Off-diagonal coefficients of the DeWitt-Schwinger and Hadamard representations of the Feynman propagator

Yves Décanini* and Antoine Folacci†

UMR CNRS 6134 SPE, Equipe Physique Semi-Classique (et) de la Matière Condensée, Université de Corse, Faculté des Sciences, BP 52, 20250 Corte, France

(Received 23 November 2005; published 22 February 2006)

Having in mind applications to gravitational wave theory (in connection with the radiation reaction problem), stochastic semiclassical gravity (in connection with the regularization of the noise kernel) and quantum field theory in higher-dimensional curved spacetime (in connection with the Hadamard regularization of the stress-energy tensor), we improve the DeWitt-Schwinger and Hadamard representations of the Feynman propagator of a massive scalar field theory defined on an arbitrary gravitational background by deriving higher-order terms for the covariant Taylor series expansions of the geometrical coefficients—i.e., the DeWitt and Hadamard coefficients—that define them.

DOI: 10.1103/PhysRevD.73.044027 PACS numbers: 04.62.+v

I. INTRODUCTION

The short-distance behavior of the Green functions of a field theory defined on a curved spacetime is of fundamental importance at both the classical and the quantum level. This has been emphasized at the beginning of the 1960s by DeWitt in his pioneering works dealing with (i) the radiation emission of a particle moving in a gravitational background and the radiation reaction force or self-force felt by this particle [1] and (ii) the general problem of the quantization of fields in curved spacetime [2]. In order to describe this short-distance behavior, DeWitt, extending some ideas developed in Refs. [3–5], introduced the so-called DeWitt-Schwinger and Hadamard representations of the Green functions. These two tools have since been extensively and successfully used to analyze and understand various aspects of gravitational physics, from gravitational wave theory to renormalization in quantum gravity. We refer to the monographs of Birrell and Davies [6], Fulling [7], Wald [8], Avramidi [9] and to the recent review articles by Vassilevich [10] and Poisson [11] as well as to references therein for a nonexhaustive state of affairs of the literature concerning the status and the use of these two representations.

On a curved spacetime, the DeWitt-Schwinger representation is constructed from the sequence of the DeWitt coefficients (also called heat-kernel coefficients in the Riemannian framework) $A_n(x, x')$ with $n \in \mathbb{N}$ which are purely geometrical two-point objects formally independent of the dimension d of spacetime and defined by a recursion relation. The DeWitt coefficients $A_n(x, x')$ of lowest orders encode the short-distance singular behavior of the Green functions and, as a consequence, their determination is an important problem. Unfortunately, in general, these coefficients cannot be determined exactly. It is however possible to look for them in the form of a covariant Taylor series expansion for x' in the neighborhood of x

$$A_n(x, x') = a_n(x) - a_{n\mu_1}(x)\sigma^{;\mu_1}(x, x') + \frac{1}{2!} a_{n\mu_1\mu_2}(x)\sigma^{;\mu_1}(x, x')\sigma^{;\mu_2}(x, x') + \cdots$$

[here $\sigma(x, x')$ denotes the geodetic interval between x and x' [1,2]] and to try then to "solve" the recursion relation defining them. This is not an easy task and computational complications increase very rapidly with the orders n and p of the coefficient $a_{n\mu_1\cdots\mu_p}(x)$ which is a scalar of order R^{n+p} constructed from the Riemann tensor R and its derivatives. In fact, during the last 40 years, it is mainly the determination of the so-called diagonal DeWitt coefficients $a_n(x) = A_n(x, x)$ with $n \geq 1$ [we have the trivial result $a_0(x) = 1$] which has attracted the attention of theoretical physicists in connection with renormalization in the effective action for quantum field theories and quantum gravity and with gravitational anomalies. In addition, mathematicians have calculated these coefficients in connection with spectral geometry, topology of manifolds and the Atiyah-Singer index theorem [12]. Among the numerous important results obtained by very different technical approaches, it is worthwhile pointing out the derivation of the following:

(i) $a_1(x)$ and $a_2(x)$ by DeWitt [2] for a scalar field (in the presence of a Yang-Mills background) and for the Dirac spinor field both propagating on an arbitrary curved spacetime;

(ii) $a_3(x)$ by Sakai [13] for an ordinary scalar field theory defined on an arbitrary curved space and by Gilkey [14] for the general case, i.e. for tensorial field theories defined on Riemannian manifolds in the presence of external gauge fields;

(iii) $a_4(x)$ by Amsterdamski, Berkin and O'Connors [15] for an ordinary scalar field and by Avramidi, in Ref. [16] (see also the corresponding erratum [17]), for the general case;

(iv) $a_5(x)$ by van de Ven [18] for the general case.

*Electronic address: decanini@univ-corse.fr
†Electronic address: folacci@univ-corse.fr

The importance, from the physical point of view, of the off-diagonal DeWitt coefficients has been clearly realized in the mid-1970s when interest in the regularization and renormalization of the stress-energy tensor associated with a quantum field propagating on a curved spacetime began to grow [19–21]. Indeed, it appeared that, in this context, the knowledge of the first terms of the covariant Taylor series expansions of the DeWitt coefficients of lowest orders was crucial. Christensen then derived the covariant Taylor series expansions of the DeWitt coefficients $A_0(x, x')$, $A_1(x, x')$, and $A_2(x, x')$ up to orders σ^2, σ^1 and σ^0 respectively for an ordinary scalar field theory in 1976 [20] and for the spin-1/2 and spin-1 theories in 1978 [21]. Christensen's work has had a great impact on quantum field theory in curved spacetime: in connection with the point-splitting prescription [2], it has provided a general technique for the regularization and renormalization of the stress-energy tensor. However, it is important to note that Christensen's results have a limited domain of applicability: they have been used to regularize the stress-energy tensor in a four-dimensional curved spacetime and they could also permit us to develop the regularization process in a three-dimensional curved spacetime but, to our knowledge, this has never been explicitly realized. Nowadays, supergravity theories, string theories and M theory predict that spacetime has more dimensions than the four we observe. In this context, it is therefore necessary to extend Christensen's method taking into account the possible extra dimensions: in order to be able to work in five dimensions, it is necessary to derive the DeWitt coefficients $A_0(x, x')$, $A_1(x, x')$ and $A_2(x, x')$ up to orders $\sigma^{5/2}$, $\sigma^{3/2}$ and $\sigma^{1/2}$; in order to be able to work in six dimensions, it is necessary to derive the DeWitt coefficients $A_0(x, x')$, $A_1(x, x')$, $A_2(x, x')$ and $A_3(x, x')$ up to orders σ^3, σ^2, σ^1 and σ^0 ... in order to be able to work in ten dimensions, it is necessary to derive the DeWitt coefficients $A_0(x, x')$, $A_1(x, x')$, $A_2(x, x')$, $A_3(x, x')$, $A_4(x, x')$ and $A_5(x, x')$ up to orders σ^5, σ^4, σ^3, σ^2 , σ^1 and σ^0; and in order to be able to work in 11 dimensions, it is necessary to derive the DeWitt coefficients $A_0(x, x')$, $A_1(x, x')$, $A_2(x, x')$, $A_3(x, x')$, $A_4(x, x')$ and $A_5(x, x')$ up to orders $\sigma^{11/2}$, $\sigma^{9/2}$, $\sigma^{7/2}$, $\sigma^{5/2}$, $\sigma^{3/2}$ and $\sigma^{1/2}$.

In fact, we do not need to appeal to supergravity theories, string theories and M theory as well as the possible extra dimensions of spacetime to justify the necessity to go beyond Christensen's results. In recent works dealing with four-dimensional gravitational physics, such a necessity has clearly appeared in two different contexts: in the quantum domain of stochastic semiclassical gravity, in connection with the regularization of the noise kernel, but also in the classical domain of gravitational wave theory, in connection with the radiation reaction force. As far as the noise kernel is concerned, it should be recalled that it is a measure of the fluctuations of the stress-energy tensor associated with a quantum field theory defined on a curved spacetime. It is defined as the vacuum expectation value of a bitensor constructed by taking the product of the stress-energy-tensor operator with itself [22,23]. It plays a central role in stochastic semiclassical gravity (see Ref. [24] for a review on this topic) permitting us to define the stochastic part of the source in the Einstein-Langevin equations. Its regularization, in the coincidence limit, necessitates the knowledge of the divergent part of the Feynman propagator up to order σ^2 [23] and therefore the knowledge of the DeWitt coefficients $A_0(x, x')$, $A_1(x, x')$, $A_2(x, x')$ and $A_3(x, x')$ up to orders σ^3, σ^2, σ^1 and σ^0. As far as the radiation reaction force is concerned, it should be recalled that its computation in Schwarzschild and Kerr spacetimes for arbitrary orbits is now an urgent problem of gravitational wave theory (see Ref. [11] for a review as well as Refs. [25,26] for recent important progress). In particular, the computation of the nonlocal part of this force (it is an integral of the retarded Green function over the past trajectory of the particle moving in the gravitational background and which is the source of the gravitational radiation) has been considered in recent works [27,28] and the necessity, in this context, to go beyond Christensen's expansions of the DeWitt coefficients has been pointed out.

On a d-dimensional curved spacetime, the Hadamard representation is constructed from a set of two-point coefficients, the so-called Hadamard coefficients, which are also defined by recursion relations. For d even, there exists 3 families of Hadamard coefficients noted $U_n(x, x')$ with $n = 0, 1, \ldots, d/2 - 2$ and $V_n(x, x')$ and $W_n(x, x')$ with $n \in \mathbb{N}$, while for d odd, there exists 2 families of Hadamard coefficients noted $U_n(x, x')$ and $W_n(x, x')$ with $n \in \mathbb{N}$. The Hadamard coefficients $U_n(x, x')$ and $V_n(x, x')$ are, like the DeWitt coefficients $A_n(x, x')$, purely geometrical objects and here again those of lowest orders encode the short-distance singular behavior of the Green functions. In fact, the Hadamard coefficients $U_n(x, x')$ and $V_n(x, x')$ can be constructed from the DeWitt coefficients $A_n(x, x')$. Thus, the knowledge of the covariant Taylor series expansions for x' in the neighborhood of x of the DeWitt coefficients permits us to construct immediately the corresponding expansions of the geometrical Hadamard coefficients. As far as the coefficients $W_n(x, x')$ are concerned, it is important to note that they correspond to a finite part of the Green functions and that they are neither determined in terms of the local geometry nor uniquely defined by a recursion relation. As a consequence, they can be used to encode supplementary physical information concerning the studied field (boundary conditions, quantum state dependence, ...). Because of that property, the Hadamard representation is in our opinion more interesting than the DeWitt-Schwinger one. Moreover, in the context of the regularization of the stress-energy tensor, the Christensen approach has been replaced by a variant based on the Hadamard representation, the so-called Hadamard method

[29–39]. It is more general than the original method and more efficient. Furthermore, because of its axiomatic foundations [8,29,32], it is more rigorous. It has been developed in a four-dimensional framework and its extension in higher dimensions necessitates the derivation of the covariant Taylor series expansions of the Hadamard coefficients beyond the orders reached in Refs. [29–39].

In the present article, we shall consider the DeWitt-Schwinger and Hadamard representations of the Feynman propagator of a massive scalar field theory and we shall improve these two representations by obtaining higher-order terms for the covariant Taylor series expansions of the coefficients—i.e., the DeWitt and geometrical Hadamard coefficients—that define them. More precisely, we shall first provide the covariant Taylor series expansions of the DeWitt coefficients $A_0(x, x')$, $A_1(x, x')$, $A_2(x, x')$ and $A_3(x, x')$ up to orders σ^3, σ^2, σ^1 and σ^0 respectively. We shall then provide the following:

(i) in three dimensions, the covariant Taylor series expansions of the geometrical Hadamard coefficients $U_0(x, x')$, $U_1(x, x')$, $U_2(x, x')$ and $U_3(x, x')$ up to orders σ^3, σ^2, σ^1 and σ^0 respectively or, in other words, the covariant Taylor series expansion of the divergent part $U(x, x')/\sigma^{1/2}(x, x')$ of the Hadamard representation up to order $\sigma^{5/2}$;

(ii) in four dimensions, the covariant Taylor series expansions of the geometrical Hadamard coefficients $U_0(x, x')$, $V_0(x, x')$, $V_1(x, x')$, and $V_2(x, x')$ up to orders σ^3, σ^2, σ^1 and σ^0 respectively or, in other words, the covariant Taylor series expansions of the divergent parts $U(x, x')/\sigma(x, x')$ and $V(x, x') \times \ln\sigma(x, x')$ of the Hadamard representation up to order σ^2 and $\sigma^2 \ln\sigma$ respectively;

(iii) in five dimensions, the covariant Taylor series expansions of the geometrical Hadamard coefficients $U_0(x, x')$, $U_1(x, x')$, $U_2(x, x')$ and $U_3(x, x')$ up to orders σ^3, σ^2, σ^1 and σ^0 respectively or, in other words, the covariant Taylor series expansion of the divergent part $U(x, x')/\sigma^{3/2}(x, x')$ of the Hadamard representation up to order $\sigma^{3/2}$;

(iv) in six dimensions, the covariant Taylor series expansions of the geometrical Hadamard coefficients $U_0(x, x')$, $U_1(x, x')$, $V_0(x, x')$ and $V_1(x, x')$ up to orders σ^3, σ^2, σ^1 and σ^0 respectively or, in other words, the covariant Taylor series expansions of the divergent parts $U(x, x')/\sigma^2(x, x')$ and $V(x, x') \times \ln\sigma(x, x')$ of the Hadamard representation up to order σ and $\sigma \ln\sigma$ respectively.

Our article is organized as follows. In Sec. II, we establish the framework of our study as well as our notations. In particular, we establish the relationship linking the DeWitt and the geometrical Hadamard coefficients and we also prove that the DeWitt-Schwinger representation possesses the Hadamard form. In Sec. III, by combining the old covariant recursive method of DeWitt [1,2] with results obtained from the modern covariant nonrecursive approach of Avramidi [9,40], we explicitly construct the covariant Taylor series expansions of the DeWitt coefficients $A_0(x, x')$, $A_1(x, x')$, $A_2(x, x')$ and $A_3(x, x')$ up to orders σ^3, σ^2, σ^1 and σ^0 respectively. In Sec. IV, we translate the results previously obtained in the framework of the Hadamard formalism and we provide the explicit expressions for the covariant Taylor series expansions of the corresponding geometrical Hadamard coefficients. Finally, in Sec. V, we discuss possible extensions of our work as well as immediate applications. In five appendixes, we gather some technical details which have been used to derive our results. In these appendixes we have also provided the covariant Taylor series expansions of the bitensors $\Delta^{1/2}$, $\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$, $\sigma_{;\mu\nu}$ and $g_{\nu\nu'}\sigma_{;\mu}{}^{\nu'}$ beyond the orders needed in the present article, i.e. up to orders $\sigma^{11/2}$, $\sigma^{9/2}$, $\sigma^{9/2}$ and $\sigma^{9/2}$ respectively. We think that these results could be very useful in the near future for people working in the field of gravitational physics.

It should be noted that we shall use the geometrical conventions of Hawking and Ellis [41] concerning the definitions of the scalar curvature R, the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\rho\sigma}$ and we shall extensively use the commutation of covariant derivatives in the form

$$T^{\rho\cdots}_{\sigma\cdots;\nu\mu} - T^{\rho\cdots}_{\sigma\cdots;\mu\nu} = +R^{\rho}{}_{\tau\mu\nu}T^{\tau\cdots}_{\sigma\cdots} + \cdots - R^{\tau}{}_{\sigma\mu\nu}T^{\rho\cdots}_{\tau\cdots} - \cdots. \quad (1)$$

II. DEWITT-SCHWINGER AND HADAMARD REPRESENTATIONS

We shall consider a massive scalar field Φ propagating on a d-dimensional curved spacetime $(\mathcal{M}, g)$ and obeying the wave equation

$$(\Box - m^2 - \xi R)\Phi = 0. \quad (2)$$

Here m is the mass of the scalar field, ξ is a dimensionless factor which accounts for the possible coupling between the scalar field and the gravitational background and we shall assume that $d > 2$. We shall focus our attention on the Feynman propagator $G^{\mathrm{F}}(x, x')$ solution of

$$(\Box_x - m^2 - \xi R)G^{\mathrm{F}}(x, x') = -\delta^d(x, x') \quad (3)$$

with $\delta^d(x, x') = [-g(x)]^{-1/2}(x)\delta^d(x - x')$, or more precisely on the way in which its DeWitt-Schwinger and Hadamard representations encode its short-distance behavior. It should be noted that our presentation does not pretend to be mathematically rigorous. It is however possible to find precisions concerning the mathematical status of the DeWitt-Schwinger and Hadamard representations as well as the nature of the series defining them in Refs. [5,7,8,42–45].

A. DeWitt-Schwinger representation of $G^{\mathrm{F}}(x, x')$

We first recall that the DeWitt-Schwinger representation of the Feynman propagator $G^{\mathrm{F}}(x, x')$ is given by (see Refs. [2,6,7,46])

$$G^{\mathrm{F}}_{\mathrm{DS}}(x, x') = i\int_0^{+\infty} H(s; x, x')ds \tag{4}$$

where $H(s; x, x')$ is a function which satisfies

$$\left(i\frac{\partial}{\partial s} + \Box_x - m^2 - \xi R\right)H(s; x, x') = 0 \quad \text{for } s > 0 \tag{5a}$$

with the boundary condition

$$H(s; x, x') \to \delta^d(x, x') \quad \text{as } s \to 0, \tag{5b}$$

and which can be formally written, for $s \to 0$ and x' near x, on the form

$$H(s; x, x') = i(4\pi i s)^{-d/2} e^{(i/2s)[\sigma(x,x')+i\epsilon]-im^2 s} \times \sum_{n=0}^{+\infty} A_n(x, x')(is)^n. \tag{6}$$

Here the factor $i\epsilon$ with $\epsilon \to 0_+$ is introduced to give to $G^{\mathrm{F}}_{\mathrm{DS}}(x, x')$ a singularity structure that is consistent with the definition of the Feynman propagator as a time-ordered product. Furthermore, the DeWitt coefficients $A_n(x, x')$ labeled by $n \in \mathbb{N}$ are a sequence of biscalar functions, symmetric in the exchange of x and x', regular for $x' \to x$, and defined by the recursion relations

$$(n+1)A_{n+1} + A_{n+1;\mu}\sigma^{;\mu} - A_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} = (\Box_x - \xi R)A_n \quad \text{for } n \in \mathbb{N} \tag{7a}$$

and the boundary condition

$$A_0 = \Delta^{1/2}. \tag{7b}$$

In Eqs. (6) and (7a), $\sigma(x, x')$ is the geodetic interval—i.e., $2\sigma(x, x')$ is the square of the geodesic distance between x and x'—and we have $\sigma(x, x') < 0$ if x and x' are timelike related, $\sigma(x, x') = 0$ if x and x' are null related and $\sigma(x, x') > 0$ if x and x' are spacelike related. It is a biscalar function that satisfies

$$2\sigma = \sigma^{;\mu}\sigma_{;\mu}. \tag{8}$$

In Eqs. (7a) and (7b), $\Delta(x, x')$ is the biscalar form of the Van Vleck-Morette determinant [2]. It is defined by

$$\Delta(x, x') = -[-g(x)]^{-1/2}\det(-\sigma_{;\mu\nu'}(x, x'))[-g(x')]^{-1/2} \tag{9}$$

and it satisfies the partial differential equation

$$\Box_x\sigma = d - 2\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \tag{10a}$$

and the boundary condition

$$\lim_{x'\to x}\Delta(x, x') = 1. \tag{10b}$$

The recursion relations (7a), the boundary condition (7b) and the relations (8), (10a), and (10b) insure that the function $H(s; x, x')$ given by (6) is a solution of (5a) and (5b) and therefore that (4) solves the wave equation (3). The DeWitt coefficients $A_n(x, x')$ can be formally obtained by solving the recursion relations (7a) taking into account the boundary condition (7b). This can be realized by integrating along the geodesic joining x to x' (it is unique for x' near x or more generally for x' in a convex normal neighborhood of x). As a consequence, the DeWitt coefficients are determined uniquely and are purely geometrical objects, i.e. they only depend on the geometry along this geodesic.

B. Hadamard representation of $G^{\mathrm{F}}(x, x')$

As far as the structure of the Hadamard representation of the Feynman propagator $G^{\mathrm{F}}(x, x')$ is concerned, we recall that it depends on whether the dimension d of spacetime is even or odd. For d even, it is given by (here we extend considerations developed in Refs. [5,42,43])

$$G^{\mathrm{F}}_{\mathrm{H}}(x, x') = i\frac{(d/2-2)!}{2(2\pi)^{d/2}}\left[\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^{d/2-1}} + V(x, x') \times \ln[\sigma(x, x') + i\epsilon] + W(x, x')\right], \tag{11}$$

where $U(x, x')$, $V(x, x')$ and $W(x, x')$ are symmetric biscalars, regular for $x' \to x$ and which possess expansions of the form

$$U(x, x') = \sum_{n=0}^{d/2-2} U_n(x, x')\sigma^n(x, x'), \tag{12a}$$

$$V(x, x') = \sum_{n=0}^{+\infty} V_n(x, x')\sigma^n(x, x'), \tag{12b}$$

$$W(x, x') = \sum_{n=0}^{+\infty} W_n(x, x')\sigma^n(x, x'). \tag{12c}$$

For d odd, it is given by (see Refs. [5,42,43])

$$G^{\mathrm{F}}_{\mathrm{H}}(x, x') = i\frac{\Gamma(d/2-1)}{2(2\pi)^{d/2}}\left[\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^{d/2-1}} + W(x, x')\right], \tag{13}$$

where $U(x, x')$ and $W(x, x')$ are again symmetric and regular biscalar functions which now possess expansions of the form

$$U(x, x') = \sum_{n=0}^{+\infty} U_n(x, x')\sigma^n(x, x'), \tag{14a}$$

$$W(x, x') = \sum_{n=0}^{+\infty} W_n(x, x')\sigma^n(x, x'). \tag{14b}$$

In Eqs. (11) and (13), the factor $i\epsilon$ with $\epsilon \to 0_+$ is again introduced to give to $G_{\rm H}^{\rm F}(x,x')$ a singularity structure that is consistent with the definition of the Feynman propagator as a time-ordered product.

For d even, the Hadamard coefficients $U_n(x,x')$, $V_n(x,x')$ and $W_n(x,x')$ are symmetric and regular biscalar functions. The coefficients $U_n(x,x')$ satisfy the recursion relations

$$\begin{aligned}(n+1)(2n+4-d)U_{n+1} &+ (2n+4-d)U_{n+1;\mu}\sigma^{;\mu} \\ &- (2n+4-d)U_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \\ &+ (\Box_x - m^2 - \xi R)U_n = 0 \\ &\text{for } n = 0, 1, \ldots, d/2-3 \end{aligned} \quad (15a)$$

with the boundary condition

$$U_0 = \Delta^{1/2}. \quad (15b)$$

The coefficients $V_n(x,x')$ satisfy the recursion relations

$$\begin{aligned}(n+1)(2n+d)V_{n+1} &+ 2(n+1)V_{n+1;\mu}\sigma^{;\mu} \\ &- 2(n+1)V_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \\ &+ (\Box_x - m^2 - \xi R)V_n = 0 \quad \text{for } n \in \mathbb{N} \end{aligned} \quad (16a)$$

with the boundary condition

$$\begin{aligned}(d-2)V_0 + 2V_{0;\mu}\sigma^{;\mu} - 2V_0\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \\ + (\Box_x - m^2 - \xi R)U_{d/2-2} = 0. \end{aligned} \quad (16b)$$

The coefficients $W_n(x,x')$ satisfy the recursion relations

$$\begin{aligned}(n+1)(2n+d)W_{n+1} &+ 2(n+1)W_{n+1;\mu}\sigma^{;\mu} \\ &- 2(n+1)W_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \\ &+ (4n+2+d)V_{n+1} + 2V_{n+1;\mu}\sigma^{;\mu} \\ &- 2V_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \\ &+ (\Box_x - m^2 - \xi R)W_n = 0 \quad \text{for } n \in \mathbb{N}. \end{aligned} \quad (17)$$

From the recursion relations (15a), (16a), and (17), the boundary conditions (15b) and (16b) and the relations (8), (10a), and (10b) it is possible to prove that the Hadamard representation (11) and (12) solves the wave equation (3). This can be done easily by noting that we have

$$(\Box_x - m^2 - \xi R)V = 0 \quad (18)$$

as a consequence of (16a) and

$$\begin{aligned}\sigma(\Box_x - m^2 - \xi R)W = -(\Box_x - m^2 - \xi R)U_{d/2-2} \\ - (d-2)V - 2V_{;\mu}\sigma^{;\mu} \\ + 2V\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \end{aligned} \quad (19)$$

as a consequence of (16b) and (17). The Hadamard coefficients $U_n(x,x')$ can be formally obtained by integrating the recursion relations (15a) along the geodesic joining x to x'. *Mutatis mutandis*, the Hadamard coefficients $V_n(x,x')$ can be obtained by solving the recursion relations (16a). As a consequence, the Hadamard coefficients $U_n(x,x')$ and $V_n(x,x')$ are purely geometric biscalars. As far as the Hadamard coefficients $W_n(x,x')$ are concerned, it should be noted that the biscalar $W_0(x,x')$ is unrestrained by the recursion relations (17). These relations only determine the $W_n(x,x')$ with $n \geq 1$ once $W_0(x,x')$ is specified.

For d odd, the Hadamard coefficients $U_n(x,x')$ and $W_n(x,x')$ are symmetric and regular biscalar functions. The coefficients $U_n(x,x')$ satisfy the recursion relations

$$\begin{aligned}(n+1)(2n+4-d)U_{n+1} &+ (2n+4-d)U_{n+1;\mu}\sigma^{;\mu} \\ &- (2n+4-d)U_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \\ &+ (\Box_x - m^2 - \xi R)U_n = 0 \quad \text{for } n \in \mathbb{N} \end{aligned} \quad (20a)$$

with the boundary condition

$$U_0 = \Delta^{1/2}. \quad (20b)$$

The coefficients $W_n(x,x')$ satisfy the recursion relations

$$\begin{aligned}(n+1)(2n+d)W_{n+1} &+ 2(n+1)W_{n+1;\mu}\sigma^{;\mu} \\ &- 2(n+1)W_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \\ &+ (\Box_x - m^2 - \xi R)W_n = 0 \quad \text{for } n \in \mathbb{N}. \end{aligned} \quad (21)$$

From the recursion relations (20a) and (21), the boundary conditions (20b) and the relations (8), (10a), and (10b) it is possible to prove that the Hadamard representation (13) and (14) solves the wave equation (3). This can be done easily from

$$(\Box_x - m^2 - \xi R)W = 0 \quad (22)$$

which is a consequence of (21). Here again, it should be noted that the Hadamard coefficients $U_n(x,x')$ are purely geometric biscalars which can be formally obtained by integrating the recursion relations (20a) along the geodesic joining x to x'. Here again the biscalar $W_0(x,x')$ is unrestrained by the recursion relations (21).

C. From the DeWitt coefficients $A_n(x,x')$ to the geometrical Hadamard coefficients $U_n(x,x')$ and $V_n(x,x')$

It is possible to establish the relationship linking the DeWitt coefficients $A_n(x,x')$ and the geometrical Hadamard coefficients $U_n(x,x')$ and $V_n(x,x')$. In order to do this, we first introduce a new sequence $\tilde{A}_n(m^2;x,x')$ with $n \in \mathbb{N}$ of geometrical coefficients which we shall call the mass-dependent DeWitt coefficients. They are defined as the sequence of biscalar functions, symmetric in the exchange of x and x', regular for $x' \to x$, which satisfy the recursion relations

$$\begin{aligned}(n+1)\tilde{A}_{n+1} + \tilde{A}_{n+1;\mu}\sigma^{;\mu} - \tilde{A}_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \\ = (\Box_x - m^2 - \xi R)\tilde{A}_n \quad \text{for } n \in \mathbb{N} \end{aligned} \quad (23a)$$

and the boundary condition

$$\tilde{A}_0 = \Delta^{1/2}. \tag{23b}$$

Of course, they are linked to the (ordinary) DeWitt coefficients $A_n(x,x')$. We have

$$A_n(x,x') = \tilde{A}_n(m^2 = 0; x, x') \tag{24}$$

and a direct comparison of Eqs. (7a), (7b), (23a), and (23b) permits us to obtain easily

$$\tilde{A}_n(m^2; x, x') = \sum_{k=0}^{n} \frac{(-1)^k}{k!}(m^2)^k A_{n-k}(x,x'). \tag{25}$$

Now, by comparing the equations (15a), (15b), (16a), and (16b) defining the Hadamard coefficients for d even with the equations (23a) and (23b) defining the mass-dependent DeWitt coefficients, we can obtain

$$U_n(x,x') = \frac{(d/2-2-n)!}{2^n(d/2-2)!}\tilde{A}_n(m^2;x,x') \quad \text{for } n = 0,1,\ldots,d/2-2, \tag{26a}$$

$$V_n(x,x') = \frac{(-1)^{n+1}}{2^{n+d/2-1}n!(d/2-2)!} \times \tilde{A}_{n+d/2-1}(m^2;x,x') \quad \text{for } n \in \mathbb{N}, \tag{26b}$$

and therefore we establish from (25) the relations

$$U_n(x,x') = \frac{(d/2-2-n)!}{2^n(d/2-2)!}\sum_{k=0}^{n}\frac{(-1)^k}{k!}(m^2)^k A_{n-k}(x,x') \quad \text{for } n = 0,1,\ldots,d/2-2, \tag{27a}$$

$$V_n(x,x') = \frac{(-1)^{n+1}}{2^{n+d/2-1}n!(d/2-2)!}\sum_{k=0}^{n+d/2-1}\frac{(-1)^k}{k!} \times (m^2)^k A_{n+d/2-1-k}(x,x') \quad \text{for } n \in \mathbb{N}. \tag{27b}$$

Similarly, by comparing the equations (20a) and (20b) defining the Hadamard coefficients for d odd with the equations (23a) and (23b) defining the mass-dependent DeWitt coefficients, we can obtain

$$U_n(x,x') = \frac{\Gamma(d/2-1-n)}{2^n\Gamma(d/2-1)}\tilde{A}_n(m^2;x,x') \quad \text{for } n \in \mathbb{N} \tag{28}$$

and therefore we establish from (25) the relation

$$U_n(x,x') = \frac{\Gamma(d/2-1-n)}{2^n\Gamma(d/2-1)}\sum_{k=0}^{n}\frac{(-1)^k}{k!}(m^2)^k A_{n-k}(x,x') \quad \text{for } n \in \mathbb{N}. \tag{29}$$

D. Hadamard form of the DeWitt-Schwinger representation

The short-distance behavior of the DeWitt-Schwinger representation $G^{\rm F}_{\rm DS}(x,x')$ of the Feynman propagator does not explicitly appear in its expression given by Eqs. (4), (5a), (5b), and (6). In fact, this behavior is of the same form as that of the Hadamard representation $G^{\rm F}_{\rm H}(x,x')$. Indeed, it is possible to prove that the DeWitt-Schwinger representation is a particular case of the Hadamard one (see Appendix A for details). It corresponds to the Hadamard representation constructed from the biscalar $W_0(x,x')$ given by

$$W_0(x,x') = [\ln(m^2/2) + \gamma - \psi(d/2)]V_0(x,x') - \frac{1}{2^{d/2-1}(d/2-2)!}\Bigg[\sum_{k=0}^{d/2-2}\frac{(-1)^k(m^2)^k}{k!} \times \Bigg(\sum_{\ell=k+1}^{d/2-1}\frac{1}{\ell}\Bigg)A_{d/2-1-k}(x,x') - \sum_{k=0}^{+\infty}\frac{k!}{(m^2)^{k+1}}A_{d/2+k}(x,x')\Bigg] \tag{30}$$

for d even and by

$$W_0(x,x') = -\frac{1}{2^{d/2-1}\Gamma(d/2-1)}\Bigg[\sum_{k=0}^{d/2-3/2}\frac{(-1)^k(m^2)^{k+1/2}}{\Gamma(k+3/2)} \times \pi A_{d/2-3/2-k}(x,x') - \sum_{k=0}^{+\infty}\frac{\Gamma(k+1/2)}{(m^2)^{k+1/2}}A_{d/2-1/2+k}(x,x')\Bigg] \tag{31}$$

for d odd. In Eq. (30), ψ denotes the logarithm derivative of the gamma function and γ is the Euler constant. The pathological behavior for $m^2 \to 0$ (infrared divergence) of this Hadamard coefficient must be noted. Of course, such a behavior also exists for the DeWitt-Schwinger representation $G^{\rm F}_{\rm DS}(x,x')$ of the Feynman propagator. Furthermore, it should be also noted that for $d = 4$ we recover the result derived by Brown and Ottewill in Ref. [33].

III. COVARIANT TAYLOR SERIES EXPANSIONS OF THE DEWITT COEFFICIENTS

In this section, we shall solve the recursion relations (7a) and (7b) by looking for their solutions $A_n(x,x')$ with $n = 0$, 1, 2 and 3 as covariant Taylor series expansions for x' near x of the form

$$A_n(x,x') = a_n(x) + \sum_{p=1}^{+\infty}\frac{(-1)^p}{p!}a_{n(p)}(x,x') \tag{32}$$

where the $a_{n(p)}(x,x')$ with $p = 1, 2, \ldots$ are biscalars in x and x' which are of the form

$$a_{n(p)}(x,x') = a_{na_1\cdots a_p}(x)\sigma^{;a_1}(x,x')\cdots\sigma^{;a_p}(x,x'). \tag{33}$$

In fact, we shall first construct the covariant Taylor series expansions of the mass-dependent DeWitt coefficients $\tilde{A}_n(m^2;x,x')$ with $n = 0$, 1, 2 and 3 defined by (23a) and

(23b). Indeed, from these results, we shall then immediately obtain the expansions of the DeWitt coefficients $A_n(x, x')$ with $n = 0$, 1, 2 and 3 by using (24) and, in the next section, we shall be able to easily obtain the expansions of the corresponding geometrical Hadamard coefficients by using (26) and (28). We shall write the covariant Taylor series expansions of the mass-dependent DeWitt coefficients in the form

$$\tilde{A}_n(m^2; x, x') = \tilde{a}_n(m^2; x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!} \tilde{a}_{n(p)}(m^2; x, x') \tag{34}$$

where the $\tilde{a}_{n(p)}(m^2; x, x')$ with $p = 1, 2, \ldots$ are biscalars in x and x' which are of the form

$$\tilde{a}_{n(p)}(m^2; x, x') = \tilde{a}_{na_1\cdots a_p}(m^2; x)\sigma^{;a_1}(x, x') \cdots \sigma^{;a_p}(x, x'). \tag{35}$$

We shall use the covariant recursive method invented by DeWitt [1,2] and developed by many others (see Refs. [20,21,23,28] and references therein). This method requires preliminarily the knowledge of the covariant Taylor series expansions of various bitensors such as $\sigma_{;\mu\nu}$, $g_{\nu\nu'}\sigma_{;\mu}{}^{\nu'}$, $\Delta^{1/2}$, $\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$, $\Box\Delta^{1/2}$ .... Here $g_{\mu\nu'}$ denotes the bivector of parallel transport from x to x' (see Refs. [1,2]) which is defined by the partial differential equation

$$g_{\mu\nu';\rho}\sigma^{;\rho} = 0 \tag{36a}$$

and the boundary condition

$$\lim_{x'\to x} g_{\mu\nu'} = g_{\mu\nu}. \tag{36b}$$

The construction of all the previously mentioned expansions is a rather hard task. DeWitt has shown that it necessitates the knowledge of the coincidence limits

$$\lim_{x'\to x} \sigma_{;a_1\cdots a_p}. \tag{37}$$

They can be obtained by repeatedly differentiating the relation (8) and can be expressed as complicated sums of terms involving products of derivatives of the Riemann tensor. Unfortunately, obtaining the coincidence limits (37) becomes more and more difficult as the order p increases (see the discussion on pp. 180–183 of Ref. [7]) and is even a formidable computational challenge (see the "recent" analysis by Christensen in Ref. [47]). In the 1960s, DeWitt derived the coincidence limits (37) up to order $p = 4$ and the covariant Taylor series expansion of $\Delta^{1/2}$ up to order σ [1,2]. In the mid-1970s, Christensen was able to obtain them up to orders $p = 6$ and σ^2 respectively [20,21] and in the mid-1980s, Brown and Ottewill slightly improved Christensen's results by reaching the orders $p = 7$ (the corresponding results do not appear in their article but they appear in a recent article by Anderson, Flanagan and Ottewill [28]) and $\sigma^{5/2}$ respectively. It should be also noted that Phillips and Hu in Ref. [23] claim to have reached the order $p = 8$ for the coincidence limits (37) but we think that their results are not correct because they lead to covariant Taylor series expansions of $\sigma_{;\mu\nu}$ and $\Delta^{1/2}$ up to order σ^3 which are wrong (see Appendixes B and C).

Happily, as early as 1986, Avramidi introduced in his Ph.D. thesis (see Ref. [40] for the English translation and Ref. [9] for a revised and expanded version) a set of new and powerful nonrecursive techniques permitting the construction of the covariant Taylor series expansions of various bitensors needed in quantum gravity which avoid the preliminary construction of the coincidence limits (37). By using Avramidi's techniques, we have explicitly obtained all the covariant Taylor series expansions of the bitensors we need in order to solve the recursion relations (23a) and (23b) up to the orders announced in Sec. I. All our results are displayed in Appendixes B, C, and E. In these appendixes, we have also provided the covariant Taylor series expansions of $\Delta^{1/2}$, $\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$, $\sigma_{;\mu\nu}$ and $g_{\nu\nu'}\sigma_{;\mu}{}^{\nu'}$ beyond the orders needed here, i.e. up to orders $\sigma^{11/2}$, $\sigma^{9/2}$, $\sigma^{9/2}$ and $\sigma^{9/2}$ respectively. Such results show the power of Avramidi's techniques. In fact, even if we do not need them in the present article, we think that they could be very useful in the near future for other people working in the field of gravitational physics. Furthermore, we shall use them in our next article [48] where we intend to develop the Hadamard regularization of the stress-energy tensor for a quantized scalar field in a general spacetime of arbitrary dimension.

In summary, we shall now solve the recursion relations (23a) and (23b) [and therefore the recursion relations (7a) and (7b)] by combining the old covariant recursive method of DeWitt with results obtained from the modern covariant nonrecursive techniques developed by Avramidi. In order to simplify our calculations, we shall in addition use the symmetry of the mass-dependent DeWitt coefficients $\tilde{A}_n(m^2; x, x')$ with $n \in \mathbb{N}$ in the exchange of x and x'. This property induces constraints on the coefficients $\tilde{a}_{n(p)}(m^2; x, x')$ with p odd and, in Appendix D, we have obtained and displayed various associated results which will be very useful in this section. In the same appendix, we have also collected important results concerning the covariant Taylor series expansions of the covariant derivative, the second covariant derivative and the d'Alembertian of an arbitrary biscalar.

A. Covariant Taylor series expansion of $\tilde{A}_0(m^2; x, x')$

The mass-dependent DeWitt coefficient $\tilde{A}_0(m^2; x, x')$ is equal to $\Delta^{1/2}(x, x')$ [see Eq. (23b)]. Its covariant Taylor series expansion is then given by [see Appendix C and Eqs. (C7), (C8), and (C10a)–(C10e)]

$$\tilde{A}_0 = \tilde{a}_0 - \tilde{a}_{0a}\sigma^{;a} + \frac{1}{2!}\tilde{a}_{0ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}\tilde{a}_{0abc}\sigma^{;a}\sigma^{;b}\sigma^{;c} + \frac{1}{4!}\tilde{a}_{0abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} - \frac{1}{5!}\tilde{a}_{0abcde}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}\sigma^{;e} + \frac{1}{6!}\tilde{a}_{0abcdef}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}\sigma^{;e}\sigma^{;f} + O(\sigma^{7/2}) \quad (38)$$

with

$$\tilde{a}_0 = 1, \quad (39a)$$
$$\tilde{a}_{0a} = 0, \quad (39b)$$
$$\tilde{a}_{0ab} = (1/6)R_{ab}, \quad (39c)$$
$$\tilde{a}_{0abc} = (1/4)R_{(ab;c)}, \quad (39d)$$
$$\tilde{a}_{0abcd} = (3/10)R_{(ab;cd)} + (1/15)R^{\rho}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d)} + (1/12)R_{(ab}R_{cd)}, \quad (39e)$$
$$\tilde{a}_{0abcde} = (1/3)R_{(ab;cde)} + (1/3)R^{\rho}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d;e)} + (5/12)R_{(ab}R_{cd;e)}, \quad (39f)$$
$$\tilde{a}_{0abcdef} = (5/14)R_{(ab;cdef)} + (4/7)R^{\rho}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d;ef)} + (15/28)R^{\rho}{}_{(a|\tau|b;c}R^{\tau}{}_{d|\rho|e;f)} + (3/4)R_{(ab}R_{cd;ef)} + (5/8)R_{(ab;c}R_{de;f)} + (8/63)R^{\rho}{}_{(a|\tau|b}R^{\tau}{}_{c|\sigma|d}R^{\sigma}{}_{e|\rho|f)} + (1/6)R_{(ab}R^{\rho}{}_{c|\tau|d}R^{\tau}{}_{e|\rho|f)} + (5/72)R_{(ab}R_{cd}R_{ef)}, \quad (39g)$$

and we can also write

$$\tilde{A}_0 = 1 + \frac{1}{12}R_{ab}\sigma^{;a}\sigma^{;b} - \frac{1}{24}R_{ab;c}\sigma^{;a}\sigma^{;b}\sigma^{;c} + \left[\frac{1}{80}R_{ab;cd} + \frac{1}{360}R^{\rho}{}_{a\tau b}R^{\tau}{}_{c\rho d} + \frac{1}{288}R_{ab}R_{cd}\right]\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} - \left[\frac{1}{360}R_{ab;cde} + \frac{1}{360}R^{\rho}{}_{a\tau b}R^{\tau}{}_{c\rho d;e} + \frac{1}{288}R_{ab}R_{cd;e}\right]\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}\sigma^{;e} + \left[\frac{1}{2016}R_{ab;cdef} + \frac{1}{1260}R^{\rho}{}_{a\tau b}R^{\tau}{}_{c\rho d;ef} + \frac{1}{1344}R^{\rho}{}_{a\tau b;c}R^{\tau}{}_{d\rho e;f} + \frac{1}{960}R_{ab}R_{cd;ef} + \frac{1}{1152}R_{ab;c}R_{de;f} + \frac{1}{5670}R^{\rho}{}_{a\tau b}R^{\tau}{}_{c\sigma d}R^{\sigma}{}_{e\rho f} + \frac{1}{4320}R_{ab}R^{\rho}{}_{c\tau d}R^{\tau}{}_{e\rho f} + \frac{1}{10\,368}R_{ab}R_{cd}R_{ef}\right]\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}\sigma^{;e}\sigma^{;f} + O(\sigma^{7/2}). \quad (40)$$

B. Covariant Taylor series expansion of $\tilde{A}_1(m^2; x, x')$

The mass-dependent DeWitt coefficient $\tilde{A}_1(m^2; x, x')$ is the solution of Eq. (23a) with $n = 0$, i.e. it satisfies

$$\tilde{A}_1 + \tilde{A}_{1;\mu}\sigma^{;\mu} - \tilde{A}_1\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} = (\Box_x - m^2 - \xi R)\tilde{A}_0. \quad (41)$$

In this equation, we replace $\tilde{A}_1$ by its covariant Taylor series expansion for x' in the neighborhood of x given by

$$\tilde{A}_1 = \tilde{a}_1 - \tilde{a}_{1a}\sigma^{;a} + \frac{1}{2!}\tilde{a}_{1ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}\tilde{a}_{1abc}\sigma^{;a}\sigma^{;b}\sigma^{;c} + \frac{1}{4!}\tilde{a}_{1abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} + O(\sigma^{5/2}). \quad (42)$$

By using the covariant Taylor series expansion of $\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$ constructed in Appendix C [see Eq. (C18)] as well as the constraints induced by the symmetry of $\tilde{A}_1(m^2; x, x')$ under the exchange of x and x' [see Appendix D and Eqs. (D13a), (D13b), and (D15)], we can easily obtain the covariant Taylor series expansion of the left-hand side of Eq. (41) up to order σ^2. The covariant Taylor series expansion of the right-hand side of Eq. (41) up to order σ^2 can be found from the expansions of $\Box\tilde{A}_0 = \Box\Delta^{1/2}$ and $\tilde{A}_0 = \Delta^{1/2}$ respectively given by (E12) and (E13) or (E14) and by (38) and (39) or (40). The direct comparison of the expansions of the left- and right-hand sides of Eq. (41) yields the coefficients $\tilde{a}_1$, $\tilde{a}_{1a}$, $\tilde{a}_{1ab}$, $\tilde{a}_{1abc}$ and $\tilde{a}_{1abcd}$:

$$\tilde{a}_1 = -m^2 - (\xi - 1/6)R, \quad (43a)$$
$$\tilde{a}_{1a} = -(1/2)(\xi - 1/6)R_{;a}, \quad (43b)$$
$$\tilde{a}_{1ab} = (1/60)\Box R_{ab} - (1/3)(\xi - 3/20)R_{;ab} - (1/6)m^2R_{ab} - (1/6)(\xi - 1/6)RR_{ab} - (1/45)R^{\rho}{}_aR_{\rho b} + (1/90)R^{\rho\sigma}R_{\rho a\sigma b} + (1/90)R^{\rho\sigma\tau}{}_aR_{\rho\sigma\tau b}, \quad (43c)$$
$$\tilde{a}_{1abc} = -(1/4)(\xi - 2/15)R_{;(abc)} + (1/40)(\Box R_{(ab})_{;c)} - (1/4)m^2R_{(ab;c)} - (1/4)(\xi - 1/6)RR_{(ab;c)} - (1/4)(\xi - 1/6)R_{;(a}R_{bc)} - (1/15)R^{\rho}{}_{(a}R_{|\rho|b;c)} + (1/60)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;c)} + (1/60)R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c)} + (1/30)R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;c)}, \quad (43d)$$

and

$$\begin{aligned}\tilde{a}_{1abcd} = {} & (1/35)(\Box R_{(ab)})_{;cd)} - (1/5)(\xi - 5/42)R_{;(abcd)} - (3/10)m^2R_{(ab;cd)} - (3/10)(\xi - 1/6)RR_{(ab;cd)} \\ & - (1/2)(\xi - 1/6)R_{;(a}R_{bc;d)} - (1/3)(\xi - 3/20)R_{;(ab}R_{cd)} + (1/60)R_{(ab}\Box R_{cd)} - (1/12)m^2R_{(ab}R_{cd)} \\ & - (3/35)R^{\rho}{}_{(a}R_{|\rho|b;cd)} + (1/105)R^{\rho}{}_{(a}R_{bc;|\rho|d)} - (11/210)R^{\rho}{}_{(a;b}R_{|\rho|c;d)} - (3/70)R^{\rho}{}_{(a;b}R_{cd);\rho} \\ & + (17/840)R_{(ab}{}^{;\rho}R_{cd);\rho} + (2/105)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;cd)} + (1/105)R^{\rho}{}_{(a;|\sigma|}R^{\sigma}{}_{b|\rho|c;d)} + (1/30)R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c;d)} \\ & - (4/175)R^{\rho}{}_{(a;|\sigma|b}R^{\sigma}{}_{c|\rho|d)} + (11/525)R_{(ab}{}^{;\rho}{}_{\sigma}R^{\sigma}{}_{c|\rho|d)} + (11/525)R^{\rho}{}_{\sigma;(ab}R^{\sigma}{}_{c|\rho|d)} + (4/525)R^{\rho}{}_{(a|\sigma|b}\Box R^{\sigma}{}_{c|\rho|d)} \\ & - (1/15)m^2R^{\rho}{}_{(a|\sigma|b}R^{\sigma}{}_{c|\rho|d)} + (4/105)R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;cd)} + (1/140)R^{\rho}{}_{(a|\sigma|b}{}^{;\tau}R^{\sigma}{}_{c|\rho|d);\tau} + (1/28)R^{\rho\sigma\tau}{}_{(a;b}R_{|\rho\sigma\tau|c;d)} \\ & - (1/12)(\xi - 1/6)RR_{(ab}R_{cd)} - (1/45)R^{\rho}{}_{(a}R_{|\rho|b}R_{cd)} + (1/315)R^{\rho}{}_{(a}R_{|\sigma|b}R^{\sigma}{}_{c|\rho|d)} + (1/90)R^{\rho\sigma}R_{(ab}R_{|\rho|c|\sigma|d)} \\ & - (1/15)(\xi - 1/6)RR^{\rho}{}_{(a|\sigma|b}R^{\sigma}{}_{c|\rho|d)} + (1/90)R_{(ab}R^{\rho\sigma\tau}{}_{c}R_{|\rho\sigma\tau|d)} + (26/1575)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d)} \\ & + (2/63)R^{\rho}{}_{(a}R^{\sigma}{}_{b}{}^{\tau}{}_{c}R_{|\rho\sigma\tau|d)} + (4/1575)R^{\rho\sigma\tau\kappa}R_{\rho(a|\tau|b}R_{|\sigma|c|\kappa|d)} + (4/525)R^{\rho\kappa\tau}{}_{(a}R_{|\rho\tau|}{}^{\sigma}{}_{b}R_{|\sigma|c|\kappa|d)} \\ & + (16/1575)R^{\rho\kappa\tau}{}_{(a}R_{|\rho|}{}^{\sigma}{}_{|\tau|b}R_{|\sigma|c|\kappa|d)} + (8/1575)R^{\rho\tau\kappa}{}_{(a}R_{|\rho\tau|}{}^{\sigma}{}_{b}R_{|\sigma|c|\kappa|d)}. \end{aligned} \tag{43e}$$

By replacing (43) into (42) we can then write

$$\begin{aligned}\tilde{A}_1 = {} & -m^2 - \left(\xi - \frac{1}{6}\right)R + \frac{1}{2}\left(\xi - \frac{1}{6}\right)R_{;a}\sigma^{;a} + \left[\frac{1}{120}\Box R_{ab} - \frac{1}{6}\left(\xi - \frac{3}{20}\right)R_{;ab} - \frac{1}{12}m^2R_{ab} - \frac{1}{12}\left(\xi - \frac{1}{6}\right)RR_{ab} \right. \\ & \left. - \frac{1}{90}R^{\rho}{}_{a}R_{\rho b} + \frac{1}{180}R^{\rho\sigma}R_{\rho a\sigma b} + \frac{1}{180}R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b}\right]\sigma^{;a}\sigma^{;b} + \left[\frac{1}{24}\left(\xi - \frac{2}{15}\right)R_{;abc} - \frac{1}{240}(\Box R_{ab})_{;c} + \frac{1}{24}m^2R_{ab;c} \right. \\ & + \frac{1}{24}\left(\xi - \frac{1}{6}\right)RR_{ab;c} + \frac{1}{24}\left(\xi - \frac{1}{6}\right)R_{;a}R_{bc} + \frac{1}{90}R^{\rho}{}_{a}R_{\rho b;c} - \frac{1}{360}R^{\rho}{}_{\sigma}R^{\sigma}{}_{a\rho b;c} - \frac{1}{360}R^{\rho}{}_{\sigma;a}R^{\sigma}{}_{b\rho c} \\ & \left. - \frac{1}{180}R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b;c}\right]\sigma^{;a}\sigma^{;b}\sigma^{;c} + \left[\frac{1}{840}(\Box R_{ab})_{;cd} - \frac{1}{120}\left(\xi - \frac{5}{42}\right)R_{;abcd} - \frac{1}{80}m^2R_{ab;cd} - \frac{1}{80}\left(\xi - \frac{1}{6}\right)RR_{ab;cd} \right. \\ & - \frac{1}{48}\left(\xi - \frac{1}{6}\right)R_{;a}R_{bc;d} - \frac{1}{72}\left(\xi - \frac{3}{20}\right)R_{;ab}R_{cd} + \frac{1}{1440}R_{ab}\Box R_{cd} - \frac{1}{288}m^2R_{ab}R_{cd} - \frac{1}{280}R^{\rho}{}_{a}R_{\rho b;cd} \\ & + \frac{1}{2520}R^{\rho}{}_{a}R_{bc;\rho d} - \frac{11}{5040}R^{\rho}{}_{a;b}R_{\rho c;d} - \frac{1}{560}R^{\rho}{}_{a;b}R_{cd;\rho} + \frac{17}{20\,160}R_{ab}{}^{;\rho}R_{cd;\rho} + \frac{1}{1260}R^{\rho}{}_{\sigma}R^{\sigma}{}_{a\rho b;cd} \\ & + \frac{1}{2520}R^{\rho}{}_{a;\sigma}R^{\sigma}{}_{b\rho c;d} + \frac{1}{720}R^{\rho}{}_{\sigma;a}R^{\sigma}{}_{b\rho c;d} - \frac{1}{1050}R^{\rho}{}_{a;\sigma b}R^{\sigma}{}_{c\rho d} + \frac{11}{12\,600}R_{ab}{}^{;\rho}{}_{\sigma}R^{\sigma}{}_{c\rho d} + \frac{11}{12\,600}R^{\rho}{}_{\sigma;ab}R^{\sigma}{}_{c\rho d} \\ & + \frac{1}{3150}R^{\rho}{}_{a\sigma b}\Box R^{\sigma}{}_{c\rho d} - \frac{1}{360}m^2R^{\rho}{}_{a\sigma b}R^{\sigma}{}_{c\rho d} + \frac{1}{630}R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b;cd} + \frac{1}{3360}R^{\rho}{}_{a\sigma b}{}^{;\tau}R^{\sigma}{}_{c\rho d;\tau} + \frac{1}{672}R^{\rho\sigma\tau}{}_{a;b}R_{\rho\sigma\tau c;d} \\ & - \frac{1}{288}\left(\xi - \frac{1}{6}\right)RR_{ab}R_{cd} - \frac{1}{1080}R^{\rho}{}_{a}R_{\rho b}R_{cd} + \frac{1}{7560}R^{\rho}{}_{a}R_{\sigma b}R^{\sigma}{}_{c\rho d} + \frac{1}{2160}R^{\rho\sigma}R_{ab}R_{\rho c\sigma d} \\ & - \frac{1}{360}\left(\xi - \frac{1}{6}\right)RR^{\rho}{}_{a\sigma b}R^{\sigma}{}_{c\rho d} + \frac{1}{2160}R_{ab}R^{\rho\sigma\tau}{}_{c}R_{\rho\sigma\tau d} + \frac{13}{18\,900}R^{\rho}{}_{\sigma}R^{\sigma}{}_{a\tau b}R^{\tau}{}_{c\rho d} + \frac{1}{756}R^{\rho}{}_{a}R^{\sigma}{}_{b}{}^{\tau}{}_{c}R_{\rho\sigma\tau d} \\ & + \frac{1}{9450}R^{\rho\sigma\tau\kappa}R_{\rho a\tau b}R_{\sigma c\kappa d} + \frac{1}{3150}R^{\rho\kappa\tau}{}_{a}R_{\rho\tau}{}^{\sigma}{}_{b}R_{\sigma c\kappa d} + \frac{2}{4725}R^{\rho\kappa\tau}{}_{a}R_{\rho}{}^{\sigma}{}_{\tau b}R_{\sigma c\kappa d} \\ & \left. + \frac{1}{4725}R^{\rho\tau\kappa}{}_{a}R_{\rho\tau}{}^{\sigma}{}_{b}R_{\sigma c\kappa d}\right]\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} + O(\sigma^{5/2}). \end{aligned} \tag{44}$$

C. Covariant Taylor series expansion of $\tilde{A}_2(m^2; x, x')$

The mass-dependent DeWitt coefficient $\tilde{A}_2(m^2; x, x')$ is the solution of Eq. (23a) with $n = 1$, i.e. it satisfies

$$\begin{aligned} & 2\tilde{A}_2 + \tilde{A}_{2;\mu}\sigma^{;\mu} - \tilde{A}_2\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \\ & \quad = (\Box_x - m^2 - \xi R)\tilde{A}_1. \end{aligned} \tag{45}$$

In this equation, we replace $\tilde{A}_2$ by its covariant Taylor series expansion for x' in the neighborhood of x given by

$$\tilde{A}_2 = \tilde{a}_2 - \tilde{a}_{2a}\sigma^{;a} + \frac{1}{2!}\tilde{a}_{2ab}\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}). \tag{46}$$

By using the covariant Taylor series expansion of

$\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$ given in Eq. (C18) as well as the constraints induced by the symmetry of $\tilde{A}_2(x,x')$ under the exchange of x and x' [see Appendix D and Eqs. (D13a) and (D15)], we can easily obtain the covariant Taylor series expansion of the left-hand side of Eq. (45) up to order σ. The covariant Taylor series expansion of the right-hand side of Eq. (45) up to order σ can be found from the expansion $\tilde{A}_1$ given by (42) and (43) or (44) and from the expansion of $\Box\tilde{A}_1$. The latter can be constructed by using the theory developed in Appendix D and more particularly Eqs. (D5), (D8), and (D14). After the most tedious calculation of this article using extensively the commutation of covariant derivatives in the form (1) as well as the Bianchi identities (E1) and their consequences (E2)–(E4), we obtain

$$\Box\tilde{A}_1 = \tilde{a}_1'' - \tilde{a}_{1a}''\sigma^{;a} + \frac{1}{2!}\tilde{a}_{1ab}''\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}) \quad (47)$$

with

$$\tilde{a}_1'' = -(1/3)(\xi - 1/5)\Box R - (1/6)m^2R - (1/6)(\xi - 1/6)R^2 - (1/90)R_{\rho\sigma}R^{\rho\sigma} + (1/90)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}, \quad (48a)$$

$$\tilde{a}_{1a}'' = -(1/12)(\xi - 1/5)(\Box R)_{;a} - (1/12)(\xi - 1/6)RR_{;a} - (1/180)R_{\rho\sigma}R^{\rho\sigma}{}_{;a} + (1/180)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}{}_{;a}, \quad (48b)$$

and

$$\begin{aligned}
\tilde{a}_{1ab}'' = {} & (1/210)\Box\Box R_{ab} - (1/30)(\xi - 1/7)(\Box R)_{;ab} - (1/20)m^2\Box R_{ab} + (1/60)m^2R_{;ab} - (7/180)(\xi - 1/7)RR_{;ab} \\
& - (2/45)(\xi - 1/7)R_{;\rho(a}R^{\rho}{}_{b)} - (1/18)(\xi - 1/5)(\Box R)R_{ab} + (2/15)(\xi - 3/14)R_{;\rho}R^{\rho}{}_{(a;b)} \\
& - (2/15)(\xi - 17/84)R_{;\rho}R_{ab}{}^{;\rho} - (1/20)(\xi - 2/9)R\Box R_{ab} - (1/36)m^2RR_{ab} - (1/63)R_{\rho(a}\Box R^{\rho}{}_{b)} \\
& + (1/15)m^2R_{\rho a}R^{\rho}{}_{b} - (1/350)R^{\rho\sigma}R_{\rho\sigma;(ab)} - (2/525)R^{\rho\sigma}R_{\rho(a;b)\sigma} + (8/1575)R^{\rho\sigma}R_{ab;\rho\sigma} + (1/315)R^{\rho}{}_{a;\sigma}R_{\rho b}{}^{;\sigma} \\
& - (1/63)R^{\rho}{}_{a;\sigma}R^{\sigma}{}_{b;\rho} - (4/45)(\xi - 3/14)R^{;\rho\sigma}R_{\rho a\sigma b} + (2/315)(\Box R^{\rho\sigma})R_{\rho a\sigma b} - (1/30)m^2R^{\rho\sigma}R_{\rho a\sigma b} \\
& + (2/315)R^{\rho\sigma;\tau}R_{\tau\sigma\rho(a;b)} + (1/225)R^{\rho\sigma}\Box R_{\rho a\sigma b} + (1/105)R^{\rho\sigma;\tau}R_{\rho a\sigma b;\tau} - (8/1575)R^{\rho\sigma;\tau}{}_{(a}R_{|\tau\sigma\rho|b)} \\
& + (23/1575)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\tau\sigma\rho|b)} - (4/225)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\rho\sigma\tau|b)} - (2/175)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau(a;b)\kappa} + (16/1575)R^{\rho\sigma\tau}{}_{a}\Box R_{\rho\sigma\tau b} \\
& - (1/30)m^2R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} + (23/3150)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa;(ab)} + (1/105)R^{\rho\sigma\tau}{}_{a;\kappa}R_{\rho\sigma\tau b}{}^{;\kappa} + (1/1260)R^{\rho\sigma\tau\kappa}{}_{;a}R_{\rho\sigma\tau\kappa;b} \\
& - (1/36)(\xi - 1/6)R^2R_{ab} + (1/15)(\xi - 2/9)RR_{\rho a}R^{\rho}{}_{b} - (1/540)R^{\rho\sigma}R_{\rho\sigma}R_{ab} + (4/945)R^{\rho\sigma}R_{\rho a}R_{\sigma b} \\
& - (1/30)(\xi - 2/9)RR^{\rho\sigma}R_{\rho a\sigma b} - (2/945)R^{\rho\tau}R^{\sigma}{}_{\tau}R_{\rho a\sigma b} + (32/4725)R^{\rho\sigma}R^{\tau}{}_{(a}R_{|\tau\sigma\rho|b)} \\
& + (2/4725)R_{\rho\sigma}R^{\rho\kappa\sigma\lambda}R_{\kappa a\lambda b} - (1/30)(\xi - 2/9)RR^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} + (1/540)R_{ab}R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa} \\
& + (31/4725)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\kappa\lambda b} - (1/75)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\lambda\kappa b} + (17/4725)R^{\rho\sigma}R^{\kappa\lambda}{}_{\rho a}R_{\kappa\lambda\sigma b} \\
& - (17/1890)R^{\kappa}{}_{(a}R^{\rho\sigma\tau}{}_{|\kappa|}R_{|\rho\sigma\tau|b)} - (34/4725)R^{\rho\sigma\tau}{}_{\lambda}R_{\rho\sigma\tau\kappa}R^{\lambda}{}_{a}{}^{\kappa}{}_{b} + (4/189)R^{\rho\kappa\sigma\lambda}R^{\tau}{}_{\rho\sigma a}R_{\tau\kappa\lambda b} \\
& - (2/225)R^{\rho\kappa\sigma\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b} + (76/4725)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b}. \quad (48c)
\end{aligned}$$

The direct comparison of the expansions of the left- and right-hand sides of Eq. (45) then yields the coefficients $\tilde{a}_2$, $\tilde{a}_{2a}$ and $\tilde{a}_{2ab}$. We have

$$\begin{aligned}
\tilde{a}_2 = {} & (1/2)m^4 - (1/6)(\xi - 1/5)\Box R + (\xi - 1/6)m^2R + (1/2)(\xi - 1/6)^2R^2 \\
& - (1/180)R_{\rho\sigma}R^{\rho\sigma} + (1/180)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}, \quad (49a)
\end{aligned}$$

$$\begin{aligned}
\tilde{a}_{2a} = {} & -(1/12)(\xi - 1/5)(\Box R)_{;a} + (1/2)(\xi - 1/6)m^2R_{;a} + (1/2)(\xi - 1/6)^2RR_{;a} \\
& - (1/180)R_{\rho\sigma}R^{\rho\sigma}{}_{;a} + (1/180)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}{}_{;a}, \quad (49b)
\end{aligned}$$

and

$$\begin{aligned}\tilde{a}_{2ab} = {} & (1/840)\Box\Box R_{ab} - (1/20)(\xi - 4/21)(\Box R)_{;ab} - (1/60)m^2\Box R_{ab} + (1/3)(\xi - 3/20)m^2 R_{;ab} + (1/12)m^4 R_{ab} \\ & + (1/3)(\xi - 1/6)(\xi - 3/20)RR_{;ab} - (1/90)(\xi - 1/7)R_{;\rho(a}R^{\rho}{}_{b)} - (1/36)(\xi - 1/5)(\Box R)R_{ab} + (1/4)(\xi - 1/6)^2 R_{;a}R_{;b} \\ & + (1/30)(\xi - 3/14)R_{;\rho}R^{\rho}{}_{(a;b)} - (1/30)(\xi - 17/84)R_{;\rho}R_{ab}{}^{;\rho} + (1/6)(\xi - 1/6)m^2 RR_{ab} - (1/60)(\xi - 1/6)R\Box R_{ab} \\ & - (1/252)R_{\rho(a}\Box R^{\rho}{}_{b)} + (1/45)m^2 R_{\rho a}R^{\rho}{}_{b} - (11/3150)R^{\rho\sigma}R_{\rho\sigma;(ab)} - (1/360)R^{\rho\sigma}{}_{;a}R_{\rho\sigma;b} - (1/1050)R^{\rho\sigma}R_{\rho(a;b)\sigma} \\ & + (2/1575)R^{\rho\sigma}R_{ab;\rho\sigma} + (1/1260)R^{\rho}{}_{a;\sigma}R_{\rho b}{}^{;\sigma} - (1/252)R^{\rho}{}_{a;\sigma}R^{\sigma}{}_{b;\rho} - (1/45)(\xi - 3/14)R^{;\rho\sigma}R_{\rho a\sigma b} \\ & + (1/630)(\Box R^{\rho\sigma})R_{\rho a\sigma b} - (1/90)m^2 R^{\rho\sigma}R_{\rho a\sigma b} + (1/630)R^{\rho\sigma;\tau}R_{\tau\sigma\rho(a;b)} + (1/900)R^{\rho\sigma}\Box R_{\rho a\sigma b} \\ & + (1/420)R^{\rho\sigma;\tau}R_{\rho a\sigma b;\tau} - (2/1575)R^{\rho\sigma;\tau}{}_{(a}R_{|\tau\sigma\rho|b)} + (23/6300)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\tau\sigma\rho|b)} - (1/225)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\rho\sigma\tau|b)} \\ & - (1/350)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau(a;b)\kappa} + (4/1575)R^{\rho\sigma\tau}{}_{a}\Box R_{\rho\sigma\tau b} - (1/90)m^2 R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} + (29/6300)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa;(ab)} \\ & + (1/420)R^{\rho\sigma\tau}{}_{a;\kappa}R_{\rho\sigma\tau b}{}^{;\kappa} + (1/336)R^{\rho\sigma\tau\kappa}{}_{;a}R_{\rho\sigma\tau\kappa;b} + (1/12)(\xi - 1/6)^2 R^2 R_{ab} + (1/45)(\xi - 1/6)RR_{\rho a}R^{\rho}{}_{b} \\ & - (1/1080)R^{\rho\sigma}R_{\rho\sigma}R_{ab} + (1/945)R^{\rho\sigma}R_{\rho a}R_{\sigma b} - (1/90)(\xi - 1/6)RR^{\rho\sigma}R_{\rho a\sigma b} - (1/1890)R^{\rho\tau}R^{\sigma}{}_{\tau}R_{\rho a\sigma b} \\ & + (8/4725)R^{\rho\sigma}R^{\tau}{}_{(a}R_{|\tau\sigma\rho|b)} + (1/9450)R_{\rho\sigma}R^{\rho\kappa\sigma\lambda}R_{\kappa a\lambda b} - (1/90)(\xi - 1/6)RR^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} + (1/1080)R_{ab}R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa} \\ & + (31/18900)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\kappa\lambda b} - (1/300)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\lambda\kappa b} + (17/18900)R^{\rho\sigma}R^{\kappa\lambda}{}_{\rho a}R_{\kappa\lambda\sigma b} \\ & - (17/7560)R^{\kappa}{}_{(a}R^{\rho\sigma\tau}{}_{|\kappa|}R_{|\rho\sigma\tau|b)} - (17/9450)R^{\rho\sigma\tau}{}_{\lambda}R_{\rho\sigma\tau\kappa}R^{\lambda}{}_{a}{}^{\kappa}{}_{b} + (1/189)R^{\rho\kappa\sigma\lambda}R^{\tau}{}_{\rho\sigma a}R_{\tau\kappa\lambda b} \\ & - (1/450)R^{\rho\kappa\sigma\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b} + (19/4725)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b}. \end{aligned} \tag{49c}$$

By replacing (49) into (46) we can then write

$$\begin{aligned}\tilde{A}_2 = {} & \frac{1}{2}m^4 - \frac{1}{6}\left(\xi - \frac{1}{5}\right)\Box R + \left(\xi - \frac{1}{6}\right)m^2 R + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R^2 - \frac{1}{180}R_{\rho\sigma}R^{\rho\sigma} + \frac{1}{180}R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} + \left[\frac{1}{12}\left(\xi - \frac{1}{5}\right)(\Box R)_{;a}\right. \\ & \left. - \frac{1}{2}\left(\xi - \frac{1}{6}\right)m^2 R_{;a} - \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 RR_{;a} + \frac{1}{180}R_{\rho\sigma}R^{\rho\sigma}{}_{;a} - \frac{1}{180}R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}{}_{;a}\right]\sigma^{;a} + \left[\frac{1}{1680}\Box\Box R_{ab} - \frac{1}{40}\left(\xi - \frac{4}{21}\right)(\Box R)_{;ab}\right. \\ & - \frac{1}{120}m^2\Box R_{ab} + \frac{1}{6}\left(\xi - \frac{3}{20}\right)m^2 R_{;ab} + \frac{1}{24}m^4 R_{ab} + \frac{1}{6}\left(\xi - \frac{1}{6}\right)\left(\xi - \frac{3}{20}\right)RR_{;ab} - \frac{1}{180}\left(\xi - \frac{1}{7}\right)R_{;\rho a}R^{\rho}{}_{b} \\ & - \frac{1}{72}\left(\xi - \frac{1}{5}\right)(\Box R)R_{ab} + \frac{1}{8}\left(\xi - \frac{1}{6}\right)^2 R_{;a}R_{;b} + \frac{1}{60}\left(\xi - \frac{3}{14}\right)R_{;\rho}R^{\rho}{}_{a;b} - \frac{1}{60}\left(\xi - \frac{17}{84}\right)R_{;\rho}R_{ab}{}^{;\rho} + \frac{1}{12}\left(\xi - \frac{1}{6}\right)m^2 RR_{ab} \\ & - \frac{1}{120}\left(\xi - \frac{1}{6}\right)R\Box R_{ab} - \frac{1}{504}R_{\rho a}\Box R^{\rho}{}_{b} + \frac{1}{90}m^2 R_{\rho a}R^{\rho}{}_{b} - \frac{11}{6300}R^{\rho\sigma}R_{\rho\sigma;ab} - \frac{1}{720}R^{\rho\sigma}{}_{;a}R_{\rho\sigma;b} - \frac{1}{2100}R^{\rho\sigma}R_{\rho a;b\sigma} \\ & + \frac{1}{1575}R^{\rho\sigma}R_{ab;\rho\sigma} + \frac{1}{2520}R^{\rho}{}_{a;\sigma}R_{\rho b}{}^{;\sigma} - \frac{1}{504}R^{\rho}{}_{a;\sigma}R^{\sigma}{}_{b;\rho} - \frac{1}{90}\left(\xi - \frac{3}{14}\right)R^{;\rho\sigma}R_{\rho a\sigma b} + \frac{1}{1260}(\Box R^{\rho\sigma})R_{\rho a\sigma b} \\ & - \frac{1}{180}m^2 R^{\rho\sigma}R_{\rho a\sigma b} + \frac{1}{1260}R^{\rho\sigma;\tau}R_{\tau\sigma\rho a;b} + \frac{1}{1800}R^{\rho\sigma}\Box R_{\rho a\sigma b} + \frac{1}{840}R^{\rho\sigma;\tau}R_{\rho a\sigma b;\tau} - \frac{1}{1575}R^{\rho\sigma;\tau}{}_{a}R_{\tau\sigma\rho b} \\ & + \frac{23}{12600}R^{\rho}{}_{a}{}^{;\sigma\tau}R_{\tau\sigma\rho b} - \frac{1}{450}R^{\rho}{}_{a}{}^{;\sigma\tau}R_{\rho\sigma\tau b} - \frac{1}{700}R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau a;b\kappa} + \frac{2}{1575}R^{\rho\sigma\tau}{}_{a}\Box R_{\rho\sigma\tau b} - \frac{1}{180}m^2 R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} \\ & + \frac{29}{12600}R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa;ab} + \frac{1}{840}R^{\rho\sigma\tau}{}_{a;\kappa}R_{\rho\sigma\tau b}{}^{;\kappa} + \frac{1}{672}R^{\rho\sigma\tau\kappa}{}_{;a}R_{\rho\sigma\tau\kappa;b} + \frac{1}{24}\left(\xi - \frac{1}{6}\right)^2 R^2 R_{ab} + \frac{1}{90}\left(\xi - \frac{1}{6}\right)RR_{\rho a}R^{\rho}{}_{b} \\ & - \frac{1}{2160}R^{\rho\sigma}R_{\rho\sigma}R_{ab} + \frac{1}{1890}R^{\rho\sigma}R_{\rho a}R_{\sigma b} - \frac{1}{180}\left(\xi - \frac{1}{6}\right)RR^{\rho\sigma}R_{\rho a\sigma b} - \frac{1}{3780}R^{\rho\tau}R^{\sigma}{}_{\tau}R_{\rho a\sigma b} + \frac{4}{4725}R^{\rho\sigma}R^{\tau}{}_{a}R_{\tau\sigma\rho b} \\ & + \frac{1}{18900}R_{\rho\sigma}R^{\rho\kappa\sigma\lambda}R_{\kappa a\lambda b} - \frac{1}{180}\left(\xi - \frac{1}{6}\right)RR^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} + \frac{1}{2160}R_{ab}R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa} + \frac{31}{37800}R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\kappa\lambda b} \\ & - \frac{1}{600}R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\lambda\kappa b} + \frac{17}{37800}R^{\rho\sigma}R^{\kappa\lambda}{}_{\rho a}R_{\kappa\lambda\sigma b} - \frac{17}{15120}R^{\kappa}{}_{a}R^{\rho\sigma\tau}{}_{\kappa}R_{\rho\sigma\tau b} - \frac{17}{18900}R^{\rho\sigma\tau}{}_{\lambda}R_{\rho\sigma\tau\kappa}R^{\lambda}{}_{a}{}^{\kappa}{}_{b} \\ & \left. + \frac{1}{378}R^{\rho\kappa\sigma\lambda}R^{\tau}{}_{\rho\sigma a}R_{\tau\kappa\lambda b} - \frac{1}{900}R^{\rho\kappa\sigma\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b} + \frac{19}{9450}R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b}\right]\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}). \end{aligned} \tag{50}$$

D. Covariant Taylor series expansion of $\tilde{A}_3(m^2; x, x')$

The mass-dependent DeWitt coefficient $\tilde{A}_3(m^2; x, x')$ is the solution of Eq. (23a) with $n = 2$, i.e. it satisfies

$$3\tilde{A}_3 + \tilde{A}_{3;\mu}\sigma^{;\mu} - \tilde{A}_3\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} = (\Box_x - m^2 - \xi R)\tilde{A}_2. \tag{51}$$

In this equation, we replace $\tilde{A}_3$ by its covariant Taylor series expansion for x' in the neighborhood of x given by

$$\tilde{A}_3 = \tilde{a}_3 + O(\sigma^{1/2}). \tag{52}$$

By noting that $\tilde{A}_{3;\mu}\sigma^{;\mu} - \tilde{A}_3\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} = O(\sigma^{1/2})$, we can see easily that the left-hand side of Eq. (51) to order σ^0 reduces to a_3. The covariant Taylor series expansion of the right-hand side of Eq. (51) to order σ^0 can be found from the expansion $\tilde{A}_2$ given by (46) and (49) or (50) and from the expansion of $\Box\tilde{A}_2$. The latter can be constructed by using the theory developed in Appendix D and more particularly Eqs. (D5), (D8), and (D14). We easily obtain

$$\Box\tilde{A}_2 = \tilde{a}_2^{\prime\prime} + O(\sigma^{1/2}) \tag{53}$$

with

$$\begin{aligned}\tilde{a}_2^{\prime\prime} = {} & -(1/20)(\xi - 3/14)\Box\Box R + (1/3)(\xi - 1/5)m^2\Box R + (1/12)m^4 R + (1/3)(\xi - 1/4)(\xi - 1/5)R\Box R \\ & + (1/4)[\xi^2 - (2/5)\xi + 17/420]R_{;\rho}R^{;\rho} + (1/6)(\xi - 1/6)m^2R^2 - (1/30)(\xi - 3/14)R_{;\rho\sigma}R^{\rho\sigma} \\ & - (1/210)R_{\rho\sigma}\Box R^{\rho\sigma} + (1/90)m^2R_{\rho\sigma}R^{\rho\sigma} - (1/840)R_{\rho\sigma;\tau}R^{\rho\sigma;\tau} - (1/420)R_{\rho\tau;\sigma}R^{\sigma\tau;\rho} + (1/140)R_{\rho\sigma\tau\kappa}\Box R^{\rho\sigma\tau\kappa} \\ & - (1/90)m^2R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} + (3/560)R_{\rho\sigma\tau\kappa;\lambda}R^{\rho\sigma\tau\kappa;\lambda} + (1/12)(\xi - 1/6)^2R^3 + (1/90)(\xi - 1/4)RR_{\rho\sigma}R^{\rho\sigma} \\ & + (1/1890)R_{\rho\sigma}R^{\rho}{}_{\tau}R^{\sigma\tau} - (1/630)R_{\rho\sigma}R_{\kappa\lambda}R^{\rho\kappa\sigma\lambda} - (1/90)(\xi - 1/4)RR_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} - (1/315)R_{\kappa\lambda}R^{\kappa\rho\sigma\tau}R^{\lambda}{}_{\rho\sigma\tau} \\ & + (1/189)R^{\rho\kappa\sigma\lambda}R_{\rho\alpha\sigma\beta}R_{\kappa}{}^{\alpha}{}_{\lambda}{}^{\beta} + (11/3780)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\alpha\beta}R_{\kappa\lambda}{}^{\alpha\beta}.\end{aligned} \tag{54}$$

The direct comparison of the expansions of the left- and right-hand sides of Eq. (51) then yields the coefficients a_3:

$$\begin{aligned}\tilde{a}_3 = {} & -(1/6)m^6 - (1/60)(\xi - 3/14)\Box\Box R + (1/6)(\xi - 1/5)m^2\Box R - (1/2)(\xi - 1/6)m^4R + (1/6)(\xi - 1/6)(\xi - 1/5)R\Box R \\ & + (1/12)[\xi^2 - (2/5)\xi + 17/420]R_{;\rho}R^{;\rho} - (1/2)(\xi - 1/6)^2m^2R^2 - (1/90)(\xi - 3/14)R_{;\rho\sigma}R^{\rho\sigma} \\ & - (1/630)R_{\rho\sigma}\Box R^{\rho\sigma} + (1/180)m^2R_{\rho\sigma}R^{\rho\sigma} - (1/2520)R_{\rho\sigma;\tau}R^{\rho\sigma;\tau} - (1/1260)R_{\rho\tau;\sigma}R^{\sigma\tau;\rho} \\ & + (1/420)R_{\rho\sigma\tau\kappa}\Box R^{\rho\sigma\tau\kappa} - (1/180)m^2R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} + (1/560)R_{\rho\sigma\tau\kappa;\lambda}R^{\rho\sigma\tau\kappa;\lambda} - (1/6)(\xi - 1/6)^3R^3 \\ & + (1/180)(\xi - 1/6)RR_{\rho\sigma}R^{\rho\sigma} + (1/5670)R_{\rho\sigma}R^{\rho}{}_{\tau}R^{\sigma\tau} - (1/1890)R_{\rho\sigma}R_{\kappa\lambda}R^{\rho\kappa\sigma\lambda} - (1/180)(\xi - 1/6)RR_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} \\ & - (1/945)R_{\kappa\lambda}R^{\kappa\rho\sigma\tau}R^{\lambda}{}_{\rho\sigma\tau} + (1/567)R^{\rho\kappa\sigma\lambda}R_{\rho\alpha\sigma\beta}R_{\kappa}{}^{\alpha}{}_{\lambda}{}^{\beta} + (11/11340)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\alpha\beta}R_{\kappa\lambda}{}^{\alpha\beta}.\end{aligned} \tag{55}$$

This result agrees with that of Gilkey [12,14] obtained by using totally different methods, i.e. in the framework of the pseudodifferential operator theory. The comparison of our result with his own is immediate in spite of our different conventions with regard to the metric signature, the Riemann tensor and the commutation of covariant derivatives. This agreement permits us to believe in the validity of all the calculations previously carried out and therefore in the validity of the covariant Taylor series obtained for the mass-dependent DeWitt coefficients. Finally, by replacing (55) into (52) we can now write

$$\begin{aligned}\tilde{A}_3 = {} & -\frac{1}{6}m^6 - \frac{1}{60}\left(\xi - \frac{3}{14}\right)\Box\Box R + \frac{1}{6}\left(\xi - \frac{1}{5}\right)m^2\Box R - \frac{1}{2}\left(\xi - \frac{1}{6}\right)m^4R + \frac{1}{6}\left(\xi - \frac{1}{6}\right)\left(\xi - \frac{1}{5}\right)R\Box R \\ & + \frac{1}{12}\left(\xi^2 - \frac{2}{5}\xi + \frac{17}{420}\right)R_{;\rho}R^{;\rho} - \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2m^2R^2 - \frac{1}{90}\left(\xi - \frac{3}{14}\right)R_{;\rho\sigma}R^{\rho\sigma} - \frac{1}{630}R_{\rho\sigma}\Box R^{\rho\sigma} + \frac{1}{180}m^2R_{\rho\sigma}R^{\rho\sigma} \\ & - \frac{1}{2520}R_{\rho\sigma;\tau}R^{\rho\sigma;\tau} - \frac{1}{1260}R_{\rho\tau;\sigma}R^{\sigma\tau;\rho} + \frac{1}{420}R_{\rho\sigma\tau\kappa}\Box R^{\rho\sigma\tau\kappa} - \frac{1}{180}m^2R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} + \frac{1}{560}R_{\rho\sigma\tau\kappa;\lambda}R^{\rho\sigma\tau\kappa;\lambda} \\ & - \frac{1}{6}\left(\xi - \frac{1}{6}\right)^3R^3 + \frac{1}{180}\left(\xi - \frac{1}{6}\right)RR_{\rho\sigma}R^{\rho\sigma} + \frac{1}{5670}R_{\rho\sigma}R^{\rho}{}_{\tau}R^{\sigma\tau} - \frac{1}{1890}R_{\rho\sigma}R_{\kappa\lambda}R^{\rho\kappa\sigma\lambda} - \frac{1}{180}\left(\xi - \frac{1}{6}\right)RR_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} \\ & - \frac{1}{945}R_{\kappa\lambda}R^{\kappa\rho\sigma\tau}R^{\lambda}{}_{\rho\sigma\tau} + \frac{1}{567}R^{\rho\kappa\sigma\lambda}R_{\rho\alpha\sigma\beta}R_{\kappa}{}^{\alpha}{}_{\lambda}{}^{\beta} + \frac{11}{11\,340}R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\alpha\beta}R_{\kappa\lambda}{}^{\alpha\beta} + O(\sigma^{1/2}).\end{aligned} \tag{56}$$

E. Covariant Taylor series expansion of the DeWitt coefficients $A_n(x, x')$ with $n = 0, 1, 2, 3$

We think it is unnecessary to write explicitly the covariant Taylor series expansion of the DeWitt coefficients $A_n(x, x')$ with $n = 0, 1, 2, 3$. Indeed, we know that $A_n(x, x') = \tilde{A}_n(m^2 = 0; x, x')$ [see (24)]. Then, the coefficients $a_n(x)$ and $a_{na_1\cdots a_p}(x)$ defining the expansion of the DeWitt coefficients $A_n(x, x')$ [see (32) and (33)] and the coefficients $\tilde{a}_n(m^2; x)$ and $\tilde{a}_{na_1\cdots a_p}(m^2; x)$ defining the expansion of the mass-dependent DeWitt coefficients $\tilde{A}_n(m^2; x, x')$ [see (34) and (35)] are linked by

$$a_n(x) = \tilde{a}_n(m^2 = 0; x), \qquad (57a)$$

$$a_{na_1\cdots a_p}(x) = \tilde{a}_{na_1\cdots a_p}(m^2 = 0; x). \qquad (57b)$$

As a consequence:

(i) the expansion of $A_0(x, x')$ up to order σ^3 is directly given by (38) and (39) or equivalently by (40),

(ii) the expansion of $A_1(x, x')$ up to order σ^2 is obtained from (42) and (43) or equivalently from (44) by taking $m^2 = 0$ in these relations,

(iii) the expansion of $A_2(x, x')$ up to order σ is given by (46) and (49) or equivalently by (50) by taking $m^2 = 0$ in these relations,

(iv) the expansion of $A_3(x, x')$ up to order σ^0 is given by (52) and (55) or equivalently by (56) by taking $m^2 = 0$ in these relations.

IV. COVARIANT TAYLOR SERIES EXPANSIONS OF THE HADAMARD COEFFICIENTS

In this section, we shall provide the covariant Taylor series expansions of the geometrical Hadamard coefficients $U_n(x, x')$ and $V_n(x, x')$ of lowest orders for the dimensions $d = 3, 4, 5$ and 6 of spacetime. For x' near x, we shall write these expansions in the form

$$U_n(x, x') = u_n(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!} u_{n(p)}(x, x'), \qquad (58)$$

$$V_n(x, x') = v_n(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!} v_{n(p)}(x, x'), \qquad (59)$$

where $u_{n(p)}(x, x')$ and $v_{n(p)}(x, x')$ with $p = 1, 2, \ldots$ are all biscalars in x and x' which are of the form

$$u_{n(p)}(x, x') = u_{na_1\cdots a_p}(x)\sigma^{;a_1}(x, x')\cdots\sigma^{;a_p}(x, x'), \qquad (60)$$

$$v_{n(p)}(x, x') = v_{na_1\cdots a_p}(x)\sigma^{;a_1}(x, x')\cdots\sigma^{;a_p}(x, x'). \qquad (61)$$

These expansions are easily obtained from those of the mass-dependent DeWitt coefficients constructed in the previous section by using the relations (26) for d even and the relation (28) for d odd. By using the relation (28), we obtain for $d = 3$

$$U_0(x, x') = \tilde{A}_0(m^2; x, x'), \qquad (62a)$$

$$U_1(x, x') = -\tilde{A}_1(m^2; x, x'), \qquad (62b)$$

$$U_2(x, x') = (1/3)\tilde{A}_2(m^2; x, x'), \qquad (62c)$$

$$U_3(x, x') = -(1/15)\tilde{A}_3(m^2; x, x'). \qquad (62d)$$

From the relation (26), we obtain for $d = 4$

$$U_0(x, x') = \tilde{A}_0(m^2; x, x'), \qquad (63a)$$

$$V_0(x, x') = -(1/2)\tilde{A}_1(m^2; x, x'), \qquad (63b)$$

$$V_1(x, x') = (1/4)\tilde{A}_2(m^2; x, x'), \qquad (63c)$$

$$V_2(x, x') = -(1/16)\tilde{A}_3(m^2; x, x'). \qquad (63d)$$

From the relation (28), we obtain for $d = 5$

$$U_0(x, x') = \tilde{A}_0(m^2; x, x'), \qquad (64a)$$

$$U_1(x, x') = \tilde{A}_1(m^2; x, x'), \qquad (64b)$$

$$U_2(x, x') = -\tilde{A}_2(m^2; x, x'), \qquad (64c)$$

$$U_3(x, x') = (1/3)\tilde{A}_3(m^2; x, x'), \qquad (64d)$$

and finally, from the relation (26), we obtain for $d = 6$

$$U_0(x, x') = \tilde{A}_0(m^2; x, x'), \qquad (65a)$$

$$U_1(x, x') = (1/2)\tilde{A}_1(m^2; x, x'), \qquad (65b)$$

$$V_0(x, x') = -(1/4)\tilde{A}_2(m^2; x, x'), \qquad (65c)$$

$$V_1(x, x') = (1/8)\tilde{A}_3(m^2; x, x'). \qquad (65d)$$

We could stop here with this section. However, we prefer to provide the explicit expansions of all these Hadamard coefficients for the reader who simply needs them and is not specially interested in following the derivation of the expansions of the DeWitt coefficients carried out in the previous section.

For $d = 3, 4, 5$ and 6 we have

$$\begin{aligned} U_0 = {} & u_0 - u_{0a}\sigma^{;a} + \frac{1}{2!}u_{0ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}u_{0abc}\sigma^{;a}\sigma^{;b}\sigma^{;c} \\ & + \frac{1}{4!}u_{0abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} \\ & - \frac{1}{5!}u_{0abcde}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}\sigma^{;e} \\ & + \frac{1}{6!}u_{0abcdef}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}\sigma^{;e}\sigma^{;f} + O(\sigma^{7/2}) \end{aligned} \qquad (66)$$

with

$$u_0 = 1, \tag{67a}$$
$$u_{0a} = 0, \tag{67b}$$
$$u_{0ab} = (1/6)R_{ab}, \tag{67c}$$
$$u_{0abc} = (1/4)R_{(ab;c)}, \tag{67d}$$
$$u_{0abcd} = (3/10)R_{(ab;cd)} + (1/15)R^{\rho}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d)} + (1/12)R_{(ab}R_{cd)}, \tag{67e}$$
$$u_{0abcde} = (1/3)R_{(ab;cde)} + (1/3)R^{\rho}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d;e)} + (5/12)R_{(ab}R_{cd;e)}, \tag{67f}$$
$$\begin{aligned} u_{0abcdef} = {} & (5/14)R_{(ab;cdef)} + (4/7)R^{\rho}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d;ef)} + (15/28)R^{\rho}{}_{(a|\tau|b;c}R^{\tau}{}_{d|\rho|e;f)} + (3/4)R_{(ab}R_{cd;ef)} \\ & + (5/8)R_{(ab;c}R_{de;f)} + (8/63)R^{\rho}{}_{(a|\tau|b}R^{\tau}{}_{c|\sigma|d}R^{\sigma}{}_{e|\rho|f)} + (1/6)R_{(ab}R^{\rho}{}_{c|\tau|d}R^{\tau}{}_{e|\rho|f)} + (5/72)R_{(ab}R_{cd}R_{ef)}. \end{aligned} \tag{67g}$$

Furthermore, for $d = 3$, we have

$$U_1 = u_1 - u_{1a}\sigma^{;a} + \frac{1}{2!}u_{1ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}u_{1abc}\sigma^{;a}\sigma^{;b}\sigma^{;c} + \frac{1}{4!}u_{1abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} + O(\sigma^{5/2}), \tag{68}$$

$$U_2 = u_2 - u_{2a}\sigma^{;a} + \frac{1}{2!}u_{2ab}\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}), \tag{69}$$

$$U_3 = u_3 + O(\sigma^{1/2}), \tag{70}$$

with

$$u_1 = m^2 + (\xi - 1/6)R, \tag{71a}$$
$$u_{1a} = (1/2)(\xi - 1/6)R_{;a}, \tag{71b}$$
$$\begin{aligned} u_{1ab} = {} & -(1/60)\Box R_{ab} + (1/3)(\xi - 3/20)R_{;ab} + (1/6)m^2R_{ab} + (1/6)(\xi - 1/6)RR_{ab} \\ & + (1/45)R^{\rho}{}_{a}R_{\rho b} - (1/90)R^{\rho\sigma}R_{\rho a\sigma b} - (1/90)R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b}, \end{aligned} \tag{71c}$$
$$\begin{aligned} u_{1abc} = {} & (1/4)(\xi - 2/15)R_{;(abc)} - (1/40)(\Box R_{(ab})_{;c)} + (1/4)m^2R_{(ab;c)} + (1/4)(\xi - 1/6)RR_{(ab;c)} + (1/4)(\xi - 1/6)R_{;(a}R_{bc)} \\ & + (1/15)R^{\rho}{}_{(a}R_{|\rho|b;c)} - (1/60)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;c)} - (1/60)R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c)} - (1/30)R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;c)}, \end{aligned} \tag{71d}$$
$$\begin{aligned} u_{1abcd} = {} & -(1/35)(\Box R_{(ab})_{;cd)} + (1/5)(\xi - 5/42)R_{;(abcd)} + (3/10)m^2R_{(ab;cd)} + (3/10)(\xi - 1/6)RR_{(ab;cd)} \\ & + (1/2)(\xi - 1/6)R_{;(a}R_{bc;d)} + (1/3)(\xi - 3/20)R_{;(ab}R_{cd)} - (1/60)R_{(ab}\Box R_{cd)} + (1/12)m^2R_{(ab}R_{cd)} \\ & + (3/35)R^{\rho}{}_{(a}R_{|\rho|b;cd)} - (1/105)R^{\rho}{}_{(a}R_{bc;|\rho|d)} + (11/210)R^{\rho}{}_{(a;b}R_{|\rho|c;d)} + (3/70)R^{\rho}{}_{(a;b}R_{cd);\rho} - (17/840)R_{(ab}{}^{;\rho}R_{cd);\rho} \\ & - (2/105)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;cd)} - (1/105)R^{\rho}{}_{(a;|\sigma|}R^{\sigma}{}_{b|\rho|c;d)} - (1/30)R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c;d)} + (4/175)R^{\rho}{}_{(a;|\sigma|b}R^{\sigma}{}_{c|\rho|d)} \\ & - (11/525)R_{(ab}{}^{;\rho}{}_{\sigma}R^{\sigma}{}_{c|\rho|d)} - (11/525)R^{\rho}{}_{\sigma;(ab}R^{\sigma}{}_{c|\rho|d)} - (4/525)R^{\rho}{}_{(a|\sigma|b}\Box R^{\sigma}{}_{c|\rho|d)} + (1/15)m^2R^{\rho}{}_{(a|\sigma|b}R^{\sigma}{}_{c|\rho|d)} \\ & - (4/105)R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;cd)} - (1/140)R^{\rho}{}_{(a|\sigma|b}{}^{;\tau}R^{\sigma}{}_{c|\rho|d);\tau} - (1/28)R^{\rho\sigma\tau}{}_{(a;b}R_{|\rho\sigma\tau|c;d)} + (1/12)(\xi - 1/6)RR_{(ab}R_{cd)} \\ & + (1/45)R^{\rho}{}_{(a}R_{|\rho|b}R_{cd)} - (1/315)R^{\rho}{}_{(a}R_{|\sigma|b}R^{\sigma}{}_{c|\rho|d)} - (1/90)R^{\rho\sigma}R_{(ab}R_{|\rho|c|\sigma|d)} + (1/15)(\xi - 1/6)RR^{\rho}{}_{(a|\sigma|b}R^{\sigma}{}_{c|\rho|d)} \\ & - (1/90)R_{(ab}R^{\rho\sigma\tau}{}_{c}R_{|\rho\sigma\tau|d)} - (26/1575)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d)} - (2/63)R^{\rho}{}_{(a}R^{\sigma}{}_{b}{}^{\tau}{}_{c}R_{|\rho\sigma\tau|d)} \\ & - (4/1575)R^{\rho\sigma\tau\kappa}R_{\rho(a|\tau|b}R_{|\sigma|c|\kappa|d)} - (4/525)R^{\rho\kappa\tau}{}_{(a}R_{|\rho\tau|}{}^{\sigma}{}_{b}R_{|\sigma|c|\kappa|d)} - (16/1575)R^{\rho\kappa\tau}{}_{(a}R_{|\rho|}{}^{\sigma}{}_{|\tau|b}R_{|\sigma|c|\kappa|d)} \\ & - (8/1575)R^{\rho\tau\kappa}{}_{(a}R_{|\rho\tau|}{}^{\sigma}{}_{b}R_{|\sigma|c|\kappa|d)}, \end{aligned} \tag{71e}$$

and

$$\begin{aligned} u_2 = {} & (1/6)m^4 - (1/18)(\xi - 1/5)\Box R + (1/3)(\xi - 1/6)m^2R + (1/6)(\xi - 1/6)^2R^2 - (1/540)R_{\rho\sigma}R^{\rho\sigma} \\ & + (1/540)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}, \end{aligned} \tag{72a}$$

$$\begin{aligned} u_{2a} = {} & -(1/36)(\xi - 1/5)(\Box R)_{;a} + (1/6)(\xi - 1/6)m^2R_{;a} + (1/6)(\xi - 1/6)^2RR_{;a} - (1/540)R_{\rho\sigma}R^{\rho\sigma}{}_{;a} \\ & + (1/540)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}{}_{;a}, \end{aligned} \tag{72b}$$

and

$$\begin{aligned}
u_{2ab} = {} & (1/2520)\Box\Box R_{ab} - (1/60)(\xi - 4/21)(\Box R)_{;ab} - (1/180)m^2\Box R_{ab} + (1/9)(\xi - 3/20)m^2 R_{;ab} + (1/36)m^4 R_{ab} \\
& + (1/9)(\xi - 1/6)(\xi - 3/20)RR_{;ab} - (1/270)(\xi - 1/7)R_{;\rho(a}R^{\rho}{}_{b)} - (1/108)(\xi - 1/5)(\Box R)R_{ab} \\
& + (1/12)(\xi - 1/6)^2 R_{;a}R_{;b} + (1/90)(\xi - 3/14)R_{;\rho}R^{\rho}{}_{(a;b)} - (1/90)(\xi - 17/84)R_{;\rho}R_{ab}{}^{;\rho} + (1/18)(\xi - 1/6)m^2 RR_{ab} \\
& - (1/180)(\xi - 1/6)R\Box R_{ab} - (1/756)R_{\rho(a}\Box R^{\rho}{}_{b)} + (1/135)m^2 R_{\rho a}R^{\rho}{}_{b} - (11/9450)R^{\rho\sigma}R_{\rho\sigma;(ab)} \\
& - (1/1080)R^{\rho\sigma}{}_{;a}R_{\rho\sigma;b} - (1/3150)R^{\rho\sigma}R_{\rho(a;b)\sigma} + (2/4725)R^{\rho\sigma}R_{ab;\rho\sigma} + (1/3780)R^{\rho}{}_{a;\sigma}R_{\rho b}{}^{;\sigma} - (1/756)R^{\rho}{}_{a;\sigma}R^{\sigma}{}_{b;\rho} \\
& - (1/135)(\xi - 3/14)R^{;\rho\sigma}R_{\rho a\sigma b} + (1/1890)(\Box R^{\rho\sigma})R_{\rho a\sigma b} - (1/270)m^2 R^{\rho\sigma}R_{\rho a\sigma b} + (1/1890)R^{\rho\sigma;\tau}R_{\tau\sigma\rho(a;b)} \\
& + (1/2700)R^{\rho\sigma}\Box R_{\rho a\sigma b} + (1/1260)R^{\rho\sigma;\tau}R_{\rho a\sigma b;\tau} - (2/4725)R^{\rho\sigma;\tau}{}_{(a}R_{|\tau\sigma\rho|b)} + (23/18900)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\tau\sigma\rho|b)} \\
& - (1/675)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\rho\sigma\tau|b)} - (1/1050)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau(a;b)\kappa} + (4/4725)R^{\rho\sigma\tau}{}_{a}\Box R_{\rho\sigma\tau b} - (1/270)m^2 R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} \\
& + (29/18900)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa;(ab)} + (1/1260)R^{\rho\sigma\tau}{}_{a;\kappa}R_{\rho\sigma\tau b}{}^{;\kappa} + (1/1008)R^{\rho\sigma\tau\kappa}{}_{;a}R_{\rho\sigma\tau\kappa;b} \\
& + (1/36)(\xi - 1/6)^2 R^2 R_{ab} + (1/135)(\xi - 1/6)RR_{\rho a}R^{\rho}{}_{b} - (1/3240)R^{\rho\sigma}R_{\rho\sigma}R_{ab} + (1/2835)R^{\rho\sigma}R_{\rho a}R_{\sigma b} \\
& - (1/270)(\xi - 1/6)RR^{\rho\sigma}R_{\rho a\sigma b} - (1/5670)R^{\rho\tau}R^{\sigma}{}_{\tau}R_{\rho a\sigma b} + (8/14175)R^{\rho\sigma}R^{\tau}{}_{(a}R_{|\tau\sigma\rho|b)} + (1/28350)R_{\rho\sigma}R^{\rho\kappa\sigma\lambda}R_{\kappa a\lambda b} \\
& - (1/270)(\xi - 1/6)RR^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} + (1/3240)R_{ab}R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa} + (31/56700)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\kappa\lambda b} \\
& - (1/900)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\lambda\kappa b} + (17/56700)R^{\rho\sigma}R^{\kappa\lambda}{}_{\rho a}R_{\kappa\lambda\sigma b} - (17/22680)R^{\kappa}{}_{(a}R^{\rho\sigma\tau}{}_{|\kappa|}R_{|\rho\sigma\tau|b)} \\
& - (17/28350)R^{\rho\sigma\tau}{}_{\lambda}R_{\rho\sigma\tau\kappa}R^{\lambda}{}_{a}{}^{\kappa}{}_{b} + (1/567)R^{\rho\kappa\sigma\lambda}R^{\tau}{}_{\rho\sigma a}R_{\tau\kappa\lambda b} - (1/1350)R^{\rho\kappa\sigma\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b} \\
& + (19/14175)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b},
\end{aligned} \tag{72c}$$

and

$$\begin{aligned}
u_3 = {} & (1/90)m^6 + (1/900)(\xi - 3/14)\Box\Box R - (1/90)(\xi - 1/5)m^2\Box R + (1/30)(\xi - 1/6)m^4 R - (1/90)(\xi - 1/6)(\xi \\
& - 1/5)R\Box R - (1/180)[\xi^2 - (2/5)\xi + 17/420]R_{;\rho}R^{;\rho} + (1/30)(\xi - 1/6)^2 m^2 R^2 + (1/1350)(\xi - 3/14)R_{;\rho\sigma}R^{\rho\sigma} \\
& + (1/9450)R_{\rho\sigma}\Box R^{\rho\sigma} - (1/2700)m^2 R_{\rho\sigma}R^{\rho\sigma} + (1/37\,800)R_{\rho\sigma;\tau}R^{\rho\sigma;\tau} + (1/18\,900)R_{\rho\tau;\sigma}R^{\sigma\tau;\rho} \\
& - (1/6300)R_{\rho\sigma\tau\kappa}\Box R^{\rho\sigma\tau\kappa} + (1/2700)m^2 R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} - (1/8400)R_{\rho\sigma\tau\kappa;\lambda}R^{\rho\sigma\tau\kappa;\lambda} + (1/90)(\xi - 1/6)^3 R^3 \\
& - (1/2700)(\xi - 1/6)RR_{\rho\sigma}R^{\rho\sigma} - (1/85\,050)R_{\rho\sigma}R^{\rho}{}_{\tau}R^{\sigma\tau} + (1/28\,350)R_{\rho\sigma}R_{\kappa\lambda}R^{\rho\kappa\sigma\lambda} + (1/2700)(\xi \\
& - 1/6)RR_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} + (1/14\,175)R_{\kappa\lambda}R^{\kappa\rho\sigma\tau}R^{\lambda}{}_{\rho\sigma\tau} - (1/8505)R^{\rho\kappa\sigma\lambda}R_{\rho\alpha\sigma\beta}R_{\kappa}{}^{\alpha}{}_{\lambda}{}^{\beta} \\
& - (11/170\,100)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\alpha\beta}R_{\kappa\lambda}{}^{\alpha\beta}.
\end{aligned} \tag{73}$$

For $d = 4$, we have

$$V_0 = v_0 - v_{0a}\sigma^{;a} + \frac{1}{2!}v_{0ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}v_{0abc}\sigma^{;a}\sigma^{;b}\sigma^{;c} + \frac{1}{4!}v_{0abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} + O(\sigma^{5/2}), \tag{74}$$

$$V_1 = v_1 - v_{1a}\sigma^{;a} + \frac{1}{2!}v_{1ab}\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}), \tag{75}$$

$$V_2 = v_2 + O(\sigma^{1/2}), \tag{76}$$

with

$$v_0 = (1/2)m^2 + (1/2)(\xi - 1/6)R, \tag{77a}$$

$$v_{0a} = (1/4)(\xi - 1/6)R_{;a}, \tag{77b}$$

$$\begin{aligned} v_{0ab} = {} & -(1/120)\Box R_{ab} + (1/6)(\xi - 3/20)R_{;ab} + (1/12)m^2 R_{ab} + (1/12)(\xi - 1/6)RR_{ab} + (1/90)R^{\rho}{}_{a}R_{\rho b} \\ & - (1/180)R^{\rho\sigma}R_{\rho a\sigma b} - (1/180)R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b}, \end{aligned} \tag{77c}$$

$$\begin{aligned} v_{0abc} = {} & (1/8)(\xi - 2/15)R_{;(abc)} - (1/80)(\Box R_{(ab})_{;c)} + (1/8)m^2 R_{(ab;c)} + (1/8)(\xi - 1/6)RR_{(ab;c)} + (1/8)(\xi - 1/6)R_{;(a}R_{bc)} \\ & + (1/30)R^{\rho}{}_{(a}R_{|\rho|b;c)} - (1/120)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;c)} - (1/120)R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c)} - (1/60)R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;c)}, \end{aligned} \tag{77d}$$

$$\begin{aligned} v_{0abcd} = {} & -(1/70)(\Box R_{(ab})_{;cd)} + (1/10)(\xi - 5/42)R_{;(abcd)} + (3/20)m^2 R_{(ab;cd)} + (3/20)(\xi - 1/6)RR_{(ab;cd)} \\ & + (1/4)(\xi - 1/6)R_{;(a}R_{bc;d)} + (1/6)(\xi - 3/20)R_{;(ab}R_{cd)} - (1/120)R_{(ab}\Box R_{cd)} + (1/24)m^2 R_{(ab}R_{cd)} \\ & + (3/70)R^{\rho}{}_{(a}R_{|\rho|b;cd)} - (1/210)R^{\rho}{}_{(a}R_{bc;|\rho|d)} + (11/420)R^{\rho}{}_{(a;b}R_{|\rho|c;d)} + (3/140)R^{\rho}{}_{(a;b}R_{cd);\rho} \\ & - (17/1680)R_{(ab}{}^{;\rho}R_{cd);\rho} - (1/105)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;cd)} - (1/210)R^{\rho}{}_{(a;|\sigma|}R^{\sigma}{}_{b|\rho|c;d)} - (1/60)R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c;d)} \\ & + (2/175)R^{\rho}{}_{(a;|\sigma|b}R^{\sigma}{}_{c|\rho|d)} - (11/1050)R_{(ab}{}^{;\rho}{}_{\sigma}R^{\sigma}{}_{c|\rho|d)} - (11/1050)R^{\rho}{}_{\sigma;(ab}R^{\sigma}{}_{c|\rho|d)} - (2/525)R^{\rho}{}_{(a|\sigma|b}\Box R^{\sigma}{}_{c|\rho|d)} \\ & + (1/30)m^2 R^{\rho}{}_{(a|\sigma|b}R^{\sigma}{}_{c|\rho|d)} - (2/105)R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;cd)} - (1/280)R^{\rho}{}_{(a|\sigma|b}{}^{;\tau}R^{\sigma}{}_{c|\rho|d);\tau} - (1/56)R^{\rho\sigma\tau}{}_{(a;b}R_{|\rho\sigma\tau|c;d)} \\ & + (1/24)(\xi - 1/6)RR_{(ab}R_{cd)} + (1/90)R^{\rho}{}_{(a}R_{|\rho|b}R_{cd)} - (1/630)R^{\rho}{}_{(a}R_{|\sigma|b}R^{\sigma}{}_{c|\rho|d)} - (1/180)R^{\rho\sigma}R_{(ab}R_{|\rho|c|\sigma|d)} \\ & + (1/30)(\xi - 1/6)RR^{\rho}{}_{(a|\sigma|b}R^{\sigma}{}_{c|\rho|d)} - (1/180)R_{(ab}R^{\rho\sigma\tau}{}_{c}R_{|\rho\sigma\tau|d)} - (13/1575)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d)} \\ & - (1/63)R^{\rho}{}_{(a}R^{\sigma}{}_{b}{}^{\tau}{}_{c}R_{|\rho\sigma\tau|d)} - (2/1575)R^{\rho\sigma\tau\kappa}R_{\rho(a|\tau|b}R_{|\sigma|c|\kappa|d)} - (2/525)R^{\rho\kappa\tau}{}_{(a}R_{|\rho\tau|}{}^{\sigma}{}_{b}R_{|\sigma|c|\kappa|d)} \\ & - (8/1575)R^{\rho\kappa\tau}{}_{(a}R_{|\rho|}{}^{\sigma}{}_{|\tau|b}R_{|\sigma|c|\kappa|d)} - (4/1575)R^{\rho\tau\kappa}{}_{(a}R_{|\rho\tau|}{}^{\sigma}{}_{b}R_{|\sigma|c|\kappa|d)}, \end{aligned} \tag{77e}$$

and

$$\begin{aligned} v_1 = {} & (1/8)m^4 - (1/24)(\xi - 1/5)\Box R + (1/4)(\xi - 1/6)m^2 R + (1/8)(\xi - 1/6)^2 R^2 - (1/720)R_{\rho\sigma}R^{\rho\sigma} \\ & + (1/720)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}, \end{aligned} \tag{78a}$$

$$\begin{aligned} v_{1a} = {} & -(1/48)(\xi - 1/5)(\Box R)_{;a} + (1/8)(\xi - 1/6)m^2 R_{;a} + (1/8)(\xi - 1/6)^2 RR_{;a} - (1/720)R_{\rho\sigma}R^{\rho\sigma}{}_{;a} \\ & + (1/720)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}{}_{;a}, \end{aligned} \tag{78b}$$

and

$$\begin{aligned} v_{1ab} = {} & (1/3360)\Box\Box R_{ab} - (1/80)(\xi - 4/21)(\Box R)_{;ab} - (1/240)m^2\Box R_{ab} + (1/12)(\xi - 3/20)m^2 R_{;ab} + (1/48)m^4 R_{ab} \\ & + (1/12)(\xi - 1/6)(\xi - 3/20)RR_{;ab} - (1/360)(\xi - 1/7)R_{;\rho(a}R^{\rho}{}_{b)} - (1/144)(\xi - 1/5)(\Box R)R_{ab} \\ & + (1/16)(\xi - 1/6)^2 R_{;a}R_{;b} + (1/120)(\xi - 3/14)R_{;\rho}R^{\rho}{}_{(a;b)} - (1/120)(\xi - 17/84)R_{;\rho}R_{ab}{}^{;\rho} + (1/24)(\xi - 1/6)m^2 RR_{ab} \\ & - (1/240)(\xi - 1/6)R\Box R_{ab} - (1/1008)R_{\rho(a}\Box R^{\rho}{}_{b)} + (1/180)m^2 R_{\rho a}R^{\rho}{}_{b} - (11/12600)R^{\rho\sigma}R_{\rho\sigma;(ab)} \\ & - (1/1440)R^{\rho\sigma}{}_{;a}R_{\rho\sigma;b} - (1/4200)R^{\rho\sigma}R_{\rho(a;b)\sigma} + (1/3150)R^{\rho\sigma}R_{ab;\rho\sigma} + (1/5040)R^{\rho}{}_{a;\sigma}R_{\rho b}{}^{;\sigma} - (1/1008)R^{\rho}{}_{a;\sigma}R^{\sigma}{}_{b;\rho} \\ & - (1/180)(\xi - 3/14)R^{;\rho\sigma}R_{\rho a\sigma b} + (1/2520)(\Box R^{\rho\sigma})R_{\rho a\sigma b} - (1/360)m^2 R^{\rho\sigma}R_{\rho a\sigma b} + (1/2520)R^{\rho\sigma;\tau}R_{\tau\sigma\rho(a;b)} \\ & + (1/3600)R^{\rho\sigma}\Box R_{\rho a\sigma b} + (1/1680)R^{\rho\sigma;\tau}R_{\rho a\sigma b;\tau} - (1/3150)R^{\rho\sigma;\tau}{}_{(a}R_{|\tau\sigma\rho|b)} + (23/25200)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\tau\sigma\rho|b)} \\ & - (1/900)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\rho\sigma\tau|b)} - (1/1400)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau(a;b)\kappa} + (1/1575)R^{\rho\sigma\tau}{}_{a}\Box R_{\rho\sigma\tau b} - (1/360)m^2 R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} \\ & + (29/25200)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa;(ab)} + (1/1680)R^{\rho\sigma\tau}{}_{a;\kappa}R_{\rho\sigma\tau b}{}^{;\kappa} + (1/1344)R^{\rho\sigma\tau\kappa}{}_{;a}R_{\rho\sigma\tau\kappa;b} + (1/48)(\xi - 1/6)^2 R^2 R_{ab} \\ & + (1/180)(\xi - 1/6)RR_{\rho a}R^{\rho}{}_{b} - (1/4320)R^{\rho\sigma}R_{\rho\sigma}R_{ab} + (1/3780)R^{\rho\sigma}R_{\rho a}R_{\sigma b} - (1/360)(\xi - 1/6)RR^{\rho\sigma}R_{\rho a\sigma b} \\ & - (1/7560)R^{\rho\tau}R^{\sigma}{}_{\tau}R_{\rho a\sigma b} + (2/4725)R^{\rho\sigma}R^{\tau}{}_{(a}R_{|\tau\sigma\rho|b)} + (1/37800)R_{\rho\sigma}R^{\rho\kappa\sigma\lambda}R_{\kappa a\lambda b} \\ & - (1/360)(\xi - 1/6)RR^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} + (1/4320)R_{ab}R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa} + (31/75600)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\kappa\lambda b} \\ & - (1/1200)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\lambda\kappa b} + (17/75600)R^{\rho\sigma}R^{\kappa\lambda}{}_{\rho a}R_{\kappa\lambda\sigma b} - (17/30240)R^{\kappa}{}_{(a}R^{\rho\sigma\tau}{}_{|\kappa|}R_{|\rho\sigma\tau|b)} \\ & - (17/37800)R^{\rho\sigma\tau}{}_{\lambda}R_{\rho\sigma\tau\kappa}R^{\lambda}{}_{a}{}^{\kappa}{}_{b} + (1/756)R^{\rho\kappa\sigma\lambda}R^{\tau}{}_{\rho\sigma a}R_{\tau\kappa\lambda b} - (1/1800)R^{\rho\kappa\sigma\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b} \\ & + (19/18900)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b}, \end{aligned} \tag{78c}$$

and

$$\begin{aligned}
v_2 = {} & (1/96)m^6 + (1/960)(\xi - 3/14)\Box\Box R - (1/96)(\xi - 1/5)m^2\Box R + (1/32)(\xi - 1/6)m^4 R \\
& - (1/96)(\xi - 1/6)(\xi - 1/5)R\Box R - (1/192)[\xi^2 - (2/5)\xi + 17/420]R_{;\rho}R^{;\rho} + (1/32)(\xi - 1/6)^2 m^2 R^2 \\
& + (1/1440)(\xi - 3/14)R_{;\rho\sigma}R^{\rho\sigma} + (1/10\,080)R_{\rho\sigma}\Box R^{\rho\sigma} - (1/2880)m^2 R_{\rho\sigma}R^{\rho\sigma} + (1/40\,320)R_{\rho\sigma;\tau}R^{\rho\sigma;\tau} \\
& + (1/20\,160)R_{\rho\tau;\sigma}R^{\sigma\tau;\rho} - (1/6720)R_{\rho\sigma\tau\kappa}\Box R^{\rho\sigma\tau\kappa} + (1/2880)m^2 R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} - (1/8960)R_{\rho\sigma\tau\kappa;\lambda}R^{\rho\sigma\tau\kappa;\lambda} \\
& + (1/96)(\xi - 1/6)^3 R^3 - (1/2880)(\xi - 1/6)RR_{\rho\sigma}R^{\rho\sigma} - (1/90\,720)R_{\rho\sigma}R^{\rho}{}_{\tau}R^{\sigma\tau} + (1/30\,240)R_{\rho\sigma}R_{\kappa\lambda}R^{\rho\kappa\sigma\lambda} \\
& + (1/2880)(\xi - 1/6)RR_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} + (1/15\,120)R_{\kappa\lambda}R^{\kappa\rho\sigma\tau}R^{\lambda}{}_{\rho\sigma\tau} - (1/9072)R^{\rho\kappa\sigma\lambda}R_{\rho\alpha\sigma\beta}R_{\kappa}{}^{\alpha}{}_{\lambda}{}^{\beta} \\
& - (11/181\,440)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\alpha\beta}R_{\kappa\lambda}{}^{\alpha\beta}.
\end{aligned} \tag{79}$$

It should be noted that our expressions of the coefficients v_0, v_{0ab} and v_1 are in agreement with those existing in the literature (see, for example, Ref. [35]). In contrast, our expressions of the coefficients v_{0abcd}, v_{1ab} and v_2 disagree with the only known results, i.e. those obtained by Phillips and Hu in Ref. [23]. The comparison of our results with theirs is far from being obvious. Indeed, contrary to Phillips and Hu we have systematically used the Bianchi identities (E1) and their consequences (E2)–(E4) in order to simplify all our calculations. As a consequence, our results are more compact while we consider a more general scalar theory (Phillips and Hu have limited their study to the conformally invariant theory, i.e. they have worked with $m^2 = 0$ and $\xi = 1/6$). For example, our expressions of v_{0abcd} and v_{1ab} have, respectively, 36 and 54 terms while those of Phillips and Hu have, respectively, 52 and 71 terms. Because of that, we have been obligated to first simplify their results and then we have emphasized the disagreement with ours. In fact, we are sure that the results of Phillips and Hu are wrong. Indeed, we know that in the four-dimensional framework the coefficients v_2 and a_3 must be proportional and we have found that the result of Phillips and Hu does not reproduce that of Gilkey. This is not really surprising: they have constructed the covariant Taylor series expansions of the Hadamard coefficients from the expansions of $\sigma_{;\mu\nu}$ and $\Delta^{1/2}$ and we have noted in Appendixes B and C that the expansions they have obtained for these two bitensors are incorrect.

For $d = 5$, we have

$$U_1 = u_1 - u_{1a}\sigma^{;a} + \frac{1}{2!}u_{1ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}u_{1abc}\sigma^{;a}\sigma^{;b}\sigma^{;c} + \frac{1}{4!}u_{1abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} + O(\sigma^{5/2}), \tag{80}$$

$$U_2 = u_2 - u_{2a}\sigma^{;a} + \frac{1}{2!}u_{2ab}\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}), \tag{81}$$

$$U_3 = u_3 + O(\sigma^{1/2}), \tag{82}$$

with

$$u_1 = -m^2 - (\xi - 1/6)R, \tag{83a}$$

$$u_{1a} = -(1/2)(\xi - 1/6)R_{;a}, \tag{83b}$$

$$\begin{aligned}
u_{1ab} = {} & (1/60)\Box R_{ab} - (1/3)(\xi - 3/20)R_{;ab} - (1/6)m^2 R_{ab} - (1/6)(\xi - 1/6)RR_{ab} - (1/45)R^{\rho}{}_{a}R_{\rho b} + (1/90)R^{\rho\sigma}R_{\rho a\sigma b} \\
& + (1/90)R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b},
\end{aligned} \tag{83c}$$

$$\begin{aligned}
u_{1abc} = {} & -(1/4)(\xi - 2/15)R_{;(abc)} + (1/40)(\Box R_{(ab})_{;c)} - (1/4)m^2 R_{(ab;c)} - (1/4)(\xi - 1/6)RR_{(ab;c)} - (1/4)(\xi - 1/6)R_{;(a}R_{bc)} \\
& - (1/15)R^{\rho}{}_{(a}R_{|\rho|b;c)} + (1/60)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;c)} + (1/60)R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c)} + (1/30)R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;c)},
\end{aligned} \tag{83d}$$

$$\begin{aligned}
u_{1abcd} = {} & (1/35)(\Box R_{(ab})_{;cd)} - (1/5)(\xi - 5/42)R_{;(abcd)} - (3/10)m^2 R_{(ab;cd)} - (3/10)(\xi - 1/6)RR_{(ab;cd)} \\
& - (1/2)(\xi - 1/6)R_{;(a}R_{bc;d)} - (1/3)(\xi - 3/20)R_{;(ab}R_{cd)} + (1/60)R_{(ab}\Box R_{cd)} - (1/12)m^2 R_{(ab}R_{cd)} \\
& - (3/35)R^{\rho}{}_{(a}R_{|\rho|b;cd)} + (1/105)R^{\rho}{}_{(a}R_{bc;|\rho|d)} - (11/210)R^{\rho}{}_{(a;b}R_{|\rho|c;d)} - (3/70)R^{\rho}{}_{(a;b}R_{cd);\rho} + (17/840)R_{(ab}{}^{;\rho}R_{cd);\rho} \\
& + (2/105)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;cd)} + (1/105)R^{\rho}{}_{(a;|\sigma|}R^{\sigma}{}_{b|\rho|c;d)} + (1/30)R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c;d)} - (4/175)R^{\rho}{}_{(a;|\sigma|b}R^{\sigma}{}_{c|\rho|d)} \\
& + (11/525)R_{(ab}{}^{;\rho}{}_{\sigma}R^{\sigma}{}_{c|\rho|d)} + (11/525)R^{\rho}{}_{\sigma;(ab}R^{\sigma}{}_{c|\rho|d)} + (4/525)R^{\rho}{}_{(a|\sigma|b}\Box R^{\sigma}{}_{c|\rho|d)} - (1/15)m^2 R^{\rho}{}_{(a|\sigma|b}R^{\sigma}{}_{c|\rho|d)} \\
& + (4/105)R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;cd)} + (1/140)R^{\rho}{}_{(a|\sigma|b}{}^{;\tau}R^{\sigma}{}_{c|\rho|d);\tau} + (1/28)R^{\rho\sigma\tau}{}_{(a;b}R_{|\rho\sigma\tau|c;d)} - (1/12)(\xi - 1/6)RR_{(ab}R_{cd)} \\
& - (1/45)R^{\rho}{}_{(a}R_{|\rho|b}R_{cd)} + (1/315)R^{\rho}{}_{(a}R_{|\sigma|b}R^{\sigma}{}_{c|\rho|d)} + (1/90)R^{\rho\sigma}R_{(ab}R_{|\rho|c|\sigma|d)} - (1/15)(\xi - 1/6)RR^{\rho}{}_{(a|\sigma|b}R^{\sigma}{}_{c|\rho|d)} \\
& + (1/90)R_{(ab}R^{\rho\sigma\tau}{}_{c}R_{|\rho\sigma\tau|d)} + (26/1575)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d)} + (2/63)R^{\rho}{}_{(a}R^{\sigma}{}_{b}{}^{\tau}{}_{c}R_{|\rho\sigma\tau|d)} \\
& + (4/1575)R^{\rho\sigma\tau\kappa}R_{\rho(a|\tau|b}R_{|\sigma|c|\kappa|d)} + (4/525)R^{\rho\kappa\tau}{}_{(a}R_{|\rho\tau|}{}^{\sigma}{}_{b}R_{|\sigma|c|\kappa|d)} \\
& + (16/1575)R^{\rho\kappa\tau}{}_{(a}R_{|\rho|}{}^{\sigma}{}_{|\tau|b}R_{|\sigma|c|\kappa|d)} + (8/1575)R^{\rho\tau\kappa}{}_{(a}R_{|\rho\tau|}{}^{\sigma}{}_{b}R_{|\sigma|c|\kappa|d)},
\end{aligned} \tag{83e}$$

and

$$\begin{aligned} u_2 = {} & -(1/2)m^4 + (1/6)(\xi - 1/5)\Box R - (\xi - 1/6)m^2 R - (1/2)(\xi - 1/6)^2 R^2 \\ & + (1/180)R_{\rho\sigma}R^{\rho\sigma} - (1/180)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}, \end{aligned} \tag{84a}$$

$$\begin{aligned} u_{2a} = {} & (1/12)(\xi - 1/5)(\Box R)_{;a} - (1/2)(\xi - 1/6)m^2 R_{;a} - (1/2)(\xi - 1/6)^2 R R_{;a} \\ & + (1/180)R_{\rho\sigma}R^{\rho\sigma}{}_{;a} - (1/180)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}{}_{;a}, \end{aligned} \tag{84b}$$

$$\begin{aligned} u_{2ab} = {} & -(1/840)\Box\Box R_{ab} + (1/20)(\xi - 4/21)(\Box R)_{;ab} + (1/60)m^2\Box R_{ab} - (1/3)(\xi - 3/20)m^2 R_{;ab} - (1/12)m^4 R_{ab} \\ & - (1/3)(\xi - 1/6)(\xi - 3/20)R R_{;ab} + (1/90)(\xi - 1/7)R_{;\rho(a}R^{\rho}{}_{b)} + (1/36)(\xi - 1/5)(\Box R)R_{ab} \\ & - (1/4)(\xi - 1/6)^2 R_{;a}R_{;b} - (1/30)(\xi - 3/14)R_{;\rho}R^{\rho}{}_{(a;b)} + (1/30)(\xi - 17/84)R_{;\rho}R_{ab}{}^{;\rho} - (1/6)(\xi - 1/6)m^2 R R_{ab} \\ & + (1/60)(\xi - 1/6)R\Box R_{ab} + (1/252)R_{\rho(a}\Box R^{\rho}{}_{b)} - (1/45)m^2 R_{\rho a}R^{\rho}{}_{b} + (11/3150)R^{\rho\sigma}R_{\rho\sigma;(ab)} \\ & + (1/360)R^{\rho\sigma}{}_{;a}R_{\rho\sigma;b} + (1/1050)R^{\rho\sigma}R_{\rho(a;b)\sigma} - (2/1575)R^{\rho\sigma}R_{ab;\rho\sigma} - (1/1260)R^{\rho}{}_{a;\sigma}R_{\rho b}{}^{;\sigma} + (1/252)R^{\rho}{}_{a;\sigma}R^{\sigma}{}_{b;\rho} \\ & + (1/45)(\xi - 3/14)R^{;\rho\sigma}R_{\rho a\sigma b} - (1/630)(\Box R^{\rho\sigma})R_{\rho a\sigma b} + (1/90)m^2 R^{\rho\sigma}R_{\rho a\sigma b} - (1/630)R^{\rho\sigma;\tau}R_{\tau\sigma\rho(a;b)} \\ & - (1/900)R^{\rho\sigma}\Box R_{\rho a\sigma b} - (1/420)R^{\rho\sigma;\tau}R_{\rho a\sigma b;\tau} + (2/1575)R^{\rho\sigma;\tau}{}_{(a}R_{|\tau\sigma\rho|b)} - (23/6300)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\tau\sigma\rho|b)} \\ & + (1/225)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\rho\sigma\tau|b)} + (1/350)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau(a;b)\kappa} - (4/1575)R^{\rho\sigma\tau}{}_{a}\Box R_{\rho\sigma\tau b} + (1/90)m^2 R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} \\ & - (29/6300)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa;(ab)} - (1/420)R^{\rho\sigma\tau}{}_{a;\kappa}R_{\rho\sigma\tau b}{}^{;\kappa} - (1/336)R^{\rho\sigma\tau\kappa}{}_{;a}R_{\rho\sigma\tau\kappa;b} - (1/12)(\xi - 1/6)^2 R^2 R_{ab} \\ & - (1/45)(\xi - 1/6)R R_{\rho a}R^{\rho}{}_{b} + (1/1080)R^{\rho\sigma}R_{\rho\sigma}R_{ab} - (1/945)R^{\rho\sigma}R_{\rho a}R_{\sigma b} + (1/90)(\xi - 1/6)R R^{\rho\sigma}R_{\rho a\sigma b} \\ & + (1/1890)R^{\rho\tau}R^{\sigma}{}_{\tau}R_{\rho a\sigma b} - (8/4725)R^{\rho\sigma}R^{\tau}{}_{(a}R_{|\tau\sigma\rho|b)} - (1/9450)R_{\rho\sigma}R^{\rho\kappa\sigma\lambda}R_{\kappa a\lambda b} + (1/90)(\xi - 1/6)R R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} \\ & - (1/1080)R_{ab}R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa} - (31/18\,900)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\kappa\lambda b} + (1/300)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\lambda\kappa b} - (17/18\,900)R^{\rho\sigma}R^{\kappa\lambda}{}_{\rho a}R_{\kappa\lambda\sigma b} \\ & + (17/7560)R^{\kappa}{}_{(a}R^{\rho\sigma\tau}{}_{|\kappa|}R_{|\rho\sigma\tau|b)} + (17/9450)R^{\rho\sigma\tau}{}_{\lambda}R_{\rho\sigma\tau\kappa}R^{\lambda}{}_{a}{}^{\kappa}{}_{b} - (1/189)R^{\rho\kappa\sigma\lambda}R^{\tau}{}_{\rho\sigma a}R_{\tau\kappa\lambda b} \\ & + (1/450)R^{\rho\kappa\sigma\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b} - (19/4725)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b}, \end{aligned} \tag{84c}$$

and

$$\begin{aligned} u_3 = {} & -(1/18)m^6 - (1/180)(\xi - 3/14)\Box\Box R + (1/18)(\xi - 1/5)m^2\Box R - (1/6)(\xi - 1/6)m^4 R \\ & + (1/18)(\xi - 1/6)(\xi - 1/5)R\Box R + (1/36)[\xi^2 - (2/5)\xi + 17/420]R_{;\rho}R^{;\rho} - (1/6)(\xi - 1/6)^2 m^2 R^2 \\ & - (1/270)(\xi - 3/14)R_{;\rho\sigma}R^{\rho\sigma} - (1/1890)R_{\rho\sigma}\Box R^{\rho\sigma} + (1/540)m^2 R_{\rho\sigma}R^{\rho\sigma} - (1/7560)R_{\rho\sigma;\tau}R^{\rho\sigma;\tau} \\ & - (1/3780)R_{\rho\tau;\sigma}R^{\sigma\tau;\rho} + (1/1260)R_{\rho\sigma\tau\kappa}\Box R^{\rho\sigma\tau\kappa} - (1/540)m^2 R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} + (1/1680)R_{\rho\sigma\tau\kappa;\lambda}R^{\rho\sigma\tau\kappa;\lambda} \\ & - (1/18)(\xi - 1/6)^3 R^3 + (1/540)(\xi - 1/6)R R_{\rho\sigma}R^{\rho\sigma} + (1/17\,010)R_{\rho\sigma}R^{\rho}{}_{\tau}R^{\sigma\tau} - (1/5670)R_{\rho\sigma}R_{\kappa\lambda}R^{\rho\kappa\sigma\lambda} \\ & - (1/540)(\xi - 1/6)R R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} - (1/2835)R_{\kappa\lambda}R^{\kappa\rho\sigma\tau}R^{\lambda}{}_{\rho\sigma\tau} + (1/1701)R^{\rho\kappa\sigma\lambda}R_{\rho\alpha\sigma\beta}R_{\kappa}{}^{\alpha}{}_{\lambda}{}^{\beta} \\ & + (11/34\,020)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\alpha\beta}R_{\kappa\lambda}{}^{\alpha\beta}. \end{aligned} \tag{85}$$

For $d = 6$, we have

$$\begin{aligned} U_1 = {} & u_1 - u_{1a}\sigma^{;a} + \frac{1}{2!}u_{1ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}u_{1abc}\sigma^{;a}\sigma^{;b}\sigma^{;c} \\ & + \frac{1}{4!}u_{1abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} + O(\sigma^{5/2}), \end{aligned} \tag{86}$$

$$V_0 = v_0 - v_{0a}\sigma^{;a} + \frac{1}{2!}v_{0ab}\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}), \tag{87}$$

$$V_1 = v_1 + O(\sigma^{1/2}), \tag{88}$$

with

$$u_1 = -(1/2)m^2 - (1/2)(\xi - 1/6)R, \tag{89a}$$

$$u_{1a} = -(1/4)(\xi - 1/6)R_{;a}, \tag{89b}$$

$$\begin{aligned} u_{1ab} = {} & (1/120)\Box R_{ab} - (1/6)(\xi - 3/20)R_{;ab} \\ & - (1/12)m^2 R_{ab} - (1/12)(\xi - 1/6)R R_{ab} \\ & - (1/90)R^{\rho}{}_{a}R_{\rho b} + (1/180)R^{\rho\sigma}R_{\rho a\sigma b} \\ & + (1/180)R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b}, \end{aligned} \tag{89c}$$

$$\begin{aligned} u_{1abc} = {} & -(1/8)(\xi - 2/15)R_{;(abc)} + (1/80)(\Box R_{(ab})_{;c)} \\ & - (1/8)m^2 R_{(ab;c)} - (1/8)(\xi - 1/6)R R_{(ab;c)} \\ & - (1/8)(\xi - 1/6)R_{;(a}R_{bc)} - (1/30)R^{\rho}{}_{(a}R_{|\rho|b;c)} \\ & + (1/120)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;c)} + (1/120)R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c)} \\ & + (1/60)R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;c)}, \end{aligned} \tag{89d}$$

and

$$\begin{aligned}
u_{1abcd} = {}& (1/70)(\Box R_{(ab})_{;cd)} - (1/10)(\xi - 5/42)R_{;(abcd)} - (3/20)m^2 R_{(ab;cd)} - (3/20)(\xi - 1/6)RR_{(ab;cd)} \\
& - (1/4)(\xi - 1/6)R_{;(a}R_{bc;d)} - (1/6)(\xi - 3/20)R_{;(ab}R_{cd)} + (1/120)R_{(ab}\Box R_{cd)} - (1/24)m^2 R_{(ab}R_{cd)} \\
& - (3/70)R^{\rho}{}_{(a}R_{|\rho|b;cd)} + (1/210)R^{\rho}{}_{(a}R_{bc;|\rho|d)} - (11/420)R^{\rho}{}_{(a;b}R_{|\rho|c;d)} - (3/140)R^{\rho}{}_{(a;b}R_{cd);\rho} \\
& + (17/1680)R_{(ab}{}^{;\rho}R_{cd);\rho} + (1/105)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;cd)} + (1/210)R^{\rho}{}_{(a;|\sigma|}R^{\sigma}{}_{b|\rho|c;d)} + (1/60)R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c;d)} \\
& - (2/175)R^{\rho}{}_{(a;|\sigma|b}R^{\sigma}{}_{c|\rho|d)} + (11/1050)R_{(ab}{}^{;\rho}{}_{\sigma}R^{\sigma}{}_{c|\rho|d)} + (11/1050)R^{\rho}{}_{\sigma;(ab}R^{\sigma}{}_{c|\rho|d)} + (2/525)R^{\rho}{}_{(a|\sigma|b}\Box R^{\sigma}{}_{c|\rho|d)} \\
& - (1/30)m^2 R^{\rho}{}_{(a|\sigma|b}R^{\sigma}{}_{c|\rho|d)} + (2/105)R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;cd)} + (1/280)R^{\rho}{}_{(a|\sigma|b}{}^{;\tau}R^{\sigma}{}_{c|\rho|d);\tau} + (1/56)R^{\rho\sigma\tau}{}_{(a;b}R_{|\rho\sigma\tau|c;d)} \\
& - (1/24)(\xi - 1/6)RR_{(ab}R_{cd)} - (1/90)R^{\rho}{}_{(a}R_{|\rho|b}R_{cd)} + (1/630)R^{\rho}{}_{(a}R_{|\sigma|b}R^{\sigma}{}_{c|\rho|d)} + (1/180)R^{\rho\sigma}R_{(ab}R_{|\rho|c|\sigma|d)} \\
& - (1/30)(\xi - 1/6)RR^{\rho}{}_{(a|\sigma|b}R^{\sigma}{}_{c|\rho|d)} + (1/180)R_{(ab}R^{\rho\sigma\tau}{}_{c}R_{|\rho\sigma\tau|d)} + (13/1575)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\tau|b}R^{\tau}{}_{c|\rho|d)} \\
& + (1/63)R^{\rho}{}_{(a}R^{\sigma}{}_{b}{}^{\tau}{}_{c}R_{|\rho\sigma\tau|d)} + (2/1575)R^{\rho\sigma\tau\kappa}R_{\rho(a|\tau|b}R_{|\sigma|c|\kappa|d)} + (2/525)R^{\rho\kappa\tau}{}_{(a}R_{|\rho\tau|}{}^{\sigma}{}_{b}R_{|\sigma|c|\kappa|d)} \\
& + (8/1575)R^{\rho\kappa\tau}{}_{(a}R_{|\rho|}{}^{\sigma}{}_{|\tau|b}R_{|\sigma|c|\kappa|d)} + (4/1575)R^{\rho\tau\kappa}{}_{(a}R_{|\rho\tau|}{}^{\sigma}{}_{b}R_{|\sigma|c|\kappa|d)}, \qquad (89e)
\end{aligned}$$

and

$$\begin{aligned}
v_0 = {}& -(1/8)m^4 + (1/24)(\xi - 1/5)\Box R - (1/4)(\xi - 1/6)m^2 R - (1/8)(\xi - 1/6)^2 R^2 + (1/720)R_{\rho\sigma}R^{\rho\sigma} \\
& - (1/720)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}, \qquad (90a)
\end{aligned}$$

$$\begin{aligned}
v_{0a} = {}& (1/48)(\xi - 1/5)(\Box R)_{;a} - (1/8)(\xi - 1/6)m^2 R_{;a} - (1/8)(\xi - 1/6)^2 RR_{;a} + (1/720)R_{\rho\sigma}R^{\rho\sigma}{}_{;a} \\
& - (1/720)R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}{}_{;a}, \qquad (90b)
\end{aligned}$$

and

$$\begin{aligned}
v_{0ab} = {}& -(1/3360)\Box\Box R_{ab} + (1/80)(\xi - 4/21)(\Box R)_{;ab} + (1/240)m^2\Box R_{ab} - (1/12)(\xi - 3/20)m^2 R_{;ab} - (1/48)m^4 R_{ab} \\
& - (1/12)(\xi - 1/6)(\xi - 3/20)RR_{;ab} + (1/360)(\xi - 1/7)R_{;\rho(a}R^{\rho}{}_{b)} + (1/144)(\xi - 1/5)(\Box R)R_{ab} \\
& - (1/16)(\xi - 1/6)^2 R_{;a}R_{;b} - (1/120)(\xi - 3/14)R_{;\rho}R^{\rho}{}_{(a;b)} + (1/120)(\xi - 17/84)R_{;\rho}R_{ab}{}^{;\rho} \\
& - (1/24)(\xi - 1/6)m^2 RR_{ab} + (1/240)(\xi - 1/6)R\Box R_{ab} + (1/1008)R_{\rho(a}\Box R^{\rho}{}_{b)} - (1/180)m^2 R_{\rho a}R^{\rho}{}_{b} \\
& + (11/12\,600)R^{\rho\sigma}R_{\rho\sigma;(ab)} + (1/1440)R^{\rho\sigma}{}_{;a}R_{\rho\sigma;b} + (1/4200)R^{\rho\sigma}R_{\rho(a;b)\sigma} - (1/3150)R^{\rho\sigma}R_{ab;\rho\sigma} \\
& - (1/5040)R^{\rho}{}_{a;\sigma}R_{\rho b}{}^{;\sigma} + (1/1008)R^{\rho}{}_{a;\sigma}R^{\sigma}{}_{b;\rho} + (1/180)(\xi - 3/14)R^{;\rho\sigma}R_{\rho a\sigma b} - (1/2520)(\Box R^{\rho\sigma})R_{\rho a\sigma b} \\
& + (1/360)m^2 R^{\rho\sigma}R_{\rho a\sigma b} - (1/2520)R^{\rho\sigma;\tau}R_{\tau\sigma\rho(a;b)} - (1/3600)R^{\rho\sigma}\Box R_{\rho a\sigma b} - (1/1680)R^{\rho\sigma;\tau}R_{\rho a\sigma b;\tau} \\
& + (1/3150)R^{\rho\sigma;\tau}{}_{(a}R_{|\tau\sigma\rho|b)} - (23/25\,200)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\tau\sigma\rho|b)} + (1/900)R^{\rho}{}_{(a}{}^{;\sigma\tau}R_{|\rho\sigma\tau|b)} + (1/1400)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau(a;b)\kappa} \\
& - (1/1575)R^{\rho\sigma\tau}{}_{a}\Box R_{\rho\sigma\tau b} + (1/360)m^2 R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} - (29/25\,200)R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa;(ab)} - (1/1680)R^{\rho\sigma\tau}{}_{a;\kappa}R_{\rho\sigma\tau b}{}^{;\kappa} \\
& - (1/1344)R^{\rho\sigma\tau\kappa}{}_{;a}R_{\rho\sigma\tau\kappa;b} - (1/48)(\xi - 1/6)^2 R^2 R_{ab} - (1/180)(\xi - 1/6)RR_{\rho a}R^{\rho}{}_{b} + (1/4320)R^{\rho\sigma}R_{\rho\sigma}R_{ab} \\
& - (1/3780)R^{\rho\sigma}R_{\rho a}R_{\sigma b} + (1/360)(\xi - 1/6)RR^{\rho\sigma}R_{\rho a\sigma b} + (1/7560)R^{\rho\tau}R^{\sigma}{}_{\tau}R_{\rho a\sigma b} - (2/4725)R^{\rho\sigma}R^{\tau}{}_{(a}R_{|\tau\sigma\rho|b)} \\
& - (1/37\,800)R_{\rho\sigma}R^{\rho\kappa\sigma\lambda}R_{\kappa a\lambda b} + (1/360)(\xi - 1/6)RR^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} - (1/4320)R_{ab}R^{\rho\sigma\tau\kappa}R_{\rho\sigma\tau\kappa} \\
& - (31/75\,600)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\kappa\lambda b} + (1/1200)R_{\rho\sigma}R^{\rho\kappa\lambda}{}_{a}R^{\sigma}{}_{\lambda\kappa b} - (17/75\,600)R^{\rho\sigma}R^{\kappa\lambda}{}_{\rho a}R_{\kappa\lambda\sigma b} \\
& + (17/30\,240)R^{\kappa}{}_{(a}R^{\rho\sigma\tau}{}_{|\kappa|}R_{|\rho\sigma\tau|b)} + (17/37\,800)R^{\rho\sigma\tau}{}_{\lambda}R_{\rho\sigma\tau\kappa}R^{\lambda}{}_{a}{}^{\kappa}{}_{b} - (1/756)R^{\rho\kappa\sigma\lambda}R^{\tau}{}_{\rho\sigma a}R_{\tau\kappa\lambda b} \\
& + (1/1800)R^{\rho\kappa\sigma\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b} - (19/18\,900)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\tau a}R_{\kappa\lambda}{}^{\tau}{}_{b}, \qquad (90c)
\end{aligned}$$

and

$$
\begin{aligned}
v_1 = & -(1/48)m^6 - (1/480)(\xi - 3/14)\Box\Box R + (1/48)(\xi - 1/5)m^2\Box R - (1/16)(\xi - 1/6)m^4 R \\
& + (1/48)(\xi - 1/6)(\xi - 1/5)R\Box R + (1/96)[\xi^2 - (2/5)\xi + 17/420]R_{;\rho}R^{;\rho} - (1/16)(\xi - 1/6)^2 m^2 R^2 \\
& - (1/720)(\xi - 3/14)R_{;\rho\sigma}R^{\rho\sigma} - (1/5040)R_{\rho\sigma}\Box R^{\rho\sigma} + (1/1440)m^2 R_{\rho\sigma}R^{\rho\sigma} - (1/20\,160)R_{\rho\sigma;\tau}R^{\rho\sigma;\tau} \\
& - (1/10\,080)R_{\rho\tau;\sigma}R^{\sigma\tau;\rho} + (1/3360)R_{\rho\sigma\tau\kappa}\Box R^{\rho\sigma\tau\kappa} - (1/1440)m^2 R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} + (1/4480)R_{\rho\sigma\tau\kappa;\lambda}R^{\rho\sigma\tau\kappa;\lambda} \\
& - (1/48)(\xi - 1/6)^3 R^3 + (1/1440)(\xi - 1/6)RR_{\rho\sigma}R^{\rho\sigma} + (1/45\,360)R_{\rho\sigma}R^{\rho}{}_{\tau}R^{\sigma\tau} - (1/15\,120)R_{\rho\sigma}R_{\kappa\lambda}R^{\rho\kappa\sigma\lambda} \\
& - (1/1440)(\xi - 1/6)RR_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} - (1/7560)R_{\kappa\lambda}R^{\kappa\rho\sigma\tau}R^{\lambda}{}_{\rho\sigma\tau} + (1/4536)R^{\rho\kappa\sigma\lambda}R_{\rho\alpha\sigma\beta}R_{\kappa}{}^{\alpha}{}_{\lambda}{}^{\beta} \\
& + (11/90\,720)R^{\rho\sigma\kappa\lambda}R_{\rho\sigma\alpha\beta}R_{\kappa\lambda}{}^{\alpha\beta}. \qquad (91)
\end{aligned}
$$

V. CONCLUSION AND PERSPECTIVES

In this article, we have considered for a massive scalar field theory defined on an arbitrary curved spacetime the DeWitt-Schwinger and Hadamard representations of the associated Feynman propagator $G^{\mathrm{F}}(x, x')$. By combining the old covariant recursive method invented by DeWitt [1,2] with the modern covariant nonrecursive techniques introduced and developed by Avramidi (see Refs. [9,40] and references therein), we have obtained the covariant Taylor series expansions of the DeWitt coefficients $A_0(x, x')$, $A_1(x, x')$, $A_2(x, x')$ and $A_3(x, x')$ up to orders σ^3, σ^2, σ^1 and σ^0 respectively. We have then constructed the corresponding geometrical Hadamard coefficients for the dimensions $d = 3$, 4, 5 and 6 of spacetime. It should be noted that the DeWitt and Hadamard coefficients do not formally depend on the signature of the manifold on which the field theory is defined. As a consequence, all our results remain valid, *mutatis mutandis*, in the Riemannian framework, i.e. when the metric of the gravitational background is a Riemannian one.

As an immediate first application of the results obtained in this article, we intend now to develop the Hadamard regularization of the stress-energy tensor for a quantized scalar field in a general spacetime of arbitrary dimension [48], emphasizing more particularly the cases corresponding to the dimensions $d = 3, 5, 6$ of spacetime which have not been treated explicitly till now.

Our results could be also immediately used in stochastic semiclassical gravity. Indeed, as we have noted in Sec. IV, the results obtained by Phillips and Hu in Ref. [23] which concern the covariant Taylor series expansion of the four-dimensional Hadamard representation are incorrect. As a consequence, our own results could be useful to test some of the conclusions of Ref. [23] concerning the behavior of the noise kernel in the Schwarzschild spacetime and to emphasize those which remain valid and those which are wrong.

Keeping in mind the various applications in classical and quantum gravitational physics mentioned in Sec. I, it seems to us also interesting to extend the present work (i) by going beyond the orders reached here for the scalar field theory, (ii) for the graviton field propagating on a curved vacuum spacetime and (iii) for more general field theories, i.e., for tensorial field theories coupled to external gauge fields. Of course, it is obvious that we shall not be able to realize such a program in the technical framework developed in this article, i.e., by partially using the old covariant recursive method of DeWitt. This method has permitted us to go beyond existing results but at the cost of odious calculations. Even if it presents the advantage to provide, at each step of the work, explicit results which can be controlled, it has certainly reached its limits here. In fact, it seems to us that the program we have proposed could be certainly realized by fully working in the framework of the covariant nonrecursive approach of Avramidi or by using the treatment developed in Ref. [12] by Gilkey which is based on the pseudodifferential operator theory (see Ref. [49] for a covariant version).

ACKNOWLEDGMENTS

We are grateful to Bruce Jensen for help with the English as well as for various discussions concerning quantum field theory in curved spacetime during the last 18 years.

APPENDIX A: HADAMARD FORM OF THE DEWITT-SCHWINGER REPRESENTATION

In this Appendix, we shall prove that the DeWitt-Schwinger representation $G^{\mathrm{F}}_{\mathrm{DS}}(x, x')$ of the Feynman propagator given by Eqs. (4), (5a), (5b), and (6) is a particular case of Hadamard representation. We shall use the method developed by Christensen in Refs. [20,21] for the four-dimensional theory and by DeWitt in Ref. [46] for the d-dimensional one in order to obtain the first divergent terms of the DeWitt-Schwinger representation. This method can be used to obtain the full expansion of $G^{\mathrm{F}}_{\mathrm{DS}}(x, x')$ and to show that it is of the Hadamard form. This was done in the four-dimensional context by Brown and Ottewill in Ref. [33] and we shall here extend the proceeding for an arbitrary dimension.

We first substitute (6) into (4). Then, by assuming that it is possible to exchange the summation and integration in the resulting expression we find that

$$
G^{\mathrm{F}}_{\mathrm{DS}}(x, x') = -(4\pi)^{-d/2} \sum_{n=0}^{+\infty} A_n(x, x') \times \int_0^{+\infty} ds\, (is)^{-d/2+n} e^{(i/2s)[\sigma(x,x')+i\epsilon] - im^2 s}. \qquad (A1)
$$

If we assume that x' is in the light cone of x, i.e. that $\sigma(x, x') < 0$, we can express the integral in Eq. (A1) in terms of the Hankel function of the second kind. By using Eqs. 8.421.7 and 8.476.8 of Ref. [50], we obtain

$$G^{\rm F}_{\rm DS}(x, x') = -\pi(4\pi)^{-d/2} i^{-d} \sum_{n=0}^{+\infty} (-1)^n A_n(x, x') \times \left(\frac{z(x, x')}{2m^2}\right)^{-d/2+1+n} H^{(2)}_{d/2-1-n}(z(x, x')) \tag{A2}$$

with

$$z(x, x') = (-2m^2[\sigma(x, x') + i\epsilon])^{1/2}. \tag{A3}$$

It should be noted that Eq. (A2) remains valid when x' is outside the light cone of x, i.e. when $\sigma(x, x') > 0$, provided we consider its analytic continuation from the fourth quadrant of the complex $z(x, x')$ plane to the first one.

Let us now assume d even. From $H^{(2)}_{-n}(z) = (-1)^n H^{(2)}_n(z)$ which is valid for $n \in \mathbb{N}$ (see Eq. 8.484.2 of Ref. [50]) we can write (A2) in the form

$$G^{\rm F}_{\rm DS}(x, x') = \pi(4\pi)^{-d/2} \sum_{n=1}^{d/2-1} (-1)^n (m^2)^n A_{d/2-1-n}(x, x') \times \left(\frac{z(x, x')}{2}\right)^{-n} H^{(2)}_n(z(x, x')) + \pi(4\pi)^{-d/2} \times \sum_{n=0}^{+\infty} \frac{1}{(m^2)^n} A_{d/2-1+n}(x, x') \left(\frac{z(x, x')}{2}\right)^n \times H^{(2)}_n(z(x, x')). \tag{A4}$$

It is then possible to insert into Eq. (A4) the series expansion

$$H^{(2)}_n(z) = \left[1 - (2i/\pi)\ln\left(\frac{z}{2}\right)\right]\left(\frac{z}{2}\right)^n \sum_{k=0}^{+\infty} (-1)^k \frac{(z/2)^{2k}}{k!(n+k)!} + (i/\pi)(1 - \delta_{n0}) \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{-n+2k} + (i/\pi) \sum_{k=0}^{+\infty} (-1)^k \frac{\psi(k+1) + \psi(n+k+1)}{k!(n+k)!} \times \left(\frac{z}{2}\right)^{n+2k} \tag{A5}$$

which is valid for $n \in \mathbb{N}$ and $|\arg z| < \pi$ (see Eqs. 8.402 and 8.403 of Ref. [50]). Here ψ denotes the logarithm derivative of the gamma function and it is given by (see Eq. 8.362.1 of Ref. [50])

$$\psi(z) = -\gamma - \sum_{\ell=0}^{+\infty} \left(\frac{1}{z+\ell} - \frac{1}{1+\ell}\right) \tag{A6}$$

where γ is the Euler constant. A tedious calculation permits us to prove that $G^{\rm F}_{\rm DS}(x, x')$ has the Hadamard form (11) and (12) with the Hadamard coefficients $U_n(x, x')$ and $V_n(x, x')$ respectively given by (27a) and (27b) and with the Hadamard coefficients $W_n(x, x')$ given by

$$W_n(x, x') = \ln(m^2/2) V_n(x, x') - [\psi(n+1) + \psi(n+d/2)] \times V_n(x, x') - \frac{(-1)^n}{2^{n+d/2-1} n!(d/2-2)!} \left[\sum_{k=0}^{n+d/2-2} \frac{(-1)^k (m^2)^k}{k!} \left(\sum_{\ell=k+1}^{n+d/2-1} \frac{1}{\ell} \right) A_{n+d/2-1-k}(x, x') - \sum_{k=0}^{+\infty} \frac{k!}{(m^2)^{k+1}} A_{n+d/2+k}(x, x') \right]. \tag{A7}$$

Another tedious calculation using Eqs. (7a), (7b), (16a), (16b), (27b), and (A6) permits us to verify that these coefficients satisfy the recursion relations (17). It is finally interesting to note the pathological behavior of the Hadamard coefficients $W_n(x, x')$ for $m^2 \to 0$ (infrared divergence).

Let us now assume d odd. From $H^{(2)}_{-n-1/2}(z) = -i(-1)^n H^{(2)}_{n+1/2}(z)$ valid for $n \in \mathbb{N}$ (see Eq. 8.484.2 of Ref. [50]) we can write (A2) in the form

$$G^{\rm F}_{\rm DS}(x, x') = -i\pi(4\pi)^{-d/2} \sum_{n=0}^{d/2-3/2} (-1)^n (m^2)^{n+1/2} \times A_{d/2-3/2-n}(x, x') \left(\frac{2}{z(x, x')}\right)^{n+1/2} \times H^{(2)}_{n+1/2}(z(x, x')) + \pi(4\pi)^{-d/2} \times \sum_{n=0}^{+\infty} \frac{1}{(m^2)^{n+1/2}} A_{d/2-1/2+n}(x, x') \times \left(\frac{z(x, x')}{2}\right)^{n+1/2} H^{(2)}_{n+1/2}(z(x, x')). \tag{A8}$$

It is then possible to insert into Eq. (A8) the series expansion

$$H^{(2)}_{n+1/2}(z) = (-1)^n i \left(\frac{2}{z}\right)^{n+1/2} \left[\sum_{k=0}^{+\infty} \frac{1}{k!\Gamma(k-n+1/2)} \left(\frac{iz}{2}\right)^{2k} - \sum_{k=n}^{+\infty} \frac{1}{(k-n)!\Gamma(k+3/2)} \left(\frac{iz}{2}\right)^{2k+1} \right] \tag{A9}$$

which is valid for $n \in \mathbb{N}$ and $|\arg z| < \pi$. We have not found this useful expansion in the literature. We have constructed it from

$$H^{(2)}_{n+1/2}(z) = \sqrt{\frac{2}{\pi z}} i^{n+1} e^{-iz} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} \frac{1}{(2iz)^k} \tag{A10}$$

which is valid for $n \in \mathbb{N}$ and $|\arg z| < \pi$ (see Eqs. 8.466.2 of Ref. [50]) by replacing e^{-iz} by its series expansion. Then, an easy calculation permits us to prove that $G^{\rm F}_{\rm DS}(x, x')$ has the Hadamard form (13) and (14) with the

Hadamard coefficients $U_n(x,x')$ given by (29) and with the Hadamard coefficients $W_n(x,x')$ given by

$$W_n(x,x') = -\frac{(-1)^n}{2^{n+d/2-1} n! \Gamma(d/2-1)} \times \Bigg[\sum_{k=0}^{n+d/2-3/2} \frac{(-1)^k (m^2)^{k+1/2}}{\Gamma(k+3/2)} \times \pi A_{n+d/2-3/2-k}(x,x') - \sum_{k=0}^{+\infty} \frac{\Gamma(k+1/2)}{(m^2)^{k+1/2}} A_{n+d/2-1/2+k}(x,x') \Bigg]. \quad \text{(A11)}$$

Using Eqs. (7a) and (7b), it is easy to verify that these coefficients satisfy the recursion relations (21). Here again, it is interesting to note the pathological behavior of the Hadamard coefficients $W_n(x,x')$ for $m^2 \to 0$ (infrared divergence).

APPENDIX B: COVARIANT TAYLOR SERIES EXPANSIONS OF THE BITENSORS $\Lambda^{\mu\nu} = \sigma^{;\mu\nu}$ AND $H^{\mu\nu} = g^{\nu}{}_{\nu'}\sigma^{;\mu\nu'}$

In this Appendix, we shall provide, up to order $\sigma^{9/2}$, the covariant Taylor series expansions of the bitensors $\Lambda(x,x')$ and $H(x,x')$ which are both tensors of type (1,1) in x and scalars in x' and of which components are, respectively, given by $\Lambda^{\mu}{}_{\nu} = \sigma^{;\mu}{}_{\nu}$ and $H^{\mu}{}_{\nu} = g_{\nu\nu'}\sigma^{;\mu\nu'}$. The notations used in this Appendix as well as the results obtained will be extensively used in Appendixes C, D, and E.

With Avramidi [9,40], we first introduce the bitensors $K_{(p)}(x,x')$ and $\gamma_{(p)}(x,x')$ with $p \geq 2$ which are all tensors of type (1,1) in x and scalars in x'. The bitensors $K_{(p)}(x,x')$ are defined from their components

$$K_{(p)}{}^{\mu}{}_{\nu}(x,x') = K^{\mu}{}_{\nu a_1 \cdots a_p}(x) \sigma^{;a_1}(x,x') \cdots \sigma^{;a_p}(x,x') \quad \text{(B1a)}$$

with

$$K^{\mu}{}_{\nu a_1 a_2 a_3 \cdots a_p} = R^{\mu}{}_{(a_1|\nu|a_2;a_3\cdots a_p)}. \quad \text{(B1b)}$$

The bitensors $\gamma_{(p)}(x,x')$ are constructed from the bitensors $K_{(p)}(x,x')$. They are defined by the relation

$$\gamma_{(p)} = \sum_{1\leq k \leq [p/2]} (-1)^{k+1}(2k)! \binom{p}{2k} \times \sum_{\substack{p_1,\ldots,p_k \geq 2 \\ p_1+\cdots+p_k = p}} \binom{p-2k}{p_1-2, \quad \cdots \quad, p_k-2} \frac{K_{(p_k)}}{p(p+1)} \times \frac{K_{(p_{k-1})}}{(p_1+\cdots+p_{k-1})(p_1+\ldots+p_{k-1}+1)} \cdots \times \frac{K_{(p_2)}}{(p_1+p_2)(p_1+p_2+1)} \frac{K_{(p_1)}}{p_1(p_1+1)} \quad \text{(B2)}$$

where

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} \quad \text{and} \quad \binom{p}{p_1, \quad \cdots \quad, p_k} = \frac{p!}{p_1! \cdots p_k!} \quad \text{if } p_1 + \cdots + p_k = p. \quad \text{(B3)}$$

The components of the $\gamma_{(p)}(x,x')$ are therefore of the form

$$\gamma_{(p)}{}^{\mu}{}_{\nu}(x,x') = \gamma^{\mu}{}_{\nu a_1 \cdots a_p}(x) \sigma^{;a_1}(x,x') \cdots \sigma^{;a_p}(x,x'). \quad \text{(B4)}$$

We have obtained the expressions of the bitensors $\gamma_{(p)}$ for $p = 2, \ldots, 11$. The results are

$$\gamma_{(2)} = (1/3)K_{(2)}, \quad \text{(B5a)}$$

$$\gamma_{(3)} = (1/2)K_{(3)}, \quad \text{(B5b)}$$

$$\gamma_{(4)} = (3/5)K_{(4)} - (1/5){K_{(2)}}^2, \quad \text{(B5c)}$$

$$\gamma_{(5)} = (2/3)K_{(5)} - (1/3)K_{(2)}K_{(3)} - (2/3)K_{(3)}K_{(2)}, \quad \text{(B5d)}$$

$$\gamma_{(6)} = (5/7)K_{(6)} - (3/7)K_{(2)}K_{(4)} - (10/7){K_{(3)}}^2 - (10/7)K_{(4)}K_{(2)} + (1/7){K_{(2)}}^3, \quad \text{(B5e)}$$

$$\gamma_{(7)} = (3/4)K_{(7)} - (1/2)K_{(2)}K_{(5)} - (9/4)K_{(3)}K_{(4)} - (15/4)K_{(4)}K_{(3)} - (5/2)K_{(5)}K_{(2)} + (1/4){K_{(2)}}^2K_{(3)} + (1/2)K_{(2)}K_{(3)}K_{(2)} + (3/4)K_{(3)}{K_{(2)}}^2, \quad \text{(B5f)}$$

$$\gamma_{(8)} = (7/9)K_{(8)} - (5/9)K_{(2)}K_{(6)} - (28/9)K_{(3)}K_{(5)} - 7{K_{(4)}}^2 - (70/9)K_{(5)}K_{(3)} - (35/9)K_{(6)}K_{(2)} + (1/3){K_{(2)}}^2K_{(4)} + (10/9)K_{(2)}{K_{(3)}}^2 + (10/9)K_{(2)}K_{(4)}K_{(2)} + (14/9)K_{(3)}K_{(2)}K_{(3)} + (28/9){K_{(3)}}^2K_{(2)} + (7/3)K_{(4)}{K_{(2)}}^2 - (1/9){K_{(2)}}^4, \quad \text{(B5g)}$$

and

$$\gamma_{(9)} = (4/5)K_{(9)} - (3/5)K_{(2)}K_{(7)} - 4K_{(3)}K_{(6)} - (56/5)K_{(4)}K_{(5)} - (84/5)K_{(5)}K_{(4)} - 14K_{(6)}K_{(3)} - (28/5)K_{(7)}K_{(2)} + (2/5){K_{(2)}}^2K_{(5)} + (9/5)K_{(2)}K_{(3)}K_{(4)} + 3K_{(2)}K_{(4)}K_{(3)} + 2K_{(2)}K_{(5)}K_{(2)} + (12/5)K_{(3)}K_{(2)}K_{(4)} + 8{K_{(3)}}^3 + 8K_{(3)}K_{(4)}K_{(2)} + (28/5)K_{(4)}K_{(2)}K_{(3)} + (56/5)K_{(4)}K_{(3)}K_{(2)} + (28/5)K_{(5)}{K_{(2)}}^2 - (1/5){K_{(2)}}^3K_{(3)} - (2/5){K_{(2)}}^2K_{(3)}K_{(2)} - (3/5)K_{(2)}K_{(3)}{K_{(2)}}^2 - (4/5)K_{(3)}{K_{(2)}}^3, \tag{B5h}$$

$$\gamma_{(10)} = (9/11)K_{(10)} - (7/11)K_{(2)}K_{(8)} - (54/11)K_{(3)}K_{(7)} - (180/11)K_{(4)}K_{(6)} - (336/11){K_{(5)}}^2 - (378/11)K_{(6)}K_{(4)} - (252/11)K_{(7)}K_{(3)} - (84/11)K_{(8)}K_{(2)} + (5/11){K_{(2)}}^2K_{(6)} + (28/11)K_{(2)}K_{(3)}K_{(5)} + (63/11)K_{(2)}{K_{(4)}}^2 + (70/11)K_{(2)}K_{(5)}K_{(3)} + (35/11)K_{(2)}K_{(6)}K_{(2)} + (36/11)K_{(3)}K_{(2)}K_{(5)} + (162/11){K_{(3)}}^2K_{(4)} + (270/11)K_{(3)}K_{(4)}K_{(3)} + (180/11)K_{(3)}K_{(5)}K_{(2)} + (108/11)K_{(4)}K_{(2)}K_{(4)} + (360/11)K_{(4)}{K_{(3)}}^2 + (360/11){K_{(4)}}^2K_{(2)} + (168/11)K_{(5)}K_{(2)}K_{(3)} + (336/11)K_{(5)}K_{(3)}K_{(2)} + (126/11)K_{(6)}{K_{(2)}}^2 - (3/11){K_{(2)}}^3K_{(4)} - (10/11){K_{(2)}}^2{K_{(3)}}^2 - (10/11){K_{(2)}}^2K_{(4)}K_{(2)} - (14/11)K_{(2)}K_{(3)}K_{(2)}K_{(3)} - (28/11)K_{(2)}{K_{(3)}}^2K_{(2)} - (21/11)K_{(2)}K_{(4)}{K_{(2)}}^2 - (18/11)K_{(3)}{K_{(2)}}^2K_{(3)} - (36/11)K_{(3)}K_{(2)}K_{(3)}K_{(2)} - (54/11){K_{(3)}}^2{K_{(2)}}^2 - (36/11)K_{(4)}{K_{(2)}}^3 + (1/11){K_{(2)}}^5, \tag{B5i}$$

and

$$\gamma_{(11)} = (5/6)K_{(11)} - (2/3)K_{(2)}K_{(9)} - (35/6)K_{(3)}K_{(8)} - (45/2)K_{(4)}K_{(7)} - 50K_{(5)}K_{(6)} - 70K_{(6)}K_{(5)} - 63K_{(7)}K_{(4)} - 35K_{(8)}K_{(3)} - 10K_{(9)}K_{(2)} + (1/2){K_{(2)}}^2K_{(7)} + (10/3)K_{(2)}K_{(3)}K_{(6)} + (28/3)K_{(2)}K_{(4)}K_{(5)} + 14K_{(2)}K_{(5)}K_{(4)} + (35/3)K_{(2)}K_{(6)}K_{(3)} + (14/3)K_{(2)}K_{(7)}K_{(2)} + (25/6)K_{(3)}K_{(2)}K_{(6)} + (70/3){K_{(3)}}^2K_{(5)} + (105/2)K_{(3)}{K_{(4)}}^2 + (175/3)K_{(3)}K_{(5)}K_{(3)} + (175/6)K_{(3)}K_{(6)}K_{(2)} + 15K_{(4)}K_{(2)}K_{(5)} + (135/2)K_{(4)}K_{(3)}K_{(4)} + (225/2){K_{(4)}}^2K_{(3)} + 75K_{(4)}K_{(5)}K_{(2)} + 30K_{(5)}K_{(2)}K_{(4)} + 100K_{(5)}{K_{(3)}}^2 + 100K_{(5)}K_{(4)}K_{(2)} + 35K_{(6)}K_{(2)}K_{(3)} + 70K_{(6)}K_{(3)}K_{(2)} + 21K_{(7)}{K_{(2)}}^2 - (1/3){K_{(2)}}^3K_{(5)} - (3/2){K_{(2)}}^2K_{(3)}K_{(4)} - (5/2){K_{(2)}}^2K_{(4)}K_{(3)} - (5/3){K_{(2)}}^2K_{(5)}K_{(2)} - 2K_{(2)}K_{(3)}K_{(2)}K_{(4)} - (20/3)K_{(2)}{K_{(3)}}^3 - (20/3)K_{(2)}K_{(3)}K_{(4)}K_{(2)} - (14/3)K_{(2)}K_{(4)}K_{(2)}K_{(3)} - (28/3)K_{(2)}K_{(4)}K_{(3)}K_{(2)} - (14/3)K_{(2)}K_{(5)}{K_{(2)}}^2 - (5/2)K_{(3)}{K_{(2)}}^2K_{(4)} - (25/3)K_{(3)}K_{(2)}{K_{(3)}}^2 - (25/3)K_{(3)}K_{(2)}K_{(4)}K_{(2)} - (35/3){K_{(3)}}^2K_{(2)}K_{(3)} - (70/3){K_{(3)}}^3K_{(2)} - (35/2)K_{(3)}K_{(4)}{K_{(2)}}^2 - (15/2)K_{(4)}{K_{(2)}}^2K_{(3)} - 15K_{(4)}K_{(2)}K_{(3)}K_{(2)} - (45/2)K_{(4)}K_{(3)}{K_{(2)}}^2 - 10K_{(5)}{K_{(2)}}^3 + (1/6){K_{(2)}}^4K_{(3)} + (1/3){K_{(2)}}^3K_{(3)}K_{(2)} + (1/2){K_{(2)}}^2K_{(3)}{K_{(2)}}^2 + (2/3)K_{(2)}K_{(3)}{K_{(2)}}^3 + (5/6)K_{(3)}{K_{(2)}}^4. \tag{B5j}$$

In the present Appendix, we shall need only the $\gamma_{(p)}$ with $p = 2, \ldots, 9$ in order to construct the expansions of Λ and H up to order $\sigma^{9/2}$, but in Appendix C, we shall need also their expressions for $p = 10, 11$ in order to obtain the expansion of $\Delta^{1/2}$ up to order $\sigma^{11/2}$. It should be noted that Eqs. (B5a)–(B5j) provide compact expressions of the bitensors $\gamma_{(p)}$ with $p = 2, \ldots, 11$. By using (B1a), (B1b), and (B4) into (B5), it is also possible to reexpress these relations at the level of the components ${\gamma^\mu}_{\nu a_1 \cdots a_p}$ of the bitensors $\gamma_{(p)}$. Of course, the results are much more heavy. For example, the components of the lowest order bitensors $\gamma_{(p)}$ take the form

$${\gamma^\mu}_{\nu a_1 a_2} = (1/3){R^\mu}_{(a_1|\nu|a_2)}, \tag{B6a}$$

$${\gamma^\mu}_{\nu a_1 a_2 a_3} = (1/2){R^\mu}_{(a_1|\nu|a_2;a_3)}, \tag{B6b}$$

$${\gamma^\mu}_{\nu a_1 a_2 a_3 a_4} = (3/5){R^\mu}_{(a_1|\nu|a_2;a_3a_4)} - (1/5){R^\mu}_{(a_1|\rho|a_2}{R^\rho}_{a_3|\nu|a_4)}, \tag{B6c}$$

$${\gamma^\mu}_{\nu a_1 a_2 a_3 a_4 a_5} = (2/3){R^\mu}_{(a_1|\nu|a_2;a_3a_4a_5)} - (1/3){R^\mu}_{(a_1|\rho|a_2}{R^\rho}_{a_3|\nu|a_4;a_5)} - (2/3){R^\mu}_{(a_1|\rho|a_2;a_3}{R^\rho}_{a_4|\nu|a_5)}, \tag{B6d}$$

$${\gamma^\mu}_{\nu a_1 a_2 a_3 a_4 a_5 a_6} = (5/7){R^\mu}_{(a_1|\nu|a_2;a_3a_4a_5a_6)} - (3/7){R^\mu}_{(a_1|\rho|a_2}{R^\rho}_{a_3|\nu|a_4;a_5a_6)} - (10/7){R^\mu}_{(a_1|\rho|a_2;a_3}{R^\rho}_{a_4|\nu|a_5;a_6)} - (10/7){R^\mu}_{(a_1|\rho|a_2;a_3a_4}{R^\rho}_{a_5|\nu|a_6)} + (1/7){R^\mu}_{(a_1|\rho|a_2}{R^\rho}_{a_3|\tau|a_4}{R^\tau}_{a_5|\nu|a_6)}. \tag{B6e}$$

The bitensors $\gamma_{(p)}(x, x')$ permit us to construct the covariant Taylor series expansions of the bitensor $H(x, x')$ and of its inverse denoted by $\Gamma(x, x')$. Of course, the latter satisfies

$$H\Gamma = \Gamma H = 1 \tag{B7}$$

and is also a tensor of type (1,1) in x and a scalar in x'. We have [9,40]

$$\Gamma(x, x') = -1 + \sum_{p=2}^{+\infty} \frac{(-1)^p}{p!} \gamma_{(p)}(x, x') \tag{B8}$$

and

$$H(x, x') = -1 + \sum_{p=2}^{+\infty} \frac{(-1)^p}{p!} \eta_{(p)}(x, x') \tag{B9}$$

with

$$\eta_{(p)} = - \sum_{1\le k \le [p/2]} \sum_{\substack{p_1,\ldots,p_k \ge 2 \\ p_1+\cdots+p_k=p}} \begin{pmatrix} & p & \\ p_1, & \ldots, & p_k \end{pmatrix} \gamma_{(p_k)} \cdots \gamma_{(p_1)}. \tag{B10}$$

The components of the bitensors $\eta_{(p)}(x, x')$ are therefore of the form

$$\eta_{(p)}{}^{\mu}{}_{\nu}(x, x') = \eta^{\mu}{}_{\nu a_1 \ldots a_p}(x) \sigma^{;a_1}(x, x') \cdots \sigma^{;a_p}(x, x'), \tag{B11}$$

with this result being a direct consequence of (B4) and (B10). It is possible to express the bitensors $\eta_{(p)}$ in terms of the bitensors $K_{(p)}$. From (B5) and (B10), we obtain the $\eta_{(p)}$ for $p = 2, \ldots, 9$. We have

$$\eta_{(2)} = -(1/3)K_{(2)}, \tag{B12a}$$
$$\eta_{(3)} = -(1/2)K_{(3)}, \tag{B12b}$$
$$\eta_{(4)} = -(3/5)K_{(4)} - (7/15){K_{(2)}}^2, \tag{B12c}$$
$$\eta_{(5)} = -(2/3)K_{(5)} - (4/3)K_{(2)}K_{(3)} - K_{(3)}K_{(2)}, \tag{B12d}$$
$$\eta_{(6)} = -(5/7)K_{(6)} - (18/7)K_{(2)}K_{(4)} - (25/7){K_{(3)}}^2 - (11/7)K_{(4)}K_{(2)} - (31/21){K_{(2)}}^3, \tag{B12e}$$
$$\begin{aligned}\eta_{(7)} = {}& -(3/4)K_{(7)} - (25/6)K_{(2)}K_{(5)} - (33/4)K_{(3)}K_{(4)} - (27/4)K_{(4)}K_{(3)} - (13/6)K_{(5)}K_{(2)} - (73/12){K_{(2)}}^2K_{(3)} \\ & - (31/6)K_{(2)}K_{(3)}K_{(2)} - (17/4)K_{(3)}{K_{(2)}}^2,\end{aligned} \tag{B12f}$$
$$\begin{aligned}\eta_{(8)} = {}& -(7/9)K_{(8)} - (55/9)K_{(2)}K_{(6)} - (140/9)K_{(3)}K_{(5)} - (91/5){K_{(4)}}^2 - (98/9)K_{(5)}K_{(3)} - (25/9)K_{(6)}K_{(2)} \\ & - (239/15){K_{(2)}}^2K_{(4)} - (226/9)K_{(2)}{K_{(3)}}^2 - (106/9)K_{(2)}K_{(4)}K_{(2)} - (182/9)K_{(3)}K_{(2)}K_{(3)} - (160/9){K_{(3)}}^2K_{(2)} \\ & - (43/5)K_{(4)}{K_{(2)}}^2 - (127/15){K_{(2)}}^4,\end{aligned} \tag{B12g}$$

and

$$\begin{aligned}\eta_{(9)} = {}& -(4/5)K_{(9)} - (42/5)K_{(2)}K_{(7)} - 26K_{(3)}K_{(6)} - (196/5)K_{(4)}K_{(5)} - (168/5)K_{(5)}K_{(4)} - 16K_{(6)}K_{(3)} - (17/5)K_{(7)}K_{(2)} \\ & - (168/5){K_{(2)}}^2K_{(5)} - (378/5)K_{(2)}K_{(3)}K_{(4)} - 66K_{(2)}K_{(4)}K_{(3)} - 22K_{(2)}K_{(5)}K_{(2)} - 60K_{(3)}K_{(2)}K_{(4)} - 98{K_{(3)}}^3 \\ & - 47K_{(3)}K_{(4)}K_{(2)} - (232/5)K_{(4)}K_{(2)}K_{(3)} - (209/5)K_{(4)}K_{(3)}K_{(2)} - (74/5)K_{(5)}{K_{(2)}}^2 - (226/5){K_{(2)}}^3K_{(3)} \\ & - (197/5){K_{(2)}}^2K_{(3)}K_{(2)} - (184/5)K_{(2)}K_{(3)}{K_{(2)}}^2 - 31K_{(3)}{K_{(2)}}^3.\end{aligned} \tag{B12h}$$

By using (B1a), (B1b), and (B11) into (B12), we can also obtain the expressions of the components $\eta^{\mu}{}_{\nu a_1 \cdots a_p}$ of the bitensors $\eta_{(p)}$. The components of the lowest order bitensors $\eta_{(p)}$ take the form

$$\eta^{\mu}{}_{\nu a_1 a_2} = -(1/3)R^{\mu}{}_{(a_1|\nu|a_2)}, \tag{B13a}$$
$$\eta^{\mu}{}_{\nu a_1 a_2 a_3} = -(1/2)R^{\mu}{}_{(a_1|\nu|a_2;a_3)}, \tag{B13b}$$
$$\eta^{\mu}{}_{\nu a_1 a_2 a_3 a_4} = -(3/5)R^{\mu}{}_{(a_1|\nu|a_2;a_3a_4)} - (7/15)R^{\mu}{}_{(a_1|\rho|a_2}R^{\rho}{}_{a_3|\nu|a_4)}, \tag{B13c}$$
$$\eta^{\mu}{}_{\nu a_1 a_2 a_3 a_4 a_5} = -(2/3)R^{\mu}{}_{(a_1|\nu|a_2;a_3a_4a_5)} - (4/3)R^{\mu}{}_{(a_1|\rho|a_2}R^{\rho}{}_{a_3|\nu|a_4;a_5)} - R^{\mu}{}_{(a_1|\rho|a_2;a_3}R^{\rho}{}_{a_4|\nu|a_5)}, \tag{B13d}$$
$$\begin{aligned}\eta^{\mu}{}_{\nu a_1 a_2 a_3 a_4 a_5 a_6} = {}& -(5/7)R^{\mu}{}_{(a_1|\nu|a_2;a_3a_4a_5a_6)} - (18/7)R^{\mu}{}_{(a_1|\rho|a_2}R^{\rho}{}_{a_3|\nu|a_4;a_5a_6)} - (25/7)R^{\mu}{}_{(a_1|\rho|a_2;a_3}R^{\rho}{}_{a_4|\nu|a_5;a_6)} \\ & - (11/7)R^{\mu}{}_{(a_1|\rho|a_2;a_3a_4}R^{\rho}{}_{a_5|\nu|a_6)} - (31/21)R^{\mu}{}_{(a_1|\rho|a_2}R^{\rho}{}_{a_3|\tau|a_4}R^{\tau}{}_{a_5|\nu|a_6)}.\end{aligned} \tag{B13e}$$

Finally, we can now construct the covariant Taylor series expansion of the bitensor $\Lambda(x, x')$ from the covariant Taylor series expansions of the bitensors $H(x, x')$ and $\Gamma(x, x')$. Indeed, by differentiating the identity (8) in x and in x', we obtain the relation $\sigma^{;\nu\rho'} = \sigma_{;\mu}{}^{\nu}\sigma^{;\mu\rho'} + \sigma_{;\mu}\sigma^{;\mu\nu\rho'}$. We then multiply this result by $g_{\rho\rho'}$ and taking into account (36a), we can write

$$g_{\rho\rho'}\sigma^{;\nu\rho'} = \sigma_{;\mu}{}^{\nu}(g_{\rho\rho'}\sigma^{;\mu\rho'}) + \sigma^{;\mu}\nabla_{\mu}(g_{\rho\rho'}\sigma^{;\nu\rho'}). \tag{B14}$$

This relation links the components of the bitensors H and Λ. It can be rewritten in the form

$$H = \Lambda H + DH, \tag{B15}$$

where we have introduced the differential operator

$$D = \sigma^{;\mu}\nabla_{\mu}. \tag{B16}$$

As a consequence, we have

$$\Lambda = 1 - (DH)\Gamma. \tag{B17}$$

The covariant Taylor series expansion of $\Lambda(x, x')$ is therefore given by

$$\Lambda(x, x') = 1 + \sum_{p=2}^{+\infty} \frac{(-1)^p}{p!} \lambda_{(p)}(x, x') \tag{B18}$$

where the $\lambda_{(p)}(x, x')$ are also tensors of type (1,1) in x and scalars in x' of which components are of the form

$$\lambda_{(p)}{}^{\mu}{}_{\nu}(x, x') = \lambda^{\mu}{}_{\nu a_1 \cdots a_p}(x)\sigma^{;a_1}(x, x') \cdots \sigma^{;a_p}(x, x') \tag{B19}$$

and it can be constructed from the covariant Taylor series expansions of $H(x, x')$ and $\Gamma(x, x')$. By noting the identities

$$D[K_{(p)}] = pK_{(p)} + K_{(p+1)}, \tag{B20a}$$

$$D[K_{(p)}K_{(q)}] = (p+q)K_{(p)}K_{(q)} + K_{(p+1)}K_{(q)} + K_{(p)}K_{(q+1)}, \tag{B20b}$$

$$D[K_{(p)}K_{(q)}K_{(r)}] = (p+q+r)K_{(p)}K_{(q)}K_{(r)} + K_{(p+1)}K_{(q)}K_{(r)} + K_{(p)}K_{(q+1)}K_{(r)} + K_{(p)}K_{(q)}K_{(r+1)}, \tag{B20c}$$

$$\begin{aligned} D[K_{(p)}K_{(q)}K_{(r)}K_{(s)}] = {} & (p+q+r+s)K_{(p)}K_{(q)}K_{(r)}K_{(s)} + K_{(p+1)}K_{(q)}K_{(r)}K_{(s)} + K_{(p)}K_{(q+1)}K_{(r)}K_{(s)} + K_{(p)}K_{(q)}K_{(r+1)}K_{(s)} \\ & + K_{(p)}K_{(q)}K_{(r)}K_{(s+1)} \end{aligned} \tag{B20d}$$

which follow from (8), (B1a), and (B1b), a tedious calculation using (B17) and (B18) as well as (B5), (B8), (B9), and (B12) permits us to obtain the $\lambda_{(p)}$ for $p = 2, \ldots, 9$. We have

$$\lambda_{(2)} = -(2/3)K_{(2)}, \tag{B21a}$$

$$\lambda_{(3)} = -(1/2)K_{(3)}, \tag{B21b}$$

$$\lambda_{(4)} = -[(2/5)K_{(4)} + (8/15)K_{(2)}{}^2], \tag{B21c}$$

$$\lambda_{(5)} = -[(1/3)K_{(5)} + K_{(2)}K_{(3)} + K_{(3)}K_{(2)}], \tag{B21d}$$

$$\lambda_{(6)} = -[(2/7)K_{(6)} + (10/7)K_{(2)}K_{(4)} + (17/7)K_{(3)}{}^2 + (10/7)K_{(4)}K_{(2)} + (32/21)K_{(2)}{}^3], \tag{B21e}$$

$$\begin{aligned} \lambda_{(7)} = {} & -[(1/4)K_{(7)} + (11/6)K_{(2)}K_{(5)} + (17/4)K_{(3)}K_{(4)} + (17/4)K_{(4)}K_{(3)} + (11/6)K_{(5)}K_{(2)} + (17/4)K_{(2)}{}^2K_{(3)} \\ & + (29/6)K_{(2)}K_{(3)}K_{(2)} + (17/4)K_{(3)}K_{(2)}{}^2], \end{aligned} \tag{B21f}$$

$$\begin{aligned} \lambda_{(8)} = {} & -[(2/9)K_{(8)} + (20/9)K_{(2)}K_{(6)} + (58/9)K_{(3)}K_{(5)} + (44/5)K_{(4)}{}^2 + (58/9)K_{(5)}K_{(3)} + (20/9)K_{(6)}K_{(2)} \\ & + (42/5)K_{(2)}{}^2K_{(4)} + (146/9)K_{(2)}K_{(3)}{}^2 + (92/9)K_{(2)}K_{(4)}K_{(2)} + (124/9)K_{(3)}K_{(2)}K_{(3)} + (146/9)K_{(3)}{}^2K_{(2)} \\ & + (42/5)K_{(4)}K_{(2)}{}^2 + (128/15)K_{(2)}{}^4], \end{aligned} \tag{B21g}$$

and

$$\begin{aligned} \lambda_{(9)} = {} & -[(1/5)K_{(9)} + (13/5)K_{(2)}K_{(7)} + 9K_{(3)}K_{(6)} + (77/5)K_{(4)}K_{(5)} + (77/5)K_{(5)}K_{(4)} + 9K_{(6)}K_{(3)} + (13/5)K_{(7)}K_{(2)} \\ & + (71/5)K_{(2)}{}^2K_{(5)} + (187/5)K_{(2)}K_{(3)}K_{(4)} + 40K_{(2)}K_{(4)}K_{(3)} + 18K_{(2)}K_{(5)}K_{(2)} + 31K_{(3)}K_{(2)}K_{(4)} + 62K_{(3)}{}^3 \\ & + 40K_{(3)}K_{(4)}K_{(2)} + 31K_{(4)}K_{(2)}K_{(3)} + (187/5)K_{(4)}K_{(3)}K_{(2)} + (71/5)K_{(5)}K_{(2)}{}^2 + 31K_{(2)}{}^3K_{(3)} \\ & + (181/5)K_{(2)}{}^2K_{(3)}K_{(2)} + (181/5)K_{(2)}K_{(3)}K_{(2)}{}^2 + 31K_{(3)}K_{(2)}{}^3]. \end{aligned} \tag{B21h}$$

By using (B1a), (B1b), and (B19) into (B21), we can also obtain the expressions of the components $\gamma^{\mu}{}_{\nu a_1 \cdots a_p}$ of the bitensors $\gamma_{(p)}$. The components of the lowest order bitensors $\lambda_{(p)}$ take the form

$$\lambda^{\mu}{}_{\nu a_1 a_2} = -(2/3)R^{\mu}{}_{(a_1|\nu|a_2)}, \tag{B22a}$$

$$\lambda^{\mu}{}_{\nu a_1 a_2 a_3} = -(1/2)R^{\mu}{}_{(a_1|\nu|a_2;a_3)}, \tag{B22b}$$

$$\lambda^{\mu}{}_{\nu a_1 a_2 a_3 a_4} = -[(2/5)R^{\mu}{}_{(a_1|\nu|a_2;a_3a_4)} + (8/15)R^{\mu}{}_{(a_1|\rho|a_2}R^{\rho}{}_{a_3|\nu|a_4)}], \tag{B22c}$$

$$\lambda^{\mu}{}_{\nu a_1 a_2 a_3 a_4 a_5} = -[(1/3)R^{\mu}{}_{(a_1|\nu|a_2;a_3a_4a_5)} + R^{\mu}{}_{(a_1|\rho|a_2}R^{\rho}{}_{a_3|\nu|a_4;a_5)} + R^{\mu}{}_{(a_1|\rho|a_2;a_3}R^{\rho}{}_{a_4|\nu|a_5)}], \tag{B22d}$$

$$\begin{aligned}\lambda^{\mu}{}_{\nu a_1 a_2 a_3 a_4 a_5 a_6} = &-[(2/7)R^{\mu}{}_{(a_1|\nu|a_2;a_3a_4a_5a_6)} + (10/7)R^{\mu}{}_{(a_1|\rho|a_2}R^{\rho}{}_{a_3|\nu|a_4;a_5a_6)} + (17/7)R^{\mu}{}_{(a_1|\rho|a_2;a_3}R^{\rho}{}_{a_4|\nu|a_5;a_6)} \\ &+ (10/7)R^{\mu}{}_{(a_1|\rho|a_2;a_3a_4}R^{\rho}{}_{a_5|\nu|a_6)} + (32/21)R^{\mu}{}_{(a_1|\rho|a_2}R^{\rho}{}_{a_3|\tau|a_4}R^{\tau}{}_{a_5|\nu|a_6)}].\end{aligned} \tag{B22e}$$

The relations (B18) and (B21) provide a compact form for the covariant Taylor series expansion up to order $\sigma^{9/2}$ of the bitensor Λ. Similarly, the relations (B9) and (B12) provide a compact form for the covariant Taylor series expansion up to order $\sigma^{9/2}$ of the bitensor H. It is also possible to provide the covariant Taylor series expansions of the bitensors Λ and H in a more explicit form, i.e. by working at the level of their components $\Lambda^{\mu}{}_{\nu} = \sigma^{;\mu}{}_{\nu}$ and $H^{\mu}{}_{\nu} = g_{\nu\nu'}\sigma^{;\mu\nu'}$. Of course, the corresponding results are much more heavy. From (B18), (B19), and (B22) we obtain

$$\begin{aligned}\sigma_{;\mu\nu} = &\, g_{\mu\nu} - \frac{1}{3}R_{\mu a_1 \nu a_2}\sigma^{;a_1}\sigma^{;a_2} + \frac{1}{12}R_{\mu a_1\nu a_2;a_3}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} - \left[\frac{1}{60}R_{\mu a_1\nu a_2;a_3a_4} + \frac{1}{45}R_{\mu a_1\rho a_2}R^{\rho}{}_{a_3\nu a_4}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} \\ &+ \left[\frac{1}{360}R_{\mu a_1\nu a_2;a_3a_4a_5} + \frac{1}{120}R_{\mu a_1\rho a_2}R^{\rho}{}_{a_3\nu a_4;a_5} + \frac{1}{120}R_{\mu a_1\rho a_2;a_3}R^{\rho}{}_{a_4\nu a_5}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5} \\ &- \left[\frac{1}{2520}R_{\mu a_1\nu a_2;a_3a_4a_5a_6} + \frac{1}{504}R_{\mu a_1\rho a_2}R^{\rho}{}_{a_3\nu a_4;a_5a_6} + \frac{17}{5040}R_{\mu a_1\rho a_2;a_3}R^{\rho}{}_{a_4\nu a_5;a_6} + \frac{1}{504}R_{\mu a_1\rho a_2;a_3a_4}R^{\rho}{}_{a_5\nu a_6}\right. \\ &\left.+ \frac{2}{945}R_{\mu a_1\rho a_2}R^{\rho}{}_{a_3\tau a_4}R^{\tau}{}_{a_5\nu a_6}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5}\sigma^{;a_6} + O(\sigma^{7/2})\end{aligned} \tag{B23}$$

[here we have not included the term of order $\sigma^{7/2}$ corresponding to (B21f), the term of order σ^4 corresponding to (B21g) and the term of order $\sigma^{9/2}$ corresponding to (B21h)] while from (B9), (B11), and (B13) we obtain

$$\begin{aligned}g_{\nu\nu'}\sigma_{;\mu}{}^{\nu'} = &-g_{\mu\nu} - \frac{1}{6}R_{\mu a_1\nu a_2}\sigma^{;a_1}\sigma^{;a_2} + \frac{1}{12}R_{\mu a_1\nu a_2;a_3}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} - \left[\frac{1}{40}R_{\mu a_1\nu a_2;a_3a_4} + \frac{7}{360}R_{\mu a_1\rho a_2}R^{\rho}{}_{a_3\nu a_4}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} \\ &+ \left[\frac{1}{180}R_{\mu a_1\nu a_2;a_3a_4a_5} + \frac{1}{90}R_{\mu a_1\rho a_2}R^{\rho}{}_{a_3\nu a_4;a_5} + \frac{1}{120}R_{\mu a_1\rho a_2;a_3}R^{\rho}{}_{a_4\nu a_5}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5} \\ &- \left[\frac{1}{1008}R_{\mu a_1\nu a_2;a_3a_4a_5a_6} + \frac{1}{280}R_{\mu a_1\rho a_2}R^{\rho}{}_{a_3\nu a_4;a_5a_6} + \frac{5}{1008}R_{\mu a_1\rho a_2;a_3}R^{\rho}{}_{a_4\nu a_5;a_6} + \frac{11}{5040}R_{\mu a_1\rho a_2;a_3a_4}R^{\rho}{}_{a_5\nu a_6}\right. \\ &\left.+ \frac{31}{15120}R_{\mu a_1\rho a_2}R^{\rho}{}_{a_3\tau a_4}R^{\tau}{}_{a_5\nu a_6}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5}\sigma^{;a_6} + O(\sigma^{7/2})\end{aligned} \tag{B24}$$

[here we have not included the term of order $\sigma^{7/2}$ corresponding to (B12f), the term of order σ^4 corresponding to (B12g) and the term of order $\sigma^{9/2}$ corresponding to (B12h)].

It should be noted that the previous expansions were obtained by DeWitt [1,2] up to order σ and by Christensen [20,21] up to order σ^2. They have been recently improved by Anderson, Flanagan and Ottewill [28] who have obtained the terms corresponding to the order $\sigma^{5/2}$. In Ref. [23], Phillips and Hu calculated the term of order σ^3 for the expansion of $\sigma_{;\mu\nu}$ but their result is incorrect: even if we simplify their equation (B25d), there remain three terms of which coefficients disagree with our own results in (B23).

APPENDIX C: COVARIANT TAYLOR SERIES EXPANSIONS OF THE BISCALARS $\Delta^{1/2}$ AND $\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$

In this Appendix, we shall construct the covariant Taylor series expansions of the biscalars $\Delta^{1/2}$ and $\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$ up to orders $\sigma^{11/2}$ and $\sigma^{9/2}$ respectively.

We first consider the biscalar Z which is defined as the logarithm of $\Delta^{1/2}$. The general form of its covariant Taylor series expansion has been obtained by Avramidi [9,40]. It is given by

$$Z(x, x') = \sum_{p=2}^{+\infty}\frac{(-1)^p}{p!}\zeta_{(p)}(x, x'), \tag{C1}$$

where the $\zeta_{(p)}(x, x')$ are biscalars of the form

$$\zeta_{(p)}(x, x') = \zeta_{a_1 \cdots a_p}(x)\sigma^{;a_1}(x, x') \cdots \sigma^{;a_p}(x, x') \tag{C2}$$

which can be constructed from the bitensors $\gamma_{(p)}(x, x')$ by using the relation

$$\zeta_{(p)} = \sum_{1\le k\le[p/2]} \frac{1}{2k} \sum_{\substack{p_1,\ldots,p_k\ge 2\\ p_1+\cdots+p_k=p}} \binom{p}{p_1,\ \ldots,\ p_k} \mathrm{tr}(\gamma_{(p_1)}\cdots\gamma_{(p_k)}). \tag{C3}$$

From this relation and from the relations (B5) which express the $\gamma_{(p)}$ in terms of the $K_{(p)}$, we can obtain the $\zeta_{(p)}$. After a long calculation, we have found that the terms corresponding to $p = 2, \ldots, 11$ are given by

$$\zeta_{(2)} = (1/6)\mathrm{tr}K_{(2)}, \tag{C4a}$$

$$\zeta_{(3)} = (1/4)\mathrm{tr}K_{(3)}, \tag{C4b}$$

$$\zeta_{(4)} = (3/10)\mathrm{tr}K_{(4)} + (1/15)\mathrm{tr}{K_{(2)}}^2, \tag{C4c}$$

$$\zeta_{(5)} = (1/3)\mathrm{tr}K_{(5)} + (1/3)\mathrm{tr}K_{(2)}K_{(3)}, \tag{C4d}$$

$$\zeta_{(6)} = (5/14)\mathrm{tr}K_{(6)} + (4/7)\mathrm{tr}K_{(2)}K_{(4)} + (15/28)\mathrm{tr}{K_{(3)}}^2 + (8/63)\mathrm{tr}{K_{(2)}}^3, \tag{C4e}$$

$$\zeta_{(7)} = (3/8)\mathrm{tr}K_{(7)} + (5/6)\mathrm{tr}K_{(2)}K_{(5)} + (9/4)\mathrm{tr}K_{(3)}K_{(4)} + (4/3)\mathrm{tr}{K_{(2)}}^2K_{(3)}, \tag{C4f}$$

$$\begin{aligned}\zeta_{(8)} = {}& (7/18)\mathrm{tr}K_{(8)} + (10/9)\mathrm{tr}K_{(2)}K_{(6)} + (35/9)\mathrm{tr}K_{(3)}K_{(5)} + (14/5)\mathrm{tr}{K_{(4)}}^2 + (136/45)\mathrm{tr}{K_{(2)}}^2K_{(4)} \\ & + (50/9)\mathrm{tr}K_{(2)}{K_{(3)}}^2 + (8/15)\mathrm{tr}{K_{(2)}}^4,\end{aligned} \tag{C4g}$$

$$\begin{aligned}\zeta_{(9)} = {}& (2/5)\mathrm{tr}K_{(9)} + (7/5)\mathrm{tr}K_{(2)}K_{(7)} + 6\,\mathrm{tr}K_{(3)}K_{(6)} + (56/5)\mathrm{tr}K_{(4)}K_{(5)} + (28/5)\mathrm{tr}{K_{(2)}}^2K_{(5)} + (73/5)\mathrm{tr}K_{(2)}K_{(3)}K_{(4)} \\ & + (73/5)\mathrm{tr}K_{(2)}K_{(4)}K_{(3)} + 9\,\mathrm{tr}{K_{(3)}}^3 + (48/5)\mathrm{tr}{K_{(2)}}^3K_{(3)},\end{aligned} \tag{C4h}$$

$$\begin{aligned}\zeta_{(10)} = {}& (9/22)\mathrm{tr}K_{(10)} + (56/33)\mathrm{tr}K_{(2)}K_{(8)} + (189/22)\mathrm{tr}K_{(3)}K_{(7)} + (216/11)\mathrm{tr}K_{(4)}K_{(6)} + (140/11)\mathrm{tr}{K_{(5)}}^2 \\ & + (304/33)\mathrm{tr}{K_{(2)}}^2K_{(6)} + (1015/33)\mathrm{tr}K_{(2)}K_{(3)}K_{(5)} + (480/11)\mathrm{tr}K_{(2)}{K_{(4)}}^2 + (1015/33)\mathrm{tr}K_{(2)}K_{(5)}K_{(3)} + 81\,\mathrm{tr}{K_{(3)}}^2K_{(4)} \\ & + (896/33)\mathrm{tr}{K_{(2)}}^3K_{(4)} + (149/3)\mathrm{tr}{K_{(2)}}^2{K_{(3)}}^2 + (805/33)\mathrm{tr}K_{(2)}K_{(3)}K_{(2)}K_{(3)} + (128/33)\mathrm{tr}{K_{(2)}}^5,\end{aligned} \tag{C4i}$$

and

$$\begin{aligned}\zeta_{(11)} = {}& (5/12)\mathrm{tr}K_{(11)} + 2\,\mathrm{tr}K_{(2)}K_{(9)} + (35/3)\mathrm{tr}K_{(3)}K_{(8)} + (63/2)\mathrm{tr}K_{(4)}K_{(7)} + 50\,\mathrm{tr}K_{(5)}K_{(6)} + 14\,\mathrm{tr}{K_{(2)}}^2K_{(7)} \\ & + (170/3)\mathrm{tr}K_{(2)}K_{(3)}K_{(6)} + 103\,\mathrm{tr}K_{(2)}K_{(4)}K_{(5)} + 103\,\mathrm{tr}K_{(2)}K_{(5)}K_{(4)} + (170/3)\mathrm{tr}K_{(2)}K_{(6)}K_{(3)} + (575/3)\mathrm{tr}{K_{(3)}}^2K_{(5)} \\ & + 273\,\mathrm{tr}K_{(3)}{K_{(4)}}^2 + (184/3)\mathrm{tr}{K_{(2)}}^3K_{(5)} + (317/2)\mathrm{tr}{K_{(2)}}^2K_{(3)}K_{(4)} + (317/2)\mathrm{tr}{K_{(2)}}^2K_{(4)}K_{(3)} \\ & + (461/3)\mathrm{tr}K_{(2)}K_{(3)}K_{(2)}K_{(4)} + (860/3)\mathrm{tr}K_{(2)}{K_{(3)}}^3 + (320/3)\mathrm{tr}{K_{(2)}}^4K_{(3)}.\end{aligned} \tag{C4j}$$

By using (B1a), (B1b), and (C2) into (C4), we can also obtain the expressions of the components $\zeta_{a_1\cdots a_p}$ of the biscalars $\zeta_{(p)}$. The components of the lowest order biscalars $\zeta_{(p)}$ take the form

$$\zeta_{a_1a_2} = (1/6)R_{a_1a_2}, \tag{C5a}$$

$$\zeta_{a_1a_2a_3} = (1/4)R_{(a_1a_2;a_3)}, \tag{C5b}$$

$$\zeta_{a_1a_2a_3a_4} = (3/10)R_{(a_1a_2;a_3a_4)} + (1/15)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4)}, \tag{C5c}$$

$$\zeta_{a_1a_2a_3a_4a_5} = (1/3)R_{(a_1a_2;a_3a_4a_5)} + (1/3)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5)}, \tag{C5d}$$

$$\begin{aligned}\zeta_{a_1a_2a_3a_4a_5a_6} = {}& (5/14)R_{(a_1a_2;a_3a_4a_5a_6)} + (4/7)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5a_6)} + (15/28)R^{\rho}{}_{(a_1|\tau|a_2;a_3}R^{\tau}{}_{a_4|\rho|a_5;a_6)} \\ & + (8/63)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4}R^{\sigma}{}_{a_5|\rho|a_6)},\end{aligned} \tag{C5e}$$

$$\begin{aligned}\zeta_{a_1a_2a_3a_4a_5a_6a_7} = {}& (3/8)R_{(a_1a_2;a_3a_4a_5a_6a_7)} + (5/6)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5a_6a_7)} + (9/4)R^{\rho}{}_{(a_1|\tau|a_2;a_3}R^{\tau}{}_{a_4|\rho|a_5;a_6a_7)} \\ & + (4/3)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4}R^{\sigma}{}_{a_5|\rho|a_6;a_7)},\end{aligned} \tag{C5f}$$

$$\begin{aligned}\zeta_{a_1a_2a_3a_4a_5a_6a_7a_8} = {}& (7/18)R_{(a_1a_2;a_3a_4a_5a_6a_7a_8)} + (10/9)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5a_6a_7a_8)} + (35/9)R^{\rho}{}_{(a_1|\tau|a_2;a_3}R^{\tau}{}_{a_4|\rho|a_5;a_6a_7a_8)} \\ & + (14/5)R^{\rho}{}_{(a_1|\tau|a_2;a_3a_4}R^{\tau}{}_{a_5|\rho|a_6;a_7a_8)} + (136/45)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4}R^{\sigma}{}_{a_5|\rho|a_6;a_7a_8)} \\ & + (50/9)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4;a_5}R^{\sigma}{}_{a_6|\rho|a_7;a_8)} + (8/15)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4}R^{\sigma}{}_{a_5|\kappa|a_6}R^{\kappa}{}_{a_7|\rho|a_8)}.\end{aligned} \tag{C5g}$$

The covariant Taylor series expansion of the biscalar $\Delta^{1/2}$ can now be constructed from (C1) and (C4). By noting that

$$\Delta^{1/2} = e^Z, \tag{C6}$$

we can write this expansion in the form

$$\Delta^{1/2}(x, x') = 1 + \sum_{p=2}^{+\infty} \frac{(-1)^p}{p!} \delta^{1/2}{}_{(p)}(x, x'), \tag{C7}$$

where the biscalars $\delta^{1/2}{}_{(p)}(x, x')$ are of the form

$$\delta^{1/2}{}_{(p)}(x, x') = \delta^{1/2}{}_{a_1 \cdots a_p}(x) \sigma^{;a_1}(x, x') \cdots \sigma^{;a_p}(x, x'). \tag{C8}$$

Here, it should be noted that $\delta^{1/2}{}_{(p)}$ is a notation and nothing else. The symbol $1/2$ simply recalls that $\delta^{1/2}{}_{(p)}$ is a "component" of the covariant Taylor series expansion of $\Delta^{1/2}$: we do not attribute any meaning to the symbol $\delta_{(p)}$ itself. Using (C1) and (C4) into (C6), we obtain after another long calculation the expressions of the $\delta^{1/2}{}_{(p)}$ for $p = 2, \ldots, 11$. We have

$$\delta^{1/2}{}_{(2)} = (1/6)\mathrm{tr}K_{(2)}, \tag{C9a}$$

$$\delta^{1/2}{}_{(3)} = (1/4)\mathrm{tr}K_{(3)}, \tag{C9b}$$

$$\delta^{1/2}{}_{(4)} = (3/10)\mathrm{tr}K_{(4)} + (1/15)\mathrm{tr}K_{(2)}{}^2 + (1/12)(\mathrm{tr}K_{(2)})^2, \tag{C9c}$$

$$\delta^{1/2}{}_{(5)} = (1/3)\mathrm{tr}K_{(5)} + (1/3)\mathrm{tr}K_{(2)}K_{(3)} + (5/12)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}, \tag{C9d}$$

$$\begin{aligned}\delta^{1/2}{}_{(6)} = {}& (5/14)\mathrm{tr}K_{(6)} + (4/7)\mathrm{tr}K_{(2)}K_{(4)} + (15/28)\mathrm{tr}K_{(3)}{}^2 + (3/4)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(4)} + (5/8)(\mathrm{tr}K_{(3)})^2 + (8/63)\mathrm{tr}K_{(2)}{}^3 \\ & + (1/6)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}{}^2 + (5/72)(\mathrm{tr}K_{(2)})^3,\end{aligned} \tag{C9e}$$

$$\begin{aligned}\delta^{1/2}{}_{(7)} = {}& (3/8)\mathrm{tr}K_{(7)} + (5/6)\mathrm{tr}K_{(2)}K_{(5)} + (9/4)\mathrm{tr}K_{(3)}K_{(4)} + (7/6)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(5)} + (21/8)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(4)} \\ & + (4/3)\mathrm{tr}K_{(2)}{}^2K_{(3)} + (7/6)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}K_{(3)} + (7/12)\mathrm{tr}K_{(3)}\mathrm{tr}K_{(2)}{}^2 + (35/48)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(3)},\end{aligned} \tag{C9f}$$

$$\begin{aligned}\delta^{1/2}{}_{(8)} = {}& (7/18)\mathrm{tr}K_{(8)} + (10/9)\mathrm{tr}K_{(2)}K_{(6)} + (35/9)\mathrm{tr}K_{(3)}K_{(5)} + (14/5)\mathrm{tr}K_{(4)}{}^2 + (5/3)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(6)} + (14/3)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(5)} \\ & + (63/20)(\mathrm{tr}K_{(4)})^2 + (136/45)\mathrm{tr}K_{(2)}{}^2K_{(4)} + (50/9)\mathrm{tr}K_{(2)}K_{(3)}{}^2 + (8/3)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}K_{(4)} + (5/2)\mathrm{tr}K_{(2)}\mathrm{tr}K_{(3)}{}^2 \\ & + (14/3)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}K_{(3)} + (7/5)\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(2)}{}^2 + (7/4)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(4)} + (35/12)\mathrm{tr}K_{(2)}(\mathrm{tr}K_{(3)})^2 + (8/15)\mathrm{tr}K_{(2)}{}^4 \\ & + (16/27)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}{}^3 + (7/45)(\mathrm{tr}K_{(2)}{}^2)^2 + (7/18)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(2)}{}^2 + (35/432)(\mathrm{tr}K_{(2)})^4,\end{aligned} \tag{C9g}$$

$$\begin{aligned}\delta^{1/2}{}_{(9)} = {}& (2/5)\mathrm{tr}K_{(9)} + (7/5)\mathrm{tr}K_{(2)}K_{(7)} + 6\,\mathrm{tr}K_{(3)}K_{(6)} + (56/5)\mathrm{tr}K_{(4)}K_{(5)} + (9/4)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(7)} + (15/2)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(6)} \\ & + (63/5)\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(5)} + (28/5)\mathrm{tr}K_{(2)}{}^2K_{(5)} + (73/5)\mathrm{tr}K_{(2)}K_{(3)}K_{(4)} + (73/5)\mathrm{tr}K_{(2)}K_{(4)}K_{(3)} + 9\,\mathrm{tr}K_{(3)}{}^3 \\ & + 5\,\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}K_{(5)} + (27/2)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}K_{(4)} + 12\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}K_{(4)} + (45/4)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(3)}{}^2 + (63/5)\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(2)}K_{(3)} \\ & + (14/5)\mathrm{tr}K_{(5)}\,\mathrm{tr}K_{(2)}{}^2 + (7/2)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(5)} + (63/4)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(4)} + (35/8)(\mathrm{tr}K_{(3)})^3 + (48/5)\mathrm{tr}K_{(2)}{}^3K_{(3)} \\ & + 8\,\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}{}^2K_{(3)} + (8/3)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}{}^3 + (14/5)\mathrm{tr}K_{(2)}{}^2\,\mathrm{tr}K_{(2)}K_{(3)} + (7/2)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(2)}K_{(3)} \\ & + (7/2)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}\mathrm{tr}K_{(2)}{}^2 + (35/24)(\mathrm{tr}K_{(2)})^3\mathrm{tr}K_{(3)},\end{aligned} \tag{C9h}$$

and

$$\delta^{1/2}{}_{(10)} = (9/22)\mathrm{tr}K_{(10)} + (56/33)\mathrm{tr}K_{(2)}K_{(8)} + (189/22)\mathrm{tr}K_{(3)}K_{(7)} + (216/11)\mathrm{tr}K_{(4)}K_{(6)} + (140/11)\mathrm{tr}K_{(5)}{}^2$$
$$+ (35/12)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(8)} + (45/4)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(7)} + (45/2)\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(6)} + 14(\mathrm{tr}K_{(5)})^2 + (304/33)\mathrm{tr}K_{(2)}{}^2K_{(6)}$$
$$+ (1015/33)\mathrm{tr}K_{(2)}K_{(3)}K_{(5)} + (480/11)\mathrm{tr}K_{(2)}K_{(4)}{}^2 + (1015/33)\mathrm{tr}K_{(2)}K_{(5)}K_{(3)} + 81\,\mathrm{tr}K_{(3)}{}^2K_{(4)}$$
$$+ (25/3)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}K_{(6)} + (175/6)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}K_{(5)} + 21\,\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(4)}{}^2 + 25\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}K_{(5)}$$
$$+ (135/2)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(3)}K_{(4)} + 36\,\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(2)}K_{(4)} + (135/4)\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(3)}{}^2 + 28\,\mathrm{tr}K_{(5)}\,\mathrm{tr}K_{(2)}K_{(3)} + 5\,\mathrm{tr}K_{(6)}\,\mathrm{tr}K_{(2)}{}^2$$
$$+ (25/4)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(6)} + 35\,\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(5)} + (189/8)\mathrm{tr}K_{(2)}(\mathrm{tr}K_{(4)})^2 + (315/8)(\mathrm{tr}K_{(3)})^2\mathrm{tr}K_{(4)}$$
$$+ (896/33)\mathrm{tr}K_{(2)}{}^3K_{(4)} + (149/3)\mathrm{tr}K_{(2)}{}^2K_{(3)}{}^2 + (805/33)\mathrm{tr}K_{(2)}K_{(3)}K_{(2)}K_{(3)} + (68/3)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}{}^2K_{(4)}$$
$$+ (125/3)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}K_{(3)}{}^2 + 40\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}{}^2K_{(3)} + 8\,\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(2)}{}^3 + 8\,\mathrm{tr}K_{(2)}{}^2\,\mathrm{tr}K_{(2)}K_{(4)} + (15/2)\mathrm{tr}K_{(2)}{}^2\,\mathrm{tr}K_{(3)}{}^2$$
$$+ 14(\mathrm{tr}K_{(2)}K_{(3)})^2 + 10(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(2)}K_{(4)} + (75/8)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(3)}{}^2 + 35\,\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}K_{(3)}$$
$$+ (21/2)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(2)}{}^2 + (35/4)(\mathrm{tr}K_{(3)})^2\mathrm{tr}K_{(2)}{}^2 + (35/8)(\mathrm{tr}K_{(2)})^3\mathrm{tr}K_{(4)} + (175/16)(\mathrm{tr}K_{(2)})^2(\mathrm{tr}K_{(3)})^2$$
$$+ (128/33)\mathrm{tr}K_{(2)}{}^5 + 4\,\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}{}^4 + (16/9)\mathrm{tr}K_{(2)}{}^2\,\mathrm{tr}K_{(2)}{}^3 + (7/6)\mathrm{tr}K_{(2)}(\mathrm{tr}K_{(2)}{}^2)^2 + (20/9)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(2)}{}^3$$
$$+ (35/36)(\mathrm{tr}K_{(2)})^3\mathrm{tr}K_{(2)}{}^2 + (35/288)(\mathrm{tr}K_{(2)})^5, \quad \text{(C9i)}$$

and

$$\delta^{1/2}{}_{(11)} = (5/12)\mathrm{tr}K_{(11)} + 2\,\mathrm{tr}K_{(2)}K_{(9)} + (35/3)\mathrm{tr}K_{(3)}K_{(8)} + (63/2)\mathrm{tr}K_{(4)}K_{(7)} + 50\,\mathrm{tr}K_{(5)}K_{(6)} + (11/3)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(9)}$$
$$+ (385/24)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(8)} + (297/8)\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(7)} + 55\,\mathrm{tr}K_{(5)}\,\mathrm{tr}K_{(6)} + 14\,\mathrm{tr}K_{(2)}{}^2K_{(7)} + (170/3)\mathrm{tr}K_{(2)}K_{(3)}K_{(6)}$$
$$+ 103\,\mathrm{tr}K_{(2)}K_{(4)}K_{(5)} + 103\,\mathrm{tr}K_{(2)}K_{(5)}K_{(4)} + (170/3)\mathrm{tr}K_{(2)}K_{(6)}K_{(3)} + (575/3)\mathrm{tr}K_{(3)}{}^2K_{(5)} + 273\,\mathrm{tr}K_{(3)}K_{(4)}{}^2$$
$$+ (77/6)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}K_{(7)} + 55\,\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}K_{(6)} + (308/3)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(4)}K_{(5)} + (275/6)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}K_{(6)}$$
$$+ (1925/12)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(3)}K_{(5)} + (231/2)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(4)}{}^2 + (165/2)\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(2)}K_{(5)} + (891/4)\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(3)}K_{(4)}$$
$$+ 88\,\mathrm{tr}K_{(5)}\,\mathrm{tr}K_{(2)}K_{(4)} + (165/2)\mathrm{tr}K_{(5)}\,\mathrm{tr}K_{(3)}{}^2 + 55\,\mathrm{tr}K_{(6)}\,\mathrm{tr}K_{(2)}K_{(3)} + (33/4)\mathrm{tr}K_{(7)}\,\mathrm{tr}K_{(2)}{}^2 + (165/16)$$
$$\times(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(7)} + (275/4)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(6)} + (231/2)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(5)} + (385/4)(\mathrm{tr}K_{(3)})^2\mathrm{tr}K_{(5)}$$
$$+ (2079/16)\mathrm{tr}K_{(3)}(\mathrm{tr}K_{(4)})^2 + (184/3)\mathrm{tr}K_{(2)}{}^3K_{(5)} + (317/2)\mathrm{tr}K_{(2)}{}^2K_{(3)}K_{(4)} + (317/2)\mathrm{tr}K_{(2)}{}^2K_{(4)}K_{(3)}$$
$$+ (461/3)\mathrm{tr}K_{(2)}K_{(3)}K_{(2)}K_{(4)} + (860/3)\mathrm{tr}K_{(2)}K_{(3)}{}^3 + (154/3)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}{}^2K_{(5)} + (803/6)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}K_{(3)}K_{(4)}$$
$$+ (803/6)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}K_{(4)}K_{(3)} + (165/2)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}{}^3 + (374/3)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}{}^2K_{(4)} + (1375/6)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}K_{(3)}{}^2$$
$$+ 132\,\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(2)}{}^2K_{(3)} + (176/9)\mathrm{tr}K_{(5)}\,\mathrm{tr}K_{(2)}{}^3 + (55/3)\mathrm{tr}K_{(2)}{}^2\,\mathrm{tr}K_{(2)}K_{(5)} + (99/2)\mathrm{tr}K_{(2)}{}^2\,\mathrm{tr}K_{(3)}K_{(4)}$$
$$+ 88\,\mathrm{tr}K_{(2)}K_{(3)}\,\mathrm{tr}K_{(2)}K_{(4)} + (165/2)\mathrm{tr}K_{(2)}K_{(3)}\,\mathrm{tr}K_{(3)}{}^2 + (275/12)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(2)}K_{(5)} + (495/8)$$
$$\times(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(3)}K_{(4)} + 110\,\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}K_{(4)} + (825/8)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(3)}{}^2 + (231/2)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(2)}K_{(3)}$$
$$+ (77/3)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(5)}\,\mathrm{tr}K_{(2)}{}^2 + (385/4)(\mathrm{tr}K_{(3)})^2\mathrm{tr}K_{(2)}K_{(3)} + (231/4)\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(4)}\,\mathrm{tr}K_{(2)}{}^2 + (385/36)$$
$$\times(\mathrm{tr}K_{(2)})^3\mathrm{tr}K_{(5)} + (1155/16)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(4)} + (1925/48)\mathrm{tr}K_{(2)}(\mathrm{tr}K_{(3)})^3 + (320/3)\mathrm{tr}K_{(2)}{}^4K_{(3)}$$
$$+ 88\,\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}{}^3K_{(3)} + 22\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}{}^4 + (88/3)\mathrm{tr}K_{(2)}{}^2\,\mathrm{tr}K_{(2)}{}^2K_{(3)} + (176/9)\mathrm{tr}K_{(2)}K_{(3)}\,\mathrm{tr}K_{(2)}{}^3 + (110/3)$$
$$\times(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(2)}{}^2K_{(3)} + (220/9)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}{}^3 + (77/3)\mathrm{tr}K_{(2)}\,\mathrm{tr}K_{(2)}{}^2\,\mathrm{tr}K_{(2)}K_{(3)} + (77/12)\mathrm{tr}K_{(3)}(\mathrm{tr}K_{(2)}{}^2)^2$$
$$+ (385/36)(\mathrm{tr}K_{(2)})^3\mathrm{tr}K_{(2)}K_{(3)} + (385/24)(\mathrm{tr}K_{(2)})^2\mathrm{tr}K_{(3)}\,\mathrm{tr}K_{(2)}{}^2 + (1925/576)(\mathrm{tr}K_{(2)})^4\mathrm{tr}K_{(3)}. \quad \text{(C9j)}$$

By using (B1a), (B1b), and (C8) into (C9), we can also obtain the expressions of the components $\delta^{1/2}{}_{a_1\cdots a_p}$ of the biscalars $\delta^{1/2}{}_{(p)}$. The components of the lowest order biscalars $\delta^{1/2}{}_{(p)}$ take the form

$$\delta^{1/2}{}_{a_1a_2} = (1/6)R_{a_1a_2}, \tag{C10a}$$

$$\delta^{1/2}{}_{a_1a_2a_3} = (1/4)R_{(a_1a_2;a_3)}, \tag{C10b}$$

$$\delta^{1/2}{}_{a_1a_2a_3a_4} = (3/10)R_{(a_1a_2;a_3a_4)} + (1/15)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4)} + (1/12)R_{(a_1a_2}R_{a_3a_4)}, \tag{C10c}$$

$$\delta^{1/2}{}_{a_1a_2a_3a_4a_5} = (1/3)R_{(a_1a_2;a_3a_4a_5)} + (1/3)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5)} + (5/12)R_{(a_1a_2}R_{a_3a_4;a_5)}, \tag{C10d}$$

$$\begin{aligned}\delta^{1/2}{}_{a_1a_2a_3a_4a_5a_6} &= (5/14)R_{(a_1a_2;a_3a_4a_5a_6)} + (4/7)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5a_6)} + (15/28)R^{\rho}{}_{(a_1|\tau|a_2;a_3}R^{\tau}{}_{a_4|\rho|a_5;a_6)} \\ &\quad + (3/4)R_{(a_1a_2}R_{a_3a_4;a_5a_6)} + (5/8)R_{(a_1a_2;a_3}R_{a_4a_5;a_6)} + (8/63)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4}R^{\sigma}{}_{a_5|\rho|a_6)} \\ &\quad + (1/6)R_{(a_1a_2}R^{\rho}{}_{a_3|\tau|a_4}R^{\tau}{}_{a_5|\rho|a_6)} + (5/72)R_{(a_1a_2}R_{a_3a_4}R_{a_5a_6)},\end{aligned} \tag{C10e}$$

$$\begin{aligned}\delta^{1/2}{}_{a_1a_2a_3a_4a_5a_6a_7} &= (3/8)R_{(a_1a_2;a_3a_4a_5a_6a_7)} + (5/6)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5a_6a_7)} + (9/4)R^{\rho}{}_{(a_1|\tau|a_2;a_3}R^{\tau}{}_{a_4|\rho|a_5;a_6a_7)} \\ &\quad + (7/6)R_{(a_1a_2}R_{a_3a_4;a_5a_6a_7)} + (21/8)R_{(a_1a_2;a_3}R_{a_4a_5;a_6a_7)} + (4/3)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4}R^{\sigma}{}_{a_5|\rho|a_6;a_7)} \\ &\quad + (7/6)R_{(a_1a_2}R^{\rho}{}_{a_3|\tau|a_4}R^{\tau}{}_{a_5|\rho|a_6;a_7)} + (7/12)R_{(a_1a_2;a_3}R^{\rho}{}_{a_4|\tau|a_5}R^{\tau}{}_{a_6|\rho|a_7)} + (35/48)R_{(a_1a_2}R_{a_3a_4}R_{a_5a_6;a_7)},\end{aligned} \tag{C10f}$$

$$\begin{aligned}\delta^{1/2}{}_{a_1a_2a_3a_4a_5a_6a_7a_8} &= (7/18)R_{(a_1a_2;a_3a_4a_5a_6a_7a_8)} + (10/9)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5a_6a_7a_8)} + (35/9)R^{\rho}{}_{(a_1|\tau|a_2;a_3}R^{\tau}{}_{a_4|\rho|a_5;a_6a_7a_8)} \\ &\quad + (14/5)R^{\rho}{}_{(a_1|\tau|a_2;a_3a_4}R^{\tau}{}_{a_5|\rho|a_6;a_7a_8)} + (5/3)R_{(a_1a_2}R_{a_3a_4;a_5a_6a_7a_8)} + (14/3)R_{(a_1a_2;a_3}R_{a_4a_5;a_6a_7a_8)} \\ &\quad + (63/20)R_{(a_1a_2;a_3a_4}R_{a_5a_6;a_7a_8)} + (136/45)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4}R^{\sigma}{}_{a_5|\rho|a_6;a_7a_8)} \\ &\quad + (50/9)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4;a_5}R^{\sigma}{}_{a_6|\rho|a_7;a_8)} + (8/3)R_{(a_1a_2}R^{\rho}{}_{a_3|\tau|a_4}R^{\tau}{}_{a_5|\rho|a_6;a_7a_8)} \\ &\quad + (5/2)R_{(a_1a_2}R^{\rho}{}_{a_3|\tau|a_4;a_5}R^{\tau}{}_{a_6|\rho|a_7;a_8)} + (14/3)R_{(a_1a_2;a_3}R^{\rho}{}_{a_4|\tau|a_5}R^{\tau}{}_{a_6|\rho|a_7;a_8)} \\ &\quad + (7/5)R_{(a_1a_2;a_3a_4}R^{\rho}{}_{a_5|\tau|a_6}R^{\tau}{}_{a_7|\rho|a_8)} + (7/4)R_{(a_1a_2}R_{a_3a_4}R_{a_5a_6;a_7a_8)} + (35/12)R_{(a_1a_2}R_{a_3a_4;a_5}R_{a_6a_7;a_8)} \\ &\quad + (8/15)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4}R^{\sigma}{}_{a_5|\kappa|a_6}R^{\kappa}{}_{a_7|\rho|a_8)} + (16/27)R_{(a_1a_2}R^{\rho}{}_{a_3|\tau|a_4}R^{\tau}{}_{a_5|\sigma|a_6}R^{\sigma}{}_{a_7|\rho|a_8)} \\ &\quad + (7/18)R_{(a_1a_2}R_{a_3a_4}R^{\rho}{}_{a_5|\tau|a_6}R^{\tau}{}_{a_7|\rho|a_8)} + (7/45)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4}R^{\kappa}{}_{a_5|\lambda|a_6}R^{\lambda}{}_{a_7|\kappa|a_8)} \\ &\quad + (35/432)R_{(a_1a_2}R_{a_3a_4}R_{a_5a_6}R_{a_7a_8)}.\end{aligned} \tag{C10g}$$

The relations (C7) and (C9) provide a compact form for the covariant Taylor series expansion of the biscalar $\Delta^{1/2}$. It is also possible to give this expansion in a more explicit form by using (C7), (C8), and (C10). Of course, the corresponding result is very heavy. For this explicit expansion up to order σ^4, we have

$$\begin{aligned}\Delta^{1/2} = 1 &+ \frac{1}{12}R_{a_1a_2}\sigma^{;a_1}\sigma^{;a_2} - \frac{1}{24}R_{a_1a_2;a_3}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} + \left[\frac{1}{80}R_{a_1a_2;a_3a_4} + \frac{1}{360}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\rho a_4}\right. \\ &\left. + \frac{1}{288}R_{a_1a_2}R_{a_3a_4}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} - \left[\frac{1}{360}R_{a_1a_2;a_3a_4a_5} + \frac{1}{360}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\rho a_4;a_5}\right. \\ &\left. + \frac{1}{288}R_{a_1a_2}R_{a_3a_4;a_5}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5} + \left[\frac{1}{2016}R_{a_1a_2;a_3a_4a_5a_6} + \frac{1}{1260}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\rho a_4;a_5a_6}\right. \\ &+ \frac{1}{1344}R^{\rho}{}_{a_1\tau a_2;a_3}R^{\tau}{}_{a_4\rho a_5;a_6} + \frac{1}{960}R_{a_1a_2}R_{a_3a_4;a_5a_6} + \frac{1}{1152}R_{a_1a_2;a_3}R_{a_4a_5;a_6} + \frac{1}{5670}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\sigma a_4}R^{\sigma}{}_{a_5\rho a_6} \\ &\left. + \frac{1}{4320}R_{a_1a_2}R^{\rho}{}_{a_3\tau a_4}R^{\tau}{}_{a_5\rho a_6} + \frac{1}{10\,368}R_{a_1a_2}R_{a_3a_4}R_{a_5a_6}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5}\sigma^{;a_6} - \left[\frac{1}{13\,440}R_{a_1a_2;a_3a_4a_5a_6a_7}\right. \\ &+ \frac{1}{6048}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\rho a_4;a_5a_6a_7} + \frac{1}{2240}R^{\rho}{}_{a_1\tau a_2;a_3}R^{\tau}{}_{a_4\rho a_5;a_6a_7} + \frac{1}{4320}R_{a_1a_2}R_{a_3a_4;a_5a_6a_7} + \frac{1}{1920}R_{a_1a_2;a_3}R_{a_4a_5;a_6a_7}\end{aligned}$$

$$+\frac{1}{3780}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\sigma a_4}R^{\sigma}{}_{a_5\rho a_6;a_7}+\frac{1}{4320}R_{a_1a_2}R^{\rho}{}_{a_3\tau a_4}R^{\tau}{}_{a_5\rho a_6;a_7}+\frac{1}{8640}R_{a_1a_2;a_3}R^{\rho}{}_{a_4\tau a_5}R^{\tau}{}_{a_6\rho a_7}$$
$$+\frac{1}{6912}R_{a_1a_2}R_{a_3a_4}R_{a_5a_6;a_7}\Big]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5}\sigma^{;a_6}\sigma^{;a_7}+\Big[\frac{1}{103680}R_{a_1a_2;a_3a_4a_5a_6a_7a_8}$$
$$+\frac{1}{36288}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\rho a_4;a_5a_6a_7a_8}+\frac{1}{10368}R^{\rho}{}_{a_1\tau a_2;a_3}R^{\tau}{}_{a_4\rho a_5;a_6a_7a_8}+\frac{1}{14400}R^{\rho}{}_{a_1\tau a_2;a_3a_4}R^{\tau}{}_{a_5\rho a_6;a_7a_8}$$
$$+\frac{1}{24192}R_{a_1a_2}R_{a_3a_4;a_5a_6a_7a_8}+\frac{1}{8640}R_{a_1a_2;a_3}R_{a_4a_5;a_6a_7a_8}+\frac{1}{12800}R_{a_1a_2;a_3a_4}R_{a_5a_6;a_7a_8}$$
$$+\frac{17}{226800}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\sigma a_4}R^{\sigma}{}_{a_5\rho a_6;a_7a_8}+\frac{5}{36288}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\sigma a_4;a_5}R^{\sigma}{}_{a_6\rho a_7;a_8}+\frac{1}{15120}R_{a_1a_2}R^{\rho}{}_{a_3\tau a_4}R^{\tau}{}_{a_5\rho a_6;a_7a_8}$$
$$+\frac{1}{16128}R_{a_1a_2}R^{\rho}{}_{a_3\tau a_4;a_5}R^{\tau}{}_{a_6\rho a_7;a_8}+\frac{1}{8640}R_{a_1a_2;a_3}R^{\rho}{}_{a_4\tau a_5}R^{\tau}{}_{a_6\rho a_7;a_8}+\frac{1}{28800}R_{a_1a_2;a_3a_4}R^{\rho}{}_{a_5\tau a_6}R^{\tau}{}_{a_7\rho a_8}$$
$$+\frac{1}{23040}R_{a_1a_2}R_{a_3a_4}R_{a_5a_6;a_7a_8}+\frac{1}{13824}R_{a_1a_2}R_{a_3a_4;a_5}R_{a_6a_7;a_8}+\frac{1}{75600}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\sigma a_4}R^{\sigma}{}_{a_5\kappa a_6}R^{\kappa}{}_{a_7\rho a_8}$$
$$+\frac{1}{68040}R_{a_1a_2}R^{\rho}{}_{a_3\tau a_4}R^{\tau}{}_{a_5\sigma a_6}R^{\sigma}{}_{a_7\rho a_8}+\frac{1}{103680}R_{a_1a_2}R_{a_3a_4}R^{\rho}{}_{a_5\tau a_6}R^{\tau}{}_{a_7\rho a_8}+\frac{1}{259200}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\rho a_4}R^{\kappa}{}_{a_5\lambda a_6}R^{\lambda}{}_{a_7\kappa a_8}$$
$$+\frac{1}{497664}R_{a_1a_2}R_{a_3a_4}R_{a_5a_6}R_{a_7a_8}\Big]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5}\sigma^{;a_6}\sigma^{;a_7}\sigma^{;a_8}+O(\sigma^{9/2}) \quad \text{(C11)}$$

(here we have not included the enormous terms given in (C9h)–(C9j) which correspond, respectively, to the orders $\sigma^{9/2}$, σ^5 and $\sigma^{11/2}$ of the expansion).

It should be noted that this expansion was obtained up to order σ by DeWitt [1,2], up to order σ^2 by Christensen [20,21] and up to order $\sigma^{5/2}$ by Brown and Ottewill [35]. The term of order σ^3 disagrees with the recent calculation of Phillips and Hu in Ref. [23]: the signs of the last three terms in their equation (C12c) are incorrect.

Let us now consider the covariant Taylor series expansion of the biscalar $T := \Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$. By noting that we have

$$T = Z_{;\mu}\sigma^{;\mu} = DZ \quad \text{(C12)}$$

[this is an immediate consequence of (C6)], it is obvious that this expansion is necessarily given by

$$T(x,x') = \sum_{p=2}^{+\infty}\frac{(-1)^p}{p!}\tau_{(p)}(x,x'), \quad \text{(C13)}$$

where the $\tau_{(p)}(x,x')$ are biscalars of the form

$$\tau_{(p)}(x,x') = \tau_{a_1\cdots a_p}(x)\sigma^{;a_1}(x,x')\cdots\sigma^{;a_p}(x,x'). \quad \text{(C14)}$$

By noting the identities

$$D[\mathrm{tr}K_{(p)}] = p\,\mathrm{tr}K_{(p)} + \mathrm{tr}K_{(p+1)}, \quad \text{(C15a)}$$
$$D[\mathrm{tr}K_{(p)}K_{(q)}] = (p+q)\mathrm{tr}K_{(p)}K_{(q)} + \mathrm{tr}K_{(p+1)}K_{(q)} + \mathrm{tr}K_{(p)}K_{(q+1)}, \quad \text{(C15b)}$$
$$D[\mathrm{tr}K_{(p)}K_{(q)}K_{(r)}] = (p+q+r)\mathrm{tr}K_{(p)}K_{(q)}K_{(r)} + \mathrm{tr}K_{(p+1)}K_{(q)}K_{(r)} + \mathrm{tr}K_{(p)}K_{(q+1)}K_{(r)} + \mathrm{tr}K_{(p)}K_{(q)}K_{(r+1)}, \quad \text{(C15c)}$$
$$D[\mathrm{tr}K_{(p)}K_{(q)}K_{(r)}K_{(s)}] = (p+q+r+s)\mathrm{tr}K_{(p)}K_{(q)}K_{(r)}K_{(s)} + \mathrm{tr}K_{(p+1)}K_{(q)}K_{(r)}K_{(s)} + \mathrm{tr}K_{(p)}K_{(q+1)}K_{(r)}K_{(s)}$$
$$+ \mathrm{tr}K_{(p)}K_{(q)}K_{(r+1)}K_{(s)} + \mathrm{tr}K_{(p)}K_{(q)}K_{(r)}K_{(s+1)}, \quad \text{(C15d)}$$

which follow from (8), (B1a), and (B1b), another tedious calculation using (C12) and (C13) as well as (C1) and (C4) permits us to obtain the $\tau_{(p)}$ for $p = 2, \ldots, 9$. We have

$$\tau_{(2)} = (1/3)\mathrm{tr}K_{(2)}, \quad \text{(C16a)}$$
$$\tau_{(3)} = (1/4)\mathrm{tr}K_{(3)}, \quad \text{(C16b)}$$
$$\tau_{(4)} = (1/5)\mathrm{tr}K_{(4)} + (4/15)\mathrm{tr}K_{(2)}{}^2, \quad \text{(C16c)}$$
$$\tau_{(5)} = (1/6)\mathrm{tr}K_{(5)} + \mathrm{tr}K_{(2)}K_{(3)}, \quad \text{(C16d)}$$
$$\tau_{(6)} = (1/7)\mathrm{tr}K_{(6)} + (10/7)\mathrm{tr}K_{(2)}K_{(4)} + (17/14)\mathrm{tr}K_{(3)}{}^2 + (16/21)\mathrm{tr}K_{(2)}{}^3, \quad \text{(C16e)}$$
$$\tau_{(7)} = (1/8)\mathrm{tr}K_{(7)} + (11/6)\mathrm{tr}K_{(2)}K_{(5)} + (17/4)\mathrm{tr}K_{(3)}K_{(4)} + (20/3)\mathrm{tr}K_{(2)}{}^2K_{(3)}, \quad \text{(C16f)}$$
$$\tau_{(8)} = (1/9)\mathrm{tr}K_{(8)} + (20/9)\mathrm{tr}K_{(2)}K_{(6)} + (58/9)\mathrm{tr}K_{(3)}K_{(5)} + (22/5)\mathrm{tr}K_{(4)}{}^2 + (608/45)\mathrm{tr}K_{(2)}{}^2K_{(4)}$$
$$+ (208/9)\mathrm{tr}K_{(2)}K_{(3)}{}^2 + (64/15)\mathrm{tr}K_{(2)}{}^4, \quad \text{(C16g)}$$

and

$$\tau_{(9)} = (1/10)\mathrm{tr}K_{(9)} + (13/5)\mathrm{tr}K_{(2)}K_{(7)} + 9\,\mathrm{tr}K_{(3)}K_{(6)} + (77/5)\mathrm{tr}K_{(4)}K_{(5)} + (116/5)\mathrm{tr}{K_{(2)}}^2K_{(5)} + (271/5)\mathrm{tr}K_{(2)}K_{(3)}K_{(4)} + (271/5)\mathrm{tr}K_{(2)}K_{(4)}K_{(3)} + 31\,\mathrm{tr}{K_{(3)}}^3 + (336/5)\mathrm{tr}{K_{(2)}}^3K_{(3)}. \tag{C16h}$$

By using (B1a), (B1b), and (C14) into (C16), we can also obtain the expressions of the components $\tau_{a_1\cdots a_p}$ of the biscalars $\tau_{(p)}$. The components of the lowest order biscalars $\tau_{(p)}$ take the form

$$\tau_{a_1a_2} = (1/3)R_{a_1a_2}, \tag{C17a}$$

$$\tau_{a_1a_2a_3} = (1/4)R_{(a_1a_2;a_3)}, \tag{C17b}$$

$$\tau_{a_1a_2a_3a_4} = (1/5)R_{(a_1a_2;a_3a_4)} + (4/15)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4)}, \tag{C17c}$$

$$\tau_{a_1a_2a_3a_4a_5} = (1/6)R_{(a_1a_2;a_3a_4a_5)} + R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5)}, \tag{C17d}$$

$$\tau_{a_1a_2a_3a_4a_5a_6} = (1/7)R_{(a_1a_2;a_3a_4a_5a_6)} + (10/7)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5a_6)} + (17/14)R^{\rho}{}_{(a_1|\tau|a_2;a_3}R^{\tau}{}_{a_4|\rho|a_5;a_6)} + (16/21)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\sigma|a_4}R^{\sigma}{}_{a_5|\rho|a_6)}. \tag{C17e}$$

The relations (C13) and (C16) provide a compact form for the covariant Taylor series expansion of the biscalar $T = \Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$. Of course, it is also possible to give that covariant Taylor series expansion in a more explicit form by replacing into (C13) and (C16) the $K_{(p)}$ by their expressions (B1a) and (B1b). We then have

$$\begin{aligned} T = \Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} &= \frac{1}{6}R_{a_1a_2}\sigma^{;a_1}\sigma^{;a_2} - \frac{1}{24}R_{a_1a_2;a_3}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} + \left[\frac{1}{120}R_{a_1a_2;a_3a_4} + \frac{1}{90}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\rho a_4}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} \\ &\quad - \left[\frac{1}{720}R_{a_1a_2;a_3a_4a_5} + \frac{1}{120}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\rho a_4;a_5}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5} + \left[\frac{1}{5040}R_{a_1a_2;a_3a_4a_5a_6}\right. \\ &\quad + \frac{1}{504}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\rho a_4;a_5a_6} + \frac{17}{10\,080}R^{\rho}{}_{a_1\tau a_2;a_3}R^{\tau}{}_{a_4\rho a_5;a_6} \\ &\quad \left. + \frac{1}{945}R^{\rho}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\sigma a_4}R^{\sigma}{}_{a_5\rho a_6}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5}\sigma^{;a_6} + O(\sigma^{7/2}) \end{aligned} \tag{C18}$$

(here we have not included the terms given in (C16f) and (C16g) and (C16h) which correspond, respectively, to the orders $\sigma^{7/2}$, σ^4 and $\sigma^{9/2}$ of the expansion).

APPENDIX D: COVARIANT TAYLOR SERIES EXPANSIONS OF THE BITENSORS F, $F_{;\mu}$, $F_{;\mu\nu}$ AND $\Box F$ WHEN F IS A SYMMETRIC BISCALAR

In this Appendix, we shall first construct the covariant Taylor series expansions of the covariant derivative, the second covariant derivative and the d'Alembertian of an arbitrary biscalar $F(x, x')$ from that of this biscalar. We shall then express the constraints induced on all these expansions by the symmetry of the biscalar $F(x, x')$ in the exchange of x and x'.

Let us first consider an arbitrary biscalar $F(x, x')$. Its covariant Taylor series expansion is given by

$$F(x, x') = f(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!} f_{(p)}(x, x'), \tag{D1}$$

where the $f_{(p)}(x, x')$ are biscalars in x and x' which are of the form

$$f_{(p)}(x, x') = f_{a_1\cdots a_p}(x)\sigma^{;a_1}(x, x')\cdots\sigma^{;a_p}(x, x'). \tag{D2}$$

Its covariant derivative $(\nabla F)(x, x')$, its second covariant derivative $(\nabla\nabla F)(x, x')$ and its d'Alembertian $(\Box F)(x, x')$ possess covariant Taylor series expansions given by

$$(\nabla F)(x, x') = \overline{f}(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!} \overline{f}_{(p)}(x, x'), \tag{D3}$$

$$(\nabla\nabla F)(x, x') = \overline{\overline{f}}(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!} \overline{\overline{f}}_{(p)}(x, x'), \tag{D4}$$

$$(\Box F)(x, x') = f''(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!} f''_{(p)}(x, x'). \tag{D5}$$

In Eq. (D3), $\overline{f}(x)$ is a tensor of type (0,1) in x of which components are of the form $\overline{f}_{\mu}(x)$ while the $\overline{f}_{(p)}(x, x')$ with $p = 1, 2, \ldots$ are tensors of type (0,1) in x and scalars in x' of which components are of the form

$$\bar{f}_{\mu a_1\cdots a_p}(x)\sigma^{;a_1}(x,x')\cdots\sigma^{;a_p}(x,x'). \tag{D6}$$

In Eq. (D4), $\bar{\bar{f}}(x)$ is a tensor of type (0,2) in x of which components are of the form $\bar{\bar{f}}_{\mu\nu}(x)$ while the $\bar{\bar{f}}_{(p)}(x,x')$ with $p=1,2,\ldots$ are tensors of type (0,2) in x and scalars in x' of which components are of the form

$$\bar{\bar{f}}_{\mu\nu a_1\cdots a_p}(x)\sigma^{;a_1}(x,x')\cdots\sigma^{;a_p}(x,x'). \tag{D7}$$

In Eq. (D5), $f''(x)$ is a scalar in x while the $f''_{(p)}(x,x')$ with $p=1,2,\ldots$ are biscalars of the form

$$f''_{(p)}(x,x') = f''_{a_1\cdots a_p}(x)\sigma^{;a_1}(x,x')\cdots\sigma^{;a_p}(x,x'). \tag{D8}$$

Of course, we can establish relationships between the components of the covariant Taylor series expansions of $F(x,x')$, $(\nabla F)(x,x')$, $(\nabla\nabla F)(x,x')$ and $(\Box F)(x,x')$. By taking the covariant derivative of (D1) and by using (D2) as well as the covariant Taylor series expansions of $\sigma^{;\mu}{}_{\nu}$ given in Eqs. (B18) and (B19), we can link the components of the covariant Taylor series expansions of $(\nabla F)(x,x')$ and $F(x,x')$. We have

$$\bar{f}_{\mu} = f_{;\mu} - f_{\mu}, \tag{D9a}$$
$$\bar{f}_{\mu a_1} = f_{a_1;\mu} - f_{\mu a_1}, \tag{D9b}$$
$$\bar{f}_{\mu a_1 a_2} = f_{a_1 a_2;\mu} - f_{\mu a_1 a_2} - f_{\rho}\lambda^{\rho}{}_{\mu a_1 a_2}, \tag{D9c}$$
$$\bar{f}_{\mu a_1\cdots a_p} = f_{a_1\cdots a_p;\mu} - f_{\mu a_1\cdots a_p} - f_{\rho}\lambda^{\rho}{}_{\mu a_1\cdots a_p} - \sum_{\substack{r+s=p\\ r\geq 1, s\geq 2}}\binom{p}{rs} f_{\rho(a_1\cdots a_r}\lambda^{\rho}{}_{|\mu|a_{r+1}\cdots a_p)} \quad \text{for } p\geq 3. \tag{D9d}$$

Similarly, by taking the covariant derivative of (D3) and by using (D6) as well as the covariant Taylor series expansions of $\sigma^{;\mu}{}_{\nu}$ given in Eqs. (B18) and (B19), we can link the components of the covariant Taylor series expansions of $(\nabla\nabla F)(x,x')$ and $(\nabla F)(x,x')$. We have

$$\bar{\bar{f}}_{\mu\nu} = \bar{f}_{\mu;\nu} - \bar{f}_{\mu\nu}, \tag{D10a}$$
$$\bar{\bar{f}}_{\mu\nu a_1} = \bar{f}_{\mu a_1;\nu} - \bar{f}_{\mu\nu a_1}, \tag{D10b}$$
$$\bar{\bar{f}}_{\mu\nu a_1 a_2} = \bar{f}_{\mu a_1 a_2;\nu} - \bar{f}_{\mu\nu a_1 a_2} - \bar{f}_{\mu\rho}\lambda^{\rho}{}_{\nu a_1 a_2}, \tag{D10c}$$
$$\bar{\bar{f}}_{\mu\nu a_1\cdots a_p} = \bar{f}_{\mu a_1\cdots a_p;\nu} - \bar{f}_{\mu\nu a_1\cdots a_p} - \bar{f}_{\mu\rho}\lambda^{\rho}{}_{\nu a_1\cdots a_p} - \sum_{\substack{r+s=p\\ r\geq 1, s\geq 2}}\binom{p}{rs}\bar{f}_{\mu\rho(a_1\cdots a_r}\lambda^{\rho}{}_{|\nu|a_{r+1}\cdots a_p)} \quad \text{for } p\geq 3. \tag{D10d}$$

It is also possible to link the components of the covariant Taylor series expansions of $(\nabla\nabla F)(x,x')$ and $F(x,x')$. This can be done by using (D9) into (D10). The resulting relations are rather complicated and we do not display them here. However, it should be noted that they permit us to show the symmetry of $\bar{\bar{f}}_{\mu\nu}$ and of the $\bar{\bar{f}}_{\mu\nu a_1\cdots a_n}$ in the exchange of the indices μ and ν, a result which does not explicitly appear in Eq. (D10). It is also possible to link the components of the covariant Taylor series expansion of $(\Box F)(x,x')$ and $F(x,x')$. This can be done from the previous results by noting that

$$f'' = g^{\mu\nu}\bar{\bar{f}}_{\mu\nu} \quad\text{and}\quad f''_{a_1\cdots a_p} = g^{\mu\nu}\bar{\bar{f}}_{\mu\nu a_1\cdots a_p} \quad \text{for } p\geq 1. \tag{D11}$$

We now assume that the biscalar $F(x,x')$ is symmetric in the exchange of x and x'. It is well known that this property induces constraints on the coefficients $f_{a_1\cdots a_p}(x)$ of the covariant Taylor series expansion of $F(x,x')$ (see, for example, Ref. [33]). These constraints are of the form

$$f_{a_1\cdots a_p} = (-1)^p f_{a_1\cdots a_p} + \sum_{k=0}^{p-1}(-1)^k\binom{p}{k} f_{(a_1\cdots a_k;a_{k+1}\cdots a_p)} \quad \text{for } p\geq 1. \tag{D12}$$

They permit us to determine the odd coefficients of the covariant Taylor series expansion of $F(x,x')$ in terms of the even ones. We have for the odd coefficients of lowest orders

$$f_{a_1} = (1/2)f_{;a_1}, \tag{D13a}$$
$$f_{a_1a_2a_3} = (3/2)f_{(a_1a_2;a_3)} - (1/4)f_{;(a_1a_2a_3)}, \tag{D13b}$$
$$f_{a_1a_2a_3a_4a_5} = (5/2)f_{(a_1a_2a_3a_4;a_5)} - (5/2)f_{(a_1a_2;a_3a_4a_5)} + (1/2)f_{;(a_1a_2a_3a_4a_5)}, \tag{D13c}$$
$$f_{a_1a_2a_3a_4a_5a_6a_7} = (7/2)f_{(a_1a_2a_3a_4a_5a_6;a_7)} - (35/4)f_{(a_1a_2a_3a_4;a_5a_6a_7)} + (21/2)f_{(a_1a_2;a_3a_4a_5a_6a_7)} - (17/8)f_{;(a_1a_2a_3a_4a_5a_6a_7)}, \tag{D13d}$$
$$f_{a_1a_2a_3a_4a_5a_6a_7a_8a_9} = (9/2)f_{(a_1a_2a_3a_4a_5a_6a_7a_8;a_9)} - 21f_{(a_1a_2a_3a_4a_5a_6;a_7a_8a_9)} + 63f_{(a_1a_2a_3a_4;a_5a_6a_7a_8a_9)} - (153/2)f_{(a_1a_2;a_3a_4a_5a_6a_7a_8a_9)} + (31/2)f_{;(a_1a_2a_3a_4a_5a_6a_7a_8a_9)}. \tag{D13e}$$

Of course, the constraints (D13) permit us to "simplify" the covariant Taylor series expansions of $(\nabla F)(x,x')$, $(\nabla\nabla F)(x,x')$ and $(\Box F)(x,x')$. As far as the latter is concerned, it is given by (D5) and by using (D9)–(D11) as well as (D13a) and (D13b) we obtain for its coefficients of lowest orders:

$$f'' = f^\rho{}_\rho, \tag{D14a}$$

$$f''_{a_1} = [\tfrac{1}{4}(\Box f)_{;a_1} + \tfrac{1}{2}f^\rho{}_{\rho;a_1} - f^\rho{}_{a_1;\rho} + \tfrac{1}{2}R^\rho{}_{a_1}f_{;\rho}], \tag{D14b}$$

$$f''_{a_1a_2} = \bigg[f^\rho{}_{\rho a_1 a_2} + \frac{1}{2}(\Box f)_{;a_1a_2} - 2f^\rho{}_{a_1;\rho a_2} + \frac{1}{2}R_{\rho a_1;a_2}f^{;\rho} + \frac{1}{12}R_{a_1a_2;\rho}f^{;\rho} - \frac{4}{3}R^\rho{}_{a_1}f_{\rho a_2} + R^\rho{}_{a_1}f_{;\rho a_2} + \frac{2}{3}R_{\rho a_1 \sigma a_2}f^{\rho\sigma}\bigg]. \tag{D14c}$$

Finally, it is interesting to construct the covariant Taylor series expansion of the biscalar $F_{;\mu}\sigma^{;\mu}$. It can be obtained from (D3), (D6), (D9), and (D13) by noting that $\lambda^\rho{}_{\mu a_1\cdots a_p}\sigma^{;\mu}\sigma^{;a_1}\cdots\sigma^{;a_p} = 0$ (this relation being a direct consequence of the symmetries of the Riemann tensor). We have

$$F_{;\mu}\sigma^{;\mu} = (1/2)f_{;a_1}\sigma^{;a_1} + [f_{a_1a_2} - (1/2)f_{;a_1a_2}]\sigma^{;a_1}\sigma^{;a_2} - [(1/4)f_{a_1a_2;a_3} - (1/8)f_{;a_1a_2a_3}]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} + [(1/6)f_{a_1a_2a_3a_4} - (1/4)f_{a_1a_2;a_3a_4} + (1/24)f_{;a_1a_2a_3a_4}]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} + O(\sigma^{5/2}). \tag{D15}$$

APPENDIX E: COVARIANT TAYLOR SERIES EXPANSIONS OF THE BISCALAR $\Box\Delta^{1/2}$

In this Appendix, we shall obtain the covariant Taylor series expansion of the biscalar $\Box\Delta^{1/2}$ up to order σ^2. We shall derive these results from three intermediate long calculations concerning the covariant Taylor series expansions of $Z_{;\mu}$, $Z_{;\mu\nu}$ and $\Box Z$ up to orders $\sigma^{5/2}$, σ^2 and σ^2 respectively. In those calculations, we have extensively used the commutation of covariant derivatives in the form (1) as well as the Bianchi identities

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{E1a}$$

$$R_{abcd;e} + R_{abec;d} + R_{abde;c} = 0, \tag{E1b}$$

and their consequences

$$R_{abcd}{}^{;d} = R_{ac;b} - R_{bc;a}, \tag{E2a}$$

$$R_{ab}{}^{;b} = (1/2)R_{;a}, \tag{E2b}$$

and

$$R^{\rho\sigma\tau}{}_a R_{\tau\sigma\rho b} = (1/2)R^{\rho\sigma\tau}{}_a R_{\rho\sigma\tau b}, \tag{E3a}$$

$$R^{\rho\sigma\tau}{}_a R_{\tau\sigma\rho b;c} = (1/2)R^{\rho\sigma\tau}{}_a R_{\rho\sigma\tau b;c}, \tag{E3b}$$

$$R^{\rho\sigma\tau}{}_a R_{\tau\sigma\rho b;cd} = (1/2)R^{\rho\sigma\tau}{}_a R_{\rho\sigma\tau b;cd}, \tag{E3c}$$

$$R^{\rho\sigma\tau}{}_{a;b} R_{\tau\sigma\rho c;d} = (1/2)R^{\rho\sigma\tau}{}_{a;b} R_{\rho\sigma\tau c;d}, \tag{E3d}$$

and

$$R^{\rho\sigma\tau}{}_a R_{\tau b\sigma c;\rho} = -(1/2)R^{\rho\sigma\tau}{}_a R_{\rho\sigma\tau b;c}, \tag{E4a}$$

$$R^{\rho\sigma\tau}{}_a R_{\tau b\sigma c;\rho d} = -(1/2)R^{\rho\sigma\tau}{}_a R_{\rho\sigma\tau b;cd}, \tag{E4b}$$

$$R^{\rho\sigma\tau}{}_{a;b} R_{\tau c\sigma d;\rho} = -(1/2)R^{\rho\sigma\tau}{}_{a;b} R_{\rho\sigma\tau c;d}. \tag{E4c}$$

Let us first consider the covariant Taylor series expansion of $Z_{;\mu}$. Up to order $\sigma^{5/2}$, it is of the form

$$Z_{;\mu} = -\bar{\zeta}_{\mu a_1}\sigma^{;a_1} + \frac{1}{2!}\bar{\zeta}_{\mu a_1a_2}\sigma^{;a_1}\sigma^{;a_2} - \frac{1}{3!}\bar{\zeta}_{\mu a_1a_2a_3}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} + \frac{1}{4!}\bar{\zeta}_{\mu a_1a_2a_3a_4}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} - \frac{1}{5!}\bar{\zeta}_{\mu a_1a_2a_3a_4a_5}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5} + O(\sigma^3), \tag{E5}$$

where the coefficients $\bar{\zeta}_{\mu a_1\cdots a_p}$ with $p = 1, \ldots, 5$ are obtained from (C5a)–(C5e) by using (D9) and (B22a)–(B22c) and are given by

$$\bar{\zeta}_{\mu a_1} = -(1/6)R_{\mu a_1}, \tag{E6a}$$

$$\bar{\zeta}_{\mu a_1a_2} = (1/12)R_{a_1a_2;\mu} - (1/6)R_{\mu(a_1;a_2)}, \tag{E6b}$$

$$\bar{\zeta}_{\mu a_1a_2a_3} = (1/10)R_{(a_1a_2;|\mu|a_3)} - (3/20)R_{\mu(a_1;a_2a_3)} - (1/15)R^\rho{}_{\mu\tau(a_1}R^\tau{}_{a_2|\rho|a_3)} - (1/60)R_{\rho(a_1}R^\rho{}_{a_2|\mu|a_3)}, \tag{E6c}$$

$$\bar{\zeta}_{\mu a_1a_2a_3a_4} = (1/10)R_{(a_1a_2;|\mu|a_3a_4)} - (2/15)R_{\mu(a_1;a_2a_3a_4)} + (1/15)R^\rho{}_{(a_1|\tau|a_2}R^\tau{}_{a_3|\rho|a_4);\mu} - (2/15)R^\rho{}_{\mu\tau(a_1}R^\tau{}_{a_2|\rho|a_3;a_4)} - (2/15)R^\rho{}_{(a_1|\tau|a_2}R^\tau{}_{|\mu\rho|a_3;a_4)} - (2/15)R_{\rho(a_1;a_2}R^\rho{}_{a_3|\mu|a_4)} + (1/10)R_{(a_1a_2;|\rho|}R^\rho{}_{a_3|\mu|a_4)}, \tag{E6d}$$

and

$$\begin{aligned}\bar{\zeta}_{\mu a_1a_2a_3a_4a_5} = {} & (2/21)R_{(a_1a_2;|\mu|a_3a_4a_5)} - (5/42)R_{\mu(a_1;a_2a_3a_4a_5)} + (1/7)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4;|\mu|a_5)} - (4/21)R^{\rho}{}_{\mu\tau(a_1}R^{\tau}{}_{a_2|\rho|a_3;a_4a_5)} \\ & - (4/21)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{|\mu\rho|a_3;a_4a_5)} + (13/84)R^{\rho}{}_{(a_1|\tau|a_2|;\mu|}R^{\tau}{}_{a_3|\rho|a_4;a_5)} - (5/14)R^{\rho}{}_{\mu\tau(a_1;a_2}R^{\tau}{}_{a_3|\rho|a_4;a_5)} \\ & + (1/42)R_{\rho(a_1}R^{\rho}{}_{a_2|\mu|a_3;a_4a_5)} - (3/14)R_{\rho(a_1;a_2}R^{\rho}{}_{a_3|\mu|a_4;a_5)} + (17/84)R_{(a_1a_2;|\rho|}R^{\rho}{}_{a_3|\mu|a_4;a_5)} \\ & - (2/7)R_{\rho(a_1;a_2a_3}R^{\rho}{}_{a_4|\mu|a_5)} + (5/21)R_{(a_1a_2;|\rho|a_3}R^{\rho}{}_{a_4|\mu|a_5)} - (1/126)R_{\rho(a_1}R^{\rho}{}_{a_2|\tau|a_3}R^{\tau}{}_{a_4|\mu|a_5)} \\ & - (8/63)R^{\rho}{}_{\mu\tau(a_1}R^{\tau}{}_{a_2|\sigma|a_3}R^{\sigma}{}_{a_4|\rho|a_5)} - (2/63)R^{\sigma}{}_{\rho\tau(a_1}R^{\tau}{}_{a_2|\sigma|a_3}R^{\rho}{}_{a_4|\mu|a_5)}. \end{aligned} \tag{E6e}$$

Let us now consider the covariant Taylor series expansion of $Z_{;\mu\nu}$. Up to order σ^2, it is of the form

$$Z_{;\mu\nu} = \bar{\bar{\zeta}}_{\mu\nu} - \bar{\bar{\zeta}}_{\mu\nu a_1}\sigma^{;a_1} + \frac{1}{2!}\bar{\bar{\zeta}}_{\mu\nu a_1a_2}\sigma^{;a_1}\sigma^{;a_2} - \frac{1}{3!}\bar{\bar{\zeta}}_{\mu\nu a_1a_2a_3}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} + \frac{1}{4!}\bar{\bar{\zeta}}_{\mu\nu a_1a_2a_3a_4}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} + O(\sigma^{5/2}), \tag{E7}$$

where the coefficients $\bar{\bar{\zeta}}_{\mu\nu}$ and $\bar{\bar{\zeta}}_{\mu\nu a_1\cdots a_p}$ with $p = 1, \ldots, 4$ can be obtained from (E6) by using (D10) and (B22a)–(B22c) and are given by

$$\bar{\bar{\zeta}}^{\mu\nu} = (1/6)R^{\mu\nu}, \tag{E8a}$$

$$\bar{\bar{\zeta}}^{\mu\nu}{}_{a_1} = (1/12)R^{\mu\nu}{}_{;a_1} - (1/6)R^{(\mu}{}_{a_1}{}^{;\nu)}, \tag{E8b}$$

$$\begin{aligned}\bar{\bar{\zeta}}^{\mu\nu}{}_{a_1a_2} = {} & (1/20)R_{a_1a_2}{}^{;(\mu\nu)} + (1/20)R^{\mu\nu}{}_{;(a_1a_2)} - (2/15)R^{(\mu}{}_{(a_1}{}^{;\nu)}{}_{a_2)} + (1/90)R^{\rho(\mu}R_{\rho(a_1}{}^{\nu)}{}_{a_2)} + (11/90)R_{\rho(a_1}R^{\rho(\mu\nu)}{}_{a_2)} \\ & + (1/45)R^{\mu}{}_{\rho}{}^{\nu}{}_{\tau}R^{\rho}{}_{(a_1}{}^{\tau}{}_{a_2)} + (1/45)R^{\mu}{}_{\rho\tau(a_1}R^{\nu\rho\tau}{}_{a_2)} + (1/45)R^{\mu}{}_{\rho\tau(a_1}R^{\nu\tau\rho}{}_{a_2)}, \end{aligned} \tag{E8c}$$

$$\begin{aligned}\bar{\bar{\zeta}}^{\mu\nu}{}_{a_1a_2a_3} = {} & (1/30)R^{\mu\nu}{}_{;(a_1a_2a_3)} + (1/20)R_{(a_1a_2}{}^{;(\mu\nu)}{}_{a_3)} - (1/10)R_{(a_1}{}^{(\mu;\nu)}{}_{a_2a_3)} + (1/12)R_{\rho(a_1}R^{\rho(\mu\nu)}{}_{a_2;a_3)} \\ & + (1/60)R_{\rho(a_1}R^{\rho}{}_{a_2a_3)}{}^{(\mu;\nu)} + (1/30)R_{\rho(a_1}{}^{;(\mu}R^{\nu)}{}_{a_2}{}^{\rho}{}_{a_3)} + (1/15)R^{(\mu}{}_{\rho;(a_1}R^{\nu)}{}_{a_2}{}^{\rho}{}_{a_3)} - (1/10)R^{(\mu}{}_{(a_1;|\rho|}R^{\nu)}{}_{a_2}{}^{\rho}{}_{a_3)} \\ & + (7/30)R_{\rho(a_1;a_2}R^{\rho(\mu\nu)}{}_{a_3)} - (1/10)R_{(a_1a_2;|\rho|}R^{\rho(\mu\nu)}{}_{a_3)} - (1/15)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3)\rho}{}^{(\mu;\nu)} - (1/15)R^{\rho}{}_{(a_1|\tau|a_2}{}^{;(\mu}R^{|\tau|}{}_{a_3)\rho}{}^{\nu)} \\ & + (1/15)R^{(\mu}{}_{\rho\tau(a_1}R^{\nu)\tau\rho}{}_{a_2;a_3)} + (1/15)R^{(\mu}{}_{\rho\tau(a_1}R^{\nu)\rho\tau}{}_{a_2;a_3)} + (1/30)R^{\mu}{}_{\rho}{}^{\nu}{}_{\tau;(a_1}R^{\rho}{}_{a_2}{}^{\tau}{}_{a_3)} + (1/30)R^{\mu}{}_{\rho}{}^{\nu}{}_{\tau}R^{\rho}{}_{(a_1}{}^{\tau}{}_{a_2;a_3)}, \end{aligned} \tag{E8d}$$

and

$$\begin{aligned}\bar{\bar{\zeta}}^{\mu\nu}{}_{a_1a_2a_3a_4} = {} & (3/70)R_{(a_1a_2}{}^{;(\mu\nu)}{}_{a_3a_4)} + (1/42)R^{\mu\nu}{}_{;(a_1a_2a_3a_4)} - (8/105)R^{(\mu}{}_{(a_1}{}^{;\nu)}{}_{a_2a_3a_4)} - (1/105)R^{\rho(\mu}R^{\nu)}{}_{(a_1|\rho|a_2;a_3a_4)} \\ & + (3/35)R^{\rho}{}_{(a_1}{}^{;(\mu}R^{\nu)}{}_{a_2|\rho|a_3;a_4)} + (3/35)R^{\rho(\mu}{}_{;(a_1}R^{\nu)}{}_{a_2|\rho|a_3;a_4)} - (17/105)R^{(\mu}{}_{(a_1}{}^{;|\rho|}R^{\nu)}{}_{a_2|\rho|a_3;a_4)} \\ & - (4/105)R^{\rho}{}_{(a_1}{}^{;(\mu}{}_{a_2}R^{\nu)}{}_{a_3|\rho|a_4)} + (11/105)R_{(a_1a_2}{}^{;(\mu|\rho|}R^{\nu)}{}_{a_3|\rho|a_4)} + (4/35)R^{\rho(\mu}{}_{;(a_1a_2}R^{\nu)}{}_{a_3|\rho|a_4)} \\ & - (4/21)R^{(\mu}{}_{(a_1}{}^{;|\rho|}{}_{a_2}R^{\nu)}{}_{a_3|\rho|a_4)} + (2/35)R^{\rho}{}_{(a_1}R_{|\rho|}{}^{(\mu\nu)}{}_{a_2;a_3a_4)} + (1/105)R^{\rho}{}_{(a_1}R_{|\rho|a_2a_3}{}^{(\mu;\nu)}{}_{a_4)} \\ & + (53/210)R^{\rho}{}_{(a_1;a_2}R_{|\rho|}{}^{(\mu\nu)}{}_{a_3;a_4)} + (19/210)R^{\rho}{}_{(a_1;a_2}R_{|\rho|a_3a_4)}{}^{(\mu;\nu)} + (11/35)R^{\rho}{}_{(a_1;a_2a_3}R_{|\rho|}{}^{(\mu\nu)}{}_{a_4)} \\ & - (43/420)R_{(a_1a_2}{}^{;\rho}R_{|\rho|}{}^{(\mu\nu)}{}_{a_3;a_4)} - (5/84)R_{(a_1a_2}{}^{;\rho}R_{|\rho|a_3a_4)}{}^{(\mu;\nu)} - (4/21)R_{(a_1a_2}{}^{;\rho}{}_{a_3}R_{|\rho|}{}^{(\mu\nu)}{}_{a_4)} \\ & + (4/105)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4)}{}^{;(\mu\nu)} + (1/28)R^{\rho}{}_{(a_1|\tau|a_2}{}^{;\mu}R^{\tau}{}_{a_3|\rho|a_4)}{}^{;\nu} + (4/105)R^{\rho\mu}{}_{\tau}{}^{\nu}R^{\tau}{}_{(a_1|\rho|a_2;a_3a_4)} \\ & + (4/105)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau\mu}{}_{\rho}{}^{\nu}{}_{;a_3a_4)} + (1/14)R^{\rho\mu}{}_{\tau}{}^{\nu}{}_{;(a_1}R^{\tau}{}_{a_2|\rho|a_3;a_4)} - (4/35)R^{\rho}{}_{(a_1|\tau|a_2}{}^{;(\mu}{}_{a_3}R^{|\tau|\nu)}{}_{|\rho|a_4)} \\ & - (4/35)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau(\mu}{}_{|\rho|a_3}{}^{;\nu)}{}_{a_4)} - (13/105)R^{\rho(\mu}{}_{\tau(a_1}{}^{;\nu)}R^{\tau}{}_{a_2|\rho|a_3;a_4)} - (13/105)R^{\rho}{}_{(a_1|\tau|a_2}{}^{;(\mu}R^{|\tau|\nu)}{}_{|\rho|a_3;a_4)} \end{aligned}$$

$$+ (8/105)R^{\rho(\mu}{}_{\tau(a_1}R^{|\tau|\nu)}{}_{|\rho|a_2;a_3a_4)} + (8/105)R^{(\mu}{}_{\rho\tau(a_1}R^{\nu)\rho\tau}{}_{a_2;a_3a_4)} + (1/14)R^{\mu}{}_{\rho\tau(a_1;a_2}R^{\nu\tau\rho}{}_{a_3;a_4)}$$
$$+ (1/14)R^{\mu}{}_{\rho\tau(a_1;a_2}R^{\nu\rho\tau}{}_{a_3;a_4)} + (17/315)R^{\rho\tau}R^{\mu}{}_{(a_1|\rho|a_2}R^{\nu}{}_{a_3|\tau|a_4)} + (1/315)R^{(\mu}{}_{\rho}R^{|\rho|}{}_{(a_1|\tau|a_2}R^{|\tau|}{}_{a_3}{}^{\nu)}{}_{a_4)}$$
$$- (68/315)R_{\rho(a_1}R^{\rho}{}_{a_2}{}^{(\mu}{}_{|\tau|}R^{|\tau|}{}_{a_3}{}^{\nu)}{}_{a_4)} + (46/315)R_{\rho(a_1}R^{\rho}{}_{|\tau|}{}^{(\mu}{}_{a_2}R^{|\tau|}{}_{a_3}{}^{\nu)}{}_{a_4)} + (2/315)R_{\rho(a_1}R^{\rho}{}_{a_2|\tau|a_3}R^{\tau(\mu\nu)}{}_{a_4)}$$
$$+ (8/315)R^{\rho(\mu}{}_{\tau}{}^{\nu)}R^{\tau}{}_{(a_1|\sigma|a_2}R^{\sigma}{}_{a_3|\rho|a_4)} + (32/315)R^{\sigma}{}_{\rho\tau(a_1}R^{\tau}{}_{a_2|\sigma|a_3}R^{\rho(\mu\nu)}{}_{a_4)} + (4/315)R^{\rho}{}_{(a_1|\tau|a_2}R^{\tau(\mu}{}_{|\rho\sigma|}R^{|\sigma|}{}_{a_3}{}^{\nu)}{}_{a_4)}$$
$$+ (4/315)R^{\rho(\mu}{}_{\tau(a_1}R^{|\sigma|}{}_{a_2}{}^{\nu)}{}_{a_3}R^{\tau}{}_{|\sigma\rho|a_4)} + (4/315)R^{\rho(\mu}{}_{\tau(a_1}R^{|\sigma|}{}_{a_2}{}^{\nu)}{}_{a_3}R^{\tau}{}_{a_4)\rho\sigma} + (16/315)R^{\rho(\mu}{}_{\tau(a_1}R^{|\tau|\nu)}{}_{|\sigma|a_2}R^{\sigma}{}_{a_3|\rho|a_4)}$$
$$+ (8/315)R^{\mu}{}_{\rho\tau(a_1}R^{\nu}{}_{|\sigma|}{}^{\tau}{}_{a_2}R^{\rho}{}_{a_3}{}^{\sigma}{}_{a_4)} + (8/315)R^{\mu}{}_{\rho\tau(a_1}R^{\nu\rho}{}_{|\sigma|a_2}R^{\sigma}{}_{a_3}{}^{\tau}{}_{a_4)}. \tag{E8e}$$

Here, the symmetry of all these coefficients in the exchange of the indices μ and ν should be noted.

The covariant Taylor series expansion of $\Box Z$ up to order σ^2 can be now obtained. It is of the form

$$\Box Z = \zeta'' - \zeta''_{a_1}\sigma^{;a_1} + \frac{1}{2!}\zeta''_{a_1a_2}\sigma^{;a_1}\sigma^{;a_2} - \frac{1}{3!}\zeta''_{a_1a_2a_3}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} + \frac{1}{4!}\zeta''_{a_1a_2a_3a_4}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} + O(\sigma^{5/2}), \tag{E9}$$

where the coefficients ζ'' and $\zeta''_{a_1\cdots a_p}$ with $p = 1, \ldots, 4$ can be obtained from (E8) by using (D11) and are given by

$$\zeta'' = (1/6)R, \tag{E10a}$$
$$\zeta''_{a_1} = 0, \tag{E10b}$$
$$\zeta''_{a_1a_2} = (1/20)\Box R_{a_1a_2} - (1/60)R_{;a_1a_2} - (11/90)R^{\rho}{}_{a_1}R_{\rho a_2} + (1/30)R_{\rho\tau}R^{\rho}{}_{a_1}{}^{\tau}{}_{a_2} + (1/30)R^{\rho\sigma\tau}{}_{a_1}R_{\rho\sigma\tau a_2}, \tag{E10c}$$
$$\zeta''_{a_1a_2a_3} = -(1/60)R_{;(a_1a_2a_3)} + (1/20)(\Box R_{(a_1a_2})_{;a_3)} - (3/10)R^{\rho}{}_{(a_1}R_{|\rho|a_2;a_3)} + (1/12)R^{\rho}{}_{(a_1}R_{a_2a_3);\rho} + (1/30)R^{\rho}{}_{\sigma;(a_1}R^{\sigma}{}_{a_2|\rho|a_3)}$$
$$+ (1/30)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a_1|\rho|a_2;a_3)} + (1/15)R^{\rho\sigma\tau}{}_{(a_1}R_{|\rho\sigma\tau|a_2;a_3)}, \tag{E10d}$$
$$\zeta''_{a_1a_2a_3a_4} = (3/70)(\Box R_{(a_1a_2})_{;a_3a_4)} - (1/70)R_{;(a_1a_2a_3a_4)} - (38/105)R^{\rho}{}_{(a_1}R_{|\rho|a_2;a_3a_4)} + (19/105)R^{\rho}{}_{(a_1}R_{a_2a_3;|\rho|a_4)}$$
$$- (17/105)R^{\rho}{}_{(a_1;a_2}R_{|\rho|a_3;a_4)} - (1/21)R^{\rho}{}_{(a_1;a_2}R_{a_3a_4);\rho} + (5/84)R_{(a_1a_2}{}^{;\rho}R_{a_3a_4);\rho} + (1/35)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a_1|\rho|a_2;a_3a_4)}$$
$$+ (1/21)R^{\rho}{}_{(a_1;|\sigma|}R^{\sigma}{}_{a_2|\rho|a_3;a_4)} + (1/30)R^{\rho}{}_{\sigma;(a_1}R^{\sigma}{}_{a_2|\rho|a_3;a_4)} - (4/35)R^{\rho}{}_{(a_1;|\sigma|a_2}R^{\sigma}{}_{a_3|\rho|a_4)}$$
$$+ (11/105)R_{(a_1a_2}{}^{;\rho}{}_{\sigma}R^{\sigma}{}_{a_3|\rho|a_4)} + (4/105)R^{\rho}{}_{\sigma;(a_1a_2}R^{\sigma}{}_{a_3|\rho|a_4)} + (4/105)R^{\rho}{}_{(a_1|\sigma|a_2}\Box R^{\sigma}{}_{a_3|\rho|a_4)}$$
$$+ (2/35)R^{\rho\sigma\tau}{}_{(a_1}R_{|\rho\sigma\tau|a_2;a_3a_4)} + (1/28)R^{\rho}{}_{(a_1|\sigma|a_2}{}^{;\tau}R^{\sigma}{}_{a_3|\rho|a_4);\tau} + (19/420)R^{\rho\sigma\tau}{}_{(a_1;a_2}R_{|\rho\sigma\tau|a_3;a_4)}$$
$$- (2/315)R^{\rho}{}_{(a_1}R_{|\sigma|a_2}R^{\sigma}{}_{a_3|\rho|a_4)} + (26/315)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4)} + (26/105)R^{\rho}{}_{(a_1}R^{\sigma}{}_{a_2}{}^{\tau}{}_{a_3}R_{|\rho\sigma\tau|a_4)}$$
$$+ (4/315)R^{\rho\sigma\tau\kappa}R_{\rho(a_1|\tau|a_2}R_{|\sigma|a_3|\kappa|a_4)} + (4/105)R^{\rho\kappa\tau}{}_{(a_1}R_{|\rho\tau|}{}^{\sigma}{}_{a_2}R_{|\sigma|a_3|\kappa|a_4)} + (16/315)R^{\rho\kappa\tau}{}_{(a_1}R_{|\rho|}{}^{\sigma}{}_{|\tau|a_2}R_{|\sigma|a_3|\kappa|a_4)}$$
$$+ (8/315)R^{\rho\tau\kappa}{}_{(a_1}R_{|\rho\tau|}{}^{\sigma}{}_{a_2}R_{|\sigma|a_3|\kappa|a_4)}. \tag{E10e}$$

Finally, by noting that

$$\Box\Delta^{1/2} = (\Box Z + Z_{;\mu}Z^{;\mu})\Delta^{1/2} \tag{E11}$$

we can construct the covariant Taylor series expansion of $\Box\Delta^{1/2}$ up to order σ^2. We have

$$\Box\Delta^{1/2} = \delta^{1/2''} - \delta^{1/2''}{}_{a_1}\sigma^{;a_1} + \frac{1}{2!}\delta^{1/2''}{}_{a_1a_2}\sigma^{;a_1}\sigma^{;a_2} - \frac{1}{3!}\delta^{1/2''}{}_{a_1a_2a_3}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}$$
$$+ \frac{1}{4!}\delta^{1/2''}{}_{a_1a_2a_3a_4}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} + O(\sigma^{5/2}), \tag{E12}$$

where the coefficients $\delta^{1/2''}$ and $\delta^{1/2''}{}_{a_1\cdots a_p}$ with $p = 1, \ldots, 4$ can be obtained from (E11) by using the expansions of $\Delta^{1/2}$ [see Eq. (C11)], $Z_{;\mu}$ [see Eqs. (E5) and (E6)] and $\Box Z$ [see Eqs. (E9) and (E10)]. They are given by

$$\delta^{1/2''} = (1/6)R, \tag{E13a}$$

$$\delta^{1/2''}{}_{a_1} = 0, \tag{E13b}$$

$$\delta^{1/2''}{}_{a_1a_2} = (1/20)\Box R_{a_1a_2} - (1/60)R_{;a_1a_2} + (1/36)RR_{a_1a_2} - (1/15)R^{\rho}{}_{a_1}R_{\rho a_2} + (1/30)R_{\rho\tau}R^{\rho}{}_{a_1}{}^{\tau}{}_{a_2} + (1/30)R^{\rho\sigma\tau}{}_{a_1}R_{\rho\sigma\tau a_2}, \tag{E13c}$$

$$\begin{aligned}\delta^{1/2''}{}_{a_1a_2a_3} = {}& -(1/60)R_{;(a_1a_2a_3)} + (1/20)(\Box R_{(a_1a_2})_{;a_3)} + (1/24)RR_{(a_1a_2;a_3)} - (2/15)R^{\rho}{}_{(a_1}R_{|\rho|a_2;a_3)} + (1/30)R^{\rho}{}_{\sigma;(a_1}R^{\sigma}{}_{a_2|\rho|a_3)}\\ &+ (1/30)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a_1|\rho|a_2;a_3)} + (1/15)R^{\rho\sigma\tau}{}_{(a_1}R_{|\rho\sigma\tau|a_2;a_3)},\end{aligned} \tag{E13d}$$

$$\begin{aligned}\delta^{1/2''}{}_{a_1a_2a_3a_4} = {}& (3/70)(\Box R_{(a_1a_2})_{;a_3a_4)} - (1/70)R_{;(a_1a_2a_3a_4)} - (1/60)R_{;(a_1a_2}R_{a_3a_4)} + (1/20)RR_{(a_1a_2;a_3a_4)} + (1/20)R_{(a_1a_2}\Box R_{a_3a_4)}\\ &- (17/105)R^{\rho}{}_{(a_1}R_{|\rho|a_2;a_3a_4)} + (1/21)R^{\rho}{}_{(a_1}R_{a_2a_3;|\rho|a_4)} + (1/210)R^{\rho}{}_{(a_1;a_2}R_{|\rho|a_3;a_4)} - (3/14)R^{\rho}{}_{(a_1;a_2}R_{a_3a_4);\rho}\\ &+ (17/168)R_{(a_1a_2}{}^{;\rho}R_{a_3a_4);\rho} + (1/35)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a_1|\rho|a_2;a_3a_4)} + (1/21)R^{\rho}{}_{(a_1;|\sigma|}R^{\sigma}{}_{a_2|\rho|a_3;a_4)}\\ &+ (1/30)R^{\rho}{}_{\sigma;(a_1}R^{\sigma}{}_{a_2|\rho|a_3;a_4)} - (4/35)R^{\rho}{}_{(a_1;|\sigma|a_2}R^{\sigma}{}_{a_3|\rho|a_4)} + (11/105)R_{(a_1a_2}{}^{;\rho}{}_{\sigma}R^{\sigma}{}_{a_3|\rho|a_4)}\\ &+ (4/105)R^{\rho}{}_{\sigma;(a_1a_2}R^{\sigma}{}_{a_3|\rho|a_4)} + (4/105)R^{\rho}{}_{(a_1|\sigma|a_2}\Box R^{\sigma}{}_{a_3|\rho|a_4)} + (2/35)R^{\rho\sigma\tau}{}_{(a_1}R_{|\rho\sigma\tau|a_2;a_3a_4)}\\ &+ (1/28)R^{\rho}{}_{(a_1|\sigma|a_2}{}^{;\tau}R^{\sigma}{}_{a_3|\rho|a_4);\tau} + (19/420)R^{\rho\sigma\tau}{}_{(a_1;a_2}R_{|\rho\sigma\tau|a_3;a_4)} + (1/72)RR_{(a_1a_2}R_{a_3a_4)}\\ &- (1/15)R^{\rho}{}_{(a_1}R_{|\rho|a_2}R_{a_3a_4)} + (1/63)R^{\rho}{}_{(a_1}R_{|\sigma|a_2}R^{\sigma}{}_{a_3|\rho|a_4)} + (1/30)R^{\rho\sigma}R_{(a_1a_2}R_{|\rho|a_3|\sigma|a_4)}\\ &+ (1/90)RR^{\rho}{}_{(a_1|\sigma|a_2}R^{\sigma}{}_{a_3|\rho|a_4)} + (1/30)R_{(a_1a_2}R^{\rho\sigma\tau}{}_{a_3}R_{|\rho\sigma\tau|a_4)} + (26/315)R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a_1|\tau|a_2}R^{\tau}{}_{a_3|\rho|a_4)}\\ &+ (10/63)R^{\rho}{}_{(a_1}R^{\sigma}{}_{a_2}{}^{\tau}{}_{a_3}R_{|\rho\sigma\tau|a_4)} + (4/315)R^{\rho\sigma\tau\kappa}R_{\rho(a_1|\tau|a_2}R_{|\sigma|a_3|\kappa|a_4)} + (4/105)R^{\rho\kappa\tau}{}_{(a_1}R_{|\rho\tau|}{}^{\sigma}{}_{a_2}R_{|\sigma|a_3|\kappa|a_4)}\\ &+ (16/315)R^{\rho\kappa\tau}{}_{(a_1}R_{|\rho|}{}^{\sigma}{}_{|\tau|a_2}R_{|\sigma|a_3|\kappa|a_4)} + (8/315)R^{\rho\tau\kappa}{}_{(a_1}R_{|\rho\tau|}{}^{\sigma}{}_{a_2}R_{|\sigma|a_3|\kappa|a_4)}.\end{aligned} \tag{E13e}$$

The coefficients (E13a)–(E13c) were calculated by Christensen [20,21]. In Ref. [33], Brown and Ottewill obtained some of the terms of the coefficient (E13d) and their result has been corrected in the recent article by Anderson, Flanagan and Ottewill [28]. To our knowledge, the expression of the coefficient (E13e) is new. Finally, from (E12) and (E13) we can write

$$\begin{aligned}\Box\Delta^{1/2} = {}& \frac{1}{6}R + \left[\frac{1}{40}\Box R_{a_1a_2} - \frac{1}{120}R_{;a_1a_2} + \frac{1}{72}RR_{a_1a_2} - \frac{1}{30}R^{\rho}{}_{a_1}R_{\rho a_2} + \frac{1}{60}R^{\rho\tau}R_{\rho a_1\tau a_2} + \frac{1}{60}R^{\rho\sigma\tau}{}_{a_1}R_{\rho\sigma\tau a_2}\right]\sigma^{;a_1}\sigma^{;a_2}\\ &- \left[-\frac{1}{360}R_{;a_1a_2a_3} + \frac{1}{120}(\Box R_{a_1a_2})_{;a_3} + \frac{1}{144}RR_{a_1a_2;a_3} - \frac{1}{45}R^{\rho}{}_{a_1}R_{\rho a_2;a_3} + \frac{1}{180}R^{\rho}{}_{\sigma;a_1}R^{\sigma}{}_{a_2\rho a_3} + \frac{1}{180}R^{\rho}{}_{\sigma}R^{\sigma}{}_{a_1\rho a_2;a_3}\right.\\ &\left.+ \frac{1}{90}R^{\rho\sigma\tau}{}_{a_1}R_{\rho\sigma\tau a_2;a_3}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} + \left[\frac{1}{560}(\Box R_{a_1a_2})_{;a_3a_4} - \frac{1}{1680}R_{;a_1a_2a_3a_4} - \frac{1}{1440}R_{;a_1a_2}R_{a_3a_4} + \frac{1}{480}RR_{a_1a_2;a_3a_4}\right.\\ &+ \frac{1}{480}R_{a_1a_2}\Box R_{a_3a_4} - \frac{17}{2520}R^{\rho}{}_{a_1}R_{\rho a_2;a_3a_4} + \frac{1}{504}R^{\rho}{}_{a_1}R_{a_2a_3;\rho a_4} + \frac{1}{5040}R^{\rho}{}_{a_1;a_2}R_{\rho a_3;a_4} - \frac{1}{112}R^{\rho}{}_{a_1;a_2}R_{a_3a_4;\rho}\\ &+ \frac{17}{4032}R_{a_1a_2}{}^{;\rho}R_{a_3a_4;\rho} + \frac{1}{840}R^{\rho}{}_{\sigma}R^{\sigma}{}_{a_1\rho a_2;a_3a_4} + \frac{1}{504}R^{\rho}{}_{a_1;\sigma}R^{\sigma}{}_{a_2\rho a_3;a_4} + \frac{1}{720}R^{\rho}{}_{\sigma;a_1}R^{\sigma}{}_{a_2\rho a_3;a_4}\\ &- \frac{1}{210}R^{\rho}{}_{a_1;\sigma a_2}R^{\sigma}{}_{a_3\rho a_4} + \frac{11}{2520}R_{a_1a_2}{}^{;\rho}{}_{\sigma}R^{\sigma}{}_{a_3\rho a_4} + \frac{1}{630}R^{\rho}{}_{\sigma;a_1a_2}R^{\sigma}{}_{a_3\rho a_4} + \frac{1}{630}R^{\rho}{}_{a_1\sigma a_2}\Box R^{\sigma}{}_{a_3\rho a_4}\\ &+ \frac{1}{420}R^{\rho\sigma\tau}{}_{a_1}R_{\rho\sigma\tau a_2;a_3a_4} + \frac{1}{672}R^{\rho}{}_{a_1\sigma a_2}{}^{;\tau}R^{\sigma}{}_{a_3\rho a_4;\tau} + \frac{19}{10080}R^{\rho\sigma\tau}{}_{a_1;a_2}R_{\rho\sigma\tau a_3;a_4} + \frac{1}{1728}RR_{a_1a_2}R_{a_3a_4}\\ &- \frac{1}{360}R^{\rho}{}_{a_1}R_{\rho a_2}R_{a_3a_4} + \frac{1}{1512}R^{\rho}{}_{a_1}R_{\sigma a_2}R^{\sigma}{}_{a_3\rho a_4} + \frac{1}{720}R^{\rho\sigma}R_{a_1a_2}R_{\rho a_3\sigma a_4} + \frac{1}{2160}RR^{\rho}{}_{a_1\sigma a_2}R^{\sigma}{}_{a_3\rho a_4}\\ &+ \frac{1}{720}R_{a_1a_2}R^{\rho\sigma\tau}{}_{a_3}R_{\rho\sigma\tau a_4} + \frac{13}{3780}R^{\rho}{}_{\sigma}R^{\sigma}{}_{a_1\tau a_2}R^{\tau}{}_{a_3\rho a_4} + \frac{5}{756}R^{\rho}{}_{a_1}R^{\sigma}{}_{a_2}{}^{\tau}{}_{a_3}R_{\rho\sigma\tau a_4} + \frac{1}{1890}R^{\rho\sigma\tau\kappa}R_{\rho a_1\tau a_2}R_{\sigma a_3\kappa a_4}\\ &\left.+ \frac{1}{630}R^{\rho\kappa\tau}{}_{a_1}R_{\rho\tau}{}^{\sigma}{}_{a_2}R_{\sigma a_3\kappa a_4} + \frac{2}{945}R^{\rho\kappa\tau}{}_{a_1}R_{\rho}{}^{\sigma}{}_{\tau a_2}R_{\sigma a_3\kappa a_4} + \frac{1}{945}R^{\rho\tau\kappa}{}_{a_1}R_{\rho\tau}{}^{\sigma}{}_{a_2}R_{\sigma a_3\kappa a_4}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} + O(\sigma^{5/2}).\end{aligned} \tag{E14}$$

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