Formalism for testing theories of gravity using lensing by compact objects. II. Probing post-post-Newtonian metrics

Charles R. Keeton[*]

Department of Physics & Astronomy, Rutgers University, 136 Frelinghuysen Road, Piscataway, New Jersey 08854, USA

A. O. Petters[†]

Departments of Mathematics and Physics, Duke University, Science Drive, Durham, North Carolina 27708-0320, USA

We study gravitational lensing by compact objects in gravity theories that can be written in a post-post-Newtonian (PPN) framework: i.e., the metric is static and spherically symmetric, and can be written as a Taylor series in $\mathsf{m}_\bullet/r$, where $\mathsf{m}_\bullet$ is the gravitational radius of the compact object. Working invariantly, we compute corrections to standard weak-deflection lensing observables at first and second order in the perturbation parameter $\varepsilon = \vartheta_\bullet/\vartheta_E$, where $\vartheta_\bullet$ is the angular gravitational radius and ϑ_E is the angular Einstein ring radius of the lens. We show that the first-order corrections to the total magnification and centroid position vanish universally for gravity theories that can be written in the PPN framework. This arises from some surprising, fundamental relations among the lensing observables in PPN gravity models. We derive these relations for the image positions, magnifications, and time delays. A deep consequence is that any violation of the universal relations would signal the need for a gravity model outside the PPN framework (provided that some basic assumptions hold). In practical terms, the relations will guide observational programs to test general relativity, modified gravity theories, and possibly the cosmic censorship conjecture. We use the new relations to identify lensing observables that are accessible to current or near-future technology, and to find combinations of observables that are most useful for probing the spacetime metric. We give explicit applications to the galactic black hole, microlensing, and the binary pulsar J0737-3039.

I. INTRODUCTION

Gravitational lensing is now a central field of astronomy with wide-ranging applications that relate to extra-solar planets, dark matter substructures, and cosmological parameters (including dark energy) [1–3]. Theoretical studies have also explored lensing by black holes within the context of general relativity [4–15], braneworld gravity [16–19], and string theory [20,21]. Most black hole lensing studies have focused on "relativistic" images corresponding to light rays that loop around a black hole, which probe gravity in the strong-deflection limit, but are extremely difficult to detect observationally [8,13]. We are proposing and evaluating new possibilities for using gravitational lensing by compact objects (including black holes) to test various theories of gravity using current or near-future technology.

In Paper I of this series [21], we introduced an analytic framework for studying gravitational lensing by a compact deflector with mass $M_\bullet$, in which the lensing scenario satisfies three basic assumptions:

[A1] The gravitational lens is compact, static, and spherically symmetric, with an asymptotically flat spacetime geometry sufficiently far from the lens [22]. The spacetime is vacuum outside the lens and flat in the absence of the lens.

[A2] The observer and source lie in the asymptotically flat regime of the spacetime.

[A3] The light ray's distance of closest approach r_0 and impact parameter b both lie well outside the gravitational radius $\mathsf{m}_\bullet = GM_\bullet/c^2$, namely, $\mathsf{m}_\bullet/r_0 \ll 1$ and $\mathsf{m}_\bullet/b \ll 1$. The bending angle can then be expressed as a series expansion in $\mathsf{m}_\bullet/b$, as follows:

$$\hat{\alpha}(b) = A_1\left(\frac{\mathsf{m}_\bullet}{b}\right) + A_2\left(\frac{\mathsf{m}_\bullet}{b}\right)^2 + A_3\left(\frac{\mathsf{m}_\bullet}{b}\right)^3 + \mathcal{O}\left(\frac{\mathsf{m}_\bullet}{b}\right)^4. \tag{1}$$

The coefficients A_i are independent of $\mathsf{m}_\bullet/b$, but may include other fixed parameters of the spacetime. Since b and $\mathsf{m}_\bullet$ are invariants of the light ray, (1) is independent of coordinates. Note that the subscript of A_i conveniently indicates that the component is affiliated with a term of order i in $\mathsf{m}_\bullet/b$.

We applied the lensing framework to gravity theories that can be written in a post-post-Newtonian (PPN) expansion, meaning that the metric can be written as a Taylor series in $\mathsf{m}_\bullet/r$. Working invariantly, we computed the observable properties of the primary (positive-parity) and secondary (negative-parity) lensed images. Assuming that

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[*]Email address: keeton@physics.rutgers.edu
[†]Email address: petters@math.duke.edu

[A3] holds, we wrote the lensing observables as series expansions in $\varepsilon = \vartheta_\bullet/\vartheta_E$, where $\vartheta_\bullet$ is the angular gravitational radius and ϑ_E is the angular Einstein ring radius. For gravity theories that agree with general relativity in the weak-deflection limit [$A_1 = 4$ in Eq. (1)], the zeroth-order terms in the ε expansions give the familiar results for lensing by a point mass in the weak-deflection limit of general relativity. The higher-order terms give corrections to the lensing observables, which differ for different gravity theories. We studied general third-order PPN models, which allowed us to compute the weak-deflection lensing results plus the first two correction terms (order ε and ε^2).

In Paper I we found the interesting result that the first-order corrections to two lensing observables—the total magnification and the magnification-weighted centroid position—vanish in PPN gravity models that agree with general relativity in the weak-deflection limit (i.e., $A_1 = 4$). More generally, we found that these corrections depend on $A_1 - 4$, suggesting that they nearly vanish in gravity theories that agree only approximately with general relativity in the weak-deflection regime (i.e., $A_1 \approx 4$). Note that existing observations constrain this parameter to $A_1 = 3.999\,66 \pm 0.000\,90$ [23].

Working more generally than in Paper I, we have now discovered the surprising result that the first-order corrections to the total magnification and centroid in fact vanish exactly in all PPN models. This depends on precise cancellations that are striking because the PPN framework covers quite a broad range of gravity theories. Understanding why the cancellations occur has led us to identify some new fundamental relations between lensing observables in the PPN framework. Some of the relations are universal for PPN models, in the sense that they hold for all values of the invariant PPN parameters of the light bending angle.

These new lensing relations will play key roles in planning observing missions to test theories of gravity. First, the relations allow us to determine which observables are most accessible to current or near-future instrumentation. We give explicit applications to the galactic black hole, galactic microlensing, and the binary pulsar J0737-3039. Second, the lensing relations help us identify combinations of lensing observables that are most useful for probing the spacetime metric by constraining invariant PPN parameters. One of the measurable parameters is connected to the existence of naked singularities in certain gravity models (see Paper I), so we may have an observational test of the cosmic censorship conjecture [24]. Third, the universal relations provide a powerful means to test the entire PPN framework. Any violation of relations that are universal in the PPN framework would suggest that a fundamentally different theory of gravity is needed (provided that assumptions [A1]–[A3] hold).

In this paper we present the new lensing relations and use them to assess prospects for using lensing observations to test PPN gravity theories. Section II reviews the third-order PPN lensing framework. Section III derives the new relations between the image positions, magnifications, and time delays, and discusses their conceptual implications. Section IV considers applications of the relations to various astrophysical settings.

II. LENSING IN THE PPN FRAMEWORK

In this section we review our results for lensing in the PPN framework. See Paper I for the complete analysis.

Consider a compact body of mass $M_\bullet$, perhaps a black hole or neutron star, that is described by a geometric theory of gravity. By assumptions [A1]–[A3], it suffices to analyze an equatorial metric of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\varphi^2. \quad (2)$$

We study metrics whose coefficients can be expressed in third-order PPN expansions,

$$A(r) = 1 + 2a_1\left(\frac{\phi}{c^2}\right) + 2a_2\left(\frac{\phi}{c^2}\right)^2 + 2a_3\left(\frac{\phi}{c^2}\right)^3 + \cdots,$$
$$(3)$$

$$B(r) = 1 - 2b_1\left(\frac{\phi}{c^2}\right) + 4b_2\left(\frac{\phi}{c^2}\right)^2 - 8b_3\left(\frac{\phi}{c^2}\right)^3 + \cdots,$$
$$(4)$$

where ϕ is the three-dimensional Newtonian potential with

$$\frac{\phi}{c^2} = -\frac{\mathsf{m}_\bullet}{r}. \quad (5)$$

Section III C of Paper I derives the light bending angle for this metric. The invariant expression for the bending angle takes the form of (1) with coefficients

$$A_1 = 2(a_1 + b_1), \quad (6)$$

$$A_2 = \left(2a_1^2 - a_2 + a_1 b_1 - \frac{b_1^2}{4} + b_2\right)\pi, \quad (7)$$

$$A_3 = \frac{2}{3}[35a_1^3 + 15a_1^2 b_1 - 3a_1(10a_2 + b_1^2 - 4b_2) + 6a_3 + b_1^3 - 6a_2 b_1 - 4b_1 b_2 + 8b_3]. \quad (8)$$

The coordinate independent quantities A_i will be called the invariant PPN parameters of the light bending angle. For reference, the Schwarzschild metric in general relativity has $a_1 = b_1 = b_2 = b_3 = 1$ and $a_2 = a_3 = 0$, and hence $A_1 = 4$, $A_2 = 15\pi/4$, and $A_3 = 128/3$.

Figure 1 displays the gravitational lensing scenario. Elementary trigonometry establishes the relationship (see [8])

$$\tan\mathcal{B} = \tan\vartheta - D(\tan\vartheta + \tan(\hat{\alpha} - \vartheta)), \quad (9)$$

where $D = d_{LS}/d_S$. This equation agrees well with the full relativistic formalism for light propagation [9], so we take
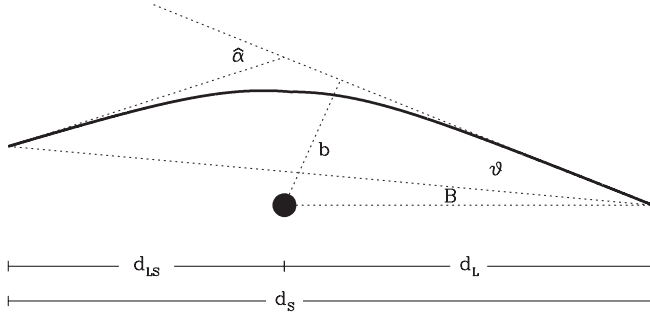
FIG. 1. Schematic diagram of the lensing geometry. Standard quantities are identified: $\mathcal{B}$ is the angular position of the unlensed source; ϑ is the angular position of an image; $\hat{\alpha}$ is the bending angle; and d_L, d_S, and d_{LS} are angular diameter distances between the observer, lens, and source. The impact parameter b is an invariant of the light ray and is related to the angular image position by $\vartheta = \sin^{-1}(b/d_L)$.

it as the general form of the lens equation. The equation has two primary (weak-deflection) solutions [25]: one corresponding to an image on the same side of the lens as the source; and one on the opposite side. By convention, angles describing image positions are taken to be positive. This forces the source's angular position to have different signs: $\mathcal{B}$ is positive when we are studying an image on the same side of the lens as the source (as depicted in Fig. 1); while $\mathcal{B}$ is negative when we are studying an image on the opposite of the lens from the source.

Following Paper I, we convert to scaled variables

$$\beta = \frac{\mathcal{B}}{\vartheta_E}, \qquad \theta = \frac{\vartheta}{\vartheta_E}, \qquad \hat{\tau} = \frac{\tau}{\tau_E}, \qquad \varepsilon = \frac{\vartheta_\bullet}{\vartheta_E} = \frac{\vartheta_E}{4D}, \tag{10}$$

where $\vartheta_\bullet = \tan^{-1}(m_\bullet/d_L)$ is the angle subtended by the gravitational radius, and τ is the lensing time delay. The natural angular scale is given by the angular Einstein ring radius,

$$\vartheta_E = \sqrt{\frac{4GM_\bullet d_{LS}}{c^2 d_L d_S}}, \tag{11}$$

while the natural time scale is

$$\tau_E \equiv \frac{d_L d_S}{c d_{LS}} \vartheta_E^2 = 4\frac{m_\bullet}{c}. \tag{12}$$

We then assume that solutions of the lens equation can be written as a series of the form

$$\theta = \theta_0 + \theta_1 \varepsilon + \theta_2 \varepsilon^2 + \mathcal{O}(\varepsilon)^3, \tag{13}$$

where θ_0 represents the image position in the weak-deflection limit, while θ_1 and θ_2 give the first- and second-order correction terms. With these substitutions, the lens equation becomes

$$0 = D\left[-4\beta + 4\theta_0 - \frac{A_1}{\theta_0}\right]\varepsilon + \frac{D}{\theta_0^2}[-A_2 + (A_1 + 4\theta_0^2)\theta_1]\varepsilon^2$$

$$+ \frac{D}{3\theta_0^3}[-A_1^3 - 3A_3 + 12A_1^2 D\theta_0^2 - A_1(56D^2\theta_0^4 + 3\theta_1^2$$

$$- 3\theta_0\theta_2) + 64D^2\theta_0^3(\theta_0^3 - \beta^3) + 6A_2\theta_1 + 12\theta_0^3\theta_2]\varepsilon^3$$

$$+ \mathcal{O}(\varepsilon)^4. \tag{14}$$

We solve for θ_0, θ_1, and θ_2 by finding the values that make each term of the lens equation vanish. From the vanishing of the first term we obtain the relation

$$\beta = \theta_0 - \frac{A_1}{4\theta_0}. \tag{15}$$

This is the generalization of the familiar weak-deflection lens equation for a point mass to $A_1 \neq 4$. Its solution is

$$\theta_0 = \tfrac{1}{2}(\sqrt{A_1 + \beta^2} + \beta). \tag{16}$$

Then requiring that the second and third terms in (14) vanish yields the correction terms

$$\theta_1 = \frac{A_2}{A_1 + 4\theta_0^2}, \tag{17}$$

$$\theta_2 = \frac{1}{3\theta_0(A_1 + 4\theta_0^2)^3}[A_1(-3A_2^2 + 3A_1A_3 - A_1^4(D^2 - 1))$$

$$+ 4(-6A_2^2 + 6A_1A_3 + A_1^4(D - 2)(D - 1))\theta_0^2$$

$$+ 8(6A_3 + A_1^3(2 + D(11D - 12)))\theta_0^4$$

$$+ 64A_1^2 D(4D - 3)\theta_0^6 + 128A_1 D^2\theta_0^8]. \tag{18}$$

Notice that the $\mathcal{O}(\varepsilon)^2$ term in the lens equation (14) does not explicitly involve β; so θ_1 depends on the source position only implicitly through θ_0. By contrast, the $\mathcal{O}(\varepsilon)^3$ term in the lens equation does involve β. In writing the expression for θ_2, we have found it convenient to substitute for β using (15). Then the two correction terms θ_1 and θ_2 are both written only in terms of θ_0. The source position dependence could be made explicit by substituting for θ_0 using (16).

The signed magnification μ of a lensed image at angular position ϑ is given by

$$\mu(\vartheta) = \left[\frac{\sin\mathcal{B}(\vartheta)}{\sin\vartheta}\frac{d\mathcal{B}(\vartheta)}{d\vartheta}\right]^{-1}. \tag{19}$$

After taking the derivative, we change to our scaled angular variables from (10) and (13), and substitute for θ_1 and θ_2 using (17) and (18). This yields a series expansion for the magnification,

$$\mu = \mu_0 + \mu_1 \varepsilon + \mu_2 \varepsilon^2 + \mathcal{O}(\varepsilon)^3, \tag{20}$$

where [see Eqs. (77)–(79) of Paper I]

$$\mu_0 = \frac{16\theta_0^4}{16\theta_0^4 - A_1^2}, \tag{21}$$

$$\mu_1 = -\frac{16A_2\theta_0^3}{(A_1 + 4\theta_0^2)^3}, \tag{22}$$

$$
\mu_2 = \frac{8\theta_0^2}{3(A_1 - 4\theta_0^2)(A_1 + 4\theta_0^2)^5} \\
\times [-A_1^6 D^2 + 8A_1^2(6A_3 + A_1^3(2 + 6D - 9D^2))\theta_0^2 \\
- 32(18A_2^2 - 12A_1A_3 + A_1^4(D(17D - 12) - 4))\theta_0^4 \\
+ 128(6A_3 + A_1^3(2 + 6D - 9D^2))\theta_0^6 - 256A^2D^2\theta_0^8]. \tag{23}
$$

We have again substituted for β using (15).

The time delay (relative to an undeflected light ray) was derived in Sec. V C of Paper I. It can be written as a series of the form

$$\hat{\tau} = \hat{\tau}_0 + \hat{\tau}_1\varepsilon + \mathcal{O}(\varepsilon)^2, \tag{24}$$

where

$$\hat{\tau}_0 = \frac{1}{2}\left[a_1 + \beta^2 - \theta_0^2 - \frac{A_1}{4}\ln\left(\frac{d_L\theta_0^2\vartheta_E^2}{4d_{LS}}\right)\right], \tag{25}$$

$$\hat{\tau}_1 = \frac{A_2}{4\theta_0}. \tag{26}$$

It is possible to derive the second-order correction to the time delay, but we have found that to be less important than the second-order corrections to the position and magnification.

Remark.—In Paper I, we gave the relation between β and θ_0 only for the case $A_1 = 4$. As a result, there are minor changes in the expressions for θ_2 and μ_2 in Paper I when $A_1 \neq 4$. Equations (16), (18), and (23) give the correct expressions for the general case in which A_1 can take on any value. Note, however, that observationally $A_1 = 3.99966 \pm 0.00090$ [23].

III. NEW RELATIONS BETWEEN LENSING OBSERVABLES

In this section we uncover several new relations between the perturbation coefficients of the fundamental lensing observables, namely, the positions, magnifications, and time delays of the two primary images. Some of the relations are universal among PPN models in the sense that they hold for any values of the PPN parameters. Others hold for all source positions.

A. Position relations

Starting from (16) and recalling that the positive- and negative-parity images correspond to $\beta > 0$ and $\beta < 0$, respectively, we can write the positions of the two images in the weak-deflection limit as

$$\theta_0^\pm = \tfrac{1}{2}(\sqrt{A_1 + \beta^2} \pm |\beta|). \tag{27}$$

We can immediately identify two interesting relations:

$$\theta_0^+ - \theta_0^- = |\beta|, \tag{28}$$

$$\theta_0^+\theta_0^- = \frac{A_1}{4}. \tag{29}$$

Equation (28) represents our first universal relation. This relation is familiar from standard weak-deflection lensing in general relativity (see [2]), but now we see that it holds for all PPN models, regardless of whether they agree with general relativity in the weak-deflection limit. The second equation is our first example of a relation that is independent of the source position.

Next, combining (17), (18), and (29) yields

$$\theta_1^+ + \theta_1^- = \frac{A_2}{A_1}, \tag{30}$$

$$\theta_1^+ - \theta_1^- = -\frac{A_2|\beta|}{A_1\sqrt{A_1 + \beta^2}}, \tag{31}$$

$$\theta_2^+ - \theta_2^- = \frac{2|\beta|}{3A_1^3}[6A_2^2 - 6A_1A_3 - A_1^4(2 - 3D^2)]. \tag{32}$$

The first-order relation for $\theta_1^+ + \theta_1^-$ is another source-independent relation, while the one for $\theta_1^+ - \theta_1^-$ is source dependent. Both first-order relations depend on the sign of A_2. The second-order relation is independent of the sign of A_2. The dependence on the sign of A_2 is interesting because in Paper I we found that in certain gravity theories this sign is connected to the occurrence of naked singularities.

B. Magnification relations

We take the magnification terms (21)–(23) and write θ_0 in terms of β using (27). This yields

$$\mu_0^\pm = \frac{1}{2} \pm \frac{A_1 + 2\beta^2}{4|\beta|\sqrt{A_1 + \beta^2}}, \tag{33}$$

$$\mu_1^\pm = -\frac{A_2}{4(A_1 + \beta^2)^{3/2}}, \tag{34}$$

$$\mu_2^\pm = \pm\frac{9A_2^2 + 2(A_1 + \beta^2)\{-6A_3 + 2A_1^2D^2\beta^2 + A_1^3[-2 + 3D(-2 + 3D)]\}}{24|\beta|(A_1 + \beta^2)^{5/2}}. \tag{35}$$

We can then identify three universal magnification relations:

$$\mu_0^+ + \mu_0^- = 1, \qquad \mu_1^+ - \mu_1^- = 0, \qquad \mu_2^+ + \mu_2^- = 0. \tag{36}$$

Recall that the sign of the magnification indicates the parity of an image, so $|\mu|$ actually gives the image brightness. The zeroth-order relation can be rewritten as $|\mu_0^+| - |\mu_0^-| = 1$. In other words, in the PPN framework the difference between the fluxes of the images (at zeroth order) always equals the flux of the source in the absence of lensing.

The first-order magnification relation arises because the first-order correction term μ_1 is the same for both images, but the actual magnifications have opposite signs. If A_2 is positive, then $\mu_1^\pm < 0$. This makes the positive-parity image less positive, or fainter; but it makes the negative-parity image more negative, or brighter. (If A_2 is negative, the opposite occurs.) Consequently, the magnifications of the positive- and negative-parity images are shifted by the same amount but in the opposite sense.

In addition, combining (16) and (17) with (22) and (23) yields another universal relation,

$$(\mu_0^+ \theta_1^+ + \mu_0^- \theta_1^-) + (\mu_1^+ \theta_0^+ + \mu_1^- \theta_0^-) = 0. \tag{37}$$

This relation will be useful when we analyze the centroid position below.

C. Total magnification and centroid

If the two images are not separately resolved, the main observables are the total magnification and magnification-weighted centroid position (e.g., [26]). Applying the universal magnification relations (36) to the total magnification yields

$$
\begin{aligned}
\mu_{\text{tot}} &= |\mu^+| + |\mu^-|, \\
&= (\mu_0^+ - \mu_0^-) + (\mu_1^+ - \mu_1^-)\varepsilon + (\mu_2^+ - \mu_2^-)\varepsilon^2 \\
&\quad + \mathcal{O}(\varepsilon)^3, \\
&= (2\mu_0^+ - 1) + 2\mu_2^+ \varepsilon^2 + \mathcal{O}(\varepsilon)^3.
\end{aligned}\tag{38}
$$

There is no first-order correction to the total magnification for all gravity theories within our PPN framework. This result depends on a precise cancellation between μ_1^+ and μ_1^-, so it is striking that it is universal for PPN models. An important implication is that the total magnification would have to be measured much more precisely than some other observables to find corrections to the weak-deflection limit (see Sec. IV for more discussion).

The magnification-weighted centroid position is defined by

$$\Theta_{\text{cent}} = \frac{\theta^+ |\mu^+| - \theta^- |\mu^-|}{|\mu^+| + |\mu^-|} = \frac{\theta^+ \mu^+ + \theta^- \mu^-}{\mu_+ - \mu_-}. \tag{39}$$

Writing $\theta^\pm$ and $\mu^\pm$ in terms of their series expansions, and using the magnification relations (36), yields

$$\Theta_{\text{cent}} = \Theta_0 + \Theta_1 \varepsilon + \Theta_2 \varepsilon^2 + \mathcal{O}(\varepsilon)^3, \tag{40}$$

where

$$\Theta_0 = \frac{\theta_0^+ \mu_0^+ + \theta_0^- \mu_0^-}{\mu_0^+ - \mu_0^-}, \tag{41}$$

$$\Theta_1 = \frac{\mu_0^- \theta_1^- + \mu_1^+ \theta_0^+ + \mu_1^- \theta_0^- + \mu_0^+ \theta_1^+}{\mu_0^+ - \mu_0^-}, \tag{42}$$

$$\Theta_2 = \frac{(\theta_0^+ + \theta_0^-)(\mu_0^+ \mu_2^- - \mu_0^- \mu_2^+) + (\mu_0^+ - \mu_0^-)[\mu_1^+(\theta_1^+ + \theta_1^-) + \mu_0^+ \theta_2^+ + \mu_0^- \theta_2^-]}{(\mu_0^+ - \mu_0^-)^2}. \tag{43}$$

Writing (41) and (43) in terms of β yields the zeroth-order centroid position and the second-order correction to be

$$\Theta_0 = |\beta| \frac{3A_1 + 4\beta^2}{2A_1 + 4\beta^2}, \tag{44}$$

$$\Theta_2 = -\frac{|\beta|\{9A_2^2 - 2(A_1 + \beta^2)[6A_3 + 8A_1 D^2 \beta^4 - 6A_1^2 D\beta^2(2 - 3D) + A_1^3(2 - D^2)]\}}{6(A_1 + \beta^2)(A_1 + 2\beta^2)^2}. \tag{45}$$

These results are neither universal nor source independent, but are useful generalizations of previous results to the case $A_1 \neq 4$.

The numerator of the first-order correction (42) is identical to the left-hand side of (37), yielding

$$\Theta_1 = 0. \tag{46}$$

As with the total magnification, the first-order correction to the centroid vanishes universally in the PPN framework, which also means that centroid corrections beyond zeroth order will be more challenging to observe directly.

Remark.—For the special case of the Schwarzschild metric in general relativity, Ebina *et al.* [27] and Lewis and Wang [28] found that the first-order corrections to the total magnification and centroid vanish. We have now generalized that result to *all* PPN models.

D. Differential time delay

In many cases the only observable time delay is the differential delay between the positive- and negative-parity images,

$$\Delta\hat{\tau} = \hat{\tau}^- - \hat{\tau}^+, \qquad (47)$$

which we write as a series of the form

$$\Delta\hat{\tau} = \Delta\hat{\tau}_0 + \Delta\hat{\tau}_1\varepsilon + \mathcal{O}(\varepsilon)^2. \qquad (48)$$

Starting from (24)–(26) and using (27) to write θ_0 in terms of β, we obtain

$$\Delta\hat{\tau}_0 = \frac{1}{2}|\beta|\sqrt{A_1 + \beta^2} + \frac{A_1}{4}\ln\left(\frac{\sqrt{A_1 + \beta^2} + \beta}{\sqrt{A_1 + \beta^2} - \beta}\right), \qquad (49)$$

$$\Delta\hat{\tau}_1 = \frac{A_2}{A_1}|\beta|. \qquad (50)$$

These relations are not universal in the sense that each depends on PPN parameters as well as the source position, but they are still useful for calculations.

IV. IMPLICATIONS FOR OBSERVATIONS

In this section we employ the new lensing relations to assess prospects for using lensing observations to test theories of gravity. We consider three likely astrophysical scenarios: lensing by the galactic black hole; conventional galactic microlensing; and lensing in a binary pulsar system.

A. Review of observables

To facilitate this discussion, let us review the possible lensing observables, focusing on the weak-deflection limits plus the first-order corrections which should be measurable now or in the near future. To connect with realistic observations, we revert from our convenient mathematical variables (θ, β, μ, $\hat{\tau}$) to true observable quantities (ϑ, $\mathcal{B}$, F, τ).

The traditional lensing observables are the positions, fluxes, and time delays of the images. The fluxes are related to the magnifications via the source flux: $F_i = |\mu_i|F_{\text{src}}$. (As an observable quantity, the flux is positive definite.) In principle, one could simply take the measured values, adopt the formulas derived in this paper as a model, and fit for the unknown parameters. However, with the relations found in Sec. III as a guide, we believe it is instructive to make certain combinations of observables as follows:

$$\vartheta^+ + \vartheta^- = \sqrt{A_1\vartheta_E^2 + \mathcal{B}^2} + \frac{A_2\vartheta_E}{A_1}\varepsilon + \mathcal{O}(\varepsilon)^2, \qquad (51)$$

$$\vartheta^+ - \vartheta^- = |\mathcal{B}| - \frac{A_2\vartheta_E|\mathcal{B}|}{A_1\sqrt{A_1\vartheta_E^2 + \mathcal{B}^2}}\varepsilon + \mathcal{O}(\varepsilon)^2, \qquad (52)$$

$$F_{\text{tot}} \equiv F^+ + F^- = F_{\text{src}}\frac{A_1\vartheta_E^2 + 2\mathcal{B}^2}{2|\mathcal{B}|\sqrt{A_1\vartheta_E^2 + \mathcal{B}^2}} + \mathcal{O}(\varepsilon)^2, \qquad (53)$$

$$\Delta F \equiv F^+ - F^- = F_{\text{src}} - F_{\text{src}}\frac{A_2\vartheta_E^3}{2(A_1\vartheta_E^2 + \mathcal{B}^2)^{3/2}}\varepsilon + \mathcal{O}(\varepsilon)^2, \qquad (54)$$

$$\vartheta_{\text{cent}} \equiv \frac{\vartheta^+ F^+ - \vartheta^- F^-}{F_{\text{tot}}} = |\mathcal{B}|\frac{3A_1\vartheta_E^2 + 4\mathcal{B}^2}{2A_1\vartheta_E^2 + 4\mathcal{B}^2} + \mathcal{O}(\varepsilon)^2, \qquad (55)$$

$$\begin{aligned}
\Delta\tau = \frac{d_L d_S}{c d_{LS}}\Bigg[&\frac{1}{2}|\mathcal{B}|\sqrt{A_1\vartheta_E^2 + \mathcal{B}^2} + \frac{A_1\vartheta_E^2}{4} \\
&\times \ln\left(\frac{\sqrt{A_1\vartheta_E^2 + \mathcal{B}^2} + |\mathcal{B}|}{\sqrt{A_1\vartheta_E^2 + \mathcal{B}^2} - |\mathcal{B}|}\right) + \frac{A_2\vartheta_E}{A_1}|\mathcal{B}|\varepsilon \\
&+ \mathcal{O}(\varepsilon)^2\Bigg].
\end{aligned} \qquad (56)$$

These equations summarize our results for PPN lensing, and play the key role in understanding how lensing can be used to test PPN theories of gravity.

B. The galactic black hole

The center of our Galaxy is believed to host a supermassive black hole with a mass of $M_\bullet = (3.6 \pm 0.2) \times 10^6 M_\odot$ [29]; the distance to the lens is $d_L = 7.9 \pm 0.4$ kpc [30]. Adopting the nominal values and neglecting the small uncertainties, we find the black hole's gravitational radius to be $\mathsf{m}_\bullet = 5.3 \times 10^9$ m $= 1.7 \times 10^{-7}$ pc, which corresponds to an angle of $\vartheta_\bullet = 4.5 \times 10^{-6}$ arc sec. The corresponding lensing time scale is $\tau_E = 71$ s.

We consider a source that is orbiting the black hole at a distance $d_{LS} \ll d_L$ (so $d_S \approx d_L$). In the following quantitative estimates we let $d_{LS}^* = d_{LS}/(1 \text{ pc})$ to simplify the notation. If the orbit is close to edge on, part of it will lie close enough to the black hole (in projection) that the source can be significantly lensed. The angular Einstein radius is then $\vartheta_E = 0.022(d_{LS}^*)^{1/2}$ arc sec, and our dimensionless perturbation parameter is $\varepsilon = 2.1 \times 10^{-4} \times (d_{LS}^*)^{-1/2}$.

These numbers indicate that the two lensed images could be resolved with existing technology. From (51), the angular separation of the two images is at least $\sqrt{A_1}\vartheta_E = 2\vartheta_E = 0.044(d_{LS}^*)^{1/2}$ arc sec (using $A_1 = 3.99966 \pm 0.00090$ [23]). At optical wavelengths, the

Hubble Space Telescope has a resolution of about 0.05 arc - sec, while the CHARA interferometer [31] can obtain a resolution of better than 10^{-3} arc sec. At radio wavelengths, interferometry can also achieve a resolution of $\sim 10^{-3}$ arc sec. The position uncertainties are much smaller than the resolution element; for example, radio interferometry observations of known lenses have yielded position uncertainties at the level of 10^{-6} arc sec (e.g., [32]). With the conservative assumption that the images must be separated by 10^{-2} arc sec to be well resolved, we can still consider sources as close to the black hole as $d_{LS} = 0.05$ pc, and we can expect to measure the image positions with μarc sec precision.

A single observation could therefore be expected to yield the position and flux of each of the two images. Using Eqs. (51)–(54), the four numbers (ϑ^+, ϑ^-, F^+, F^-) would allow us to solve for four unknowns, which we could take to be d_{LS}, $\mathcal{B}$, F_{src}, and A_2. (Here we imagine taking the values of $M_\bullet$, d_L, and A_1 as given above.) Thus, in principle a single observation of an appropriate source could test gravity theories by measuring the invariant PPN bending-angle parameter A_2. The position shifts that depend on A_2 are of order $\vartheta_E \varepsilon = \vartheta_\bullet = 4.5 \times 10^{-6}$ arc sec, so existing technology has sufficient precision to measure them. Note that A_2 is connected to the occurrence of naked singularities for certain gravity theories (see Sec. III D of Paper I), so lensing could provide an observational test of the cosmic censorship conjecture [24]. This analysis indicates that the main challenge for using lensing by the galactic black hole to test gravity theories is just to find a source that is lensed.

We could do even better with repeated observations, watching as the source moves and causes the images to change. We may estimate the time scale for such variations as the time it takes the source to move one linear Einstein radius, $d_S \vartheta_E$. For a circular orbit, this is $T_E = 6.5 d_{LS}^*$ yr (independent of the black hole mass, since the Einstein radius and the Keplerian orbital velocity both scale as $M_\bullet^{1/2}$). Keplerian orbital motion can be described with just five parameters: semimajor axis, period, eccentricity, inclination, and longitude of periastron. Repeated observations would thus allow us to determine a good model for $\mathcal{B}$ as a function of time. The universal relations then tell us that the quantity $\vartheta^+ - \vartheta^-$ would be the most interesting combination of observables. From (52), if $\vartheta^+ - \vartheta^-$ has any dependence on $\mathcal{B}$ that is not strictly linear, that would represent a clear detection of higher-order effects from the gravity theory. The practical value of the universal relations, then, is to identify combinations of observables that would give the most direct evidence that the measurements are probing beyond the weak-deflection limit.

The other reason to make repeated observations is of course to obtain more observables than unknowns, so the problem becomes overconstrained. In this case the data will not only determine the parameter values, but deter-

mine whether the PPN framework itself is an acceptable model.

C. Galactic microlensing

In conventional microlensing, a foreground star or compact object lenses a star in the galactic bulge. In what follows we quote $M_\bullet$ in units of the mass of the sun, and d_S in units of 8 kpc: $M_\bullet^* = M_\bullet / M_\odot$ and $d_S^* = d_S/(8 \text{ kpc})$. We consider a typical situation with the lens lying about halfway between the observer and source: $d_L \sim d_{LS} \sim d_S/2$. The angular gravitational radius is then $\vartheta_\bullet \sim 2.5 \times 10^{-12} \times (M_\bullet^*/d_S^*)$ arc sec, the angular Einstein radius is $\vartheta_E \sim 10^{-3}(M_\bullet^*/d_S^*)^{1/2}$ arc sec, and the perturbation parameter is $\varepsilon \sim 2.4 \times 10^{-9} \times (M_\bullet^*/d_S^*)^{1/2}$.

Present microlensing programs only measure the total flux as a function of time (or equivalently source position). Equation (53) shows that there is no first-order correction to the total flux, so it is not feasible to test theories of gravity with microlensing at present. Future programs may be able to resolve the images (which will be separated by a few milli-arcseconds). But in order to test gravity theories they would need to measure image positions with a precision at the level of 10^{-12} arc sec, or fluxes with a fractional uncertainty of order $\varepsilon \sim 10^{-9}$. In other words, it is not reasonable to expect to test theories of gravity with conventional microlensing in the foreseeable future.

D. Pulsars in binary systems

Hopes for using stellar-mass lenses to test theories of gravity are not lost. The amplitudes of the correction terms are governed by

$$\varepsilon = \left(\frac{GM_\bullet}{4c^2} \frac{d_S}{d_L d_{LS}} \right)^{1/2}, \tag{57}$$

so we may be able to use stellar-mass lenses if we can find systems where d_{LS} is sufficiently small. The ideal system would be a pulsar in a binary system with a compact object (another pulsar, or a black hole), in an orbit seen nearly edge on. An example of such a system was recently discovered: the binary pulsar J0737-3039 [33]. Rafikov and Lai [34] have made detailed calculations of various effects on the pulsar timing measurements, including not only multiple imaging but also relativistic aberration and latitudinal delays associated with the spin of the source. We use this system more generally to be representative of binary systems consisting of a pulsar and a compact object, and to illustrate the amplitude of lensing effects associated with different theories of gravity.

In J0737-3039, we take the fast millisecond pulsar to be the light source, and the slow pulsar with $M_\bullet = 1.25 M_\odot$ to be the lens. The binary orbit has a semimajor axis $a = 8.78 \times 10^5$ km. The orbital eccentricity is fairly small ($e = 0.088$), so for illustration purposes take $d_{LS} = a$. The lens gravitational radius is then $\mathsf{m}_\bullet = 1.8$ km, so

the lensing time scale is $\tau_E = 2.5 \times 10^{-5}$ s. Using $d_L \approx d_S$, the physical Einstein radius is $d_L \vartheta_E \approx (4GM_\bullet d_{LS}/c^2)^{1/2} = 2.5 \times 10^3$ km. (The angular Einstein radius cannot be determined because the distance to the lens is not known.) The perturbation parameter is $\varepsilon = 7.2 \times 10^{-4}$.

The two images could not be resolved spatially. They could be resolved temporally, though: when the source is behind the lens, each radio pulse would actually consist of two pulses (one from each image) separated by the lens time delay (see [34]). The amplitudes of the two pulses could be measured; and the intrinsic pulse amplitude could be measured when the source is not behind the lens. Thus, in this system the observables would be F^+, F^-, and $\Delta \tau$ as a function of source position $\mathcal{B}$, as well as $F_{\rm src}$. Should one wish to go further, analysis of the source's orbital motion would yield a prediction for the pulse arrival time in the absence of lensing and make it possible to measure the time delays $\tau^\pm$ for the two images separately.

In this scenario, the universal relations indicate that for testing gravity theories the most valuable measurement would be the flux difference $\Delta F = F^+ - F^-$ as a function of source position. From (54), this quantity is constant in the weak-deflection limit. Thus, any variation in ΔF with source position would reveal that the measurements are probing beyond the weak-deflection limit. Again we see the universal relations helping us identify combinations of observables that are best suited for testing gravity theories.

V. CONCLUSIONS

Using gravitational lensing by compact objects, we have presented new prospects for testing theories of gravity within the PPN framework. In this paper we generalized the PPN lensing formalism from Paper I to include fully general third-order PPN models. We determined the weak-deflection limits plus first- and second-order corrections in $\varepsilon = \vartheta_\bullet / \vartheta_E$ for observable properties of lensed images (positions, magnifications, and time delays).

During the PPN analysis, we discovered some surprising new fundamental relations between lensing observables. Some of the relations are universal for the entire family of PPN gravity models. A deep conceptual implication is that any observed violation of the universal lensing relations (given that assumptions [A1]–[A3] apply) would indicate that a fundamentally different theory of gravity is at work—one outside the PPN framework. The new relations have enabled us to identify combinations of lensing observables that are key to probing the spacetime metric by constraining the invariant PPN parameters of the light bending angle. The parameter A_2 is related to the existence of naked singularities in certain gravity models (see Paper I), so constraining A_2 also provides a possible observational test of the cosmic censorship conjecture. The new lensing relations will, in other words, play important roles in planning observing missions to test theories of gravity.

In a practical application, we identified lensing observables that are accessible to current or near-future instrumentation, considering three likely lensing scenarios: the galactic black hole, galactic microlensing, and the binary pulsar J0737-3039. A noted application of the new lensing relations is the ability to find combinations of observables that will yield a direct method for knowing when observations are probing beyond the standard weak-deflection regime.

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