTUM BOUNCING ABOVE MIRROR IN THE GRAVITATIONAL FIELD

Although the results of this section can be found in the handbooks [1–3] it is convenient to have them at our disposal here. We start with a well-known problem of a particle bouncing in the gravitational field above a perfect reflecting mirror. In the following we consider $\hbar = 1$. The characteristics for this problem energy scale ε_0 and length scale l_0 are

$$\varepsilon_0 = \sqrt[3]{mg^2/2}, \tag{1}$$

$$l_0 = \sqrt[3]{1/(2m^2 g)}. \tag{2}$$

Here m represents the particle mass and g free fall acceleration. In the case of neutrons, which will interest us below, these quantities are

$$\varepsilon_0 = 0.602\ \text{peV}, \qquad l_0 = 5.871\ \mu m.$$

The Schrödinger equation, which governs the wave function of the neutron, confined between mirror and gravitational field is

$$-\frac{d^2\varphi_n(\xi)}{d\xi^2} + \xi\varphi(\xi) = \lambda_n\varphi(\xi),$$

where dimensionless variable ξ is connected with the distance variable z via $\xi = z/l_0$, while quantum number λ_n determines the energy values $\varepsilon_n = \varepsilon_0\lambda_n$. The obvious boundary conditions are

$$\varphi_n(0) = 0, \tag{3}$$

$$\varphi_n(\infty) = 0. \tag{4}$$

The wave functions which satisfy the equations above are known to be

$$\varphi_n(\xi) \sim \text{Ai}(\xi - \lambda_n); \tag{5}$$

here Ai is the Airy function [46]. Substitution of (5) into (3) gives the equation for the eigenvalues λ_n:

$$\text{Ai}\,(-\lambda_n) = 0. \tag{6}$$

The semiclassical (WKB) expression for the eigenvalues is

$$\lambda_n^{\text{WKB}} = \left(\frac{3\pi}{4}(2n - 1/2)\right)^{2/3}. \tag{7}$$

This approximation gives the eigenvalues with accuracy to a few percent even for the lowest n.

The asymptotic behavior of the gravitational states' wave functions in the classically forbidden region $\xi \gg \lambda_n$ is characterized by very fast decay:

$$\text{Ai}\,(\xi - \lambda_n) \sim \exp(-2/3(\xi - \lambda_n)^{3/2}). \tag{8}$$

The fast decay of the wave functions under the gravitational barrier allows us to introduce a well-defined characteristic distance $H_n = l_0\lambda_n$ of a given state, which corresponds to the classical turning point $H_n = E_n/(Mg)$ of a bouncing particle with a given energy. Thus the quantization of energy E_n is reflected in spatial distribution of the neutron density in the above-mentioned states (here-

TABLE I. Eigenvalues, gravitational energies, and classical turning points of neutrons in the Earth's gravitational field above a mirror.

n	λ_n	λ_n^{WKB}	E_n, peV	H_n, μm
1	2.338	2.320	1.407	13.726
2	4.088	4.082	2.461	24.001
3	5.521	5.517	3.324	32.414
4	6.787	6.784	4.086	39.846
5	7.944	7.942	4.782	46.639
6	9.023	9.021	5.431	52.974
7	10.040	10.039	6.044	58.945

after referred to as gravitational states). The scanning of this ''quantized'' spatial distribution of neutron density can be used to observe neutron quantum motion experimentally in the gravitational field.

In Table I we present the first seven eigenvalues λ_n, their WKB approximation $\lambda_n^{\rm WKB}$ together with the corresponding energy values $E_n = \varepsilon_0 \lambda_n$, and classical turning points $H_n = l_0 \lambda_n$.

III. THE PRINCIPLE FOR OBSERVATION OF THE QUANTUM GRAVITATIONAL STATES

Here we discuss only the principle of the experimental observation of neutron gravitational states based on the concept of neutron tunneling through the gravitational barrier, which separates the classically allowed region and the absorber position [20,21].

A flux of neutrons with horizontal velocity V (from 4 to 10 m/s) was driven through a slit of variable height between a perfect horizontal mirror and a highly efficient absorber placed parallel to the mirror. The length L of the waveguide (which varied in different measurements from $L = 10$ to $L = 20$ cm) determined the neutron passage time $\tau^{\rm pass} = L/V \simeq 2\ 10^{-2}$ s. It was found that when the slit height H was smaller than the height of the first gravitational state H_1 (see Table I) the flux of neutrons passing through the slit was indistinguishable from the background. As soon as the absorber position was set above H_1 a rapid increase in the flux of neutrons was observed. An analogous increase, though less resolved, was observed for the slit heights close to the characteristic state height H_2. This ''steplike'' dependence faded almost completely for higher positions of the absorber, where the flux increased practically monotonously.

We will show here that such behavior of the neutron flux detected at the exit of the waveguide is what one would expect from the qualitative treatment of neutron quantum motion in the gravitational field. In fact, the transversal motion of neutrons in the waveguide can be described as a superposition of the neutron waveguide transversal modes:

$$\Phi(z, t) = \sum_n C_n \psi_n(z) \exp(-iE_n t - \Gamma_n t/2).$$

Here $\psi_n(z)$ represents the transversal state wave functions, E_n the transversal self-energies, and Γ_n the widths of these states due to the neutron interaction with an absorber. The neutron flux, detected at the exit of the waveguide is

$$F = \int_0^\infty |\Phi(z, \tau^{\rm pass})|^2 dz.$$

The WKB approach can be proposed for the estimation of the widths of transversal states:

$$\Gamma_n = P_n \omega_n, \tag{9}$$

where P_n is the probability of absorption of a neutron with energy E_n by an absorber during a ''one-time collision,'' while ω_n is the frequency of these collisions. The classical expression connecting the frequency of the bouncing particle and its classical turning point H_n is

$$\omega_n = \frac{1}{2} \varepsilon_0 \sqrt{\frac{l_0}{H_n}}. \tag{10}$$

We will use the following simple model for P_n. Namely, we will consider $P_n = 1$ when the absorber height H is below or equal H_n, so that a neutron can ''touch'' an absorber while it bounces above the mirror in the nth state. If $H > H_n$ the probability is equal to the probability of tunneling through the gravitational barrier $P = D(E_n, H)$ [47]. Such a probability has the following form in cases where $H \gg H_n$:

$$P_n = D(E_n, H) \sim \exp[-\tfrac{4}{3}((H - H_n)/l_0)^{3/2}]. \tag{11}$$

The spectrum of transversal states depends on the position of absorber H. As long as $H > H_n$ the first n states can have a long enough lifetime to pass through the waveguide:

$$\tau_n^{\rm long} = \frac{1}{\omega_n} \exp(4/3(H/l_0 - \lambda_n))^{3/2}. \tag{12}$$

The lifetime of all other states with $E > E_n$ is approximately equal to the classical time of flight of the particle with energy E from the mirror to the absorber. We consider that such a lifetime is short compared to the passage time $\tau^{\rm pass}$ through the waveguide (which is ensured by the choice of length of the waveguide L and the horizontal flux velocity V) and their contribution to the detected flux is small as far as $\tau^{\rm short} \ll \tau^{\rm pass}$.

The measured neutron flux is

$$F \simeq \sum_{n=1}^N |C_n|^2 \exp(-\tau^{\rm pass}/\tau_n^{\rm long}). \tag{13}$$

Thus the measured flux exhibits a fast increase when absorber position H is set close to H_n due to the exponential increase of the nth state lifetime (12), which enables the passage of neutrons in such a state through the waveguide. (A more accurate expression which includes the interference effects between decaying states will be obtained in a later section.) The expression presented above was used to fit the experimental data:

$$F(H) = \sum_{n=1}^N A_n \exp(-\tau^{\rm pass}/\tau_n^{\rm long}(H)), \tag{14}$$

where $\tau_n^{\rm long}(H)$ are defined by expression (12) with λ_n used as free parameters, while $A_n = |C_n|^2$ were used to fit the ''initial populations'' of transversal states. The fitted values of λ_n (taking into account the final accuracy of the height calibration) are in agreement with the expectation, as given in Table I. The values of A_n turned out to be equal, except $A_1 \simeq 0.7 A_n$ with $n \geq 2$. The reason for the approximate equality of the ''initial populations'' will be discussed in a

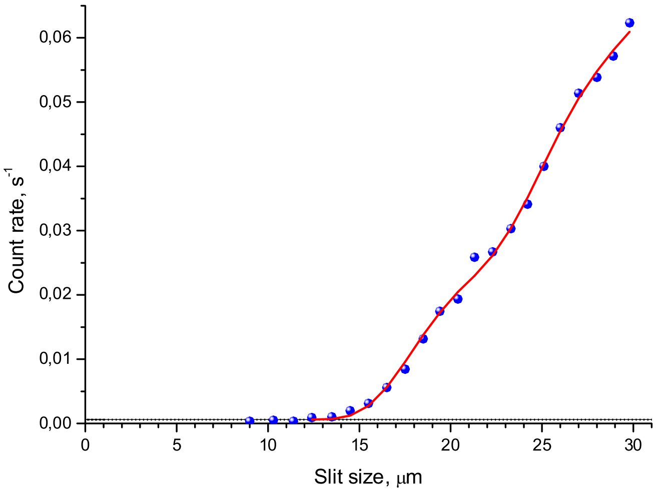

FIG. 2 (color online). Neutron count rate at the exit of the waveguide as a function of absorber position. Circles correspond to the experimental data, solid line corresponds to the theoretical fit.

later section. In Fig. 2 we show the experimental data (2002 yr run [20]) and the results of the fit.

When $H \simeq H_N$ and $N \gg 1$ a large number of states passes through the waveguide. This number can be found from the WKB expression for eigenvalues (7):

$$N^{\mathrm{WKB}} = \frac{2}{3\pi}\left(\frac{H}{l_0}\right)^{3/2} + 1/4. \tag{15}$$

Thus for big $N \gg 1$ the detected flux as a function of H turns out to be

$$F(H) \sim (H/l_0)^{3/2}. \tag{16}$$

The above-mentioned WKB expression describes well the flux behavior already for $N > 5$. The deviation of the measured flux from the above expression for small N is due to the quantum character (14) of the neutron motion in the gravitational field of Earth. Such a deviation (see Fig. 2) is clearly seen for the first state when quantum formula exhibit distinct threshold behavior at $H = H_1$. However the experimental possibility of resolving higher quantum states is restricted by the penetrability of the gravitational barrier. In fact the best resolution of the gravitational quantum states is achieved when the flux (13) has a steplike dependence on H. This means that the transition factor for a given state $\exp(-\tau^{\mathrm{pass}}/\tau_n^{\mathrm{long}}(H))$ changes from the small value to unity in the range of absorber positions $H = \tilde{H}_n \pm \delta_n$. The rate of such an increase is limited by the penetration probability through the gravitational barrier $D(E, H)$ (11).

For the clear resolution of different quantum states one needs $\delta_n \ll \tilde{H}_{n+1} - \tilde{H}_n$. Under the conditions of our experiment $\delta \simeq l_0$ and $\tilde{H}_n \simeq H_n$. Such an estimation shows that for the highly excited gravitational states (with practically $N \geq 5$) the difference $H_{n+1} - H_n$ becomes comparable with the uncertainty δ and thus the steplike behavior of the flux is suppressed. To go beyond the above-mentioned qualitative predictions of the resolution of gravitational states one needs to take into account details of the interaction of the neutron and the absorber. We will return to the discussion of the problem of the resolution of excited gravitational states in a later section.

IV. FLAT ABSORBER

The simplest approach in which the properties of the absorber could be taken into account is a model of a flat absorber, characterized by the *complex* Fermi potential. The simplification of the theory in the case of a *flat* absorber is due to the fact that, in such cases, motion in a transversal direction is independent of motion in a longitudinal direction within the waveguide.

A. Passage of the neutron through an absorbing waveguide

The Schrödinger equation, which governs the wave function $\Phi(x, z)$ of the neutron with total energy E passing through the waveguide is

$$\left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} - \frac{1}{2m}\frac{\partial^2}{\partial z^2} + mgz + V(H, z) - E\right]\Phi(x, z) = 0. \tag{17}$$

Here x is the longitudinal variable, z is the transversal variable, and $V(H, z) = V_1(H, z) - iV_2(H, z)$ is the *complex* Fermi potential of the absorber dependent on the absorber position H.

It is convenient to introduce the transversal states $\psi_n(z)$, which are the eigenstates of the transversal Hamiltonian:

$$\left[-\frac{1}{2m}\frac{\partial^2}{\partial z^2} + mgz + V(H, z) - (\varepsilon_n(H) - i\Gamma_n(H)/2)\right] \times \psi_n(z) = 0, \tag{18}$$

where $\varepsilon_n(H) - i\Gamma_n(H)/2$ are the *complex* energy eigenvalues, dependent on absorber position H. It is worth noting that due to the presence of absorption [i.e. the imaginary component of $V(z, H)$], the above-mentioned transversal Hamiltonian is no longer self-adjoint. As a consequence, the eigenfunctions $\psi_n(z)$ are substantially complex and obey the bi-orthogonality condition (see [48] and references therein):

$$\int_0^\infty \psi_k(z)\psi_n(z)dz = \delta_{kn}. \tag{19}$$

As one can see, the above expression differs from the standard orthogonality condition in the absence of complex conjugation.

From the qualitative treatment of the previous section one can expect that the lifetime of the neutron in a transversal state $\psi_n(z)$ strongly depends on the absorber position H. The first n lowest states such that $H_n \ll H$ are weakly affected by the absorber and practically coincide with gravitational states (5). Their lifetime is large compared to the passage time $\tau^{\rm pass}$. The states with $H_n \gg H$ are strongly distorted by the absorber. We will show that the corresponding lifetimes are short in comparison with $\tau^{\rm pass}$ and these states totally decay before reaching the detector. Consequently, only states with rather small transversal energy and thus small width have a chance of exiting the waveguide. When the absorber position H is reaching one of the characteristic classical turning points H_n the corresponding state lifetime (and the waveguide transition factor) undergoes fast changes with H, which allows us to monitor this quantum state in the overall flux at the exit of the waveguide. To calculate the transition factors for the given state we expand the two-dimensional wave function $\Phi(x, z)$ in the set of basis functions $\psi_n(z)$:

$$\Psi(x, z) = \sum_n \chi_n(x)\psi_n(z). \tag{20}$$

The functions $\chi_n(x)$ play the role of longitudinal wave functions of neutrons in transversal state n and can be found by substitution of (20) into the Schrödinger Eq. (17) with the use of (19):

$$\left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + \varepsilon_n(H) - i\Gamma_n(H)/2 - E\right]\chi_n(x) = 0. \tag{21}$$

The solutions of (21) corresponding to the quasifree longitudinal motion of neutrons in the waveguide are

$$\chi_n(x) \sim \exp(ip_n x). \tag{22}$$

Here $p_n = \sqrt{2m(E - \varepsilon_n(H) + i\Gamma_n(H)/2)}$ is the *complex* longitudinal momentum.

As we have already mentioned, only states with small transversal energy can reach the detector. In our case the full energy is much greater than the transversal energies:

$$|\varepsilon_n(H) + i\Gamma_n(H)/2)| \ll E.$$

Thus we can write for momentum p_n:

$$p_n \simeq \sqrt{2mE}\left(1 - \frac{\varepsilon_n(H) - i\Gamma_n(H)/2}{2E}\right) = P - \frac{\varepsilon_n(H) - i\Gamma_n(H)/2}{V},$$

where $P = \sqrt{2mE}$ and $V = P/m$. Because of the positive imaginary part of p_n, each of the longitudinal wave function decays exponentially with x inside the waveguide:

$$\chi_n(x) \sim \exp\left(i\frac{-\varepsilon_n(H)x + i\Gamma_n(H)x/2}{V}\right)\exp(iPx).$$

Taking into account that $\tau^{\rm pass} = L/V$, we obtain for the wave function $\Psi(x = L, z)$ at the exit of the waveguide:

$$\Psi(x = L, z) = \exp(iPL)\sum_n C_n \exp(-i\varepsilon_n(H)\tau^{\rm pass}) \times \exp\left(-\frac{\Gamma_n(H)\tau^{\rm pass}}{2}\right)\psi_n(z). \tag{23}$$

Here C_n are expansion factors, determined by the particular form of the wave function at the entrance of the waveguide:

$$C_n = \int_0^\infty \Psi(x = 0, z)\psi_n(z)dz. \tag{24}$$

B. Expansion factors

It is worth mentioning that strictly speaking $|C_n|^2$ cannot be interpreted as the initial population of the certain state $\psi_n(z)$. In fact, as long as the standard orthogonality condition is not valid for the eigenfunctions of the not-self-adjoint Hamiltonian $\langle\psi_n|\psi_k\rangle \neq \delta_{nk}$, we have

$$\sum_n |C_n|^2 \neq \int_0^\infty |\Psi(x = 0, z)|^2 dz.$$

Consequently, to find the measured neutron flux at the exit of the waveguide one has to take into account that

$$\begin{aligned} F &= \int_0^\infty |\Psi(x = L, z)|^2 \\ &= \sum_{n,k} C_n^* C_k \langle\psi_n|\psi_k\rangle \exp(-i(\varepsilon_k - \varepsilon_n)\tau^{\rm pass}) \\ &\quad \times \exp\left(-\frac{(\Gamma_n + \Gamma_k)\tau^{\rm pass}}{2}\right) \\ &\neq \sum_n |C_n|^2 \exp(-\Gamma_n \tau^{\rm pass}). \end{aligned} \tag{25}$$

The appearance of the interference terms $C_n^* C_k\langle\psi_n|\psi_k\rangle$ is not surprising. In fact, the states $|\psi_k\rangle$ are *not stationary* states with certain energy. Because of final decay width these states are in fact time dependent and can be expressed as superpositions of *stationary* states with *certain* energy. The contribution of the mentioned interference terms to the flux can be interpreted as oscillating in time transitions with frequency $\omega_{nk} = \varepsilon_n - \varepsilon_k$ between the true stationary states.

However, for the observation of the interference terms above, a rather narrow distribution of neutrons is required in the *longitudinal* velocity. Should such longitudinal velocity distribution be broad, the interference terms are canceled after averaging over such a distribution and the "standard" expression for the flux is restored:

$$F = \sum_n |C_n|^2 \exp(-\Gamma_n \tau^{\rm pass}). \tag{26}$$

Indeed, the interference terms $C_n^* C_k$ appear in the expression for the measured flux (25) multiplied by $\exp(i\omega_{nk}\tau^{\rm pass})$. If the initial flux has distribution $f(V)$ in the longitudinal velocity, the contribution of the interference terms averaged over such a distribution would be

$$\int C_n^* C_k \langle \psi_n | \psi_k \rangle \exp(i\omega_{nk}\tau^{\mathrm{pass}}) \times \exp\left(-\frac{(\Gamma_n + \Gamma_k)\tau^{\mathrm{pass}}}{2}\right) f(V) dV.$$

In the case of broad velocity distributions, such that

$$\frac{\Delta V}{V} \geq (\tau^{\mathrm{pass}} \omega_{nk})^{-1},$$

rhe contribution of the interference terms is canceled due to the fast oscillating term $\exp(i\omega_{nk}\tau^{\mathrm{pass}})$. To observe the interference contribution between the first and second states, the velocity resolution in the conditions of our experiment should be better than 10%. This limitation is less severe for excited states.

Let us now turn to the problem of the initial "population" of the gravitational states where the initial flux has broad distribution in *transversal* momentum. In such a case [35] the modulus square of expansion coefficient $|C_n|^2$ can be found from the following equation:

$$|C_n|^2 = \int \langle \psi_n | k \rangle \langle k | \psi_n \rangle exp(-k^2/k_0^2) dk, \tag{27}$$

where k_0 is a characteristic width of the *transversal* momentum distribution and we have used "bra-ket" notation for the matrix element $\langle \psi_n | k \rangle = \int \psi_n(x) \exp(ikx) dx$.

It has been shown in [21,35,49] that if $k_0 l_0 \gg 1$ the squares of the amplitudes of the lowest states are practically equal:

$$|C_n|^2 \sim 1 - o\left(\frac{1}{k_0 l_0}\right).$$

In the conditions of our experiment the corresponding value $k_0 l_0 \simeq 50$ and thus the approximation of a unified population of lowest states is well justified.

Indeed, having in mind fast oscillations of the integrand in (27) $|C_n|^2$ becomes very small if $k > k_c$, where $k_c \simeq 1/H_n$ is the characteristic momentum of the gravitational state with spatial extension H_n. As long as the distribution over k in the initial flux is practically uniform for $k < 1/H_n \ll k_0$, the expression (27) can be rewritten as

$$|C_n|^2 = \int \langle \psi_n | k \rangle \langle k | \psi_n \rangle dk = \langle \psi_n | \psi_n \rangle = 1,$$

and we return to the statement of the uniform distribution. It is worth mentioning that the same averaging over the initial transversal momentum distribution applied to an evaluation of the interference term $C_n^* C_k$ gives

$$C_n^* C_k = \langle \psi_n | \psi_k \rangle.$$

As we have mentioned already, this matrix element is nonzero for those states which are affected by the absorber and depends on absorber position H. Given the above arguments we can rewrite the expression for the measured flux as a function of H (25) after averaging over the transversal momentum of the initial flux as

$$F(H) = \sum_{n=1}^{N} \exp(-\Gamma_n(H)\tau^{\mathrm{pass}}) + \sum_{n,k>n}^{N} 2\,\mathrm{Re}(\langle \psi_n | \psi_k \rangle^2 \times \exp(i\omega_{nk}(H)\tau^{\mathrm{pass}})) \times \exp\left(-\frac{(\Gamma_n(H) + \Gamma_k(H))\tau^{\mathrm{pass}}}{2}\right). \tag{28}$$

In the case of a broad longitudinal velocity distribution in the incoming flux, only the first term in this expression is important.

C. Transition factor

Once the expression (28) has been obtained, the problem of calculating the neutron flux at the detector position is transformed into the problem of calculating the eigen energies ε_n and their widths $\Gamma_n(H)$ of transversal states as a function of absorber position H.

The realistic Fermi-potential $V(H, z)$ of the absorber material is characterized by the depth of order 10^{-8} eV i.e much greater than the characteristic energy 10^{-12} eV of the lowest gravitational states. The diffusion radius ρ of such a potential, i.e. the distance where the strength of potential rises from zero value in the free space to its final value inside the media, is much less than the characteristic gravitational wavelength l_0. In such a case the properties of the absorber can be precisely described by one parameter, namely, the complex scattering length a, whose imaginary part accounts for the loss of neutrons due to absorption. (We use hereafter the following definition of the scattering length $a = \lim_{k\to 0}(1 - S)/(2ik)$, where k is neutron momentum and S is the reflected wave amplitude). An analytical equation for the eigen energies of neutrons bouncing in the gravitational field between mirror and absorber which is positioned at distance H above the mirror can be derived. We refer the reader to Appendix A for the details and present here the final expression for the eigenvalues $\lambda_n(H)$:

$$\frac{\mathrm{Ai}(-\lambda_n)}{\mathrm{Bi}(-\lambda_n)} = \frac{\mathrm{Ai}(H/l_0 - \lambda_n) - \tilde{a}/l_0 \mathrm{Ai}'(H/l_0 - \lambda_n)}{\mathrm{Bi}(H/l_0 - \lambda_n) - \tilde{a}/l_0 \mathrm{Bi}'(H/l_0 - \lambda_n)}. \tag{29}$$

Here $\tilde{a} = a - H$ plays the role of "the scattering length on the diffuse tail" of the potential. Let us note that this expression is valid for *any* Fermi potential with a small diffuse radius, as long as the corresponding scattering length $\tilde{a}$ is small compared to the gravitational wavelength l_0. If the position of absorber $H \gg H_n$, the right-hand side of the Eq. (29) becomes exponentially small:

$$\frac{\mathrm{Ai}(-\lambda_n)}{\mathrm{Bi}(-\lambda_n)} \simeq \frac{1}{2}\exp\left[-\frac{4}{3}((H-H_n)/l_0)^{3/2}\right] \times \left(1+2\sqrt{(H-H_n)/l_0}\,\frac{\tilde{a}}{l_0}\right).$$

One can easily recognize in the right-hand side exponent the penetration probability through the gravitational barrier. Taking into account the smallness of such a probability, one can find the correction to the gravitational eigenvalue λ_n due to the small, but nonzero, possibility of penetration under the gravitational barrier to the absorber:

$$\Delta\lambda_n = -\frac{\mathrm{Bi}(-\lambda_n^0)}{2\mathrm{Ai}'(-\lambda_n^0)}\exp\left[-\frac{4}{3}((H-H_n)/l_0)^{3/2}\right] \times \left(2\sqrt{(H-H_n)/l_0}\,\frac{\tilde{a}}{l_0}+1\right), \tag{30}$$

where λ_n^0 are unperturbed eigenvalues determined by (6). The width of the nth state due to penetration under the gravitational barrier and absorption turns out to be

$$\Gamma_n \simeq 2\frac{|\mathrm{Im}a|}{l_0}\varepsilon_0\sqrt{\frac{l_0}{H_n}}\sqrt{(H-H_n)/l_0} \times \exp\left[-\frac{4}{3}((H-H_n)/l_0)^{3/2}\right]. \tag{31}$$

We have used in the derivation of this expression the semiclassical approximation for the Airy function. The physical sense of this expression for the decay rate becomes clear after comparison with the semiclassical expressions (9). Taking into account the expression (10) for the classical frequency ω_n we can rewrite (31) as

$$\Gamma_n \simeq 4\frac{|\mathrm{Im}a|}{l_0}\omega_n\sqrt{(H-H_n)/l_0}\exp\left[-\frac{4}{3}((H-H_n)/l_0)^{3/2}\right].$$

The neutrons penetrate through the gravitational barrier into the absorber; the corresponding probability (11) is exponentially small. This probability is multiplied by the classical bouncing frequency in given state n. The properties of the absorber itself appear in the above expression through the ratio $4\frac{|\mathrm{Im}a|}{l_0}\sqrt{(H-H_n)/l_0}$. Later we will show that it coincides with a general expression for the absorption probability of slow quantum particles on the short-range absorbing potential. Thus the intuitive formula (9) we used before is justified.

One can introduce the characteristic absorption time

$$\tau_n^{\mathrm{abs}} = \frac{l_0}{2|\mathrm{Im}a|\varepsilon_0}\sqrt{\frac{H_n}{l_0}}.$$

For efficient absorption one needs $\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}} \gg 1$, which puts the following requirement for the imaginary part of the scattering length $\mathrm{Im}\,a$:

$$|\mathrm{Im}\,a| \gg \frac{l_0}{2\varepsilon_0\tau^{\mathrm{pass}}}\sqrt{\frac{H_n}{l_0}}.$$

In the conditions of our experiment ($\tau^{\mathrm{pass}} = 2\times 10^{-2}$ s) the above requirement means that

$$\mathrm{Im}\ a \gg 0.05 l_0 \sim 0.3\ \mu\mathrm{m}. \tag{32}$$

(Let us mention that at the same time the scattering length approximation used above is valid only for values of $|\tilde{a}| \ll l_0$.)

This treatment shows that for the clear resolution of quantum states the most favorable absorbers are those with the largest possible scattering length.

Let us see how the scattering length discussed above is connected to the properties of the absorber's Fermi potential, namely, its complex depth and the diffusion radius ρ. We will study the case of the *complex* potential of the Woods-Saxon type:

$$V(z,H) = \frac{U\exp(-i\varphi)}{1+\exp((H-z)/\rho)}. \tag{33}$$

In the following $U > 0$. It can be shown [3] that the scattering length on such a potential is given by

$$a = H - \frac{1}{\kappa} + 2\rho\left(\gamma + \frac{\Gamma'(1+\rho\kappa)}{\Gamma(1+\rho\kappa)}\right). \tag{34}$$

Here $\gamma \approx 0.577$ is the Euler constant, $\kappa = \exp(-i\varphi/2)\times\sqrt{2mU}$ and $\Gamma(x)$ is the Gamma function.

An important limiting case is the case of the deep complex Fermi potential, namely, $\rho|\kappa| \gg 1$ and $\rho|\mathrm{Im}\,\kappa| \gg 1$. It follows from (34) that in such a case

$$\mathrm{Im}\ a = -\rho\varphi \tag{35}$$

and is *independent* of the depth U of the complex Fermi potential. It has been shown in [50] that such behavior of the imaginary part of the scattering length is universal for deep complex potentials with the exponential tail. For such a strong absorbing Fermi potential the neutron is completely absorbed on the tail of the complex Fermi potential; the properties of the inner part of the absorber therefore lose their importance. The only way to increase the scattering length in such a limit is to increase the diffuseness ρ.

In another limiting case, when the diffuseness is so small that $\rho\kappa \ll 1$, the scattering length becomes

$$\mathrm{Im}\ a = -\mathrm{Im}\frac{1}{\kappa} - \frac{\pi^2}{3}\rho^2\kappa.$$

The leading term in the above expression is $-1/\kappa$. One can check that it coincides with the scattering length, characterizing the low-energy reflection from the steplike potential $U\exp(-i\varphi)\Theta(z)$. Indeed, in the case of $\rho\kappa \ll 1$ the neutron can penetrate through the narrow exponential tail of the complex potential into its core without significant losses. One can see from the above expression that for *weak* absorbers with depth $U \sim \varepsilon_0$ the imaginary part of the scattering length becomes as large as l_0.

It is important to mention here that the absorption of the ultracold neutrons by the complex potential is closely related to the so-called quantum reflection [51,52] of ultraslow neutrons from the fast changing complex Fermi potential. The reflection probability R in the case of slow quantum particles [53] impinging on the absorber at normal incidence with momentum k can be written as follows:

$$R = 1 - 4k|\mathrm{Im}\, a|, \tag{36}$$

while the absorption probability P is

$$P = 1 - R = 4k|\mathrm{Im}\, a|. \tag{37}$$

The smaller the ρ (and $|\mathrm{Im}\, a|$) the better the reflection and the weaker the absorption. One can see that the limit of $\kappa\rho \to 0$, $p/\kappa \to 0$ corresponds to the case when the absorber is replaced by an absolutely reflecting mirror. The above-mentioned high reflectivity of a fast changing potential is a general quantum-mechanical property.

The numerical calculations verify the above conclusions.

The values U, ρ, and ϕ of potential (33) were chosen to be $U = 10^{-8}$ eV, $\rho = 1$ μm and $\phi = 3\pi/4$ (this corresponds to attractive potential with absorption).

In Fig. 3 we plot the lifetimes τ_n of the first three states as a function of absorber position H. In Fig. 4 we show the corresponding evolution of the real part of energy $\varepsilon_n(H)$. One can see the fast increase in the lifetimes at certain values of H, close to H_n, in agreement with the qualitative predictions of the previous section. The real part of the energy quickly approaches its limiting value equal to the energy of the gravitational state when $H > H_n$.

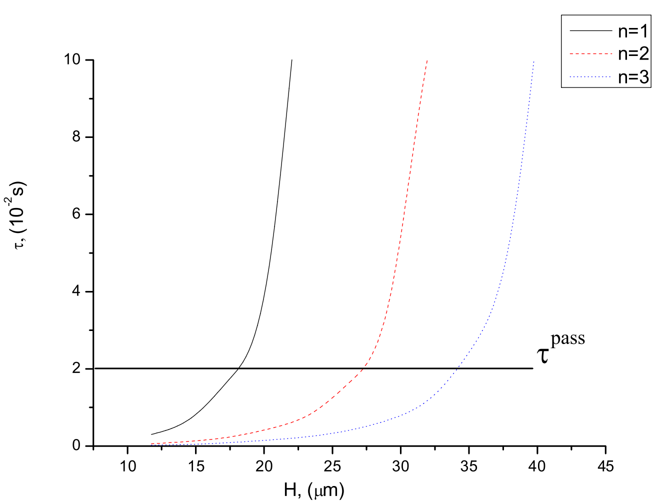

FIG. 3 (color online). Lifetime of the first three gravitational states as a function of absorber position in a typical arrangement of our experiment.

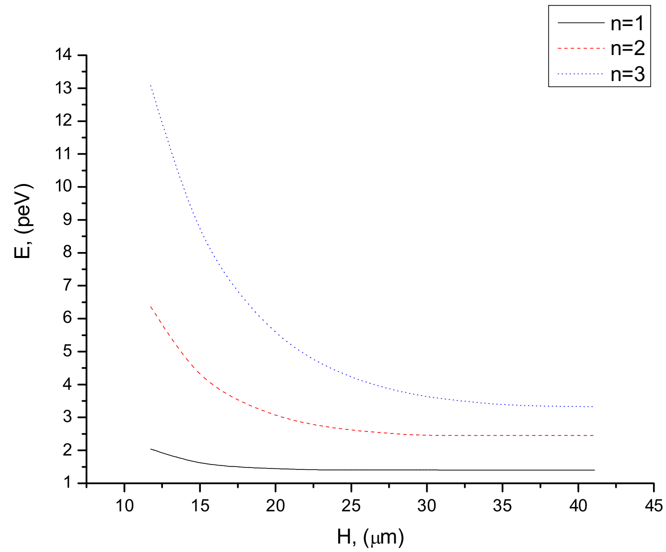

FIG. 4 (color online). Energy of the first three gravitational states as a function of absorber position in an arrangement typical for our experiment.

The fast changes in the lifetime of gravitational states as a function of absorber position are clearly seen. However, the overall plot of flux intensity, Fig. 5, where all these states are taken into account simultaneously shows that the steplike dependence is suppressed, except for the first step at $H = H_1$ and partially for the second step.

To achieve much higher absorber efficiency the diffuse radius ρ should be significantly increased. Another way is to reduce the depth of absorber Fermi potential to the level of 10^{-12} eV, the characteristic scale of gravitational states energies.

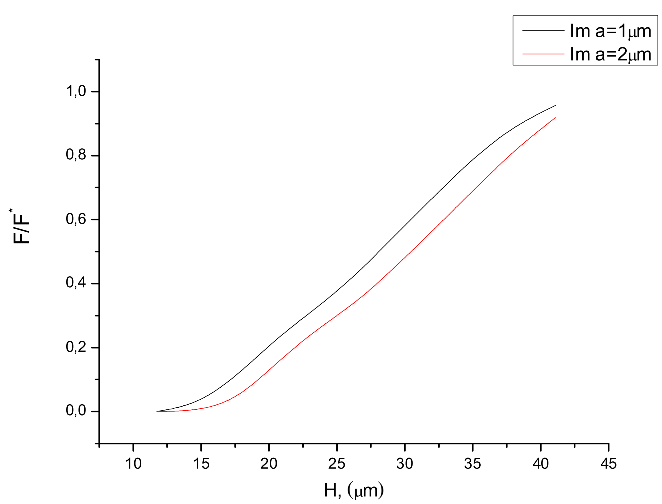

FIG. 5 (color online). The relative neutron flux as a function of absorber position for different absorber diffuseness in an arrangement typical for our experiment. F^* is the flux calculated at absorber position $H = 45$ μm and diffuseness $\rho = 1$ μm.

Absorbers with optimal parameters can be obtained if their surface is corrugated. In fact such an absorber was used in the experimental setup. The zone of such a corrugation can be considered as a low density media with an extended diffuse radius of Fermi potential. In the following we will study the neutron passage through an absorber with a rough surface. We will show, however, that the main loss mechanism in such a case is due to nonspecular reflections from the rough edges of the absorber.

D. Zero gravity experiment

We will study here the important case of the neutron passage through the waveguide formed by the mirror and the absorber in the *absence* of the gravitational field. The case is interesting from two points of view. On the one hand, a comparison of the transition factors with and without gravity clearly shows the role of the latter [22,23]. On the other hand the ''zero'' gravity experiment (which simply means the installation of the mirror and the absorber *parallel* to the gravitational field) enables independent measurement of the mirror and absorber properties.

Let us first mention that the neutrons' motion transversal to the direction of the mirror (and the absorber) is quantized. Neutron states of this type, localized between the mirror and the absorber, will be referred to as ''boxlike.'' However due to the loss of neutrons inside the absorber such states are no longer stationary states; they are quasibound states with finite lifetimes (width). The existence of quasibound states in the presence of an absorber is a consequence of the phenomenon mentioned above as quantum reflection from the fast changing absorber Fermi potential. In fact the partial reflection of neutron waves from the absorbing potential leads to the formation of the standing wave (i.e. quasibound state). The more efficient the absorber, the smaller the amplitude of the reflected wave and the shorter the neutron lifetime. In the case of full absorption of the neutron wave (which means that the amplitude of the reflected wave is exactly zero) no quasibound state can exist.

With these remarks we can now turn to the calculation of the neutron flux through the waveguide:

$$F = \sum_n |C_n|^2 \exp(-\Gamma_n \tau^{\text{pass}}). \tag{38}$$

Here n is the quantum number of the quasibound boxlike state. In the above expression we neglect for the moment the contribution of the interference terms. As we have shown, this is possible when the *longitudinal* velocity distribution is rather wide. In the following we will also assume a wide distribution of the incident flux over transversal momentum (orthogonal to the mirror and absorber). We have already established that in such a case the first N_h states are populated homogeneously. The number N_h of homogeneously populated states can be estimated from the condition that the characteristic momentum of the boxlike state $k_c \sim n/H$ is equal to the spread of the transversal momentum distribution k_0 in the incident flux:

$$N_h \simeq k_0 H. \tag{39}$$

Let us now turn to the calculation of the widths of certain neutron states, confined between mirror and absorber. As in the case of the gravitation states we assume that the absorber Fermi potential can be characterized by a complex scattering length a, which is possible when $k_n \rho \ll 1$ (where $k_n = \sqrt{2mE_n}$ is the neutron momentum in given boxlike state with energy E_n). To obtain the complex energies of the boxlike states we note that the neutron wave function in the region where the absorber Fermi potential can be neglected is

$$\Psi_b(z) \sim \sin(k_n z).$$

Such a wave function can be matched with the asymptotic form of the neutron wave function inside the absorber at distances $H - 1/k_n \ll z \ll H - \rho$, where absorber potential vanishes. The general asymptotic form of the wave function in this region is

$$\Psi_a(z) \sim 1 + \frac{H - z}{\tilde{a}},$$

where $\tilde{a} = a - H$ is the ''diffuse tail'' scattering length. The matching of the wave function and its derivative leads to

$$k_n = \frac{\pi n}{H - \tilde{a}}, \tag{40}$$

$$E_n \approx \frac{\pi^2 n^2}{2mH^2} + \frac{2\pi^2 n^2 \operatorname{Re} \tilde{a}}{2mH^3}, \tag{41}$$

$$\Gamma_n \approx 4E_n \frac{|\operatorname{Im} a|}{H} = 4\frac{\pi^2 n^2 |\operatorname{Im} a|}{2mH^3}. \tag{42}$$

We will show that the dependence (42) will play a crucial role in establishing the waveguide transition factor dependence on H. Let us note here that the expression for the width of the boxlike state (42) is a consequence of the quantum reflection from the fast changing tail of the absorber Fermi potential. To see how the quantum reflection phenomenon is connected to the width of the boxlike state let us return to the semiclassical expression for the loss rate:

$$\Gamma \sim \omega_n P,$$

where ω_n is the classic frequency of collisions with the absorbing wall and P is the probability of absorption in a ''one touch'' collision. The expression for the collision frequency with *one of two* walls is

$$\omega_n = \frac{v_n}{2H} = \frac{k_n}{2mH} = \frac{\pi n}{2mH^2}.$$

The probability of absorption P (37) turns out to be

$$P = 1 - R = 4k_n|\mathrm{Im}\, a| = \frac{4\pi n|\mathrm{Im}\, a|}{H}. \qquad (43)$$

Combining the above results for the frequency ω_n and absorption probability P we return to the expression (42). The quantum properties of neutron motion appear here through the energy dependence of the absorption probability (43) and quantization of the box-state energy (momentum).

Integrating the results for C_n and Γ_n into the expression for the flux (26) we obtain

$$F = F_0 \sum_n \exp\left(-4\frac{\pi^2 n^2 |\mathrm{Im}\, a|}{2mH^3}\tau^{\mathrm{pass}}\right). \qquad (44)$$

Here F_0 is the normalization constant, characterizing the intensity of initial flux.

One can see that the number of states passed through the waveguide is obtained from the condition:

$$\Gamma_n \tau^{\mathrm{pass}} \simeq 1,$$

which gives

$$N^{\mathrm{pass}} \simeq \frac{H^{3/2}\sqrt{2m}}{2\pi\sqrt{|\mathrm{Im}\, a|\tau^{\mathrm{pass}}}}.$$

Hereafter we expect that the number of homogeneously populated states N_h (39) is greater than N^{pass}

$$k_0 H \geq N^{\mathrm{pass}}.$$

From the expression (44) it follows that the waveguide transition coefficient is determined by the characteristic *absorption constant*:

$$\xi = 4\frac{\pi^2 \tau^{\mathrm{pass}} |\mathrm{Im}\, a|}{2mH^3},$$

which is connected with the number of states passed through the waveguide via

$$N^{\mathrm{pass}} \simeq 1/\sqrt{\xi}.$$

There are two important limiting cases.

The first case, which we call the *"strong absorption"* limit $\xi > 1$, means that a maximum of one state only can pass through the waveguide, i.e.

$$N^{\mathrm{pass}} \simeq 1/\sqrt{\xi} \leq 1.$$

In this case the neutron flux is a rapidly increasing function of H:

$$F \approx F_0 \exp\left(-4\frac{\pi^2 |\mathrm{Im}\, a|}{2mH^3}\tau^{\mathrm{pass}}\right). \qquad (45)$$

The opposite case, which we call the *"weak absorption"* limit $\xi \ll 1$ means that a large number of states can pass through the absorber:

$$N^{\mathrm{pass}} = 1/\sqrt{\xi} \gg 1.$$

In this case the summation in expression (44) can be substituted by integration, which gives

$$F \approx F_0 \frac{H^{3/2}\sqrt{2m}}{2\sqrt{2\pi|\mathrm{Im}\, a|\tau^{\mathrm{pass}}}}. \qquad (46)$$

It is worth mentioning that $H^{3/2}$ dependence is a consequence of the quantum threshold behavior that determines the energy dependence of the absorption probability of ultracold neutrons (37). This expression is valid in the so-called *anticlassical limit* $k_n\tilde{a} \ll 1$. In the opposite case, when $k_n\tilde{a} > 1$ the absorption probability energy dependence differs from (43). In particular, if the absorption occurs with unit probability for each collision $P \simeq 1$ it is easy to establish:

$$\Gamma_n \approx \pi n/(2H).$$

The substitution of this expression into (26) results in the H^2 dependence of the flux instead of $H^{3/2}$. Note that the large value of the absorption probability of ultraslow neutrons can be achieved only if there is a large imaginary part of the scattering length $k_n|\mathrm{Im}\, a| \gg 1$.

Let us also note that we restrict ourselves with the condition of a homogeneous population of boxlike states $k_0 H \geq N^{\mathrm{pass}}$; so far ξ cannot be smaller than

$$\xi_{\mathrm{min}} = \frac{1}{k_0^2 H^2}.$$

Obviously, in the limit of very small $\xi \ll \xi_{\mathrm{min}}$, when absorption can be fully neglected, the flux passed through the slit starts to be proportional to the slit size

$$F \sim H,$$

which means that all the neutrons that enter the slit pass through it without losses. So far, depending on the efficiency of the absorber one can get different flux dependence on the slit size H.

A comparison with the gravitational case results in the following conclusions.

First, for small slit sizes, both flux curves, seen as a function of H manifest fast increases in the vicinity of the characteristic value H_c. However in the case of gravitational states this critical slit size is determined by the "height" of the ground gravitational state $H_c \simeq H_1$, while in the case of zero gravity it is fully determined by the properties of the absorber, namely, the imaginary part of the scattering length $|\mathrm{Im}\, a|$ and the passage time τ^{pass}:

$$H_c = (2\pi^2 \tau^{\mathrm{pass}} |\mathrm{Im}\, a|/m)^{1/3}.$$

Second, in the presence of gravitation the flux exhibits a steplike dependence on H with increasing slit size, and tends to $H^{3/2}$ dependence for large H. Such steplike behavior is more pronounced for larger diffuseness of the absorber (larger $|\mathrm{Im}\, a|$) or for longer passage time. In case of zero gravity the flux increases with H monotonously.

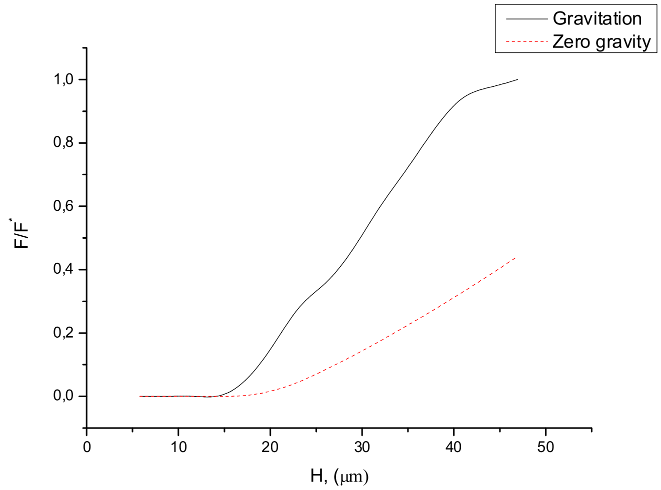

FIG. 6 (color online). The relative neutron flux in the presence of, and in the absence of, gravity. F^* is the flux with gravity calculated for absorber position $H = 45\ \mu$m, $|\mathrm{Im}\, a| = 2\ \mu$m, and $\tau^{\mathrm{pass}} = 0.02$ s.

Such an increase has power law dependence in the limit of large H. Depending on the absorber efficiency the corresponding exponent can vary from 2 (full absorption) to 1 (no absorption).

We plot on Fig. 6 the neutron flux in the presence of and in the absence of gravity for $|\mathrm{Im}\ a| = 2\ \mu$m and $\tau^{\mathrm{pass}} = 0.02$ s.

E. Inverse geometry experiment

Another way to clarify the gravitational effects and to measure the efficiency of the absorber is to exchange the position of the mirror and the absorber in the experimental setup. Here we will study such an inverse geometry experiment, in which the absorber is placed below and the mirror above.

First we will study the modification of the gravitational energy values due to the interaction with an "absorbing mirror." As we have already shown, as long as the distance ρ where absorption takes place is much smaller than the gravitational wavelength l_0, such an interaction can be characterized by only one parameter, namely, the scattering length $a \ll l_0$, regardless of certain details of the absorber Fermi potential.

The modification of the eigenvalues λ_n due to the interaction with an absorbing mirror can be obtained by matching the wave function of the neutron, reflected from the absorber, which large z asymptotic form ($z \gg a$) in the case of small neutron energies can be written as follows:

$$\psi(z) \sim 1 - z/a,$$

with the gravitational wave function $\mathrm{Ai}(z/l_0 - \tilde{\lambda})$, where $\tilde{\lambda}$ is a modified eigenvalue. We take into account that in the matching region $z/l_0 \ll 1$ we obtain the following equation for $\tilde{\lambda}$:

$$\frac{\mathrm{Ai}(-\tilde{\lambda})}{\mathrm{Ai}'(-\tilde{\lambda})} = -a/l_0. \tag{47}$$

As long as $|a/l_0| \ll 1$ we get the following expression for the modified eigenvalues $\tilde{\lambda}_n$ accurate up to the first order of small parameter $|a/l_0|$:

$$\tilde{\lambda}_n = \lambda_n + a/l_0. \tag{48}$$

From the above equation we obtain the following modified energy levels:

$$E_n = \varepsilon(\lambda_n + \mathrm{Re}\, a/l_0), \tag{49}$$

$$\Gamma_n = 2\varepsilon \frac{|\mathrm{Im}\, a|}{l_0} = 2mg|\mathrm{Im}\, a|. \tag{50}$$

If we use the expression (35) for the scattering length on the deep ($2\rho\sqrt{2mU} \gg 1$) imaginary ($\varphi = \pi/2$) exponential potential, we obtain for the width of the gravitational state:

$$\Gamma^{\mathrm{inv}} = mg\pi\rho. \tag{51}$$

We should mention that the width of the gravitational state (50) is independent of the energy (for such states that $\sqrt{2mE_n}\rho \ll 1$). This can be explained easily by the following simple arguments. The frequency of the neutron bouncing above the surface in the gravitational field is $\omega \sim 1/\sqrt{E}$, while the probability of the absorption $P = 4k|\mathrm{Im}\, a| \sim \sqrt{E}$. Combining these two variables we get the energy-independent expression for the width $\Gamma = \omega P$. This means that all the gravitational states which are not affected by the upper mirror ($H_n \ll H$) decay at the same rate (51). The corresponding lifetime in the case of $\rho = 1\ \mu$m is

$$\tau \simeq 1.7 \times 10^{-3}\ \mathrm{s},$$

which is much smaller than the passage time $\tau^{\mathrm{pass}} = 0.02$ s. The expression (50) manifests the very important property of the neutron bouncing above the absorbing mirror, namely, the factorization of gravitational properties, which appears through factor mg and the absorber properties, characterized by $|\mathrm{Im}\, a|$.

Let us now understand the behavior of the transversal states with much higher energy $E \gg mgH$. For such high energies the influence of the gravitational field can be neglected (for neutron motion between mirror and absorber). The corresponding states can be treated as the previously studied boxlike states of the free neutron, confined between the absorber and the mirror; their widths are given by (42).

As long as we study the transversal states with $E_n \gg mgH$ their lifetimes are much smaller than those of the gravitational states already considered, and their contribution to the neutron flux at the exit of the waveguide can be neglected.

The two limiting cases studied above naturally follow from the equation for the eigenvalues for the inverse geometry experiment (see Appendix A for details of the derivation):

$$a[\mathrm{Ai}(H/l_0 - \lambda_n)\mathrm{Bi}'(-\lambda_n) - \mathrm{Ai}'(-\lambda_n)\mathrm{Bi}(H/l_0 - \lambda_n)] = \mathrm{Ai}(-\lambda_n)\mathrm{Bi}(H/l_0 - \lambda_n) - \mathrm{Bi}(-\lambda_n)\mathrm{Ai}(H/l_0 - \lambda_n). \quad (52)$$

We come to the conclusion that, with mirror position $H \gg H_1$, the measured neutron flux is mainly determined by the gravitational states passed through the waveguide such that $H_n < H$. The number of such states is given by (15) and thus the dependence of the flux on H is given by (16). The ratio of the fluxes in the "direct" and in the inverse geometry experiment turns out to be

$$F_{\mathrm{inv}}/F_{\mathrm{dir}} \simeq \exp(-2mg|\mathrm{Im}\, a|\tau^{\mathrm{pass}}). \quad (53)$$

The results of comparison of neutron fluxes in direct and inverse geometry experiment is shown in Fig. 7

This difference in the fluxes clearly shows the role of gravitation in the passage of the neutrons through the waveguide. On the other hand, it enables us to measure the efficiency of the absorber. It is also interesting to note that if $|\mathrm{Im}\, a|$ is known by independent measurement (e.g. from the zero gravity experiment discussed above), the measurement of the lifetime of neutrons bouncing in the low gravitational states above an absorbing surface will give direct access to the *gravitational* mass of neutron m

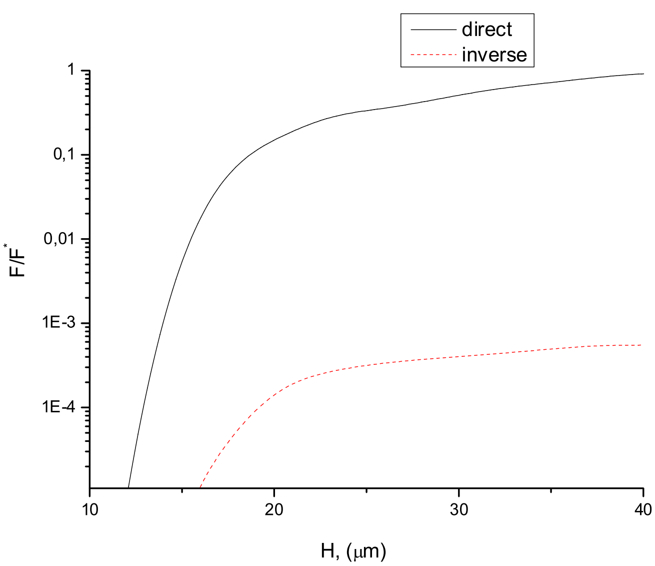

FIG. 7 (color online). The relative neutron flux in the direct and inverse geometry experiment. F^* is the flux in the direct geometry case, calculated for absorber position $H = 45\ \mu$m, $|\mathrm{Im}\, a| = 2\ \mu$m, and $\tau^{\mathrm{pass}} = 0.02$ s.

and will allow us to apply the quantum equivalence principle test.

The inverse geometry measurements were performed during one of the first runs of neutron gravitational states experiment [19]. The obtained results verify strong suppression of the flux in an inverse geometry experiment case in agreement with (53).

V. ROUGH SURFACE ABSORBERS

The previous analysis shows that in order to increase the efficiency of flat absorbers one needs to use substances either with a Fermi potential of large diffuse radius or of very small depth ($U \sim 10^{-12}$ eV). The construction of such absorbing materials is rather problematic. An alternative way to increase absorber efficiency is to use an absorber with a rough surface. In the waveguide experiments an absorber with a rough surface was used with a roughness amplitude of about 2 μm. In this section we will study the role of roughness in the neutron loss mechanism.

A. Effective potential approach

The rigorous study of neutron interactions with a rough surface requires solving the two-dimensional problem, where the neutron-surface interaction is described by a rather complicated function $V(x, z)$. The radical simplification of such a problem is possible via the introduction of *effective* one-dimensional potential $V_{\mathrm{eff}}(z)$ [49]. The simplest assumption enabling us to calculate such a potential is the following: We expect that the *longitudinal* kinetic energy of neutrons $p_0^2/2M$ is sufficiently superior to the characteristic value of the Fermi-potential $V(x, z)$ of rough edges. The first order Born correction to the longitudinal kinetic energy of neutrons due to the interaction with the rough edges would then be

$$\Delta E(z) = \frac{1}{L}\int V(x, z)dx, \quad (54)$$

where the "normalization length" L is selected to be much greater than characteristic correlation length of roughness. This correction to the longitudinal energy, being a function of z, plays the role of effective potential $V_{\mathrm{eff}}(z) = \Delta E(z)$ in the equation for the neutron transversal motion

$$-\frac{1}{2m}\frac{d^2}{dz^2}\varphi(z) + \Delta E(z)\varphi(z) = (E - p_0^2/2m)\varphi(z).$$

The physical meaning of expression (54) is transparent; it is the potential of media with reduced density. In particular if one models the roughness by the periodic gratings with z dependent width $d(z)$ and period L, than the effective Fermi potential is $V_{\mathrm{eff}}(z) = Ud(z)/L$, where U is the corresponding Fermi potential of flat surface. The benefit of this approach is the ability to connect the one-dimensional effective Fermi potential with averaged shape properties of roughness and realistic Fermi potential of absorber substance. We will not take this case any further, since the main results have been discussed already in the section devoted to the flat absorber.

The above approximative model can be justified for the longitudinal energies of neutrons much higher than the Fermi potential of rough edges. A very important effect, which is not taken into account in this simplified approach is the possibility of nonspecular reflections, i.e. the energy exchange between the horizontal and vertical motion of the neutrons. (They appear in the second order Born approximation.) In the following we develop the nonperturbative formalism in which such effects would be taken into account.

B. A time-dependent model for the neutron loss mechanism

In the previous analysis we found that only those neutrons which have sufficiently small transversal energy do not penetrate through the gravitational barrier into the absorber and thus are not absorbed in the waveguide. The role of the absorber's roughness is to transfer a significant portion of longitudinal energy into transversal energy during nonspecular reflection from the rough edges. Thus the neutron interaction with the rough surface absorber results in mixing of states with different transversal energies. As long as the states with large transversal energy have very small lifetimes, such a mixing results in a loss of neutrons. Here we will study this loss mechanism within the time-dependent model.

We will study the neutron passage through the waveguide in the frame, moving with horizontal velocity V of the incoming flux (we suppose that this velocity is well defined). The rough edges of the absorber surface can then be treated as a time-dependent variation of the *flat* absorber position. This means that the neutron loss mechanism in such a model is equivalent to the ionization of a particle, initially confined in a well with an oscillating wall.

The time-dependent Schrödinger equation for the neutron wave function is

$$i\frac{\partial\Phi(t,z)}{\partial t}=\left[-\frac{1}{2m}\frac{\partial^2}{\partial z^2}+mgz+V(z,H(t))\right]\Phi(t,z). \tag{55}$$

The time-dependence appears here through the time-dependence of the absorber position $H(t)$.

The boundary conditions are

$$\Phi(t,z=0)=0, \tag{56}$$

$$\Phi(t,z=\infty)=0. \tag{57}$$

It would be convenient here to introduce time-dependent basis functions:

$$\phi_n(t,z)=\psi_n(H(t),z)\exp\left(-\frac{i}{\hbar}\int_0^t\varepsilon_n(H(\tau)d\tau\right),$$

where $\psi_n(H,z)$ and $\varepsilon_n(H)$ are *complex* eigenfunctions and eigenvalues of transversal Hamiltonian (18) with fixed absorber position H.

The total wave function $\Phi(t,z)$ can be expanded in the set of functions:

$$\Phi(t,z)=\sum_n C_n(t)\psi_n(H(t),z)\exp\left(-i\int_0^t\varepsilon_n(H(\tau))d\tau\right). \tag{58}$$

The equation system for the expansion factor $C_n(t)$ is

$$\frac{dC_n(t)}{dt}=-\frac{dH}{dt}\sum_{k\neq n}C_k\alpha_{nk}\exp[-i\omega_{nk}(t)], \tag{59}$$

$$\alpha_{nk}=-\alpha_{kn}\equiv\int_0^\infty\psi_n(H,z)\frac{\partial\psi_k(H,z)}{\partial H}, \tag{60}$$

where

$$\omega_{nk}(t)=\int_0^t[\varepsilon_k(H(\tau))-\varepsilon_n(H(\tau))]d\tau.$$

Note that the derivation of Eqs. (59) and (60) requires the bi-orthogonal condition (19).

The initial conditions $C_k(0)$ are determined by the overlapping of the incoming flux with the basis functions:

$$C_k(0)=\int_0^\infty\Phi(t=0,z)\psi_k(H,z)dz.$$

The solution to Eq. (59) together with the above initial conditions enables us to obtain the wave function $\Phi(t,z)$ at $t=\tau^{\rm pass}$ and to calculate the measured flux:

$$F=\int_0^\infty|\Phi(\tau^{\rm pass},z)|^2dz.$$

The equation system (59) can be very much simplified under the following assumptions.

First, let us suggest that the absorber position time dependence is harmonic:

$$H(t)=H_0+b\sin(\omega t),$$

where b is the roughness amplitude and frequency $\omega=V/d$, with d being the spatial period of roughness. It is known that in such cases it is only the states in the equation system (59) obeying the "resonance" condition:

$$|\varepsilon_k-\varepsilon_n|\simeq\omega, \tag{61}$$

which are effectively coupled. As long as the transversal states have the widths this resonance can not be exact. However we will restrict our treatment to only two coupled states.

Second, we expect the roughness amplitude to be so small that the following approximation is valid:

$$\frac{d\psi_k(H(t),z)}{dt}\simeq b\cos(\omega t)\frac{\partial\psi_k(H,z)}{\partial H}|_{H=H_0}.$$

Third, we will consider that $\omega \gg |\varepsilon_n|$, so that the low-lying gravitational state ψ_n is coupled with the very highly excited state with energy $\mathrm{Re}\,\varepsilon_k \gg MgH$. The gravitational potential can be neglected in comparison with such high energy; we are thus dealing with a boxlike state. Its energy and width is given by (41) and (42). As the width of such excited states is much bigger than the width of the low-lying gravitational state ψ_n we can neglect the latter and suppose that neutrons in the low-lying gravitational state are *elastically* reflected both from the mirror and the absorber. This results in the following boundary conditions for the gravitation state wave function $\psi_n(H, z)$:

$$\psi_n(H, z = 0) = \psi_n(H, z = H) = 0.$$

The eigenfunction $\psi_n(H, z)$ and the eigenvalue $\varepsilon_n(H) = \varepsilon_0 \lambda_n(H)$, determined by the above boundary condition, are

$$\psi_n(H, z) \sim \mathrm{Bi}(-\lambda_n(H))\mathrm{Ai}(z - \lambda_n(H)) - \mathrm{Ai}(-\lambda_n(H))\mathrm{Bi}(z - \lambda_n(H)), \tag{62}$$

$$\mathrm{Ai}\,(H/l_0 - \lambda_n(H))\mathrm{Bi}(-\lambda_n(H)) = \mathrm{Ai}(-\lambda_n(H))\mathrm{Bi}(H/l_0 - \lambda_n(H)). \tag{63}$$

Finally we come to the equation system with only two coupled equations:

$$\begin{cases} \dot{C}_0(t) = -\frac{1}{2} b\omega C_1(t)\alpha(H)\exp[i(\omega - \omega_{01})t] \\ \dot{C}_1(t) = \frac{1}{2} b\omega C_0(t)\alpha(H)\exp[-i(\omega - \omega_{01})t], \end{cases} \tag{64}$$

with

$$\alpha(H) = \int_0^H \psi_0(H, z)\frac{\partial \psi_1(H, z)}{\partial H}$$

and $\omega_{01} = E_1 - E_0$.

In the above expressions index 0 labels the low-lying gravitational state, while index 1 labels the excited fast decaying boxlike state with *complex* energy $E_1 = \mathrm{Re}\,E_1 - i\Gamma/2$.

A very convenient expression [54] can be obtained for coupling matrix element $\alpha(H)$ (see Appendix B), namely,

$$\alpha(H) = \frac{\sqrt{\partial\lambda_0/\partial H \partial\lambda_1/\partial H}}{\lambda_0 - \lambda_1}. \tag{65}$$

The benefit of such a simplified equation system is that it enables an analytical solution. Taking into account initial conditions $C_0(0) = 1$ and $C_1(0) = 0$ we get

$$C_0(t) = \exp\left(-\frac{\Gamma t}{4}\right)\left(\cos(\gamma t/2) + \frac{\Gamma}{2\gamma}\sin(\gamma t/2)\right), \tag{66}$$

$$C_1(t) = -i\frac{\alpha(H)}{\gamma}\exp\left(\frac{\Gamma t}{4}\right)\sin(\gamma t/2). \tag{67}$$

Here Γ is the width of the boxlike state and $\gamma = 1/2\sqrt{b^2\omega^2\alpha^2(H) - \Gamma^2}$. [Note the exponential increase of $C_1(t)$. This does not yield in nonphysical result, as far as in the expression for the wave function (58) $C_1(t)$ is multiplied by decaying exponent $\exp(-iE_1 t)$. However, as we mentioned before, $|C_0(t)|^2$ and $|C_1(t)|^2$ cannot be interpreted as probabilities to find a system in certain quantum state.]

We are interested in the evolution of the gravitational state. Two important limiting cases are

$$|C_0(t)|^2 \to \exp\left(-\frac{\Omega^2 t}{4\Gamma}\right), \quad \text{if } \Omega^2/\Gamma^2 \ll 1, \tag{68}$$

$$|C_0(t)|^2 \to \exp\left(-\frac{\Gamma t}{2}\right)\cos^2(\Omega t/2 - \varphi)/\cos(\varphi), \quad \text{if } \Omega^2/\Gamma^2 \gg 1. \tag{69}$$

Here $\varphi = \arctan(\Gamma/(2\gamma))$ and $\Omega^2 = b^2\omega^2\alpha^2(H)$.

The quantity Ω plays the role of "transition frequency" between two states. It is proportional to the roughness amplitude b and depends on the averaged absorber position H via the coupling $\alpha(H)$. The coupling $\alpha(H)$ decays rapidly as soon as $H > H_n$, where H_n is the classical turning point for the low-lying gravitational state.

When $\Gamma \gg \Omega$ the decay rate Γ_n of the nth gravitational state , according to (68), is $\Gamma_n = \Omega^2/(4\Gamma)$. Using the asymptotic expressions for $\alpha(H)$ (see Appendix B) one can get the following expression for the decay rate in case $H \gg H_n$:

$$\Gamma_n = \varepsilon_0\sqrt{\frac{l_0}{H_n}}\frac{b^2}{8l_0|\mathrm{Im}\,a|}\sqrt{\frac{H - H_n}{l_0}} \times \exp\left[-\frac{4}{3}((H - H_n)/l_0)^{3/2}\right], \tag{70}$$

where $\mathrm{Im}\,a$ is the imaginary part of the scattering length of the neutrons on the *flat absorber* Fermi potential.

The expression (70) should be compared with the analogous formula for flat absorbers (31). One can see that

$$a_{\mathrm{eff}} = \frac{b^2}{16|\mathrm{Im}\,a|}$$

plays the role of the effective scattering length of the rough surface absorber, which is proportional to the *square* of the roughness amplitude.

The time

$$\tau_n^{\mathrm{abs}} = \frac{8l_0|\mathrm{Im}\,a|}{b^2\varepsilon_0}\sqrt{\frac{l_0}{H_n}}$$

plays the role of the characteristic absorption time in our problem.

It should be noted that the above results are true for $H > H_n$ and "weak" coupling. When the absorber position $H < H_n$ the coupling is large and another limiting case applies, namely, $\Omega \gg \Gamma$. In such cases (69) the gravitational state decay within the lifetime:

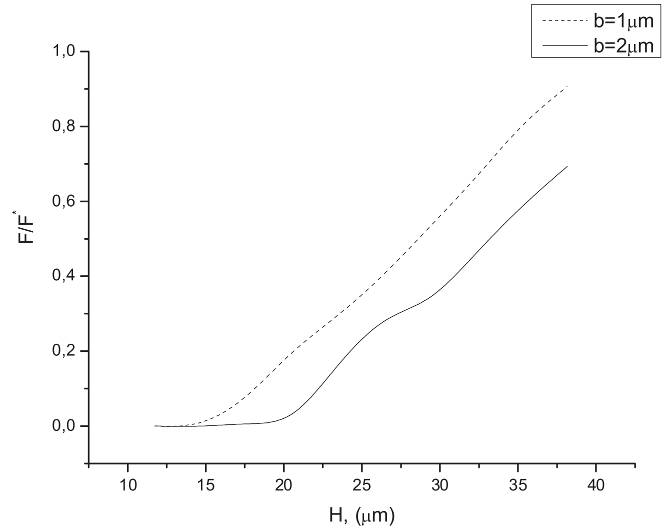

FIG. 8. The relative neutron flux as a function of the slit height in the time-dependent model. F^* is the flux calculated for absorber position $H = 40\ \mu$m, $b = 1\ \mu$m, $|\mathrm{Im}\, a| = 0.1\ \mu$m, and $\tau^{\mathrm{pass}} = 0.02$ s.

$$\tau = 2/\Gamma,$$

which is small compared to the passage time through the waveguide.

In Fig. 8 we plot the results of numerical calculations for the measured neutron flux within the time-dependent model for two values of roughness amplitude $b = 1\ \mu$m and $b = 2\ \mu$m and $|\mathrm{Im}\, a| = 0.1\ \mu$m. Better resolution of the quantum ''steps'' appears with an increase of the roughness amplitude.

In the above simple ''two-state'' model several potentially important effects are not taken into account, in particular, the ''nonresonant'' transitions between different gravitational states. However this model enables understanding of fast irregularities (steps) in the transmitted neutron flux as a function of absorber position H and naturally explains them in terms of gravitational states of neutrons. The model also establishes the dependence of the waveguide absorbing properties on roughness amplitude.

C. Resolution of gravitational states

1. Constraints on resolution

Based on the results of the previous sections we can analyze the conditions for the best resolution of gravitational states. The presented numerical calculations show that an increase in absorber efficiency (e.g. by increasing roughness amplitude) results in a shifting of the positions of ''the quantum steps'' in the neutron flux by the value Δ_n and enhancing their resolution δ_n. To perform a qualitative analysis we will accept that the steplike increase in the measured neutron flux, corresponding to the ''appearance'' of the new state, starts to be seen when the widths of this state are

$$\Gamma_n(H)\tau^{\mathrm{pass}} = e.$$

We will also accept that such a steplike increase saturates when

$$\Gamma_n(H)\tau^{\mathrm{pass}} = 1/e.$$

From expression (70) one obtains the following estimate for the shift Δ_n and the uncertainty δ_n of the nth step in the extreme limit $\ln(\tau^{pass}/\tau_n^{abs}) \gg 1$:

$$\Delta_n \simeq l_0\left(\frac{3}{4}\right)^{2/3}\left[\ln(\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}}\sqrt{\Delta_0/H_n})\right]^{2/3}, \qquad (71)$$

$$\delta_n \simeq \frac{2l_0}{3}\left(\frac{3}{4}\right)^{2/3}\left[\ln(\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}}\sqrt{\Delta_0/H_n})\right]^{-1/3}, \qquad (72)$$

where

$$\Delta_0 = l_0\left(\frac{3}{4}\right)^{2/3}[\ln(\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}})]^{2/3}.$$

The uncertainty δ_n decreases as $\ln^{-1/3}(\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}})$ with an increase in $\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}}$. The resolution of the nth state is possible if the uncertainty in the step position δ_n is much less than the distance between neighboring steps $H_{n+1} + \Delta_{n+1} - H_n - \Delta_n$. For highly excited states we can use the WKB expression (7) for the classical turning point:

$$H_n = l_0\left(\frac{3\pi}{4}(2n - 1/2)\right)^{2/3},$$

to find the *universal* limit on the number of states that can be resolved if $\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}} \gg 1$:

$$[\ln(\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}})]^{-1/3} \ll \tfrac{3}{2}(2\pi)^{2/3}[(n + 3/4)^{2/3} - (n - 1/4)^{2/3}].$$

This estimation shows that the resolution of states very slowly increases with an increase in passage time or in the efficiency of the absorber in the limit $\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}} \gg 1$, namely, $n \sim \ln(\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}})$. This law is the consequence of the linear dependence of the gravitational potential on z. Indeed due to the linearity of the gravitational potential, the level spacing decreases with n like $n^{-1/3}$, until the neighboring states' contribution to the flux starts to overlap. In particular for the value of $\tau^{\mathrm{pass}}/\tau_n^{\mathrm{abs}} = 100$ the number of states that can be resolved is around 5.

The resolution of quantum states could be improved, if the initial population of one or several of such states is artificially reduced. In this case the neighboring state would be exposed. We have studied the scenario in which the bottom mirror has a specially designed ''step'' [17,49].

2. Repopulation of states

If two bottom mirrors are shifted relative to each other by a Δ of a few μm in height, there is an additional boundary at the step position $x = L_0$ that will change the population of the eigenstates. We give here just a brief description (for the details see [49])

$$\Psi_I|_{x=L_0} = \Psi_{II}|_{x=L_0} \wedge \frac{\partial}{\partial x}\Psi_I|_{x=L_0} = \frac{\partial}{\partial x}\Psi_{II}|_{x=L_0} \forall z \in [0, H].$$

Because of the presence of the shift Δ in the bottom mirrors' position, the gravitational states are repopulated. If this step is treated as a "sudden change" in the potential, the matching at the boundary $x = L_0$ results in the following repopulation coefficients:

$$C_{jm} = \exp(-\Gamma_j\tau_0/2)) \int_0^H \varphi_j(z)\varphi_m(z + \Delta)dz. \quad (73)$$

Here $\tau_0 = L_0/V$, $\varphi_j(z)$ is the gravitational state in the presence of the absorber, positioned at height H above the first mirror. Again, note the usage of the bi-orthogonality condition.

The expression for the neutron flux at the detector position is now modified as follows:

$$F(L) \simeq \sum_{j,m} |C_{jm}|^2 \exp(-(\Gamma_j - \Gamma_m)\tau_0 - \Gamma_m\tau^{\text{pass}}). \quad (74)$$

We neglect here the interference terms, assuming wide longitudinal velocities distribution.

To illustrate the effect of repopulation, consider a simplified system consisting of just two mirrors without any absorber. The orthonormal system of eigenfunctions of the vertical motion in this case is just given by the standard bound state Airy function. Now imagine the second mirror shifted downwards compared to the first one by an amount equal to the height of the first node of the 2nd eigenstate wave function

$$(\lambda_2 - \lambda_1) \cdot l_0 \approx 1.56 \cdot l_0 \approx 9.15 \ \mu\text{m}.$$

It is clear that the 2nd eigenstate above the 2nd mirror exactly matches the ground state wave function above the 1st mirror from its edge on, while the ground state wave function of the 2nd mirror overlaps only with the exponentially decaying tail of the ground state of the 1st mirror. This implies immediately that the new ground state above the 2nd mirror will be suppressed with respect to the 2nd eigenstate above this mirror. The repopulation coefficients for the transition to the 2nd mirror, normalized to the initial population of the ground state above the 1st mirror, are given in Table II.

Figure 9 contains a plot of the population of the new ground state above the 2nd as a function of the relative shift of the two mirrors.

In Fig. 10 the neutron flux for mirror shift $\Delta = 8\mu$m is compared with the neutron flux without any shift in the bottom mirror position. Because of the depopulation of the ground state, the changes in the flux slope corresponding to the gravitational states can be seen more easily.

TABLE II. Normalized repopulation coefficients after transition from a single ground state across a mirror shift of $\approx$ 9.15 μm.

n	$(\langle\psi_n\|\psi_1\rangle)^2$
1	0.162
2	0.765
3	0.037
4	0.019
5	0.009
6	0.005
7	0.002

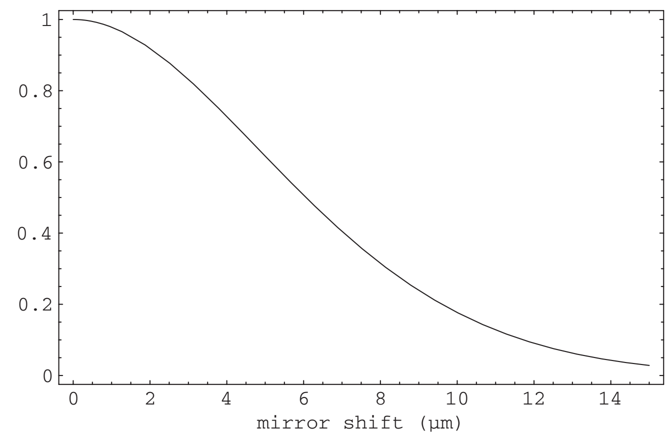

FIG. 9. Repopulation coefficient of the ground state above the 2nd mirror as a function of the relative shift of the two mirrors in μm.

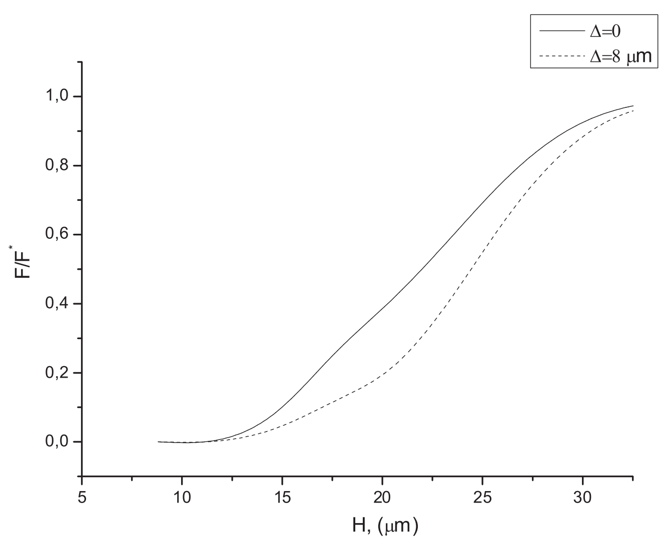

FIG. 10. The relative neutron flux as a function of slit height for two values of the bottom mirror shift. F^* is the flux calculated at absorber position $H = 35$ μm and diffuseness $\rho = 1$ μm.

VI. CONCLUSIONS

We have analyzed the problem of the passage of ultracold neutrons through an absorbing waveguide in the presence of the Earth's gravitational field, both qualitatively and numerically. We have shown that the set of existing experimental results [18–20] exhibits clear evidence for the quantum motion of neutrons in the gravitational field.

We developed the formalism describing the loss mechanism of ultracold neutrons in the waveguide with absorption. The essential role of the quantum reflection phenomenon for the loss of ultracold neutrons was established. The concept of quantum reflection enables universal description of different kinds of absorbers in terms of effective complex scattering length a. The efficiency of absorption of ultracold neutrons in the presence of the gravitational field of Earth is determined by the ratio of such a scattering length to the characteristic gravitational wavelength a/l_0.

We studied the particular case of absorbers with rough surface. It was established that in the latter case the main loss mechanism is due to the nonspecular reflection of neutrons from the rough edges of the absorber. Absorber efficiency turns out to be proportional to the square of its roughness amplitude, if this amplitude is small compared to the characteristic gravitational wavelength l_0.

We calculated the neutron flux through the waveguide in the case of zero gravity (mirror and absorber arranged parallel to the gravitational field). For large slit heights, the dependence of such a flux on the slit height H exhibits a power law. Its exponent depends on the absorber efficiency. These calculations are important for independent measurement of absorber/mirror properties.

We argue the possibility of using the "inverse geometry" experiment for measuring the lifetime of neutrons bouncing on an absorbing surface. The neutron lifetime was found to be $\tau^{\text{abs}} = 1/(2mg|\text{Im}\, a|)$. It was determined by the gravitational force mg, acting on the neutron and imaginary part of the scattering length $|\text{Im}\, a|$ of the absorbing surface Fermi potential. This experiment shows unambiguously the role of gravitation on the lifetime of ultracold neutrons.

The theory developed in this paper allows one to analyze the resolution of the gravitational spectrometer and to compare the efficiency of different kinds of absorbers/scatterers. We show that the spectrometer resolution is severely limited by a fundamental reason: finite penetrability of the gravitational barrier between the classically allowed region and the scatterer height. The resolution can be improved by a significant increase in the time of storage of neutrons in quantum states, and/or by improvement of the efficiency of the absorber/scatterer. The efficiency of best absorbers/scatterers used in actual experiments was defined mainly by the shape of their rough surface so that the efficiency is approximately proportional to the square of the roughness amplitude (when the roughness amplitude is smaller than the characteristic scale of the gravitationally bound quantum states l_0). Further increase of the roughness does not improve the efficiency; however, strict theoretical description of the case of a large amplitude roughness is not covered by the present analysis. Another way of increasing the resolution could be through the selective depopulation of certain gravitational states, for instance by applying a bottom mirror with a step.

The results obtained are rather general in character and can be applied to different physical problems, involving the transmission of quantum particles through absorbing waveguides. The development of the theoretical considerations presented would include the incorporation of large roughness amplitudes comparable to or larger than the characteristic gravitational length $l_0 \sim 6\ \mu$m—as well as the studies of long storage time case, when decay of neutron quasibound gravitational states differs from the exponential law. These are necessary if the highest resolution is to be achieved for the method considered.

ACKNOWLEDGMENTS

This work was supported by the INTAS Grant No. 99-705 and by the German Federal Ministry for Research and Education under Contract No. 06HD153I. We are sincerely grateful to all the members of our collaboration and those who have shown interest in this work and stimulated its development.

APPENDIX A

Here we derive the equation for the energies of neutrons localized between an ideal mirror and absorber in the presence of a gravitational field. We assume that the absorber Fermi potential has a diffuse radius much smaller than the characteristic gravitational wavelength $\rho \ll l_0$. In the region where the absorber potential can be fully neglected $0 \le z \ll H - \rho$ the wave function is the superposition of Airy functions:

$$\psi_b(z) \sim \text{Ai}(z/l_0 - \lambda_n) - S\text{Bi}(z/l_0 - \lambda_n).$$

The zero boundary condition on the mirror gives

$$S = \frac{\text{Ai}(-\lambda_n)}{\text{Bi}(-\lambda_n)}. \tag{A1}$$

The neutron wave function inside absorber $z > H - \rho$ is determined by the absorber Fermi potential, which is much stronger than the gravitational potential. In the range of distances $H - l_0 \ll z \ll H - \rho$ such a wave function is weakly perturbed by gravitation and can be written as

$$\psi_a(z) \sim 1 + \frac{H - z}{\tilde{a}},$$

where $\tilde{a} = a - H$, with a being the complex scattering length on the absorber Fermi potential. One can see that $\tilde{a}$

plays the role of the "scattering length of the diffuse tail" of the Fermi potential.

Now we match the wave functions $\psi_b(z)$ and $\psi_a(z)$ and their derivatives in the region $H - l_0 \ll z \ll H - \rho$. For this we use the Tailor expansion of $\psi_b(z)$ in the vicinity of H:

$$\psi_b(z) \sim \mathrm{Ai}(H/l_0 - \lambda_n) - S\mathrm{Bi}(H/l_0 - \lambda_n) + (\mathrm{Ai}'(H/l_0 - \lambda_n) - S\mathrm{Bi}'(H/l_0 - \lambda_n))(z - H).$$

The matching condition gives

$$S = \frac{\mathrm{Ai}(H/l_0 - \lambda_n) - \tilde{a}/l_0\mathrm{Ai}'(H/l_0 - \lambda_n)}{\mathrm{Bi}(H/l_0 - \lambda_n) - \tilde{a}/l_0\mathrm{Bi}'(H/l_0 - \lambda_n)}. \tag{A2}$$

Putting together (A1) and (A2) we finally get the equation for the eigenvalues λ_n:

$$\frac{\mathrm{Ai}(-\lambda_n)}{\mathrm{Bi}(-\lambda_n)} = \frac{\mathrm{Ai}(H/l_0 - \lambda_n) - \tilde{a}/l_0\mathrm{Ai}'(H/l_0 - \lambda_n)}{\mathrm{Bi}(H/l_0 - \lambda_n) - \tilde{a}/l_0\mathrm{Bi}'(H/l_0 - \lambda_n)}. \tag{A3}$$

The equation for the eigenvalues in an inverse geometry experiment can be obtained in a similar way. The wave function outside the absorber $\psi_b(z)$ now vanishes at the mirror position H, which gives for S_{inv}

$$S_{\mathrm{inv}} = \frac{\mathrm{Ai}(H/l_0 - \lambda_n)}{\mathrm{Bi}(H/l_0 - \lambda_n)}. \tag{A4}$$

The wave function $\psi_a(z)$ of the neutron inside the absorber at the asymptotic distances $z \gg \rho$ is

$$\psi_a(z) \sim 1 - z/a.$$

The matching of $\psi_a(z)$ and $\psi_b(z)$ at distances $l_0 \gg z \gg \rho$ together with (A4) results in the following equation for λ_n:

$$a[\mathrm{Ai}(H/l_0 - \lambda_n)\mathrm{Bi}'(-\lambda_n) - \mathrm{Ai}'(-\lambda_n)\mathrm{Bi}(H/l_0 - \lambda_n)] = \mathrm{Ai}(-\lambda_n)\mathrm{Bi}(H/l_0 - \lambda_n) - \mathrm{Bi}(-\lambda_n)\mathrm{Ai}(H/l_0 - \lambda_n). \tag{A5}$$

Note that the derivation of the above equations is based on the fact that $\rho \ll l_0$, so that the wave function $\psi_a(z)$ is weakly perturbed by the gravitational field in the asymptotic region $\rho \ll z \ll l_0$.

APPENDIX B

Here we derive the useful relation between the nonadiabatic coupling matrix element $\langle\varphi_j|\frac{\partial\varphi_i}{\partial H}\rangle$ and the energies of corresponding states i and j.

We will study the one-dimensional Schrödinger equation:

$$\hat{\mathbf{H}}|\varphi_i\rangle = E_i|\varphi_i\rangle. \tag{B1}$$

The eigenfunctions $\varphi_j(x)$ and $\varphi_i(x)$ obey the following boundary condition:

$$\varphi_i(x = 0) = 0, \tag{B2}$$

$$\varphi_i(x = H) = 0. \tag{B3}$$

Here the varying parameter H is a boundary. Hereafter we assume that the Hamiltonian itself is independent of H, while eigenfunctions $\varphi_i(x, H)$ and energy eigenvalues $E_i(H)$ depend on H through the boundary condition (B3) only.

Applying $\partial/\partial H$ to both sides of (B1) we get

$$\hat{\mathbf{H}}\frac{\partial\varphi_i}{\partial H} = \frac{\partial E_i}{\partial H}\varphi_i + E_i\frac{\partial\varphi_i}{\partial H}. \tag{B4}$$

Integrating the left side of (B4) with $\varphi_j(x, H)$ and taking into account boundary conditions (B2) and (B3) we get

$$\left\langle\varphi_j|\hat{\mathbf{H}}|\frac{\partial\varphi_i}{\partial H}\right\rangle = -\frac{d\varphi_j(x)}{dx}\frac{\partial\varphi_i}{\partial H}\Big\lfloor_{x=H} + \left\langle\frac{\partial\varphi_i}{\partial H}|\hat{\mathbf{H}}|\varphi_j\right\rangle.$$

Note that

$$\left\langle\frac{\partial\varphi_i}{\partial H}|\hat{\mathbf{H}}|\varphi_j\right\rangle = E_j\left\langle\frac{\partial\varphi_i}{\partial H}|\varphi_j\right\rangle.$$

Combining the above results we get for the matrix element of interest

$$\left\langle\varphi_j|\frac{\partial\varphi_i}{\partial H}\right\rangle = \frac{\frac{d\varphi_j(x)}{dx}\frac{\partial\varphi_i(x)}{\partial H}\lfloor_{x=H} + \frac{\partial E_i}{\partial H}\delta_{ij}}{E_j - E_i}. \tag{B5}$$

Now let us use the following relation

$$\frac{\partial\langle\varphi_i|\varphi_i\rangle}{\partial H} = 2\langle\partial\varphi_i/\partial H|\varphi_i\rangle = 0.$$

From (B5) we get in case $i = j$

$$\frac{d\varphi_i(x)}{dx}\frac{\partial\varphi_i(x)}{\partial H}\Big\lfloor_{x=H} = -\frac{\partial E_i}{\partial H}. \tag{B6}$$

It is clear that the expression (B5) can be expressed as

$$\left\langle\varphi_j|\frac{\partial\varphi_i}{\partial H}\right\rangle = \frac{t_i t_j}{E_j - E_i}.$$

From (B6) we finally get

$$\left\langle\varphi_j|\frac{\partial\varphi_i}{\partial H}\right\rangle = \frac{\sqrt{\partial E_i/\partial H\partial E_j/\partial H} - \partial E_i/\partial H\delta_{ij}}{E_j - E_i}. \tag{B7}$$

Applying the above result to the coupling matrix element in the time-dependent model (59) we get

$$\alpha(H) = \frac{\sqrt{\partial\lambda_n/\partial H\partial\lambda^*/\partial H}}{\lambda_n - \lambda^*}. \tag{B8}$$

Here λ_n is the eigenvalue of the low-lying gravitational state, while λ^* is the eigenvalue of the highly excited boxlike state. This expression is much more convenient for practical applications than the integral in the definition of the coupling matrix element. In particular, it can be used to obtain the asymptotic expressions for the width (70) of a given gravitational state n if $H \gg H_n$.

To obtain such an expression we first find the eigenvalue derivative $\partial\lambda_n/\partial H$ from the Eq. (63):

$$\frac{\partial\lambda_n}{\partial H} = \frac{1}{l_0}\frac{\mathrm{Ai}'(H/l_0-\lambda_n)\mathrm{Bi}(-\lambda_n)-\mathrm{Bi}'(H/l_0-\lambda_n)\mathrm{Ai}(-\lambda_n)}{\mathrm{Ai}'(H/l_0-\lambda_n)\mathrm{Bi}(-\lambda_n)-\mathrm{Bi}'(H/l_0-\lambda_n)\mathrm{Ai}(-\lambda_n)+\mathrm{Ai}(H/l_0-\lambda_n)\mathrm{Bi}'(-\lambda_n)-\mathrm{Bi}(H/l_0-\lambda_n)\mathrm{Ai}'(-\lambda_n)}. \tag{B9}$$

Taking into account the expression (63) and asymptotic properties of the Airy function of large argument $H/l_0 \gg \lambda_n$ we get

$$\frac{\partial\lambda_n}{\partial H} \approx -\frac{1}{l_0}\sqrt{\frac{H-H_n}{H_n}}\exp[-4/3(H/l_0-\lambda_n)^{3/2}].$$

For the energy E^* of the highly excited boxlike state, we can use expression (41), from which we get

$$\frac{\partial E^*}{\partial H} = -2\frac{E^*}{H}.$$

For the square of the coupling matrix element $\alpha^2(H)$ in case of large $H \gg H_n$ we get so far:

$$\alpha^2(H) = 2\frac{E^*}{H(E^*-E_n)^2}\frac{\varepsilon_0}{l_0}\sqrt{\frac{H-H_n}{H_n}} \times \exp[-4/3(H/l_0-\lambda_n)^{3/2}].$$

Taking into account the expression for the width Γ^* of the boxlike state (42) and substituting the above results into the expression for the width of gravitational state (68):

$$\Gamma_n = b^2\omega^2\alpha^2(H)/(4\Gamma^*),$$

we finally come to the expression

$$\Gamma_n = \varepsilon_0\sqrt{\frac{l_0}{H_n}}\frac{b^2}{8l_0|\mathrm{Im}\,a|}\sqrt{\frac{H-H_n}{l_0}} \times \exp\left[-\frac{4}{3}((H-H_n)/l_0)^{3/2}\right].$$

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