Second-order perturbations of a zero-pressure cosmological medium: Comoving versus synchronous gauge

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Except for the presence of gravitational wave source term, the relativistic perturbation equations of a zero-pressure irrotational fluid in a flat Friedmann world model coincide exactly with the Newtonian ones to the second order in perturbations. Such a relativistic-Newtonian correspondence is available in a special gauge condition (the comoving gauge) in which all the variables are equivalently gauge invariant. In this work we compare our results with the ones in the synchronous gauge which has been used often in the literature. Although the final equations look simpler in the synchronous gauge, the variables have remnant gauge modes. Except for the presence of the gauge mode for the perturbed-order variables, however, the equations in the synchronous gauge are gauge invariant and can be exactly identified as the Newtonian hydrodynamic equations in the Lagrangian frame. In this regard, the relativistic equations to the second order in the comoving gauge are the same as the Newtonian hydrodynamic equations in the Eulerian frame. We resolve several issues related to the two gauge conditions often to fully nonlinear orders in perturbations.

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I. INTRODUCTION

The general relativistic cosmological linear perturbation theory was first developed by Lifshitz in 1946 [1]. Lifshitz took the synchronous gauge condition in which the perturbations of the time-time part and the space-time part of the metric tensor are equal to zeros; this gauge condition can be taken to fully nonlinear order without losing any physical degree of freedom [2]. The synchronous gauge condition has been popular in the cosmological perturbation literature despite the complicating fact that, except for the zero-pressure case, there are remnant gauge modes for both the spatial and temporal gauge conditions. There exist other spatial and temporal gauge conditions which fix the gauge transformation property completely in general situation, thus without any remaining gauge mode [3–5]. This point was clarified by Bardeen [6,7]. In a zero-pressure medium the density perturbation equation in the synchronous gauge coincides with the one in the comoving gauge [1,5]. The density perturbation equation in the comoving gauge condition is known to resemble the Newtonian equation most closely [5,6], and the equations coincide in the zero-pressure case [1,8]. Thus, in the zero-pressure case the density perturbation equation in the synchronous gauge coincides with the Newtonian one to the linear order [1,8].

The synchronous gauge was also used in the nonlinear perturbation studies [9], and Kasai [10] has derived second-order differential equations for density perturbation which is valid to fully nonlinear order. Although, such an equation in the synchronous gauge naturally has proper linear limit which corresponds to the Newtonian equation, it has been unclear whether such a correspondence continues to the nonlinear situation. Recently, we have successfully shown an exact relativistic-Newtonian correspondence of scalar-type perturbations to the second order based on the comoving gauge [11–13]. In the zero-pressure case our comoving gauge condition differs from the conventional synchronous gauge in the spatial gauge condition. In this work we will investigate the case in the original synchronous gauge. We will show that although the equations in the synchronous gauge look simpler than the ones in the comoving gauge, the variables still have remaining (spurious) spatial gauge mode to the second order. The equations in the synchronous gauge, however, are gauge invariant and can be identified as the Newtonian hydrodynamic equations in the Lagrangian frame. Whereas, the equations in the comoving gauge can be identified as the Newtonian hydrodynamic equations in the Eulerian frame.

Results in Sec. II and the Appendices are valid to fully nonlinear order in perturbations, and unless mentioned otherwise results in the remaining sections are valid to the second order in perturbations. We closely follow notations used in [11–13]. We set $c \equiv 1$.

II. FULLY NONLINEAR PERTURBATIONS

The energy-conservation equation and the Raychaudhury equation give [12–14]

$$\dot{\tilde{\tilde{\mu}}} + \tilde{\mu}\,\tilde{\theta} = 0, \quad (1)$$

$$\dot{\tilde{\tilde{\theta}}} + \frac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - \tilde{\omega}^{ab}\tilde{\omega}_{ab} + 4\pi G\tilde{\mu} - \Lambda = 0, \quad (2)$$

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where $\tilde{\theta} \equiv \tilde{u}^a{}_{;a}$ is an expansion scalar based on a fluid four-vector $\tilde{u}_a$; $\tilde{\sigma}_{ab}$ and $\tilde{\omega}_{ab}$ are the shear and the rotation tensors based on $\tilde{u}_a$, respectively; tildes indicate the covariant quantities and the Latin indices indicate spacetime components. An overdot with tilde is a covariant derivative along the $\tilde{u}_a$ vector, e.g., $\tilde{\dot{\tilde{\mu}}} \equiv \tilde{\mu}_{,a}\tilde{u}^a$. By combining these equations we have

$$\left(\frac{\tilde{\dot{\tilde{\mu}}}}{\tilde{\mu}}\right)^{\tilde{\cdot}} - \frac{1}{3}\left(\frac{\tilde{\dot{\tilde{\mu}}}}{\tilde{\mu}}\right)^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + \tilde{\omega}^{ab}\tilde{\omega}_{ab} - 4\pi G\tilde{\mu} + \Lambda = 0. \tag{3}$$

Equations (1)–(3) are fully nonlinear and covariant, thus valid to all orders in perturbations.

A. Temporal Comoving Gauge

In this work we will *assume* an irrotational fluid, thus $\tilde{\omega}_{ab} \equiv 0$. We will consider two different gauge conditions. In both gauge conditions we will have

$$\tilde{u}_{\alpha} = 0, \tag{4}$$

due to a common temporal gauge condition together with the irrotational condition; the Greek indices indicate space components. If we introduce the spatial part of the four-vector as

$$\tilde{u}_{\alpha} \equiv a(-\hat{v}_{,\alpha} + \hat{v}^{(v)}_{\alpha}), \tag{5}$$

where $\hat{v}^{(v)}_{\alpha}$ is a vector-type perturbation (thus transverse), the irrotational condition sets $\hat{v}^{(v)}_{\alpha} \equiv 0$ and our temporal comoving gauge sets $\hat{v} \equiv 0$. Since $\tilde{u}_{\alpha} = 0$ the fluid four-vector in this gauge coincides with the *normal* frame four-vector $\tilde{n}_a$ with $\tilde{n}_{\alpha} \equiv 0$. Notice that our temporal comoving gauge condition $\hat{v} \equiv 0$ (together with the irrotational condition) implies $\tilde{u}_{\alpha} = 0$. This *differs* from the ordinarily known *comoving* frame condition which sets $\tilde{u}^{\alpha} \equiv 0$ [15]. In our case the normalized ($\tilde{u}^a\tilde{u}_a \equiv -1$) fluid four-vector $\tilde{u}_a$ becomes

$$\tilde{u}_0 = -\frac{1}{\sqrt{-\tilde{g}^{00}}}, \qquad \tilde{u}_{\alpha} \equiv 0;$$
$$\tilde{u}^0 = \sqrt{-\tilde{g}^{00}}, \qquad \tilde{u}^{\alpha} = -\frac{\tilde{g}^{0\alpha}}{\sqrt{-\tilde{g}^{00}}}. \tag{6}$$

In the zero-pressure case the momentum conservation equation implies $\tilde{g}^{00} = -1/a^2$ where a is the cosmic scale factor of the Friedmann background world model. In the ADM approach [16], our temporal comoving gauge $\hat{v} = 0$ together with the irrotational condition implies vanishing momentum vector $J_{\alpha} \equiv -\tilde{n}_b\tilde{T}^b_{\alpha} = 0$. The ADM momentum conservation equation in Eq. (13) of [11] gives $N_{,\alpha} = 0$ where $\tilde{g}^{00} \equiv -1/N^2$, thus $N = N(t)$. In another way, since the acceleration vector $\tilde{a}_{\alpha} \equiv \tilde{u}_{\alpha;b}\tilde{u}^b = (\ln N)_{,\alpha}$ vanishes (i.e., geodesic flow) for the zero-pressure irrotational flow, we have $N = N(t)$; see Eqs. (27) and (42) of [11]. Without losing generality we can set $N = a(t)$. Thus we have

$$\tilde{g}^{00} = -\frac{1}{a^2}. \tag{7}$$

Thus, Eq. (6) becomes

$$\tilde{u}_0 = -a, \qquad \tilde{u}_{\alpha} \equiv 0; \qquad \tilde{u}^0 = \frac{1}{a}, \qquad \tilde{u}^{\alpha} = -a\tilde{g}^{0\alpha}, \tag{8}$$

which is valid to fully nonlinear order. We can show that to all orders in perturbations the fluid quantities are independent of the spatial gauge condition which could affect $\tilde{g}^{0\alpha}$; see Appendix A.

B. Nonlinear perturbed equations

We introduce perturbations

$$\tilde{\mu} \equiv \mu + \delta\mu, \qquad \tilde{\theta} \equiv 3H - \kappa, \tag{9}$$

where $H \equiv \dot{a}/a$ and $\delta \equiv \delta\mu/\mu$; an overdot denotes a time derivative based on background proper-time t. The $\tilde{\theta}$ is an expansion scalar of the fluid four-vector which is the same as the normal four-vector because $\tilde{u}_{\alpha} = 0$ in our case. Using Eq. (8) we have

$$\tilde{\dot{\tilde{\mu}}} = \dot{\mu}(1+\delta) + \mu\left(\dot{\delta} - \frac{1}{a}N^{\alpha}\delta_{,\alpha}\right),$$
$$\tilde{\dot{\tilde{\theta}}} = 3\dot{H} - \left(\dot{\kappa} - \frac{1}{a}N^{\alpha}\kappa_{,\alpha}\right), \tag{10}$$

where N^{α} is the shift vector in the ADM notation with $N^{\alpha} \equiv a^2\tilde{g}^{0\alpha}$; the spatial indices of the ADM variables are based on $h_{\alpha\beta} \equiv \tilde{g}_{\alpha\beta}$.

Eqs. (1) and (2) give

$$(\dot{\mu} + 3H\mu)(1+\delta) + \mu\left[\dot{\delta} - \frac{1}{a}\delta_{,\alpha}N^{\alpha} - (1+\delta)\kappa\right] = 0, \tag{11}$$

$$3(\dot{H} + H^2) + 4\pi G\mu - \Lambda - \left[\dot{\kappa} - \frac{1}{a}\kappa_{,\alpha}N^{\alpha} + 2H\kappa - 4\pi G\mu\delta - \frac{1}{3}\kappa^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab}\right] = 0. \tag{12}$$

The background parts give

$$\dot{\mu} + 3H\mu = 0, \qquad 3(\dot{H} + H^2) + 4\pi G\mu - \Lambda = 0. \tag{13}$$

The perturbed parts give

$$\hat{\dot{\delta}} = (1+\delta)\kappa, \tag{14}$$

$$\hat{\dot{\kappa}} + 2H\kappa = \frac{1}{3}\kappa^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + 4\pi G\mu\delta, \tag{15}$$

where $\hat{\dot{\delta}} \equiv \dot{\delta} - a^{-1}\delta_{,\alpha}N^\alpha$. By combining these equations we have

$$\hat{\dot{\hat{\dot{\delta}}}} + 2H\hat{\dot{\delta}} - 4\pi G\mu\delta = 4\pi G\mu\delta^2 + \frac{4}{3}\frac{(\hat{\dot{\delta}})^2}{1+\delta} + (1+\delta)\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}. \quad (16)$$

These equations are valid to the fully nonlinear orders in perturbations, subject only to the temporal comoving gauge condition, the zero-pressure condition, and the irrotational condition.

C. The synchronous gauge

Under the synchronous gauge we set $\tilde{g}_{0\alpha} \equiv 0$ (thus $N^\alpha \equiv 0$) using the spatial gauge condition (together with the irrotational condition), thus

$$\tilde{\dot{\tilde{\mu}}} = \hat{\dot{\hat{\mu}}} = \dot{\mu}. \quad (17)$$

Thus, Eqs. (14)–(16) simply give

$$\dot{\delta} = (1+\delta)\kappa, \quad (18)$$

$$\dot{\kappa} + 2H\kappa = \frac{1}{3}\kappa^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + 4\pi G\mu\delta, \quad (19)$$

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\mu\delta = 4\pi G\mu\delta^2 + \frac{4}{3}\frac{\dot{\delta}^2}{1+\delta} + (1+\delta)\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}, \quad (20)$$

which are valid to the fully nonlinear order. Using $\Delta \equiv \delta/(1+\delta)$ Kasai [10] has derived

$$\ddot{\Delta} + 2H\dot{\Delta} - 4\pi G\mu\Delta = -\frac{2}{3}\frac{\dot{\Delta}^2}{1-\Delta} + (1-\Delta)\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}. \quad (21)$$

To nonlinear order in perturbations the above equations are incomplete yet because of $\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}$ term. Later we will show that these equations in the synchronous gauge differ from the equations in the comoving gauge to the second order. Furthermore, although these equations look simple, we will show that δ (thus Δ as well) and κ still have remnant gauge modes to the second order. In Sec. III B we will show that to the second order the equations are gauge invariant and can be identified with the Newtonian hydrodynamic equations in the Lagrangian frame. In this regard, the equations in the comoving gauge correspond to the Newtonian hydrodynamic equations in the Eulerian frame.

III. SECOND-ORDER PERTURBATIONS

As the metric we take

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2\beta_{,\alpha}d\eta dx^\alpha + a^2[g^{(3)}_{\alpha\beta}(1+2\varphi) + 2\gamma_{,\alpha|\beta} + 2C^{(t)}_{\alpha\beta}]dx^\alpha dx^\beta, \quad (22)$$

where α, β, γ and φ are spacetime dependent perturbed-order variables, and $C^{(t)}_{\alpha\beta}$ is a transverse and tracefree perturbed-order variable. Spatial indices of perturbed-order variables are based on $g^{(3)}_{\alpha\beta}$, and a vertical bar indicates the covariant derivative based on $g^{(3)}_{\alpha\beta}$; $g^{(3)}_{\alpha\beta}$ could become $\delta_{\alpha\beta}$ in a flat Friedmann background. We *ignored* the transverse vector-type perturbation variables. We introduce $\chi \equiv a(\beta + a\dot{\gamma})$. The perturbed variables can be regarded as nonlinearly perturbed ones to any order in perturbations.

To the second order, from Eqs. (55) and (57) of [11] we have

$$N^\alpha = -\beta^{,\alpha},$$

$$\tilde{\sigma}^{ab}\tilde{\sigma}_{ab} = \bar{K}^\alpha_\beta\bar{K}^\beta_\alpha = \frac{1}{a^4}\left[\chi^{,\alpha|\beta}\chi_{,\alpha|\beta} - \frac{1}{3}(\Delta\chi)^2\right] + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a^2}\chi_{,\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right), \quad (23)$$

where N^α is evaluated to the linear order; $\bar{K}^\alpha_\beta$ is a tracefree part of the extrinsic curvature. We note that $\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}$ is spatially gauge invariant to the second order, see Sec. IV.

Before comparing equations in the two different spatial gauges, we *compare* our $\hat{v}$ in Eq. (5) with the notation used in [11] to the second order in perturbations. In [11] we introduced the fluid quantities based on the normal-frame vector $\tilde{n}_a$ and provided the relation of fluid quantities between the energy-frame (E) and the normal-frame (N). Our fluid four-vector $\tilde{u}_a$ is based on the energy frame which sets $\tilde{q}_a \equiv 0$. The energy-frame fluid four-vector is introduced in Eq. (53) of [11], and using the relations given in Eqs. (87) and (88) of [11] we have

$$\tilde{u}_\alpha = a(V^E_\alpha - B_\alpha + AB_\alpha + 2V^\beta_E C_{\alpha\beta}) = a\left\{\frac{Q^N_\alpha}{\mu+p} - \frac{1}{(\mu+p)^2}[(\delta\mu+\delta p)Q^N_\alpha + Q^\beta_N\Pi_{\alpha\beta}]\right\}. \quad (24)$$

Using the decomposition of the normal-frame flux vector $Q^N_\alpha \equiv (\mu+p)(-v_{,\alpha} + v^{(v)}_\alpha)$ in Eq. (175) of [11] and setting $v^{(v)}_\alpha \equiv 0$ we have

$$\hat{v}_{,\alpha} = v_{,\alpha} - \frac{1}{\mu+p}[(\delta\mu+\delta p)v_{,\alpha} + v^{,\beta}\Pi_{\alpha\beta}]. \quad (25)$$

Thus, the temporal comoving gauge $v \equiv 0$ in [11] implies $\hat{v} = 0$ and vice versa.

A. The comoving gauge

In [11–13] we took the temporal comoving gauge and the spatial $\gamma = 0$ gauge

$$v \equiv 0, \qquad \gamma \equiv 0. \quad (26)$$

In this work, we call this the *comoving* gauge. Thus, we have $\beta = \chi/a$.

The momentum conservation equation in Eq. (105) of [11] gives

$$\alpha = -\frac{1}{2a^2}\chi^{,\beta}\chi_{,\beta}. \tag{27}$$

Thus, apparently, α does not vanish to the second order. Later we will show that if we take $\beta = 0$ as the spatial gauge condition instead of $\gamma = 0$, we have vanishing α. However, we prefer $\gamma \equiv 0$ as the spatial gauge condition because it fixes the spatial gauge degree of freedom completely (as long as we simultaneously take the temporal gauge which removes the temporal gauge degree of freedom completely, like our $v = 0$), see Sec. VI of [11]. Whereas, $\beta \equiv 0$ fails to fix the spatial gauge degree of freedom completely, thus having remaining gauge degree of freedom even after imposing the gauge condition, see Sec. IV B.

In our gauge the fluid four-vector in Eq. (8) becomes

$$\tilde{u}_0 = -a, \qquad \tilde{u}_\alpha = 0; \qquad \tilde{u}^0 = \frac{1}{a}, \qquad \tilde{u}^\alpha = \frac{1}{a^2}\chi^{,\beta}[(1-2\varphi)\delta^\alpha_\beta - 2C^{(t)\alpha}_\beta]. \tag{28}$$

Thus, although we prefer to call this the temporal comoving gauge (see [6,7]), because $\tilde{u}_\alpha = 0$ and $\tilde{u}^\alpha \neq 0$, our fluid four-vector corresponds to the normal four-vector rather than the comoving one.

Using Eq. (23), Eqs. (14) and (15) give

$$\dot{\delta} + \frac{1}{a^2}\delta_{,\alpha}\chi^{,\alpha} - \kappa = \delta\kappa, \tag{29}$$

$$\dot{\kappa} + \frac{1}{a^2}\kappa_{,\alpha}\chi^{,\alpha} + 2H\kappa - 4\pi G\mu\delta = \left(\frac{1}{a^2}\chi^{,\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right)\left(\frac{1}{a^2}\chi_{,\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{30}$$

These also follow from the energy-conservation equation and the trace part of ADM propagation equation in Eqs. (104), (102) of [11].

In [12,13] we *identified* to the second order

$$\delta\mu \equiv \delta\varrho, \qquad \kappa \equiv -\frac{1}{a}\nabla\cdot\mathbf{u}, \tag{31}$$

where $\delta\varrho$ and $\mathbf{u}$ are Newtonian density and velocity perturbations, respectively. As we ignore the rotational mode, the velocity is of potential type with $\mathbf{u} = \nabla u$. Apparently, we need χ to the linear order only, and to that order we have [12,13]

$$\nabla\chi = a\mathbf{u}, \tag{32}$$

where we *assume* a flat Friedmann background world model. With these identifications of the relativistic metric and energy-momentum perturbation variables (these are equivalently gauge-invariant combinations, see Sec. IV B) with the Newtonian hydrodynamic variables, Eqs. (29) and (30) give

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}), \tag{33}$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\mu\delta = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}\nabla_\beta u_\alpha + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{34}$$

By combining these we have

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}[a\nabla\cdot(\delta\mathbf{u})] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}\nabla_\beta u_\alpha + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{35}$$

which also follows from Eq. (16). Except for the presence of the gravitational waves as source terms Eqs. (33)–(35) are valid *exactly* in the Newtonian system [17].

Although our relativistic equations are valid to the second order, the Newtonian equations are valid to fully nonlinear order. Thus, all nonvanishing higher-order perturbation terms in the relativistic case are pure general relativistic corrections. Recently, we have presented such pure general relativistic correction terms appearing in the third order perturbations in [18].

B. The synchronous gauge

The synchronous and comoving gauge conditions correspond to taking [2]

$$v \equiv 0, \qquad \beta \equiv 0; \qquad \alpha = 0. \tag{36}$$

In this work, we call this simply the *synchronous* gauge. Thus, we have $\dot{\gamma} = \chi/a^2$. If we take $v \equiv 0$ and $\beta \equiv 0$ as the temporal and the spatial gauge conditions, respectively, the momentum conservation equation gives $\alpha = 0$ to *all* orders in perturbations; although this is well known in [2], we give proofs in the Appendix B . Thus, in the zero-pressure medium without rotation we can simultaneously impose the comoving ($v = 0$) and the synchronous ($\alpha = 0$) temporal gauge conditions as long as we also take $\beta \equiv 0$ as the spatial gauge condition [2]; Kasai took these conditions in his work in [10].

The original synchronous gauge used by Lifshitz [1] took $\alpha = 0$ and $\beta = 0$ as the temporal and the spatial gauge conditions, respectively. These gauge conditions are known to be *incomplete* in fixing both the temporal and the spatial gauge modes even to the linear order. Thus, even after imposing these gauge conditions we have remaining gauge modes present in the solutions, in general. Meanwhile, $v \equiv 0$ and $\gamma \equiv 0$ fix the temporal and spatial gauge degree of freedoms completely, thus no gauge mode is present in the solution, see Sec. IV. Since the original synchronous gauge implies $v = 0$ (the nonvanishing solution of v is the remnant temporal gauge mode) in the zero-pressure case, we only have to pay attention to the possible presence of the spatial gauge mode. In this gauge we have $\tilde{g}_{00} = -a^2 = 1/\tilde{g}^{00}$ and $\tilde{g}_{0\alpha} = 0 = \tilde{g}^{0\alpha}$. Thus, the fluid four-vector in Eq. (8) becomes

$$\tilde{u}_0 = -a, \qquad \tilde{u}_\alpha = 0; \qquad \tilde{u}^0 = \frac{1}{a}, \qquad \tilde{u}^\alpha = 0, \quad (37)$$

which can be compared with Eq. (28) in the comoving gauge. Thus, since $\tilde{u}^\alpha = 0$, our fluid four-vector corresponds to the conventionally known comoving four-vector [15], and simultaneously normal because $\tilde{u}_\alpha = 0$ as well. All the statements in the above two paragraphs are valid for all perturbational orders.

Using Eq. (23), Eqs. (18) and (19) give

$$\dot{\delta} - \kappa = \delta\kappa, \quad (38)$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\mu\delta = \left(\frac{1}{a^2}\chi^{,\alpha\beta} + \dot{C}^{(t)\alpha\beta}\right) \times \left(\frac{1}{a^2}\chi_{,\alpha\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \quad (39)$$

These also follow from the energy-conservation equation and the trace part of ADM propagation equation in Eqs. (104), (102) of [11]. By combining these equations we have

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\mu\delta = \dot{\delta}^2 + 4\pi G\mu\delta^2 + \left(\frac{1}{a^2}\chi^{,\alpha\beta} + \dot{C}^{(t)\alpha\beta}\right) \times \left(\frac{1}{a^2}\chi_{,\alpha\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \quad (40)$$

Apparently, these equations in the synchronous gauge look simpler than Eqs. (29) and (30) in the comoving gauge. Compared with Eqs. (29) and (30) in the comoving gauge, in Eqs. (38) and (39) we lack the convective-derivative-like terms in the left-hand-sides. By changing the time derivatives as

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \frac{1}{a^2}(\nabla\chi)\cdot\nabla = \frac{\partial}{\partial t} + \frac{1}{a}\mathbf{u}\cdot\nabla, \quad (41)$$

we can show that Eqs. (38)–(40) become the same ones in the comoving gauge in Eqs. (29), (30), and (35); in the last step of Eq. (41) we used Eq. (32) which is valid for a flat background.

If we make the same identification of the density and velocity perturbations as in Eqs. (31) and (32), thus assuming a flat background, Eqs. (38) and (39) become:

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\delta\nabla\cdot\mathbf{u}, \quad (42)$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\mu\delta = -\frac{1}{a^2}(\nabla^\beta u^\alpha)(\nabla_\beta u_\alpha) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}\nabla_\beta u_\alpha + \dot{C}^{(t)}_{\alpha\beta}\right). \quad (43)$$

Ignoring the gravitational waves, these equations can be identified as the Newtonian hydrodynamic equations in the Lagrangian frame.

Although equations in the synchronous gauge look simpler than the ones in the comoving gauge, the presence of additional convectivelike terms in the comoving gauge allows us to make exact (except for the gravitational waves) correspondence with the Newtonian hydrodynamic equations in the Eulerian frame [12,13]. Whereas, the equations in the synchronous gauge can be identified as the Newtonian equations in the Lagrangian frame. However, the variables in the synchronous gauge still have the remnant spatial gauge mode due to incomplete fixing nature of the spatial gauge condition $\beta \equiv 0$ in that gauge. That is, to the second order, δ and κ in the synchronous gauge have the remaining gauge modes, see Sec. IV B.

Now, we can relate the variables in the synchronous (S) gauge to the ones in the comoving (C) gauge. From Eqs. (59), (56), and (32) we have

$$\delta_S = \delta_C + \left(\int^t \frac{1}{a^2}\nabla\chi dt + \nabla\gamma_{S,\mathrm{Gauge}}\right)\cdot\nabla\delta_C,$$
$$\kappa_S = \kappa_C + \left(\int^t \frac{1}{a^2}\nabla\chi dt + \nabla\gamma_{S,\mathrm{Gauge}}\right)\cdot\nabla\kappa_C, \quad (44)$$

where $\gamma_{S,\mathrm{Gauge}}$ is the gauge mode present to the linear order in γ; see the next section. In a flat background, from Eq. (32) we have $\nabla\chi = a\mathbf{u}$. Notice that, even after ignoring the gauge modes δ_S and κ_S naturally differ from δ_C and κ_C, respectively, because the final equations are different. Using Eq. (44), Eqs. (38)–(40) give Eqs. (29), (30), and (35).

Although the variables in the synchronous gauge have remnant spatial gauge mode, somehow the equations in the synchronous gauge coincide with the Newtonian ones in the Lagrangian frame. Meanwhile, the Newtonian hydrodynamic equations have nothing to do with the gauge mode which appears only in the relativistic treatment. We can show that the situation is consistent in the synchronous gauge. From Eqs. (54) and (55) the gauge mode of $\delta_{S,\mathrm{Gauge}} = \xi^\alpha\nabla_\alpha\delta_C$ is proportional to the linear-order solution of δ_C; similarly, the gauge mode of $\kappa_{S,\mathrm{Gauge}} = \xi^\alpha\nabla_\alpha\kappa_C$ is proportional to the linear-order solution of κ_C. Thus, the behaviours of the gauge mode cannot be distinguished from the solutions to the linear order, and can be absorbed to the linear-order solutions. We can also check that the gauge modes in Eq. (44) cancel out in Eqs. (38) and (39). In this sense, Eqs. (38) and (39), thus Eqs. (40), (42), and (43) as well, are gauge-invariant.

Therefore, to the second order in the synchronous gauge, although the variables have remnant gauge mode, the equations are gauge invariant; this happens because the gauge mode temporally behaves exactly like one of the physical solutions.

A similar situation occurs to the linear order in the original synchronous gauge which took only $\alpha = 0 = \beta$ [1]. Under these gauge conditions Lifshitz derived

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\mu\delta = 0, \tag{45}$$

which is the LHS of Eq. (40) and concides with the later derived Newtonian equation [8]. However, under these gauge conditions (i.e., without taking $v = 0$), δ still have the remnant gauge mode due to the incomplete fixing nature of the temporal gauge condition $\alpha \equiv 0$. It happens that the temporal behavior of gauge mode of δ is proportional to H which *coincides* with one of the two physical solutions, see [19]. Thus, although δ has the gauge mode Eq. (45) is gauge invariant. In our synchronous gauge which also takes $v = 0$ the temporal gauge condition is fixed completely, but a similar situation repeats due to an incomplete fixing nature of the spatial synchronous gauge condition ($\beta \equiv 0$) now to the second order in perturbation.

IV. GAUGE ISSUE

A. Gauge transformation

Under a transformation between two coordinates $\hat{x}^a = x^a + \tilde{\xi}^a$, the gauge transformation properties of all metric and energy-momentum variables to the second order are presented in Sec. VI of [11].

Since both the synchronous gauge and the comoving gauge take $v = 0$ we have

$$\tilde{\xi}^0 = 0. \tag{46}$$

This follows from Eqs. (234) or (238) of [11]: in the normal frame by setting $Q_\alpha = 0$ (i.e., $v = 0$) in both gauges we have $\tilde{\xi}^0_{,\alpha} = 0$; or in the energy frame, by setting $V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta} = 0$ (i.e., $\hat{v} = 0$) in both gauges, we again have $\tilde{\xi}^0_{,\alpha} = 0$. Thus, without losing generality we can take $\tilde{\xi}^0 = 0$.

To the second order, with $\tilde{\xi}^0 \equiv 0$, from Eqs. (229), (232) of [11] we have

$$\hat{\alpha} = \alpha - \alpha_{,\alpha}\xi^\alpha - \beta_{,\alpha}\xi^{\alpha\prime} - \frac{1}{2}\xi^{\alpha\prime}\xi^\prime_\alpha, \quad \hat{\delta} = \delta - \delta_{,\alpha}\xi^\alpha, \quad \hat{\kappa} = \kappa - \kappa_{,\alpha}\xi^\alpha, \tag{47}$$

where $\xi^\alpha \equiv \tilde{\xi}^\alpha$ with the index of ξ^α based on $g^{(3)}_{\alpha\beta}$; a prime indicates the time derivative based on the conformal time η with $d\eta \equiv dx^0 \equiv dt/a$. The gauge transformation property of κ follows from the scalar nature of the expansion scalar $\tilde{\theta}$ with $\tilde{\theta} \equiv 3H - \kappa$ where $\tilde{\theta}$ is based on the normal frame; for the gauge transformation of a scalar quantity, see Eq. (239) of [11]. To the linear order, from Eq. (252) of [11] we have

$$\hat{\beta}_{,\alpha} = \beta_{,\alpha} + \xi^\prime_\alpha, \qquad \hat{\gamma}_{,\alpha} = \gamma_{,\alpha} - \xi_\alpha. \tag{48}$$

Thus, $\chi \equiv a(\beta + \gamma^\prime)$ is gauge invariant to the linear order, and

$$\alpha - \alpha_{,\alpha}\gamma^{,\alpha} + \frac{1}{2}\beta_{,\alpha}\beta^{,\alpha}, \qquad \delta - \delta_{,\alpha}\gamma^{,\alpha}, \qquad \kappa - \kappa_{,\alpha}\gamma^{,\alpha}, \tag{49}$$

$$\alpha - \alpha_{,\alpha}\gamma^{,\alpha} - \left(\beta + \frac{1}{2}\gamma^\prime\right)^{,\alpha}\gamma^\prime_{,\alpha}, \tag{50}$$

are gauge invariant to the second order.

B. Two gauges

In the comoving gauge, by imposing $\gamma \equiv 0$ in all coordinates (i.e., $\hat{\gamma} \equiv 0 \equiv \gamma$), from Eq. (48) we have

$$\xi_\alpha = 0. \tag{51}$$

Thus, the spatial gauge transformation property is fixed completely. From Eq. (47) we have

$$\hat{\alpha} = \alpha, \qquad \hat{\delta} = \delta, \qquad \hat{\kappa} = \kappa, \tag{52}$$

and each variable in this gauge has unique gauge-invariant counterpart as δ and κ in Eq. (49) and α in Eq. (50). Thus, we can equivalently regard all variables in this gauge as (spatially and temporally) gauge-invariant ones. For example, $\delta_{v,\gamma} \equiv \delta - \delta_{,\alpha}\gamma^{,\alpha}$ is a unique gauge-invariant combination which is the same as δ in the $v = 0 = \gamma$ gauge conditions; for an explicit form of $\delta_{v,\gamma}$ including v, see Eq. (282) in [11]. We note that these results (i.e., values remain the same in the comoving gauge conditions, complete fixing of the gauge degrees of freedom, and presence of unique corresponding gauge-invariant variables) continue to be valid even in higher-order perturbations, [11].

Whereas, in the synchronous gauge, by imposing $\beta \equiv 0$ in all coordinates (i.e., $\hat{\beta} \equiv 0 \equiv \beta$), from Eq. (48) we have

$$\xi^\prime_\alpha = 0. \tag{53}$$

Thus, even after imposing the gauge condition we have

$$\xi_\alpha = \xi_\alpha(\mathbf{x}), \tag{54}$$

which is the remaining gauge mode. Thus, under the synchronous gauge, from Eqs. (47), (48) we still have

$$\hat{\gamma}_{,\alpha} = \gamma_{,\alpha} - \xi_\alpha, \qquad \hat{\alpha} = \alpha - \alpha_{,\alpha}\xi^\alpha, \qquad \hat{\delta} = \delta - \delta_{,\alpha}\xi^\alpha, \qquad \hat{\kappa} = \kappa - \kappa_{,\alpha}\xi^\alpha, \tag{55}$$

where the transformation of γ is valid to the linear order. In this sense variables in the synchronous gauge have remaining gauge modes even after imposing the gauge condition. γ has the remaining spatial gauge mode even in the linear order, and the other variables have remaining gauge modes to the second order.

C. Transformation between the two gauges

Using the gauge transformation properties of the variables we can translate the equations and solutions in one gauge into the ones in another gauge condition. We indicate the comoving gauge and the synchronous gauge by subindices C and S, respectively. To the linear order we have

$$\beta_C = \gamma'_S = \frac{1}{a}\chi. \tag{56}$$

We present three different ways to reach the transformation properties.

First, we consider a transformation from the synchronous gauge (unhat) to the comoving gauge (hat). From Eq. (47) we have

$$\alpha_C = -\frac{1}{2}\xi^{\alpha\prime}\xi'_\alpha, \qquad \delta_C = \delta_S - \delta_{S,\alpha}\xi^\alpha, \qquad \kappa_C = \kappa_S - \kappa_{S,\alpha}\xi^\alpha. \tag{57}$$

We need to determine the gauge transformation function ξ^α, apparently, only to the linear order. From Eq. (48) we have

$$\xi_\alpha = \gamma_{S,\alpha}. \tag{58}$$

Thus, Eq. (57) becomes

$$\alpha_C = -\frac{1}{2a^2}\chi^{,\alpha}\chi_{,\alpha}, \qquad \delta_C = \delta_S - \delta_{S,\alpha}\gamma_S^{,\alpha}, \qquad \kappa_C = \kappa_S - \kappa_{S,\alpha}\gamma_S^{,\alpha}, \tag{59}$$

where we used Eq. (56).

Second, we consider a transformation from the comoving gauge (unhat) to the synchronous gauge (hat). From Eq. (47) we have

$$\alpha_C = \frac{1}{2}\xi^{\alpha\prime}\xi'_\alpha + \beta_{C,\alpha}\xi^{\alpha\prime}, \qquad \delta_S = \delta_C - \delta_{C,\alpha}\xi^\alpha, \qquad \kappa_S = \kappa_C - \kappa_{C,\alpha}\xi^\alpha. \tag{60}$$

From Eq. (48) we have

$$\xi_\alpha = -\gamma_{S,\alpha}. \tag{61}$$

Thus, Eq. (60) leads to the same results in Eq. (59).

Finally, the gauge-invariant combination in Eq. (49) provides a simpler derivation. From the gauge invariance of combinations in Eq. (49) we directly have Eq. (59).

Using these gauge transformation properties in Eq. (59) we can derive Eqs. (38) and (39) from Eqs. (29) and (30), and vice versa.

V. DISCUSSION

In this work we have compared the general relativistic weakly nonlinear cosmological perturbation equations in two different gauge conditions. In our previous works we have successfully shown that, except for the coupling with gravitational waves, the relativistic perturbation equations of a zero-pressure irrotational fluid coincide exactly with the Newtonian ones to the second order in perturbations. Such a relativistic-Newtonian correspondence was available in our special comoving gauge condition in which all the variables can be equivalently regarded as gauge-invariant ones. In this work we have compared these results with the ones in the synchronous gauge. The case in the synchronous gauge was previously studied without noticing the similarity or difference of the equations with the Newtonian ones to the nonlinear orders.

In this work we compared equations in the synchronous gauge with the ones in the comoving gauge and in the Newtonian case. Although the variables in this gauge have remnant spatial gauge modes due to the incomplete gauge fixing of the spatial gauge condition the equations are gauge invariant. Ignoring the gravitational waves, the equations in the synchronous gauge can be identified with the Newtonian hydrodynamic equations in the Lagrangian frame to the second order, whereas the equations in the comoving gauge can be identified as the Newtonian ones in the Eulerian frame. These Eulerian and Lagrangian correspondences can be understood because the fluid four-vector in our comoving gauge is in fact normal as in Eq. (28) whereas the four-vector in the synchronous gauge is both normal and comoving (thus Lagrangian) as in Eq. (37). In our way to clarify the case in the synchronous gauge we have addressed and resolved several issues related to the two gauge conditions often to fully nonlinear orders in perturbations.

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APPENDIX A: INVARIANCE OF FLUID QUANTITIES

Here, we *show* that the fluid quantities based on the fluid four-vector in Eq. (8) do not depend on the choice of $\tilde{g}^{0\alpha}$ (the spatial gauge condition) to all orders in perturbations. Using a fluid four-vector $\tilde{u}_a$ the energy-momentum tensor is decomposed into fluid quantities as [11,14]

$$\tilde{T}_{ab} \equiv \tilde{\mu}\tilde{u}_a\tilde{u}_b + \tilde{p}(\tilde{g}_{ab} + \tilde{u}_a\tilde{u}_b) + \tilde{q}_a\tilde{u}_b + \tilde{q}_b\tilde{u}_a + \tilde{\pi}_{ab}; \tag{A1}$$

$$\tilde{\mu} \equiv \tilde{T}_{ab}\tilde{u}^a\tilde{u}^b, \qquad \tilde{p} \equiv \frac{1}{3}\tilde{T}_{ab}\tilde{h}^{ab},$$
$$\tilde{q}_a \equiv -\tilde{T}_{cd}\tilde{u}^c\tilde{h}^d_a, \qquad \tilde{\pi}_{ab} \equiv \tilde{T}_{cd}\tilde{h}^c_a\tilde{h}^d_b - \tilde{p}\tilde{h}_{ab}, \tag{A2}$$

where $\tilde{h}_{ab} \equiv \tilde{g}_{ab} + \tilde{u}_a\tilde{u}_b$; we have $\tilde{u}^a\tilde{q}_a \equiv 0 \equiv \tilde{u}^a\tilde{\pi}_{ab}$, $\tilde{\pi}_{ab} \equiv \tilde{\pi}_{ba}$, and $\tilde{\pi}^a_a \equiv 0$. The variables $\tilde{\mu}$, $\tilde{p}$, $\tilde{q}_a$ and $\tilde{\pi}_{ab}$ are the energy density, the isotropic pressure (including the entropic one), the energy flux and the anisotropic pressure (stress) based on the fluid four-vector, respectively. Let us introduce another four-vector $\tilde{U}_a$ with

$$\tilde{U}_0 = -a, \qquad \tilde{U}_\alpha \equiv 0;$$
$$\tilde{U}^0 = \frac{1}{a}, \qquad \tilde{U}^\alpha = -a\tilde{g}^{0\alpha}_U. \tag{A3}$$

Thus, $\tilde{U}_a$ is subject to the same conditions as $\tilde{u}_a$ in Eq. (8), but with possibly different spatial gauge condition which could lead to $\tilde{g}^{0\alpha}_U \neq \tilde{g}^{0\alpha}$. The fluid quantities based on $\tilde{U}_a$ are similarly defined as in Eqs. (A1) and (A2) with $\tilde{U}_a$ replacing $\tilde{u}_a$:, for example, we have $\tilde{\mu}^U \equiv \tilde{T}_{ab}\tilde{U}^a\tilde{U}^b$, etc. We can easily show that if $\tilde{p} = \tilde{q}_a = \tilde{\pi}_{ab} = 0$ we have

$$\tilde{\mu}^U \equiv \tilde{T}_{ab}\tilde{U}^a\tilde{U}^b = \tilde{\mu}\tilde{u}_a\tilde{u}_b\tilde{U}^a\tilde{U}^b = \tilde{\mu}\tilde{u}_0\tilde{u}_0\tilde{U}^0\tilde{U}^0 = \tilde{\mu}, \tag{A4}$$

and $\tilde{p}^U = \tilde{q}^U_a = \tilde{\pi}^U_{ab} = 0$, and vice versa. This result is also valid to fully nonlinear order.

APPENDIX B: JUSTIFICATION OF EQ. (36)

Here, we *show* that in a zero-pressure irrotational medium we can take the original synchronous gauge ($\alpha \equiv 0 \equiv \beta$) together with the temporal comoving gauge ($v \equiv 0$) simultaneously to all orders in perturbations. This was known in [2], see Sec. 97 in [2]. Here, it is important to take $\beta \equiv 0$ as the spatial synchronous gauge although we prefer to take $\gamma \equiv 0$ as the spatial gauge condition because of the remnant gauge mode in the $\beta \equiv 0$ case. We provide two different proofs based on the ADM and the covariant formulations.

We begin by taking $v \equiv 0$ and $\beta \equiv 0$ as the temporal and spatial gauge conditions, respectively. In Eq. (7) we showed that $\tilde{g}^{00} = -1/N^2 = -1/a^2$. The spatial gauge condition $\beta = 0$ together with the irrotational condition implies $\tilde{g}_{0\alpha} \equiv N_\alpha = 0$. Thus, from Eq. (2) of [11] we have $\tilde{g}_{00} \equiv -a^2(1+2\alpha) = -N^2 = -a^2$. This implies that we have $\alpha = 0$.

Now, in the covariant approach, $v = 0$ and irrotational conditions imply $\tilde{u}_\alpha = 0$. Since $\tilde{u}_a$ is the fluid four-vector we take the energy frame, $\tilde{q}_a \equiv 0$. The momentum conservation equation in Eq. (27) of [11] gives $\tilde{a}_a = 0$. The spatial gauge condition $\beta = 0$ together with the irrotational condition implies $\tilde{g}_{0\alpha} = 0$, thus $\tilde{g}^{0\alpha} = 0$ as well. Since $\tilde{a}_\alpha \equiv \tilde{u}_{\alpha;b}\tilde{u}^b = \tilde{\Gamma}^0_{0\alpha} = \frac{1}{2}\tilde{g}^{00}\tilde{g}_{00,\alpha}$, $\tilde{a}_\alpha = 0$ implies that $\tilde{g}_{00}$ is a function of time only. Thus, we have $\alpha = 0$.

If we impose $\alpha \equiv 0$ and $\beta \equiv 0$ as the gauge condition, instead, we have nonvanishing J_α or $\tilde{u}_\alpha$, thus nonvanishing v. In a zero-pressure medium this nonvanishing v can be identified as the remnant temporal gauge mode, which can be set equal to zero without losing physical degree of freedom.

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