Supersymmetry, exact Foldy-Wouthuysen transformation, and gravity

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Gravitational effects in relativistic quantum mechanics are investigated. The Dirac particle in general space-time metrics is considered and the Dirac Hamiltonian is constructed in terms of the Newman-Penrose formalism. To discuss the physical meaning of the Dirac Hamiltonian, it is necessary to perform the Foldy-Wouthuysen transformation. In most cases this transformation exists only in an approximate form. In this paper we show that for supersymmetric Dirac Hamiltonians not depending explicitly on time the exact Foldy-Wouthuysen transformation can always be constructed. Further, we derive criteria for spin coefficients for which the accompanying Dirac Hamiltonian is supersymmetric. These criteria are fulfilled by the class of static axisymmetric space-time metrics. For the subclass of stationary metrics, the exact Foldy-Wouthuysen transformation is calculated and the transformed Dirac Hamiltonian is derived. Recently, Obukhov constructed a different exact Foldy-Wouthuysen transformation for that class of space-time metrics and calculated the Dirac Hamiltonian in the Foldy-Wouthuysen representation. We show that the expansion series in orders of $1/mc^2$ of our and Obukhov's Dirac Hamiltonians coincide.

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I. INTRODUCTION

Measurements on elementary particles like electrons are very interesting for gravitational theory because they can test the principle of equivalence lying on the basis of the relativistic theory of gravity, provide hints at the relation of gravitational and quantum physics, and can help to discriminate between different approaches to gravitational theory.

In the past two decades, experimental accuracy was increased so that it was possible to measure gravitational effects on fermions [1–3]. To use this new experimental window, one has theoretically to describe the experimental data of such measurements. To this end, one can start from the semiclassical approximation of a quantum theory of gravity, where spin-1/2 particles are described by the Dirac equation on the background of a given classical curved space-time. For the comparison of theory and experiment, one has to perform the nonrelativistic limit of the Dirac equation.

In the case of Dirac particles coupled to electromagnetic fields, it turns out that the best way to the nonrelativistic limit is to perform a Foldy-Wouthuysen (FW) transformation of the Dirac equation [4]. (Erikson and Kolsrud [5] investigated the FW transformation extensively and found some exact transformations.) So, this method was also used to calculate the nonrelativistic limit for Dirac particles that are coupled to gravity [6–8]. However, it became also clear that an approximate FW transformation can fail in this case [9,10]. Thus, it is a physically rewarding task to look for exact FW transformations.

Recently Obukhov [9] constructed an exact FW transformation for stationary metrics

$$ds^2 = V^2(\vec{x})dt^2 - W^2(\vec{x})d\vec{x}^2, \qquad (1)$$

where the functions V and W depend on $\vec{x} = (x^1, x^2, x^3)$. Since a closer examination of Obukhov's transformation shows that it is based on the supersymmetry of the Dirac Hamiltonian, it is interesting to consider the relation between supersymmetry and FW transformation in more detail. This is not done to discuss supersymmetric theories,[1] but to win a technical tool for the construction of exact FW transformations. The goal is to analyze the conditions under which, in relativistic quantum mechanics given by the general-covariant Dirac equation, there exists a canonical transformation that corresponds to an exact FW transformation. Of course, in view of the generally completely asymmetric, dynamically changing space-time structure, one cannot expect to find *the* exact FW transformation; one can only construct FW transformations for special classes of metrics.

In our analysis, we start from the anholonomic representation of the covariant Dirac equation in terms of the Newman-Penrose formalism. Although this formalism was introduced to describe null fields, of course, it can and should also be used in other cases, first of all, because it is well elaborated. In our case it is a proper means to discuss supersymmetry.

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[1] Supersymmetry (SUSY) originally was introduced in quantum field theory to describe the symmetry between bosons and fermions and by Witten represented in the framework of a toy model called "supersymmetric quantum mechanics (SSQM)," where usual quantum mechanics is equipped with two supplementary structures, the involution operator and the supercharge. Nowadays, SSQM is an established branch of quantum research.

This article proceeds as follows. In Sec. II we briefly introduce the notion of supersymmetric quantum mechanics and point out the relation between the FW transformation and supersymmetry. Further, in Sec. III we construct the Dirac Hamiltonian in Newman-Penrose formalism for general space-time metrics and give some sufficient conditions on spin coefficients leading to supersymmetric Dirac Hamiltonians. In Sec. IV we construct an exact FW transformation for stationary metrics and compare the result with Obukhov's ones. We show that the Hamiltonian obtained by Obukhov is equal to our result in all orders of the expansion series in $1/mc^2$. (This result does not conflict with the statement on an approximate approach, because it can fail in other cases.) The symmetry underlying both transformations is the supersymmetric structure of relativistic quantum mechanics and the difference consists only in the choice of the involution.

II. SUPERSYMMETRIC QUANTUM MECHANICS AND DIRAC HAMILTONIANS WITH SUPERSYMMETRY

In earlier days, SUSY was studied in quantum mechanics as a test model for symmetry breaking in field theory [11]. Today supersymmetric quantum mechanics (SSQM) is developing to a vast area with many applications [12–14]. Here we use some concepts and results of SSQM to obtain exact FW transformations for curved space-times.

SSQM consists of the quadruple $(\mathcal{H}, H_S, Q, \tau)$. The pair $(\mathcal{H}, H_S)$ defines a quantum mechanics with a self-adjoint super-Hamiltonian H_S acting on a Hilbert space $\mathcal{H}$. Furthermore, this theory requires the existence of a bounded, self-adjoint operator τ with $\tau^2 = \mathbf{1}$ (grading operator or involution) and a self-adjoint operator Q called supercharge such that $\tau Q + Q\tau = 0$. The involution τ can be used to construct the projection $P_\pm = \frac{1}{2}(1 \pm \tau)$ and therefore divides the Hilbert space into two subspaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (even and odd subspaces). Every operator A defined on $\mathcal{H}$ can be divided into its even and odd parts. The odd part $A_- = P_- A$ anticommutes with the involution τ, whereas the even part $A_+ = P_+ A$ commutes with τ. In SSQM the self-adjoint Hamilton operator H is called supersymmetric, if it consists of a supercharge Q and the operator M commuting with Q and τ such that

$$H = Q + M\tau. \tag{2}$$

The super-Hamiltonian is then given by $H_S = Q^2 + M^2$.

Dirac Hamiltonians with supersymmetry have some important properties [14]. One of them is the existence of the unitary transformation

$$U = a_+ + \tau(Q(\sqrt{Q^2})^{-1})a_- \tag{3}$$

with

$$a_\pm = \frac{1}{\sqrt{2}}\sqrt{1 \pm M(\sqrt{H^2})^{-1}}. \tag{4}$$

This transformation leads by means of the Hamiltonian (2) to the expression

$$H' = UHU^\dagger = \tau\sqrt{Q^2 + M^2}. \tag{5}$$

The Hamilton operator H' is even with respect to the involution τ. For this reason, there exists a relation between the transformation U and the exact FW transformation. Eriksen and Kolsrud [5] showed that the exact FW transformation is characterized by an operation which turns the energy-sign operator $\Lambda_H = H/\sqrt{H^2}$ in the projection β.

In the following Lemma we show that $U\Lambda_H U^\dagger = \tau$ and discuss afterwards the case when τ is identified with β.

Lemma 1.—The unitary transformation U given by Eqs. (3) and (4) turns the energy-sign operator $\Lambda_H = H/\sqrt{H^2}$ in τ, i.e. $U\Lambda_H U^\dagger = \tau$.

Proof.—Rewriting the expression under consideration in the form

$$U\Lambda_H U^\dagger = UHU^\dagger U(\sqrt{H^2})^{-1}U^\dagger$$

with the help of Eq. (5), one obtains

$$U\Lambda_H U^\dagger = \tau\sqrt{H^2}U(\sqrt{H^2})^{-1}U^\dagger.$$

Further, defining $\Lambda_Q = Q/\sqrt{Q^2}$ and transforming $(\sqrt{H^2})^{-1}$ as follows

$$\begin{aligned} U(\sqrt{H^2})^{-1}U^\dagger &= (a_+ + \tau\Lambda_Q a_-)(\sqrt{H^2})^{-1}(a_+ - \tau\Lambda_Q a_-) \\ &= (a_+ + \tau\Lambda_Q a_-)(a_+ - \tau\Lambda_Q a_-)(\sqrt{H^2})^{-1} \\ &= (a_+^2 + a_-^2)(\sqrt{H^2})^{-1} = (\sqrt{H^2})^{-1}, \end{aligned} \tag{6}$$

one gets the desired result. ∎

To show the relation between the unitary transformation (5) and the exact FW transformation, we consider the projections $P_\pm = (1/\sqrt{2})(1 \pm \Lambda_H)$ and $K_\pm = (1/\sqrt{2})(1 \pm \beta)$. In the case where $\tau = \beta$ the unitary transformation (5) turns the projector $P_\pm$ into $K_\pm$. On the other hand, $P_\pm$ project on the part of positive (negative) energy, whereas $K_\pm$ project onto big (small) spinor coefficients. This means that the positive energy states are transformed into the big spinor coefficients whereas the negative states are transformed into the small spinor coefficients and lead to the Newton-Wigner representation. It follows that the transformation (5) is an exact FW transformation.

III. THE DIRAC HAMILTONIAN IN GENERAL SPACE-TIME METRICS AND SUSY

Dirac particles in curved space-time metrics are described by the covariant Dirac equation (see, e.g. [15,16])

$$(i\hbar\gamma^{\hat\alpha}D_{\hat\alpha} - mc)\psi = 0. \tag{7}$$

Here $D_{\hat{\alpha}}$ denotes the covariant spinor derivative and $\gamma^{\hat{\alpha}}$ the point-independent gamma matrices.

We use the conventions of Bjorken and Drell [17] for the gamma matrices $\gamma^{\hat{\alpha}}$ and choose the signature -2. The space-time indices μ, ν run from 0 to 3 and the tetrad indices with a hat $\hat{\alpha}$, $\hat{\beta}$ run from 0 to 3. The covariant spinor derivative is defined as

$$D_{\hat{\alpha}} = e_{\hat{\alpha}}{}^{\mu}\left(\partial_{\mu} + \frac{i}{4}\Gamma_{\mu}\right), \tag{8}$$

with the gauge potential $\Gamma_{\mu} = \sigma^{\hat{\alpha}\hat{\beta}}\gamma_{\hat{\alpha}\hat{\beta}\hat{\gamma}}e^{\hat{\gamma}}{}_{\mu}$ ($\sigma^{\hat{\alpha}\hat{\beta}} := \frac{1}{4}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]$) and the Ricci rotation coefficients are given as $\gamma_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = e_{\hat{\alpha}}{}^{\mu}e_{\hat{\beta}\mu;\nu}e_{\hat{\gamma}}{}^{\nu}$. For the Ricci rotation coefficients and the spin coefficients we use the conventions of Chandrasekhar [18].

In the first step, we recast the Dirac equation into its Schrödinger form. For this purpose, it is necessary to specify the coordinates and to identify the x^0-coordinate with the physical time ($x^0 = ct$). By choosing an orthonormal tetrad $e_{\hat{\alpha}}{}^{\mu}(x)$, the algebra of the point-dependent gamma matrices $\gamma^{\mu}(x) = e_{\hat{\alpha}}{}^{\mu}(x)\gamma^{\hat{\alpha}}$ is given by

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2e_{\hat{\alpha}}{}^{\mu}e_{\hat{\beta}}{}^{\nu}\eta^{\hat{\alpha}\hat{\beta}} = 2g^{\mu\nu}, \tag{9}$$

where $\{.,.\}$ describes the anticommutator. Especially, we have $\{\gamma^0, \gamma^0\} = 2g^{00}$ and hence $(\gamma^0)^{-1} = \frac{1}{g^{00}}\gamma^0$. Now, by multiplying Eq. (7) with $(\gamma^0)^{-1}$ we obtain the Schrödinger form of the Dirac equation,

$$i\hbar\partial_t\psi = H\psi, \tag{10}$$

with the Dirac Hamiltonian

$$H = -\frac{i\hbar c}{g^{00}}\gamma^0\gamma^i\partial_i + \frac{\hbar c}{4g^{00}}\gamma^0\gamma^{\alpha}\Gamma_{\alpha} + \frac{mc^2}{g^{00}}\gamma^0. \tag{11}$$

Furthermore, for technical reasons, it is useful to choose the Newman-Penrose tetrad because in the literature there are many investigations of space-time structures in that framework. We choose the tetrad $(l^{\mu}, n^{\mu}, m^{\mu}, \overline{m}^{\mu})$ which is related to the orthonormal tetrad $e_{\hat{\alpha}}{}^{\mu}(x)$ as follows:

$$e_{\hat{0}}{}^{\mu} = \frac{1}{\sqrt{2}}(l^{\mu} + n^{\mu}) \tag{12}$$

$$e_{\hat{1}}{}^{\mu} = \frac{1}{\sqrt{2}}(m^{\mu} + \overline{m}^{\mu}) \tag{13}$$

$$e_{\hat{2}}{}^{\mu} = \frac{1}{\sqrt{2}i}(m^{\mu} - \overline{m}^{\mu}) \tag{14}$$

$$e_{\hat{3}}{}^{\mu} = \frac{1}{\sqrt{2}}(l^{\mu} - n^{\mu}). \tag{15}$$

The Ricci rotation coefficients can be rewritten into spin coefficients as follows:

$$\begin{aligned}\gamma_{\hat{2}\hat{1}\hat{2}} &= e_{\hat{2}}{}^{\mu}e_{\hat{1}\mu;\nu}e_{\hat{2}}{}^{\nu}\\ &= \frac{i^2}{(\sqrt{2})^3}(m^{\mu} - \overline{m}^{\mu})(m_{\mu} + \overline{m}_{\mu})_{;\nu}(m^{\nu} - \overline{m}^{\nu})\\ &= \sqrt{2}\,\mathrm{Re}(\alpha - \beta).\end{aligned} \tag{16}$$

The definition of the spin coefficients is given by Chandrasekhar [18]. The remaining Ricci rotation coefficients are given in the Appendix A.

Inserting the Ricci rotation coefficients in Eq. (11) and rearranging the gamma matrices $\gamma^{\hat{\alpha}}$, one arrives at

$$H = \frac{mc^2}{g^{00}}\gamma^{\hat{0}} + \frac{\hbar c}{g^{00}}(\vec{\alpha}\cdot\vec{\Pi} - i\vec{\Sigma}\cdot\vec{\Xi}) + \frac{\hbar c}{2g^{00}}(K\gamma^{\hat{5}} - i\Phi), \tag{17}$$

where the operators $\vec{\Xi}$ and $\vec{\Pi}$ are defined by the relations

$$\vec{\Pi} := \vec{P} - \frac{i}{2}\vec{A} \qquad \vec{\Xi} := \vec{L} - \frac{i}{2}\vec{J}. \tag{18}$$

The operators $\vec{A}, \vec{L}, \vec{J}, K, \Phi$ only consist of spin coefficients whereas the operator $\vec{P}$ is a differential operator. The explicit form of the operators is complicated and, for the following discussion, not important (see Appendix A).

We consider the case where the operators Ξ and Φ vanish and the gamma matrices appear in the same way as in the free Dirac Hamiltonian $H_0 = c\vec{\alpha}\cdot\vec{p} + \gamma^{\hat{0}}mc^2$ and construct on this condition an exact FW transformation.

For every metric we have some freedom to choose a special tetrad. A closer look at the operators $\vec{A}, \vec{L}, \vec{J}, K, \Phi$ leads to the observation that for one special tetrad called canonical [19] all operators simplify drastically, especially the operator L vanishes completely. On the condition that we choose a canonical tetrad and require that the following relations for the spin coefficient hold for a space-time metric, the operators Ξ and Φ are equal to zero:

$$J_1 = \alpha - \alpha^* - (\beta - \beta^*) + \tau - \tau^* - (\pi - \pi^*) = 0 \tag{19}$$

$$J_2 = \pi + \pi^* + \tau + \tau^* - (\alpha + \alpha^*) - (\beta + \beta^*) = 0 \tag{20}$$

$$J_3 = \rho - \rho^* + \mu - \mu^* - (\varepsilon - \varepsilon^*) - (\gamma - \gamma^*) = 0 \tag{21}$$

$$\Phi = \rho + \rho^* - (\mu + \mu^*) + (\varepsilon + \varepsilon^*) - (\gamma + \gamma^*) = 0. \tag{22}$$

In this case the Dirac Hamiltonian has the form

$$H = \frac{mc^2}{g^{00}}\gamma^{\hat{0}} + \frac{\hbar c}{g^{00}}\vec{\alpha}\cdot\vec{\Pi} + \frac{\hbar c}{2g^{00}}K\gamma^{\hat{5}}, \tag{23}$$

and the unitary transformation

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{24}$$

leads to the Hamiltonian

$$\begin{aligned} H' &= THT^\dagger \\ &= \begin{pmatrix} 0 & \frac{\hbar c}{g^{00}}(\vec{\sigma}\cdot\vec{\Pi} + \frac{1}{2}K - imc^2) \\ \frac{\hbar c}{g^{00}}(\vec{\sigma}\cdot\vec{\Pi} + \frac{1}{2}K + imc^2) & 0 \end{pmatrix} \\ &=: Q. \end{aligned}$$

The Hamiltonian H' is supersymmetric for $\tau = \beta$ [it is given by Eq. (2) with vanishing M]. Therefore, in the case of a Dirac Hamiltonian which is time independent the unitary transformation (3) describes an exact FW transformation

$$U = \frac{1}{\sqrt{2}}(1 + \beta\Lambda_Q), \tag{25}$$

with $\Lambda_Q := Q/\sqrt{Q^2}$. The Hamiltonian H' transforms according to Eq. (5) as

$$H^{\rm FW} = UH'U^\dagger = \beta\sqrt{Q^2}. \tag{26}$$

Consequently, we found that the exact FW transformation can be constructed for vanishing operators $\vec{\Xi}$, Φ and tetrad components $e_{\hat{i}}{}^0$.

Concluding this section, let us make some comments on the physical significance of our special choices. First, using a canonical tetrad does not restrict the class of describable space-time metrics. Second, we have required that the space-time is stationary, i.e., that there exists a timelike Killing field ξ^μ which ensures that the Hamiltonian is time independent. Using an appropriate coordinate system, this Killing field takes the form $\xi^\mu = \delta^\mu_0$ and with the help of the Killing equation one obtains $g_{\mu\nu,0} = 0$, i.e., the metric is time independent.

To discuss the physical meaning of this assumption let us consider some of its implications. Considering the velocity field $u^\mu = (1/\sqrt{2})(l^\mu + n^\mu)$ of comoving observers, one obtains that the expansion of this field $\Theta := u^\mu{}_{;\mu}$ is equal to zero. This quantity has an invariant meaning. Therefore, the expansion written in a Newman-Penrose tetrad also has to vanish:

$$0 = \Theta = \mu + \mu^* - \rho - \rho^* + \varepsilon + \varepsilon^* - \gamma - \gamma^*. \tag{27}$$

Furthermore, by adding Eqs. (19) and (20) and also Eqs. (21) and (22), one obtains

$$\alpha^* + \beta - \pi^* - \tau = 0, \tag{28}$$

$$\rho - \mu^* + \varepsilon^* - \gamma = 0, \tag{29}$$

and from Eq. (27) together with Eq. (29) one concludes that the following relations hold true:

$$0 = \mu + \mu^* - \rho - \rho^*, \qquad \text{and} \quad 0 = \varepsilon + \varepsilon^* - \gamma - \gamma^* \tag{30}$$

$$\Leftrightarrow \mathrm{Re}(\mu - \rho) = 0, \qquad \text{and} \quad \mathrm{Re}(\varepsilon - \gamma) = 0. \tag{31}$$

Furthermore, from Eq. (21) one also reads off the condition

$$\mathrm{Im}\,(\rho + \mu - (\varepsilon + \gamma)) = 0. \tag{32}$$

Moreover, addition and subtraction of Eq. (29) and its complex conjugate gives

$$\begin{aligned} &\mathrm{Re}\,(\alpha + \beta - (\pi + \tau)) = 0, \\ &\text{and} \quad \mathrm{Im}(\alpha - \beta + \tau - \pi) = 0. \end{aligned} \tag{33}$$

These relations can be used to relate the spin coefficients to the Ricci rotation coefficients which are given in Appendix B.

An inspection of the above relations in comparison with the kinematical description of space-times in terms of spin coefficients (see, e.g., [20]) shows that the kinematical structure, i.e., the form of shear, rotation, and acceleration is strongly restricted. But there are some important space-times fulfilling these conditions. For example, this is true for the class of space-time metrics

$$ds^2 = F^2(\vec{x})dt^2 - G^2(\vec{x})dx^2 - M^2(\vec{x})dy^2 - I^2(\vec{x})dz^2. \tag{34}$$

Here the functions F, G, M, and I depend on $\vec{x} = (x^1, x^2, x^3)$. Some important particular classes belong to this family, for example, all degenerate static vacuum solutions (classes A–C) [21] and the subclass of stationary metrics. However, we will also give important examples for space-times which do not fulfill the above conditions such that for them an exact FW transformation cannot be constructed in the described way.

IV. EXACT FOLDY-WOUTHUYSEN TRANSFORMATION FOR SOME STATIONARY METRICS

Here we construct the exact FW-transformed Dirac Hamiltonian for stationary metrics and compare the result with the exactly transformed Hamiltonian obtained by Obukhov [9]. To this end, we use the Newman-Penrose tetrad

$$\vec{l} = \frac{1}{\sqrt{2}}\left(\frac{1}{V}, 0, 0, \frac{1}{W}\right) \tag{35}$$

$$\vec{n} = \frac{1}{\sqrt{2}}\left(\frac{1}{V}, 0, 0, -\frac{1}{W}\right) \tag{36}$$

$$\vec{m} = \frac{1}{\sqrt{2}}\left(0, \frac{1}{W}, \frac{i}{W}, 0\right) \tag{37}$$

related to the metric (1), calculate the spin coefficients, and insert them into Eq. (17). This provides the relation

$$H = mc^2\gamma^{\hat{0}}V - \frac{i\hbar cV}{W}\gamma^{\hat{0}}\gamma^{\hat{\imath}}\partial_i + \frac{\hbar cV}{2W^2}\gamma^{\hat{0}}\gamma^{\hat{\imath}}\varepsilon_{ijk}\partial^j W\Sigma^{\hat{k}} - i\hbar\gamma^{\hat{0}}\frac{1}{2VW}(\vec{\alpha}\cdot\vec{\nabla}V).$$

This Dirac Hamiltonian has the same form as Obukhov's Hamiltonian.

If one defines $\psi' = \sqrt{\det(e_{ij})}\psi$, one gets a self-adjoint Hamiltonian with respect to the flat-space scalar product:

$$H' = \beta mc^2 V + \frac{c}{2}\{F, \vec{\alpha}\cdot\vec{p}\}, \tag{38}$$

where $F = V/W$ (see [9]). With the help of the unitary transformation (24), one calculates

$$TH'T^\dagger = \begin{pmatrix} 0 & \frac{c}{2}\{\vec{\sigma}\cdot\vec{p}, F\} - iVmc^2 \\ \frac{c}{2}\{\vec{\sigma}\cdot\vec{p}, F\} + iVmc^2 & 0 \end{pmatrix} \tag{39}$$

$$=: \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} =: Q. \tag{40}$$

The Hamiltonian is now a supercharge with respect to the involution $\tau = \beta$ and therefore Eq. (3) is the exact FW transformation leading to the transformed Hamiltonian

$$H^{\mathrm{FW}} = \beta\sqrt{Q^2} = \begin{pmatrix} \sqrt{D^*D} & 0 \\ 0 & -\sqrt{DD^*} \end{pmatrix}. \tag{41}$$

The spectra of D^*D and DD^* are the same, except for the zero eigenvalue [14]. The square of the supercharge

$$Q = \frac{c}{2}\{\vec{\alpha}\cdot\vec{p}, F\} + JVmc.$$

is given by

$$Q^2 = \left(\frac{c}{2}\{\vec{\alpha}\cdot\vec{p}, F\}\right)^2 + (JVmc^2)^2 + \left\{\frac{c}{2}\{\vec{\alpha}\cdot\vec{p}, F\}, JVmc^2\right\}. \tag{42}$$

After some calculations, we obtain

$$\left(\frac{c}{2}\{\vec{\alpha}\cdot\vec{p}, F\}\right)^2 = Fc^2p^2F - \frac{\hbar^2c^2}{4}\vec{f}^2 + \hbar c^2F\vec{\Sigma}\cdot(\vec{f}\times\vec{p}) + \frac{\hbar^2c^2}{2}F(\vec{\nabla}\cdot\vec{f}) \tag{43}$$

and

$$\left\{\frac{c}{2}\{\vec{\alpha}\cdot\vec{p}, F\}, JVmc^2\right\} = i\hbar mc^3F\gamma^5\vec{\Sigma}\cdot J\vec{\phi}. \tag{44}$$

Inserting Eqs. (43) and (44) in Eq. (42) we arrive at

$$Q^2 = V^2m^2c^4 + Fc^2p^2F - \frac{\hbar^2c^2}{4}\vec{f}^2 + \hbar c^2F\vec{\Sigma}\cdot(\vec{f}\times\vec{p} + i\gamma^5Jcm\vec{\phi}) + \frac{\hbar^2c^2}{2}F(\vec{\nabla}\cdot\vec{f}). \tag{45}$$

Until now the transformation is exact. Now we expand the FW Hamiltonian (41) in powers of $1/mc^2$. In the first approximation this provides

$$H^{\mathrm{FW}} = mc^2\beta V + \frac{1}{4m}\beta\left(\frac{1}{W}p^2F + Fp^2\frac{1}{W}\right) - \frac{\hbar^2}{8mV}\beta\vec{f}^2 + \frac{\hbar}{4m}\beta\vec{\Sigma}\cdot\left(\frac{1}{W}\vec{f}\times\vec{p} + \vec{f}\times\vec{p}\frac{1}{W}\right) + \frac{\hbar^2}{4mW}\beta(\vec{\nabla}\cdot\vec{f}) + \frac{\hbar c}{2W}\vec{\Sigma}\cdot\vec{\phi}, \tag{46}$$

where we regarded that $J^2 = 1$. This FW-transformed Dirac Hamiltonian is the same as the Hamiltonian obtained by Obukhov in [9].

Although Obukhov's construction differs from ours, both lead in the first order of the expansion series to the same Dirac Hamiltonian. The reason for this is that Obukhov's construction is also based on supersymmetry, as can be seen as follows: The key of Obukhov's construction is the existence of the involution operator J. The operator J anticommutes with the Dirac Hamiltonian (39) and hence the latter is a supercharge with respect to involution $\tau = J$. Therefore, the unitary transformation (3)

$$U_J = a_+ + \tau(H(\sqrt{H^2})^{-1})a_- = \frac{1}{\sqrt{2}}(1 + J\Lambda_H)$$

performed for the Dirac Hamiltonian H gives

$$U_JHU_J^\dagger = J\sqrt{H^2}. \tag{47}$$

Furthermore, the unitary operator

$$W = \tfrac{1}{2}(1 + \beta J)$$

transforms J to β, and one derives with Eq. (47) and Lemma 1 the relation

$$WU_J\Lambda U_J^\dagger W^\dagger = \beta.$$

Finally, one obtains for the FW representation of the Hamiltonian

$$H_J^{\mathrm{FW}} = WU_JHU_J^\dagger W^\dagger = P_+(\sqrt{H^2}) + JP_-(\sqrt{H^2}), \tag{48}$$

with the projections

$$P_\pm(A) := \tfrac{1}{2}(1 \pm \beta A\beta).$$

On the other hand, we used the operator (24)

$$T = \tfrac{1}{2}(1 + \beta J)$$

and the transformation (26), i.e.,

$$U_Q T = \tfrac{1}{2}(1 + \Lambda_Q)(1 + \beta J)$$

with $\Lambda_Q := Q/\sqrt{Q^2}$. This shows that in both cases the reason for the existence of exact FW transformations is the *supersymmetry* of the Dirac Hamiltonian.

However, it is not clear whether the two transformations,

$$W U_H = \tfrac{1}{2}(1 + \beta J)(1 + J\Lambda_H)$$

and

$$U_Q T = \tfrac{1}{2}(1 + \Lambda_Q)(1 + \beta J),$$

lead to the same Hamilton operator H^{FW}. But, in Appendix C we show that this is true for all orders of the expansion series of H^{FW} in powers of $1/mc^2$. But it remains unclear whether both expansions converge.

V. DISCUSSION

The connection between supersymmetry and the exact FW transformation for Dirac particles coupled to electromagnetic fields was already stressed by Romero *et al.* in [22]. Here we investigated the coupling to gravity and found a relatively wide class of space-time metrics including the class found by Obukhov for which a supercharge can be constructed. Our construction differs from Obukhov's and does not lead to the same exact FW-transformed Hamiltonian. However, it could be shown that in the expansion series in powers of $1/mc^2$ Obukhov's and our FW Hamiltonians coincide. This means that in the nonrelativistic limit we obtain the same physical result. Therefore, for the discussion of the physical relevance of these results, we can refer to Obukhov [9]. There, from the view of the principle of equivalence, the Schwarzschild space-time and accelerated observers in Minkowski space-time are compared. For a more detailed discussion of this matter, it would be desirable to find a similar construction for rotating space-times, for instance, the Kerr solution, in comparison with rotating observers in Minkowski space-time. But, unfortunately, both metrics violate the consistency relations (31)–(33) or, in other words, these relations imply that the rotation has to vanish.

Let us make this point evident in the case of the Kerr space-time. Here the relations (32) or (21) and (33) or (19) take the form

$$\mathrm{Im}\,(\rho + \mu - (\epsilon + \gamma)) = 0 \Leftrightarrow a\cos(\theta) = 0, \qquad (49)$$

$$\mathrm{Im}\,(\alpha - \beta + \tau - \pi) = 0 \Leftrightarrow a\sin(\theta) = 0, \qquad (50)$$

where a is the rotation parameter of the Kerr metric. The values of the spin coefficients can be found in [18]. Second, we consider a rotating accelerated reference frame in the Minkowski space-time

$$ds^2 = \eta_{\hat{\mu}\hat{\nu}}\theta^{\hat{\mu}}\theta^{\hat{\nu}},$$

with the co-tetrads $\theta^{\hat{0}} = (1 + \frac{\mathbf{a}\cdot\mathbf{x}}{c^2})dx^0$, $\theta^{\hat{i}} = dx^i + (\frac{\boldsymbol{\omega}}{c} \times \mathbf{x})dx^0$, where $\mathbf{a}$ denotes the 3-acceleration and $\boldsymbol{\omega}$ the 3-rotation of the observer. From the corresponding tetrad fields, one obtains for the Ricci rotation coefficients

$$\gamma_{\hat{i}\hat{j}\hat{0}} = -\gamma_{\hat{i}\hat{j}\hat{0}} = -\frac{1}{c}\epsilon_{ijk}\omega^k$$

(for details see [6,23]). From the consistency relations (31)–(33) in terms of rotation coefficients (see Appendix B), one reads off the conditions

$$\gamma_{\hat{1}\hat{3}\hat{0}} = 0 \Rightarrow \frac{1}{c}\frac{\epsilon_{13k}\omega^k}{1 + \frac{\mathbf{a}\cdot\mathbf{x}}{c^2}} = 0, \qquad \Rightarrow \omega^2 = 0, \qquad (51)$$

$$\gamma_{\hat{2}\hat{3}\hat{0}} = 0 \Rightarrow \frac{1}{c}\frac{\epsilon_{23k}\omega^k}{1 + \frac{\mathbf{a}\cdot\mathbf{x}}{c^2}} = 0, \qquad \Rightarrow \omega^1 = 0, \qquad (52)$$

$$\gamma_{\hat{2}\hat{1}\hat{0}} = 0 \Rightarrow \frac{1}{c}\frac{\epsilon_{21k}\omega^k}{1 + \frac{\mathbf{a}\cdot\mathbf{x}}{c^2}} = 0, \qquad \Rightarrow \omega^3 = 0, \qquad (53)$$

and therefore the rotation of the observer frame has to vanish. Thus, we meet in both situations the same condition, namely, that the rotation parameter has to vanish to perform an exact FW transformation of the form suggested by us. The results are insofar satisfactory as both situations lead to the same obstacle. From another point of view, these relations can be used to construct also rotating metrics which enable exact FW transformations.

Our approach is not only more general than that found in [9], it has also the advantage to get more insight into the origin of the existence of FW transformations. One sees why a number of FW transformations are possible to construct under certain circumstances (see, e.g., [24] and the references therein). The key property is the existence of the supersymmetric Hamiltonian. The FW-transformed Hamiltonian is proportional to the square root of the so-called super-Hamiltonian $H^{\mathrm{FW}} = \beta\sqrt{H_S}$, where H_S is defined as $H_S = \frac{1}{2mc^2}Q^2$. On the other hand, the super-Hamiltonian is a self-adjoint positive operator with a two-fold degenerated spectrum, except for the zero eigenstate [14].

Hence, in such space-times positron and electron are *super partners*. That means they have the same energy spectrum, except for the zero eigenstate. This can only be the case when the Dirac sea is stable [22]. We conclude that for our class of space-time metrics there exists a representation in that the Dirac sea is stable.

It should be mentioned that in supersymmetric relativistic quantum mechanics there exists a similar unitary transformation related to the exact Cini-Touscheck transformation [14]. The Cini-Touscheck transformation is used to calculate rigorously the ultrarelativistic limit.

Finally, we would like to address a problem that one meets in this context. When we spoke of an exact FW transformation, then we always thought of a block-diagonalizing transformation constructed by Eriksen-

Kolsrud [5]. But, as it was shown in [25], the equivalence of any block representation of the Hamiltonian to the FW representation has to be verified, because the Hamiltonian itself is insufficient for an analysis of observable spin effects; for this one needs to know the spin operator as well. In [25], where by another method the FW Hamiltonian is derived in a weak-field approximation as a power series, it is especially shown that, although the power series coincides with that following the form of the exact FW transformation given in [9], there are differences in the spin-gravity coupling. While in [9] a dipole spin-gravity coupling is found, this coupling (which is problematic from the view of the equivalence principle) does not occur in [25]. Thus, it would be useful to discuss our diagonalizing transformation having the same power series of the Hamiltonian, too, from the view of the complete momentum and spin equations.

APPENDIX A: THE DIRAC HAMILTONIAN IN A GENERAL SPACE-TIME METRIC

In Eq. (16) one rotation coefficient was expressed in terms of spin coefficients. Here we give the remaining expressions:

$$\left.\begin{array}{ll} \gamma_{\hat 0\hat 3\hat 1} = \sqrt{2}\,\mathrm{Re}(\alpha+\beta) & \gamma_{\hat 2\hat 1\hat 1} = \sqrt{2}\,\mathrm{Im}(\alpha+\beta) \\ \gamma_{\hat 2\hat 1\hat 2} = \sqrt{2}\,\mathrm{Re}(\alpha-\beta) & \gamma_{\hat 0\hat 3\hat 2} = -\sqrt{2}\,\mathrm{Im}(\alpha-\beta) \\ \gamma_{\hat 0\hat 3\hat 0} = \sqrt{2}\,\mathrm{Re}(\varepsilon+\gamma) & \gamma_{\hat 2\hat 1\hat 0} = \sqrt{2}\,\mathrm{Im}(\varepsilon+\gamma) \\ \gamma_{\hat 3\hat 0\hat 3} = \sqrt{2}\,\mathrm{Re}(\varepsilon-\gamma) & \gamma_{\hat 2\hat 1\hat 3} = \sqrt{2}\,\mathrm{Im}(\varepsilon-\gamma), \end{array}\right\} \tag{A1}$$

$$\left.\begin{array}{ll} \gamma_{\hat 1\hat 3\hat 0} = \frac{1}{\sqrt{2}}\,\mathrm{Re}(\tau+\nu+\kappa+\pi) & \gamma_{\hat 2\hat 0\hat 0} = \frac{1}{\sqrt{2}}\,\mathrm{Im}(\tau+\nu+\kappa+\pi) \\ \gamma_{\hat 1\hat 3\hat 3} = \frac{1}{\sqrt{2}}\,\mathrm{Re}(\kappa+\pi-\tau-\nu) & \gamma_{\hat 2\hat 0\hat 3} = \frac{1}{\sqrt{2}}\,\mathrm{Im}(\kappa+\pi-\tau-\nu) \\ \gamma_{\hat 1\hat 0\hat 3} = \frac{1}{\sqrt{2}}\,\mathrm{Re}(\kappa+\nu-\pi-\tau) & \gamma_{\hat 2\hat 3\hat 3} = \frac{1}{\sqrt{2}}\,\mathrm{Im}(\kappa+\nu-\pi-\tau) \\ \gamma_{\hat 1\hat 0\hat 0} = \frac{1}{\sqrt{2}}\,\mathrm{Re}(\kappa+\tau-\pi-\nu) & \gamma_{\hat 2\hat 3\hat 0} = \frac{1}{\sqrt{2}}\,\mathrm{Im}(\kappa+\tau-\pi-\nu) \end{array}\right\} \tag{A2}$$

$$\left.\begin{array}{ll} \gamma_{\hat 0\hat 1\hat 1} = \frac{1}{\sqrt{2}}\,\mathrm{Re}(\lambda+\mu-\sigma-\rho) & \gamma_{\hat 0\hat 1\hat 2} = \frac{1}{\sqrt{2}}\,\mathrm{Im}(\rho+\mu-\sigma-\lambda) \\ \gamma_{\hat 0\hat 2\hat 2} = \frac{1}{\sqrt{2}}\,\mathrm{Re}(\mu+\sigma-\rho-\lambda) & \gamma_{\hat 0\hat 2\hat 1} = -\frac{1}{\sqrt{2}}\,\mathrm{Im}(\sigma+\rho+\lambda+\mu) \\ \gamma_{\hat 1\hat 3\hat 1} = \frac{1}{\sqrt{2}}\,\mathrm{Re}(\sigma+\mu+\rho+\lambda) & \gamma_{\hat 1\hat 3\hat 2} = \frac{1}{\sqrt{2}}\,\mathrm{Im}(\rho+\lambda-\sigma-\mu) \\ \gamma_{\hat 2\hat 3\hat 2} = \frac{1}{\sqrt{2}}\,\mathrm{Re}(\mu+\rho-\sigma-\lambda) & \gamma_{\hat 2\hat 3\hat 1} = \frac{1}{\sqrt{2}}\,\mathrm{Im}(\lambda+\mu-\sigma-\rho) \end{array}\right\} \tag{A3}$$

In Eq. (17) the operators $\vec{P}$, $\vec{A}$, $\vec{L}$, $\vec{J}$, $\vec{K}$, and Φ appear. They are defined as

$$\vec{L} := \vec{x}\times\vec{p} \tag{A4}$$

$$(\vec{P})_i := x_0 p_i - x_i p_0 \tag{A5}$$

$$p_{\hat 0} := -ie_{\hat 0}{}^{i}\partial_i \tag{A6}$$

$$p_{\hat 1} := -ie_{\hat 1}{}^{i}\partial_i \tag{A7}$$

with

$$x_0 := e_{\hat 0}{}^{0} = \frac{1}{\sqrt{2}}(l^0+n^0) \qquad x_1 := e_{\hat 1}{}^{0} = \frac{1}{\sqrt{2}}(m^0+\bar{m}^0) \tag{A8}$$

$$\begin{aligned} x_2 &:= e_{\hat 2}{}^{0} = -\frac{i}{\sqrt{2}}(m^0-\bar{m}^0) \\ x_3 &:= e_{\hat 3}{}^{0} = \frac{1}{\sqrt{2}}(l^0-n^0). \end{aligned} \tag{A9}$$

The operator $\vec{A}$ can be expressed as

$$A_i = \sqrt{2}\sigma^0_{A\dot B}(x)\tilde{\sigma}^{A\dot B}_{\hat\alpha}T^{\hat\alpha}_i,$$

where

$$T^{\hat\alpha}_1 := \begin{pmatrix} \mathrm{Re}\,P_-(\pi-\alpha) \\ \mathrm{Re}\,P_-(\rho+\alpha) \\ -\mathrm{Im}\,P_+(\rho-\varepsilon) \\ -\mathrm{Re}\,P_+(\pi-\alpha) \end{pmatrix},$$

$$T^{\hat\alpha}_2 := \begin{pmatrix} \mathrm{Im}\,P_+(\alpha-\pi) \\ 2\,\mathrm{Im}\,P_+(\rho-\varepsilon) \\ \mathrm{Re}\,P_-(\rho+\varepsilon) \\ -\mathrm{Im}\,P_-(\alpha-\pi) \end{pmatrix},$$

$$T^{\hat\alpha}_3 := \begin{pmatrix} \mathrm{Re}\,P_+(\varepsilon-\rho) \\ \mathrm{Re}\,P_+(\pi-\alpha) \\ \mathrm{Im}\,P_-(\alpha-\pi) \\ \mathrm{Re}\,P_-(\rho+\varepsilon) \end{pmatrix}.$$

Here the coefficients $\sigma^0_{A\dot B}(x)$ are point dependent, whereas the $\tilde{\sigma}^0_{A\dot B}$ are the point-independent Infeld/van der Waerden symbols. They are given as

$$\tilde{\sigma}^{\ \mu}_{A\dot B}(x) = \begin{pmatrix} l^\mu & m^\mu \\ \bar{m}^\mu & n^\mu \end{pmatrix} \quad \text{and} \quad \tilde{\sigma}^{A\dot B}_{\hat\alpha} = \frac{1}{\sqrt{2}}(1,\vec{\sigma}),$$

where $\vec{\sigma}$ are the Pauli matrices. The operators $P_\pm$ are defined as

$$P_\pm := \tfrac{1}{2}(1\pm{}').$$

The operator $'$ was introduced in the Geroch-Held-Penrose formalism [21] and denotes the transformation $m^\mu \leftrightarrow \bar{m}^\mu$ or $l^\mu \leftrightarrow n^\mu$. Similarly, the operator $\vec{J}$ is defined as

$$J_i = \sqrt{2}\sigma^0_{A\dot{B}}(x)\tilde{\sigma}^{A\dot{B}}_{\hat{\alpha}} W^{\hat{\alpha}}_i,$$

where

$$W^{\hat{\alpha}}_1 = \begin{pmatrix} \mathrm{Im}\, P_-(\alpha - \pi) \\ -\mathrm{Im}\, P_-(\rho - \varepsilon) \\ -\mathrm{Im}\, P_+(\alpha - \pi) \\ \mathrm{Re}\, P_+(\varepsilon - \rho) \end{pmatrix},$$

$$W^{\hat{\alpha}}_2 = \begin{pmatrix} -\mathrm{Re}\, P_+(\pi - \alpha) \\ \mathrm{Re}\, P_+(\rho - \varepsilon) \\ -\mathrm{Im}\, P_-(\rho - \varepsilon) \\ \mathrm{Re}\, P_-(\pi - \alpha) \end{pmatrix},$$

$$W^{\hat{\alpha}}_3 = \begin{pmatrix} \mathrm{Im}\, P_+(\varepsilon - \rho) \\ \mathrm{Im}\, P_+(\alpha - \pi) \\ \mathrm{Re}\, P_-(\pi - \alpha) \\ -\mathrm{Im}\, P_-(\rho - \epsilon) \end{pmatrix}$$

and

$$K = \sqrt{2}\sigma^0_{A\dot{B}}(x)\tilde{\sigma}^{A\dot{B}}_{\hat{\alpha}} H^{\hat{\alpha}} \tag{A10}$$

with

$$H^{\hat{\alpha}} = \begin{pmatrix} -\mathrm{Im}\, P_-(\rho - \varepsilon) \\ \mathrm{Im}\, P_-(\alpha - \pi) \\ -\mathrm{Re}\, P_+(\pi - \alpha) \\ \mathrm{Im}\, P_+(\rho - \varepsilon) \end{pmatrix}. \tag{A11}$$

Φ is given as

$$\Phi = 2\sqrt{2}\sigma^0_{A\dot{B}}(x)\tilde{\sigma}^{A\dot{B}}_{\hat{\alpha}} R^{\hat{\alpha}}$$

$$R^{\hat{\alpha}} = \begin{pmatrix} -\mathrm{Re}\, P_-(\rho + \varepsilon) \\ \mathrm{Re}\, P_-(\alpha - \pi) \\ \mathrm{Im}\, P_+(\pi - \alpha) \\ \mathrm{Re}\, P_+(\rho - \varepsilon) \end{pmatrix}. \tag{A12}$$

APPENDIX B: RESTRICTIONS ON ROTATION COEFFICIENTS

We are also able to write down the conditions which are necessary for constructing relations between the Ricci rotation coefficients and the spin coefficients. They read

$$\text{Eq. (31)} \Leftrightarrow \gamma_{\hat{0}\hat{1}\hat{1}} + \gamma_{\hat{0}\hat{2}\hat{2}} = 0 \quad \text{and} \quad \gamma_{\hat{0}\hat{1}\hat{1}} - \gamma_{\hat{0}\hat{2}\hat{2}} = \sqrt{3}\,\mathrm{Re}(\lambda - \sigma) \quad \text{and} \quad \gamma_{\hat{3}\hat{0}\hat{3}} = 0, \tag{B1}$$

$$\text{Eq. (33)} \Leftrightarrow \gamma_{\hat{0}\hat{3}\hat{1}} - \gamma_{\hat{1}\hat{3}\hat{0}} - \gamma_{\hat{1}\hat{0}\hat{3}} = 0 \quad \text{and} \quad \gamma_{\hat{2}\hat{3}\hat{0}} - \gamma_{\hat{0}\hat{3}\hat{2}} - \gamma_{\hat{2}\hat{0}\hat{3}} = 0, \tag{B2}$$

$$\text{Eq. (32)} \Leftrightarrow \gamma_{\hat{0}\hat{1}\hat{2}} - \gamma_{\hat{0}\hat{2}\hat{1}} - \gamma_{\hat{2}\hat{1}\hat{0}} = 0. \tag{B3}$$

Further combinations with the expressions (A1)–(A3) can be given.

APPENDIX C: THE COMPARISON OF OBUKHOV'S AND OUR DIRAC HAMILTONIAN

Here we show that, for all orders in $1/mc^2$, the Hamiltonian constructed by Obukhov corresponds to the Hamiltonian Eq. (41). For this purpose, we prove the following Lemma.

Lemma 2.—If H is the Dirac Hamiltonian for a stationary metric, Q the corresponding supercharge and β the involution, then it holds the relation

$$Q^{2n} = P_+(H^{2n}) + i\gamma^{\hat{5}}P_-(H^{2n}) \quad \text{for } n \in \mathbb{N} \tag{C1}$$

if

$$P_\pm = \tfrac{1}{2}(1 \pm \gamma^{\hat{0}}).$$

The Lemma is proved by induction.

First, we show that for $n = 1$

$$Q^2 = P_+H^2 + i\gamma^{\hat{5}}P_-H^2 \tag{C2}$$

is valid. The Hamiltonian can be divided into its even and odd parts

$$H = \mathcal{E} + \mathcal{O}.$$

Its square is given by

$$H^2 = \mathcal{E}^2 + \mathcal{O}^2 + \{\mathcal{E}, \mathcal{O}\}$$

and, hence, one has

$$P_+H^2 + i\gamma^{\hat{5}}P_-H^2 = \mathcal{E}^2 + \mathcal{O}^2 + i\gamma^{\hat{5}}\{\mathcal{E}, \mathcal{O}\}. \tag{C3}$$

On the other hand, the supercharge is given by

$$Q = \mathcal{O} + i\gamma^{\hat{5}}\mathcal{E}$$

and its square reads

$$Q^2 = \mathcal{O}^2 + \mathcal{E}^2 + i\gamma^{\hat{5}}\{\mathcal{E}, \mathcal{O}\}. \tag{C4}$$

We compare Eqs. (C3) and (C4) and get

$$Q^2 = P_+H^2 + i\gamma^{\hat{5}}P_-H^2.$$

Second, we show that

$$Q^{2(n+1)} = P_+H^{2(n+1)} + i\gamma^{\hat{5}}P_-H^{2(n+1)} \tag{C5}$$

is valid, if the following condition is satisfied:

$$Q^{2n} = P_+H^{2n} + i\gamma^{\hat{5}}P_-H^{2n}. \tag{C6}$$

To this end, we rewrite the expression on the right-hand side of Eq. (C5) as follows:

$$\begin{aligned}P_+(H^{2(n+1)}) + i\gamma^{\hat 5}P_-H^{2(n+1)} &= P_+(H^{2n})P_+(H^2) + P_-(H^{2n})P_-(H^2) + i\gamma^{\hat 5}(P_+(H^{2n})P_-(H^2) + P_-(H^{2n})P_+(H^2)) \\ &= \underbrace{(P_+(H^{2n}) + i\gamma^{\hat 5}P_-(H^{2n}))}_{=Q^{2n}}P_+(H^2) + (P_-(H^{2n}) + i\gamma^{\hat 5}P_+(H^{2n}))P_-(H^2) \\ &= Q^{2n}P_+(H^2) + i\gamma^{\hat 5}(P_-(H^{2n}) - P_+(H^{2n}))P_-(H^2).\end{aligned}$$

If the relations

$$[\gamma^{\hat 5}, P_-(H^{2n})] = 0, \tag{C7}$$

$$\{\gamma^{\hat 5}, P_+(H^{2n})\} = 0 \tag{C8}$$

are valid then this expression is equal to the left-hand side of Eq. (C5):

$$\begin{aligned}&P_+(H^{2(n+1)}) + i\gamma^{\hat 5}P_-H^{2(n+1)} \\ &\quad = Q^{2n}P_+(H^2) + (P_-(H^{2n}) + P_+(H^{2n}))i\gamma^{\hat 5}P_-(H^2) \\ &\quad = Q^{2n}P_+(H^2) + Q^{2n}i\gamma^{\hat 5}P_-(H^2) = Q^{2n}Q^2 = Q^{2(n+1)}.\end{aligned}$$

Finally, we have to show that Eqs. (C7) and (C8) are valid. This can be seen as follows. Because $\gamma^{\hat 5}$ commutes with $\mathcal{O}$, it commutes also with $\mathcal{O}^n$, i.e.,

$$[\mathcal{O}^n, \gamma^{\hat 5}] = 0. \tag{C9}$$

On the other hand, $\gamma^{\hat 5}$ anticommutes with $\mathcal{E}$, hence

$$[\mathcal{E}^{2n}, \gamma^{\hat 5}] = 0.$$

The even part of H^{2n} only consists of combinations of $\mathcal{E}$ and $\mathcal{O}$ with even exponents. That is, terms like $\mathcal{O}^2\mathcal{E}^4$ or $\{\mathcal{E}, \mathcal{O}\}^2\mathcal{O}^4$, but no terms like $\mathcal{E}^3$ or $\mathcal{O}^4\mathcal{E}^5$ occur. Consequently, we have

$$[P_+(H^{2n}), \gamma^{\hat 5}] = 0. \tag{C10}$$

The odd part of H is given by

$$\begin{aligned}P_-(H^2) &= \{\mathcal{E}, \mathcal{O}\} \\ P_-(H^4) &= \{P_+(H^2), P_-(H^2)\} \\ &\vdots \\ P_-(H^{2n}) &= \{P_+(H^{2(n-1)}), P_-(H^{2(n-1)})\}.\end{aligned}$$

Because of (C9), $\gamma^{\hat 5}$ commutes with $P_+(H^{2n})$ and anticommutes with $P_-(H^2)$, $\gamma^{\hat 5}$ anticommutes with $P_-(H^4)$. If $\gamma^{\hat 5}$ anticommutes with $P_-(H^4)$, it also anticommutes with $P_-(H^6)$, etc. This leads successively to

$$\{P_-(H^{2n}), \gamma^{\hat 5}\} = 0$$

and such this Lemma is proved. ■

In the following we use Lemma 2 to show that the expansions of the Hamiltonians (41) and (48) are the same. For this purpose, we consider the square root of H^2 and its expansion

$$\sqrt{H^2} \approx Vmc^2 + \frac{1}{2}Y - \frac{1}{8}Y^2 + \frac{3}{48}Y^3 - \cdots. \tag{C11}$$

with the operator Y defined as

$$Y = \frac{1}{2mc^2}(V^{-1}\mathcal{O}^2 + \mathcal{O}^2V^{-1} + V^{-1}\{\mathcal{E}, \mathcal{O}\} + \{\mathcal{E}, \mathcal{O}\}V^{-1}) \tag{C12}$$

$$= \frac{1}{2mc^2}(V^{-1}(H^2 - m^2c^4V^2) + (H^2 - m^2c^4V^2)V^{-1}). \tag{C13}$$

For $\sqrt{Q^2}$, one has the expression

$$\sqrt{Q^2} \approx Vmc^2 + \frac{1}{2}X - \frac{1}{8}X^2 + \frac{3}{48}X^3 - \cdots$$

with

$$X = \frac{1}{2mc^2}(V^{-1}(Q^2 - m^2c^4V^2) + (Q^2 - m^2c^4V^2)V^{-1}).$$

By means of Lemma 2, one obtains from the latter relation

$$\begin{aligned}X^n &= \frac{1}{2mc^2}(V^{-1}(Q^{2n} - (m^2c^4V^2)^n) + (Q^{2n} - (m^2c^4V^2)^n)V^{-1}) \\ &= \frac{1}{2mc^2}(V^{-1}(P_+(H^{2n}) + i\gamma^{\hat 5}P_-(H^{2n}) - (m^2c^4V^2)^n) + (P_+(H^{2n}) + i\gamma^{\hat 5}P_-(H^{2n}) - (m^2c^4V^2)^n)V^{-1}) \\ &= \frac{1}{2mc^2}(V^{-1}P_+(H^{2n}) + P_+(H^{2n})V^{-1} - (m^2c^4V^2)^nV^{-1} + i\gamma^{\hat 5}(V^{-1}P_-(H^{2n}) + P_-(H^{2n})V^{-1})) \\ &= P_+(Y^n) + i\gamma^{\hat 5}P_-(Y^n)\end{aligned}$$

and hence

$$\sqrt{Q^2} \approx Vmc^2 + \frac{1}{2}X - \frac{1}{8}X^2 + \frac{3}{48}X^3 - \cdots$$
$$= Vmc^2 + \frac{1}{2}(P_+(Y) + i\gamma^{\hat{5}}P_-(Y)) - \frac{1}{8}(P_+(Y^2) + i\gamma^{\hat{5}}P_-(Y^2)) + \frac{3}{48}(P_+(Y^3) + i\gamma^{\hat{5}}P_-(Y^3)) - \cdots.$$

The terms can be rearranged as follows:

$$\sqrt{Q^2} \approx Vmc^2 + \frac{1}{2}P_+(Y) - \frac{1}{8}P_+(Y^2) + \frac{3}{48}P_+(Y^3) - \cdots + i\gamma^{\hat{5}}\Big(\frac{1}{2}P_-(Y) - \frac{1}{8}P_-(Y^2) + \frac{3}{48}P_-(Y^3)\Big) - \cdots.$$

Thus, for each order N in the expansion series we obtain

$$\sqrt{Q^2} \overset{O(N)}{=} P_+(\sqrt{H^2}) + i\gamma^{\hat{5}}P_-(\sqrt{H^2}). \quad \text{(C14)}$$

Here $O(N)$ means that the terms are equal up to order N. In this sense these Hamiltonians are equal:

$$H_Q = U_Q^{\mathrm{FW}} H_S U_Q^{\dagger\mathrm{FW}} = \beta\sqrt{Q_S^2} \overset{O(N)}{=} \beta P_+(\sqrt{H_S^2}) + \underbrace{i\beta\gamma^{\hat{5}}}_{J} P_-(\sqrt{H_S^2}) = H_J.$$

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